Toward a Quantitative General Equilibrium Asset Pricing Model with Intangible Capital

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Abstract

We model investment options as intangible capital in a production economy. In equilibrium, physical capital requires a substantially higher expected return than intangible capital. Quantitatively, our model rationalizes about 90% of the observed difference in the average return of book-to-market sorted portfolios (value premium). Our economy also produces (1) a high premium of the aggregate stock market over the risk-free interest rate; (2) a low and smooth risk-free interest rate; and (3) key features of the consumption and investment dynamics in the US data.

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Introduction

Historically, stocks with high book-to-market ratio, value stocks, earn a higher average return than those with low book-to-market ratio, i.e., growth stocks (Fama and French (1992 and 1995)). The difference in log units is approximately 4.3% per year and is known as the value premium. The book-to-market ratio of a firm is often viewed as a measure of the intensity of future growth options relative to assets currently in place. Interpreted this way, the empirical evidence on value premium suggests that the average spread between the return on physical assets in place and that on growth options is comparable to the aggregate stock market equity premium.

In this paper we propose a quantitative general equilibrium model with intangible capital. When calibrated to replicate standard statistics of the dynamics of macroeconomic quantities, our model is able to reproduce key features of asset returns data, including the difference in the average return on installed physical capital and future growth opportunities. Our model generates a high equity premium (5.66% per year for the market return, in log units) with moderate risk aversion (10) and a low and smooth risk-free interest rate. Our results are comparable to those obtained by the standard real business cycle (RBC) models in terms of the second moments of aggregate consumption, investment and hours worked. Furthermore, the expected annual log return on growth options is 4.08% lower than that on installed physical capital, about 90% of the observed value premium in the data.

We follow Ai (2009b) and model growth options as intangible capital in an otherwise standard neoclassical production economy. Growth options do not produce consumption goods directly, but represent an investment opportunity that allows its owner to build a new production unit using investment goods. This implies that the payoff of growth options is not directly linked to aggregate productivity shocks but is increasing in aggregate physical investment. High aggregate investment allows a greater fraction of growth options to be implemented, resulting in a higher payoff of claims to intangible capital. On the other hand, the payoff of physical capital is perfectly correlated with aggregate productivity shocks. This mechanism allows the payoff of growth options
and physical capital to depend on different risk factors, and hence to require different expected returns in equilibrium.

We make two major modifications to the Ai (2009b) model. First, we adopt recursive preferences and an aggregate productivity process with long-run risk as in Croce (2008). Recursive preferences and long-run risk allow us to generate a highly volatile pricing kernel. More importantly, we show that in our model physical capital endogenously has much higher exposure to long-run risk than intangible capital. The predictions of our model are thus consistent with the empirical evidence on the cross-section of equity returns. For example, Bansal, Dittmar, and Lundblad (2005); Hansen, Heaton, and Li (2008); Kiku (2006); and Koijen, Lustig, and Nieuwerburgh (2010) document that value stocks have a higher exposure to long-run economic activity risk than growth stocks. Our production-based model rationalizes these empirical findings, since, in equilibrium, tangible capital is endogenously more sensitive to variations in the predictable component of productivity growth than is intangible capital.

Second, focusing on US micro-data we document that the productivity of new vintages of capital is less sensitive to aggregate productivity shocks than that of older vintages. Based on this novel empirical finding, our model features heterogeneous productivity of vintage capital with young vintages having lower exposure to aggregate shocks, as in the data. Because of this element of our economy, the response of physical investment with respect to unexpected fluctuations in aggregate productivity (short-run shocks) is positive as in standard real business cycle (RBC) models, but negative with respect to news about future productivity shocks (long-run shocks). We demonstrate that our findings are crucial in understanding the high equity premium, the large spread between the return on growth options and assets in place, and the significant volatility of investment observed in the data.

In our setup, the elasticity of substitution between tangible investment and intangible capital is high, implying that the adjustment of tangible capital is not costly. Consequently, investment responds strongly to short-run shocks, as it does in standard RBC models. The response of invest-
ment to long-run shocks, however, is sluggish for two reasons. First, news shocks predict future productivity growth but do not affect current output. Because of consumption smoothing motives, the agent tends to avoid dramatic changes in investment as they imply fluctuations in consumption in the opposite direction. Second, since new investments are less exposed to aggregate shocks in their young age, their productivity is affected by news shocks only with a delay. The agent, therefore, finds it optimal to postpone the adjustment of investment with respect to this shocks. In equilibrium, after a long-run productivity shock, the price of physical capital responds immediately and sharply, whereas physical investment and the return on growth options do not. This feature of the model is novel and allows us to reproduce both the equity and the value premia observed in the data, while maintaining the appealing features of the traditional RBC models on the quantity side.

Our paper is related to the literature on real options and the cross-section of equity returns (see, for example, Berk, Green, and Naik (1999); Gomes, Kogan, and Zhang (2003); Carlson, Fisher, and Giammarino (2004); Cooper (2006); and Panageas, Garleanu, and Yu (2009)) and the literature on adjustment costs and value premium (Zhang (2005), Gala (2005)).

Our paper differs from the above literature along several dimensions. First, in our economy growth options are less risky than assets in place, whereas previous real options based models usually imply the exact opposite. The aforementioned papers explain the observed value premium by postulating that value firms are option intensive while growth firms are assets in place intensive. Empirical evidence, however, suggests that growth firms are option intensive. Typically, growth firms have higher R&D investment (Li and Liu (2010)) and higher capital expenditure to sales ratio (Da, Guo, and Jagannathan (2009)), two commonly used empirical proxies for firms’ growth opportunities. Growth firms also feature longer cash flow duration than value firms (for example, Dechow, Sloan, and Soliman (2004), Da (2006) and Santos and Veronesi(2009)), consistent with the interpretation that their assets consist mainly of options rather than installed physical capital. More recently, Kogan and Papanikolaou (2009) and Kogan and Papanikolaou (2010) provide direct
empirical evidence on the lower average return of growth options relative to assets in place. Our framework is consistent with the above empirical findings since in our economy assets in place have both higher returns and shorter duration than growth options.

Garleanu, Kogan, and Panageas (2009), Ai and Kiku (2009) and Ai (2009b) also presented general equilibrium models that features lower risk exposure of growth options relative to that of assets in place. None of the above papers study the quantitative implications of their model on the joint dynamics of asset prices and macroeconomic quantities as we do here.

Second, we work in general equilibrium and study the quantitative implications of our model on asset prices as well as the joint dynamics of consumption, investment, and hours worked. Many of the above papers present partial equilibrium models. Although Gomes, Kogan, and Zhang (2003) and Gala (2005) adopt a general equilibrium approach, they do not focus on standard real business cycle moments. Instead, we use the empirical evidence on the quantity side of the economy to discipline our model of the production technology and, therefore, its asset pricing implications. Our unified neoclassical framework combines the success of the real business cycle models on the quantity side with the success of long-run risk based models on the cross-section of equity returns obtained in endowment economies.

Finally, our model assumes a long-run component in productivity and endogenously generates a long-run component in consumption growth. We show that value stocks are more exposed to long-run shocks than growth assets. This feature of our model is consistent with the empirical evidence presented in Bansal, Dittmar, and Lundblad (2005); Hansen, Heaton, and Li (2008); and Kiku (2006).

Our paper builds on the literature on asset pricing in production economies that dates at least as far back as Brock and Mirman (1972). Rouwenhorst (1995) is among the first to recognize the difficulty of resolving the equity premium puzzle (Mehra and Prescott (1985)) in production economies even with extreme risk aversion. Recent attacks on this issue can be broadly classified into three classes: habit-based models (for example, Jermann (1998) and Boldrin, Christiano, and Fisher
(2001)), long-run risk–based models (for example, Croce (2008); Campanale, Castro, and Clementi (2008); Backus, Routledge, and Zin (2010); Ai (2009a); Kuehn (2009) and Kaltenbrunner and Lochstoer (2010)), and rare disaster–based models (for example, Gourio (2009)). Our work differs from the above papers in two significant ways. First, our model addresses simultaneously both the equity premium and the spread between tangible and intangible capital returns, whereas the aforementioned papers focus only on the equity premium. Second, the papers listed above typically rely on capital adjustment costs or other frictions in investment to generate variations in the price of physical capital. However, strong adjustment costs, although necessary to generate a sizeable equity premium, are often associated with either a counterfactually low volatility of investment or a counterfactually high volatility of the risk-free interest rate. Our model simultaneously produces a low volatility of the risk-free interest rate, a significant volatility of stock market returns, and a high volatility of investment, as in the data.


The remainder of the paper is organized as follows. We present the model and some analytical results in Sections I and II. In Section III, we provide empirical evidence on the lower risk exposure of new investments relative to physical capital of older vintages. We discuss the quantitative implications of the model in Section IV, and Section VI concludes. Proofs of the theorems and the
robustness analysis of the empirical results can be found in the Appendix.

I Model Setup

A Preferences

Time is discrete and infinite, \( t = 1, 2, 3, \ldots \). The representative agent has Kreps and Porteus (1978) preferences, as in Epstein and Zin (1989):

\[
V_t = \left\{(1 - \beta) u(C_t, N_t)^{1 - \frac{1}{\psi}} + \beta \left(E_t \left[V_{t+1}^{1 - \gamma}\right]\right)^{1 - \frac{1}{\psi}}\right\}^{\frac{1}{1 - \gamma}},
\]

where \( C_t \) and \( N_t \) denote, respectively, the total consumption and total hours worked at date \( t \).

For simplicity, we assume an inelastic labor supply and set \( u(C_t, N_t) = C_t \) when we present the model. We relax this assumption in the calibration part of the paper when we address the implications of our model on the dynamics of hours worked.

B Production Units

Consumption goods are perishable and are produced by production units of overlapping generations. Production units created at time \( \tau \) are called generation \( \tau \) production units and begin operation at time \( \tau + 1 \). Each generation \( \tau \) production unit uses labor, \( n^\tau \), as the only input of production. For \( t \geq \tau + 1 \), let \( A_t^\tau \) denote the time \( t \) productivity level common to all the production units belonging to generation \( \tau \). The total output of a generation \( \tau \) production unit at time \( t \), \( y_t^\tau \), is given by

\[
y_t^\tau = (A_t^\tau n_t^\tau)^{1 - \alpha}, \quad \forall t \geq \tau + 1.
\]

In each period, a production unit dies with probability \( \delta_K \) after its production activity is completed. The death shocks are i.i.d. among production units and across time. Labor market is competitive. Let \( w_t \) denote the real wage at time \( t \), the profit of a generation \( \tau \) production unit at time \( t \) is given
by:

$$\pi_t = \max_n \left\{ (A_t^\gamma n)^{1-\alpha} - w_t n \right\}.$$  

The productivity process for the initial generation of production units is denoted by $A_t$, the dynamics of which is given by:

$$\begin{align*}
\frac{A_{t+1}}{A_t} &= e^{\mu + x_t + \sigma_a \varepsilon_{a,t+1}}, \\
x_{t+1} &= \rho x_t + \sigma_x \varepsilon_x,t+1, \\
\varepsilon_{a,t+1} \varepsilon_{x,t+1} &\sim i.i.d. N\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \quad t = 0, 1, 2, \cdots.
\end{align*}$$

This specification follows Croce (2008) and captures long-run productivity risk. We further assume that production units are not exposed to aggregate risk during the first period of their lives. Specifically, the growth rate of the productivity of generation $t$ production units is given by

$$\frac{A_{t+j+1}}{A_{t+j}} = e^{\mu + x_t + \sigma_a \varepsilon_{a,t+1}}, \quad t = 1, 2, \cdots,$$

where $\phi_j = 1$ for $j = 1, 2, 3, \ldots$, and $\phi_0 = 0$. We also set $A_{t} = A_{t-1}$ in order to ensure that new production units are on average as productive as older ones.\(^{3}\)

Under the above specification, production units of all generations have the same unconditional expected growth rate. In the first period of its life, a newborn production unit is less exposed to aggregate shocks. From the second period on, its productivity grows at the same rate as all other production units of older generations. This feature of our model captures the empirical fact that exposure to aggregate productivity risk is increasing in firms age, as documented in Section III.

\(^{3}\)Generation $t$ production units are not active until period $t + 1$; therefore, the level of $A_{t}^\gamma$ does not affect the total production of the economy in period $t$. 7
C Aggregation of Production Units

As we show in Appendix A.1, our specification of the productivity process implies that the total output and profit of a generation $\tau$ production unit is $\varpi_{\tau+1}$ times those of the initial generation, where

$$\varpi_{t+1} = \left( \frac{A_t^\prime}{A_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} = e^{-\frac{1-\alpha}{\alpha}(x_t+\sigma_a\varepsilon_{a,t+1})} \text{ for all } t. \quad (1)$$

That is, in terms of total output, a generation-$t$ production unit is effectively $\varpi_{t+1}$ initial generation production units, where $\varpi_{t+1}$ adjust for productivity difference. It is convenient to measure production units of all generations in terms of their generation-zero equivalents. We use $M_t$ to denote the total measure of production units created at time $t$, and $K_t$ to denote the productivity adjusted total measure of production units. Accounting for the death shock, the law of motion of $K_t$ can be written as

$$K_1 = M_0, \quad K_{t+1} = (1 - \delta_K) K_t + \varpi_{t+1} M_t, \quad t = 1, 2, \cdots.$$

We show in Appendix A.1 that $K_t$ effectively summarizes the productivity difference of production units of all generations and can be interpreted a measure of installed physical capital of the economy. In fact, given $K_t$ and total labor input $N_t$, the aggregate production function is of the Cobb-Douglas form:

$$Y_t = K_t^\alpha (A_t N_t)^{1-\alpha}.$$

Our specification of the productivity process therefore provides a parsimonious way to incorporate the empirical fact that new investment projects have less exposure to aggregate productivity shocks than capital of older vintages, and at the same time, maintain tractability at the aggregate level.

For notational convenience, we use lower case without superscript to denote quantities associated with the initial generation production units. The equilibrium output and profit of generation-0
production units can be conveniently expressed as a function of aggregate quantities.

\[ y_t = K_t^{\alpha - 1} (A_t N_t)^{1 - \alpha}, \quad \pi_t = \alpha y_t. \]  
(2)

Intuitively, since \( K_t \) measures production units in generation-0 equivalent terms, the output and profit of initial generation production units can be obtained by dividing aggregate output and profit, respectively, by \( K_t \). We use \( p_{K,t} \) to denote the cum-dividend value of an initial generation production unit at time \( t \), and \( \Lambda_{t,t+j} \) denote the price an Arrow security that pays off one unit of consumption good on date \( t+j \) measured in date \( t \) consumption numerarie. That is, \( \{\Lambda_{t,t+j}\}_{t,j} \) is the pricing kernel of the economy. \( p_{K,t} \) is given by

\[ p_{K,t} = \alpha K_t^{\alpha - 1} (A_t N_t)^{1 - \alpha} + (1 - \delta_K) E_t [\Lambda_{t,t+1} p_{K,t+1}]. \]  
(3)

Since a generation \( t \) production unit starts operation in period \( t+1 \) and is equivalent to \( \varpi_{t+1} \) initial generation production unit, the value of a generation \( t \) production unit measured in date \( t \) consumption numerarie is \( E_t [\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1}] \).

D Blueprints

A new production unit can be constructed by implementing a blueprint. Blueprints differ in their quality. Higher quality blueprints are more efficient to implement. We use \( \theta_t \) to denote the quality of a blueprint at date \( t \) and assume that implementing a blueprint with quality \( \theta \) costs \( \frac{1}{\theta} \) units of general output. The owner of a blueprint can choose to implement it at any time. If not implemented immediately, a blueprint dies with probability \( \delta_S \). If we denote the value of a blueprint with quality \( \theta_t \) at time \( t \) as \( p_{S,t} (\theta_t) \), then

\[ p_{S,t} (\theta_t) = \max \left\{ E_t [\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1}] - \frac{1}{\theta_t}, \quad (1 - \delta_S) E_t [\Lambda_{t,t+1} p_{S,t+1} (\theta_{t+1})] \right\}. \]  
(4)
Intuitively, owners of blueprints face an optimal stopping problem. At any time $t$, they choose between implementing the blueprint in the current period or deferring implementation into the future. Implementing a blueprint immediately creates a generation $t$ production unit, the value of which is $E_t \left[ \Lambda_{t, t+1} \pi_{t+1} p_{K,t+1} \right]$. Waiting allows the owner of the blueprint to obtain a potentially better draw of $\theta_{t+1}$ in the next period. In our model, blueprints are effectively growth options that allow its owner to build a production unit at cost $\frac{1}{\theta_t}$. Intuitively, the optimal decision rule of blueprints is characterized by a threshold level $\theta^* (t)$ such that it is optimal to exercise the growth option if and only if $\theta_t > \theta^* (t)$.

In our model, production units are assets in place. They are tangible capital because creation of production units requires physical investment goods. Blueprints are growth options. They capture the key features of new innovations and new investment opportunities. They do not produce any consumption goods by themselves, but can be implemented to construct a productive asset at a cost. They are subject to substantial idiosyncratic risk ($\theta$), and only blueprints of higher quality are implemented. The stock of unimplemented blueprints is a form of capital because they can be stored and potentially utilized in the future to create a production unit. We use the term intangible capital because blueprints are mostly created by R&D related activities that are typically expensed rather than capitalized under US accounting rules. We will use the terminology blueprints and growth options, production units and assets in place interchangeably in the rest of the paper.

The timing of the cash flow of intangible and tangible capital are substantially different. Tangible capital is the claim to assets in place that produces consumption goods and pay them off to shareholders immediately as dividends. Intangible capital is the claim to growth options that do not generate any output right away, but will potential pay out dividend in the future after being exercised and turned into production units. In our setup, value stocks are tangible capital intensive and growth stocks are intangible capita intensive. Our notion of value and growth are therefore consistent with the empirical evidence on the negative relation between cash flow duration and book-to-market characteristics (for example, Dechow, Sloan, and Soliman (2004) and Da (2006)).
E  Aggregation of Blueprints

To avoid keeping track of the dynamics of the cross section distribution of the quality of blueprints as an infinite dimensional state variable, we assume $\theta_t$ are drawn from a continuous density $f$ at the beginning of each period and the random draw of $\theta$ is i.i.d. among blueprints and over time. As shown in Ai (2009b), in this case, the total measure of production units can be created at time $t$ depends on the total measure of all available blueprints, denoted $S_t$, and the total amount of investment good used to construct new production units, $I_t$:

$$M_t = G(I_t, S_t) = \max_{\theta^*} \left\{ S_t \times \int_{\theta^*}^{\infty} f(\theta) \, d\theta \right\}$$

subject to

$$S_t \times \int_{\theta^*}^{\infty} \frac{1}{\theta} f(\theta) \, d\theta \leq I_t.$$  

(5)

Intuitively, since the quality of blueprints is i.i.d., the law of large numbers implies that the density of the cross section distribution of $\theta$ is $f$. Optimal option exercise requires that all blueprints with quality above some threshold level $\theta^*$ are implemented. Since every implemented blueprint creates a production unit, the total amount of production units created equals the amount of all blueprints with quality higher than $\theta^*$, subject to the constraint that the total resources used to create production units is no more than $I_t$. Since shocks to the quality of blueprints are i.i.d. over time, the value of an ex ante identical blueprint at the beginning of period $t$, before its quality is revealed, denoted by $p_{S,t}$, is

$$p_{S,t} = \int_{0}^{\infty} p_{S,t}(\theta) f(\theta) \, d\theta.$$

In our setup, $K_t$ is a measure of the stock of physical capital, and $S_t$ measure the total stock of intangible capital in the economy. Given (5), the law of motion of $K_t$ can be written as

$$K_{t+1} = (1 - \delta_K) K_t + \varpi_{t+1} G(I_t, S_t).$$

(6)

Because creation of one production unit requires exactly one blueprint, $M_t = G(I_t, S_t)$ is also the
amount of blueprint implemented as time $t$. Let $J_t$ be the total amount of new blueprints created at time $t$, the law of motion of $S_t$ can therefore be written as

$$S_{t+1} = [S_t - G(I_t, S_t)](1 - \delta S) + J_t.$$  

(7)

Finally, we assume that general output can be transformed frictionlessly into consumption goods, investment goods used to build production units, or intangible investment goods used to develop new blueprints. The aggregate resource constraint in this economy is therefore:

$$C_t + I_t + J_t \leq K_t^\alpha (A_t N_t)^{1-\alpha}.$$  

(8)

Our modeling of intangible capital follows the general equilibrium setup in Ai (2009b) and differs from that in existing models in several aspects. First, unexercised growth options can be stored and potentially exercised in the future. The storability of growth options makes them a distinct type of capital from physical assets in place. The setup of Berk, Green, and Naik (1999) and Gomes, Kogan, and Zhang (2003), for example, assumes that options disappear immediately if not exercised. Second, the creation of new growth options in our model is endogenously determined by the optimal choice of the agent. Partial equilibrium based real options models typically assume exogenous arrival of growth options. Finally, intangible capital are growth options. They do not affect current production but represent investment opportunities that can be utilized in the future. This feature is important to address the asset pricing issues that we focus on in this paper. It captures the difference between the cash flow duration of value and growth assets and as we show in the next section, also allows their returns to be driven by different risk factors. The macroeconomic literature that focuses on the quantity dynamics of intangible capital typically models intangible capital by assuming that they affect output directly and in the same way as physical capital. For example, the aggregate production function in McGrattan and Prescott (2009a, 2009b) and Corrado, Hulten, and Sichel (2006) are of the form $Y_t = F(A_t, K_t, S_t, N_t)$, where $K_t$ is tangible.
capital and $S_t$ is the intangible capital. This specification implies that the payment to tangible and intangible capital have similar duration, and are both perfected conditionally correlated with aggregate productivity shocks and thus allow little room for difference in expected returns.

II Model Solution

A The Social Planner’s Problem

We consider a competitive equilibrium with complete markets in which claims to production units and blueprints are traded. The equilibrium allocation and prices can be constructed from the solution to the social planner’s problem that maximizes the representative agent’s utility:

$$V(K_t, S_t, x_t, A_t) = \max_{C_t, I_t, J_t \geq 0} \left\{ (1 - \beta) C_t^{1-\psi} + \beta \left( E \left[ V(K_{t+1}, S_{t+1}, x_{t+1}, A_{t+1})^{1-\gamma} \middle| x_{t+1}, A_{t+1} \right] \right)^{1/(1-\psi)} \right\}^{1/(1-\psi)},$$

subject to the resource constraint (8), the law of motion of the state variables $K_t$ and $S_t$ in (6) and (7), respectively. Despite the heterogeneity in the productivity of production units, and the quality of blueprints, our formulation of the social planner’s problem does not use the cross-section distribution of production units or blueprints as state variables. Our model therefore maintains the tractability of standard real business cycle models. At the same time, it allows us to study option exercise and the returns on physical and intangible capitals in general equilibrium. Our setup therefore provides a useful framework to link the finance literature that studies option exercise and the cross section of equity returns and the real business cycle literature that focus on macroeconomic quantity dynamics.

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4 Although the constraint (7) is nonconvex, the second welfare theorem holds in our model. We refer the reader to Ai (2009b) for a formal proof of the equivalence between the competitive equilibrium allocation and Pareto optimality, and the construction of asset prices from the solution to the social planner’s problem. Although Ai (2009b) works with time-additive preferences and does not have overlapping vintages of capital, his proof can be easily extended to our setup.
B Asset Prices

Given the equilibrium allocations, the stochastic discount factor of the economy can be represented by the ratio of marginal utilities at time $t$ and $t + 1$ under the recursive preference:

$$\Lambda_{t,t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{\frac{1}{\psi}} \left[ \frac{V_{t+1}}{(E_t \left[ V_{t+1}^{\gamma} \right])^{\frac{1}{1-\gamma}}} \right]^{\frac{1}{\psi} - \gamma}.$$  \hspace{1cm} (9)

The cum-dividend value of an initial generation asset in place at date $t$ can be expressed as a function of the partial derivatives of the value function of the social planner’s problem:

$$p_{K,t} = \frac{V_K (K_t, S_t, x_t, A_t)}{(1 - \beta) \left( C_t / V_t \right)^{-\frac{1}{\psi}}}. \hspace{1cm} (10)$$

Intuitively, $K_t$ is the total amount of physical capital in the economy measured in generation-0 production unit equivalents and $V_K (K_t, S_t, x_t, A_t)$ represents the marginal utility of an additional initial generation production unit to the planner. $V_K (K_t, S_t, x_t, A_t)$ normalized by the marginal utility of date-$t$ consumption, $(1 - \beta) \left( C_t / V_t \right)^{-\frac{1}{\psi}}$, is therefore the value of an initial generation production unit measured in date-$t$ consumption numeraire. Similarly, the value of an ex ante identical growth option at the beginning of period $t$, before the random draw of quality is

$$p_{S,t} = \frac{V_S (K_t, S_t, x_t, A_t)}{(1 - \beta) \left( C_t / V_t \right)^{-\frac{1}{\psi}}}. \hspace{1cm} (11)$$

Using (9), (10) and (11), the first order and envelope conditions of the social planner’s problem can be used to characterize the prices of growth options and assets in place. This is stated in the following proposition.

**Proposition 1** (*Equilibrium Conditions*)
The price of growth options and assets in place must satisfy

\[
E_t [\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1}] - \frac{1}{\theta^* (t)} = (1 - \delta_S) E_t [\Lambda_{t,t+1} p_{S,t+1} (\theta_{t+1})] \tag{12}
\]

and

\[
E_t [\Lambda_{t,t+1} p_{S,t+1} (\theta_{t+1})] = 1. \tag{13}
\]

Furthermore, the optimal option exercise threshold \( \theta^* (t) \) is given by:

\[
\theta^* (t) = G_I (I_t, S_t). \tag{14}
\]

**Proof.** See Ai (2009b). ■

Condition (12) is the optimality condition for option exercise. Note in equilibrium, all growth options with quality above \( \theta^* (t) \) is exercised, whereas those with quality below \( \theta^* (t) \) choose to wait until the next period. Therefore, a marginal growth option with quality \( \theta^* (t) \) must be indifferent between exercising the option immediately, and waiting to take action later. Exercising the option in the current period create a generation-\( t \) production unit, the value of which is \( E_t [\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1}] \). The left-hand side of equation (12) is therefore the net benefit of immediate option exercise. The right hand of equation (12) is the benefit of waiting. If the option is not exercise immediately, with probability \( (1 - \delta_S) \), the option survives and obtains a new draw of \( \theta_{t+1} \) in the next period. From the aggregate point of view, condition (12) is essentially the first order condition with respect to \( I_t \) in the social planner’s problem, and can be interpreted as the equalization of the marginal benefit of building an additional unit of installed physical capital \( (E_t [\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1}]) \), and its marginal cost \( (\varpi_{t+1} (t) + (1 - \delta_S) E_t [\Lambda_{t,t+1} p_{S,t+1} (\theta_{t+1})]) \).

Equation (13) implies that the value of an unexercised growth option at the end of period \( t \) after the realization of the death shocks is always 1. This is due to the nature of the production technology of new growth options. The production decision of new blueprints is determined before their quality are reviewed and one unit of consumption goods can always be transformed into one
unit of ex ante identical blueprint. Therefore, the value of a unit of ex ante identical growth option measured in current period consumption numeraire is always 1.

The last equation (14) is essentially an accounting identity that links the macro variables to the optimal option exercise decision at the micro level. From the social planner’s point of view, she choose the amount of physical investment $I_t$, to build new production units. $G_t(I_t, S_t)$ is the measure of new production units can be created with an additional amount of investment. From the micro perspective, a additional production unit costs $1/\theta^*(t)$ amount of physical investment goods as on the margin, blueprints with quality $\theta^*(t)$ are used to build new production units.

C Returns on Growth Options and Assets in Place

Note that the realized return on all existing assets in place are identical, as production units of all vintages differ in their exposure to aggregate productivity shocks only in the first period, and they face identical productivity growth rate in the future. We therefore focus on the realized return of the initial generation. It is convenient to denote

$$q_{K,t} = E_t [\Lambda_{t,t+1} p_{K,t+1}], \quad (15)$$

as the ex-dividend value of a survived initial generation production unit at the end of period $t$ after the realization of the death shock. By Equation (2), $\alpha K^\alpha (A_{t+1} N_{t+1})^{1-\alpha}$ is the profit earned by an initial generation asset in place on date $t+1$. Since an asset in place die with probability $\delta_K$ after production, the realized return on assets in place, denoted $r_K(t + 1)$ can be written as

$$r_K(t + 1) = \frac{\alpha K^\alpha (A_{t+1} N_{t+1})^{1-\alpha} + (1 - \delta_K) q_{K,t+1}}{q_{K,t}}. \quad (16)$$

Note that Equation (16) resemble the expression for return on physical capital in standard RBC models, as $\alpha K^\alpha (A_{t+1} N_{t+1})^{1-\alpha}$ can be interpreted as the marginal product of physical capital.

The realized return on growth options depends on the draw of $\theta$. Proposition 1 implies that the
price of an unexercised growth option at the end of period $t$ is 1. The realized return on holding a growth option at time $t + 1$ is therefore $p_{S,t+1}(\theta_{t+1})$. It is convenient to denote $r_{S,t+1}$ as the realized return on growth options before the realization of $\theta$:

$$
r_{S,t+1} = \int p_{S,t+1}(\theta) f(\theta) d\theta. \quad (17)$$

Conditions (4) and (13) imply

$$
\int p_{S,t+1}(\theta) f(\theta) d\theta = \int_{\theta^*(t)}^{\infty} \left\{ E_t[\Lambda_t \omega_{t+1} p_{K,t+1}] - \frac{1}{\theta} \right\} f(\theta) d\theta + (1 - \delta_S) \int_{0}^{\theta^*(t)} f(\theta) d\theta. \quad (18)
$$

Using the result of Proposition 1 and combining Equations (17) and (18), one can show

$$
r_{S,t+1} = \frac{G_S(I_{t+1}, S_{t+1})}{G_I(I_{t+1}, S_{t+1})} + (1 - \delta_S). \quad (19)
$$

Equation (19) provides a decomposition of option value into an in-the-money payoff and an out-of-the-money payoff component. By Equation (13), the value of an unexercised option is $(1 - \delta_S)$ after accounting for the death shock. The term $\frac{G_S(I_{t+1}, S_{t+1})}{G_I(I_{t+1}, S_{t+1})}$ can be interpreted as the expected payoff of an in-the-money option. Note that definition (5) implies that the aggregator $G(I, S)$ is concave and homogeneous of degree 1. Suppressing time subscripts for notational convenience, $\frac{G_S(I, S)}{G_I(I, S)}$ is an increasing function of $\frac{I}{S}$. Intuitively, concavity and homogeneity of $G$ imply that the optimal option exercise threshold $\theta^* = G_I(I, S)$ is strictly decreasing in $\frac{I}{S}$. A rise in $\frac{I}{S}$ lowers the option exercise threshold $\theta^*$ and increases the probability of growth options being exercised, therefore their payoff. From the social planner’s perspective, $\frac{G_S(I, S)}{G_I(I, S)}$ can be interpreted as the marginal product of intangible capital: $G_S(I, S)$ is the amount of new production units that can be produced by an additional growth option, and $G_I(I, S)^{-1} = \frac{1}{\theta^*}$ is the cost of a marginal

\[\text{Because shocks to } \theta \text{ is idiosyncratic, the risk premium on growth options is completely determined by the covariance of } r_{S,t+1} \text{ with the stochastic discount factor.}\]
production unit measured in current period consumption goods.

Equations (16) and (19) are the key to understanding the expected returns on tangible and intangible capital. Equation (16) implies, as is common in standard RBC models, the return on physical assets in place is an increasing function of productivity shocks. On the other hand, the return on growth options, depends on the endogenous choice of $I_{t+1}$. Our model thus allows the return on physical capital and intangible capital to depend on different risk factors, and consequently, to require different expected returns in equilibrium. In Section IV, we show that physical investment $I$ is not responsive to long-run productivity shocks, and as a result, return on intangible capital has little exposure to long-run risk, whereas physical capital is highly risky.

III Firms’ Exposure to Aggregate Risks

In our economy, new production units are less sensitive to aggregate productivity shocks than are old production units. In this section we provide empirical evidence supporting this feature of the model. A production unit in our model should be interpreted as any investment project generating cash flows. Because it is difficult to identify the productivity of individual projects within firms, we adopt an indirect approach. We hypothesize that younger firms have more new-vintage projects than older firms, and we show that the correlation between individual firms’ productivity growth and aggregate productivity growth is increasing in firm age.

In the empirical estimation, we assume that the production function at the firm level is Cobb-Douglas and allow the parameters of the production function to be industry-specific. The production function of firm $i$ in industry $j$ is given by

$$y_{i,j,t} = a_{i,j,t}k_{i,j,t}^{\alpha_1,j}n_{i,j,t}^{\alpha_2,j},$$

where $k_{i,j,t}$ and $n_{i,j,t}$ are, respectively, the capital and labor inputs of firm $i$ in industry $j$, and $a_{i,j,t}$
Table 1: Summary Statistics by Age Quantiles

<table>
<thead>
<tr>
<th>Age Quantile</th>
<th>Average Age</th>
<th>Median Age</th>
<th>Average $K_i/K_{T}^t$</th>
<th>St. Error $K_i/K_{T}^t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.32</td>
<td>07.00</td>
<td>0.07</td>
<td>0.03</td>
</tr>
<tr>
<td>2</td>
<td>20.98</td>
<td>15.00</td>
<td>0.11</td>
<td>0.05</td>
</tr>
<tr>
<td>3</td>
<td>35.11</td>
<td>27.00</td>
<td>0.15</td>
<td>0.08</td>
</tr>
<tr>
<td>4</td>
<td>90.05</td>
<td>89.00</td>
<td>0.67</td>
<td>0.16</td>
</tr>
<tr>
<td>All Firms</td>
<td>39.12</td>
<td>22.00</td>
<td>1.00</td>
<td>–</td>
</tr>
</tbody>
</table>

Notes - This table reports summary statistics of our panel. The sample ranges from 1950 to 2008 and includes about 11,100 different firms, for a total of 106,650 observations grouped into four age quantiles. Age is expressed in years and is computed using founding dates from Ritter and Loughran (2004) and Jovanovic and Rousseau (2001). We denote the aggregate physical capital in each quantile by $K$ and that of all firms by $K_{T}^t$.

is the firm-specific productivity level at time $t$. We allow for $\alpha_{1,j} + \alpha_{2,j} \neq 1$, but our results hold also when we impose constant returns to scale in the estimation, i.e., $\alpha_{1,j} + \alpha_{2,j} = 1$.

We denote aggregate productivity in the economy by $\bar{A}_t$, and we use $\Delta$ to denote first difference. In the rest of this section, we show that $Cov [\Delta \ln \bar{A}_{t+1}, \Delta \ln a_{i,j,t+1}]$ is increasing in firm age.

A Data

We consider publicly traded companies on US stock exchanges listed in both the COMPUSTAT and CRSP databases for the period 1950–2008. The output, or value added, of firm $i$ in industry $j$, $y_{i,j,t}$, is calculated as sales minus the cost of goods sold and is deflated by the aggregate GDP deflator from NIPA. We measure the capital stock of the firm, $k_{i,j,t}$, as the total value of assets minus current assets. This allows us to exclude cash and other liquid assets that may not be an appropriate component of physical capital. We use the number of employees in a firm to proxy for $n_{i,j,t}$ because data for total hours worked are not available. To measure age, we use founding years from Ritter and Loughran (2004) and Jovanovic and Rousseau (2001), for a total of about 11,100

---

6This specification is consistent with our model, even though we assume that production units use labor as the only input. Under the assumption of constant returns to scale, firm size is undetermined in our model. In the data, a firm can be viewed as a collection of production units (projects), and the observed capital stock, $k_{i,j,t}$, can be interpreted as the measure of production units owned by the firm.
Table 2: Exposure to Aggregate Risk by Firm Age

<table>
<thead>
<tr>
<th>Regression</th>
<th>$\Delta \ln A$</th>
<th>AGE</th>
<th>$AGE \times \Delta \ln A$</th>
<th>Obs.</th>
<th>Firms</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0.033</td>
<td>-0.001***</td>
<td>0.013***</td>
<td>74129</td>
<td>7371</td>
</tr>
<tr>
<td></td>
<td>(0.212)</td>
<td>(0.000)</td>
<td>(0.003)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>1.045**</td>
<td>-0.004***</td>
<td>0.016**</td>
<td>22508</td>
<td>4032</td>
</tr>
<tr>
<td></td>
<td>(0.524)</td>
<td>(0.001)</td>
<td>(0.007)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>0.549*</td>
<td>-0.001***</td>
<td>0.014***</td>
<td>62100</td>
<td>7268</td>
</tr>
<tr>
<td></td>
<td>(0.312)</td>
<td>(0.000)</td>
<td>(0.004)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes - This table reports firms’ risk exposure by age. All estimates are based on the following second-stage regression: $\Delta \ln a_{ijt} = \xi_0 + \xi_1 \Delta \ln \bar{A}_t + \xi_2 AGE_{i,j,t} + \xi_3 AGE_{i,j,t} \times \Delta \ln \bar{A}_t + \varepsilon_{ijt}$. Regression (1) is obtained using the whole sample. In order to control for exit bias, in regression (2) we use the Inverse Mills Ratio (IMR) as an additional explanatory variable. In regression (3) we exclude the years with negative aggregate productivity growth. All the estimation details are reported in Appendix B. Numbers in parentheses are standard errors. We use *, **, and *** to indicate a p-values smaller than 0.10, 0.05, and 0.01, respectively.

We thank the authors for making these data available on their web pages.

7 We use the multifactor productivity index for the private nonfarm business sector from the Bureau of Labor Statistics (BLS) as the measure of aggregate productivity $\bar{A}_t$.

We sort all the observations in our panel into four age quantiles and present summary statistics in Table 1. We first note that the fraction of physical capital owned by firms in the first quantile (7%, with a standard deviation of 3%) is comparable to the ratio of new investment to total physical capital stock in standard calibrated RBC models. In the context of our model, this is also the ratio of the measure of newborn production units to that of all operating production units in the economy. Note also that our data present substantial cross-sectional variation in age, a crucial explanatory variable in our empirical investigation.

B Empirical Strategy and Results

We follow a two-step procedure. In the first step, we estimate the parameters of the production function (20) and calculate the log productivity for each firm in our sample, i.e., $\ln a_{i,j,t}$. To address possible endogeneity issues, we estimate the industry-specific capital share, $\alpha_{1,j}$, and labor share, $\alpha_{2,j}$, using the dynamic error component model adopted in Blundell and Bond (2000). Industry is
defined at the level of two-digit SIC codes.

In the second step, we consider the following baseline regression:

\[
\Delta \ln a_{ijt} = \xi_0 + \xi_1 \Delta \ln A_t + \xi_2 AG E_{i,j,t} + \xi_3 AG E_{i,j,t} * \Delta \ln A_t + \varepsilon_{ijt},
\]

(21)

where \(\xi_0\) is a firm-specific fixed effect and \(AG E_{i,j,t}\) is the age of firm \(i\) at time \(t\). The key parameter of interest here is the coefficient \(\xi_3\), which captures the age effect on firm sensitivity to aggregate productivity growth. If the average age of investment projects is increasing in firm age, then under the null of our model \(\xi_3\) is positive.

As shown in Table 2, we find strong empirical evidence in favor of our specification of firm productivity. In our baseline estimation (regression (1)), the estimated coefficient \(\xi_3\) is both positive and statistically significant. Furthermore, we get very similar point estimates in regressions (2) and (3), where we correct for possible sample selection bias induced by firms’ exits.

If exits caused by exposure to negative aggregate productivity shocks are correlated with firm age, they can induce an upward bias in our estimate of \(\xi_3\) in regression (1). Consider a hypothetical scenario in which young firms are more exposed to negative aggregate productivity shocks than are older firms. In such a case, the estimate of \(\xi_3\) obtained from regression (1) would be biased upwards, because young firms are more likely to exit our database in years with large negative aggregate productivity shocks.

In regression (2) we correct for sample selection bias by adopting the Heckman (1979) two-stage sample selection estimator. In regression (3), we instead estimate equation (21) excluding all the observations from years with negative aggregate productivity shocks. The details of these robustness analyses can be found in Appendix B, where we also adopt an additional estimation procedure for the coefficients of the production function. Across all these different specifications, our estimates of \(\xi_3\) are very robust: they are consistently positive, significant, and comparable in magnitude.

For the sake of parsimony, in our calibration we impose that the productivity growth of new
production units differs from that of all older vintages only in the first period of their life. Since we set \( \phi_0 = 0 \) and \( \phi_j = 1 \) for all \( j > 0 \), at any time \( t \) new production units have zero exposure to aggregate productivity risk while old firms’ exposure is greater than one. To see this point, notice that the aggregate productivity growth rate is a weighted average of that of the new capital vintage, \( A_{t+1}/A_t = e^{\mu} \), and the common growth rate of all older vintages, \( A_{t+1}/A_t \), therefore the following holds:

\[
\Delta \ln \bar{A}_{t+1} = (1 - \lambda_t) \Delta \ln A_{t+1} + \lambda_t \mu, \quad 1 > \lambda_t > 0.
\]

The implied steady-state exposure of old capital vintages is \( \frac{1}{1 - \bar{\lambda}} \), where \( \bar{\lambda} \) is the average of \( \lambda_t \), and has a value of 1.12 under our benchmark calibration. We show below that this calibration conforms well to the data. To compare the empirical evidence to our calibration, we estimate the following equations:

\[
\Delta \ln a_{ijt} = \begin{cases} 
\xi_{0i} + \phi_Y \Delta \ln \bar{A}_t + \bar{\varepsilon}_{ijt} & i \in \text{Young} \\
\xi_{0i} + \phi_O \Delta \ln \bar{A}_t + \bar{\varepsilon}_{ijt} & \text{otherwise}
\end{cases}
\]

As discussed above, in each period a firm is classified as ‘Young’ if it belongs to the set of the 25% youngest firms in our sample. The exposure of the young firms to aggregate productivity shocks is denoted by \( \phi_Y \); the exposure of all the other firms is \( \phi_O \). The difference in their exposure is simply \( OMY \equiv \phi_O - \phi_Y \). The estimation results reported in Table 3 are consistent with our calibration in several respects. First, the estimated exposure of young firms, \( \phi_Y \), is small and statistically not different from zero. Second, the exposure of the other older firms is positive and statistically significant. Third, \( OMY \) is positive, statistically different from zero at a 15% confidence level and close to its model counterpart, ie, 1.12.

### IV Quantitative Implications of the Model

In this section, we calibrate our model and evaluate its ability to replicate key moments of both macroeconomic quantities and asset returns. We simulate our model and measure quantities in
### Table 3: Exposure to Aggregate Risk of Young vs. Other Firms

<table>
<thead>
<tr>
<th>Regression</th>
<th>$\phi$</th>
<th>$\phi$</th>
<th>OMY</th>
<th>Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>-0.068</td>
<td>1.053***</td>
<td>1.121***</td>
<td>15909</td>
</tr>
<tr>
<td></td>
<td>(0.343)</td>
<td>(0.096)</td>
<td>(0.338)</td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>0.113</td>
<td>1.488***</td>
<td>1.375</td>
<td>5034</td>
</tr>
<tr>
<td></td>
<td>(0.852)</td>
<td>(0.461)</td>
<td>(0.947)</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>0.264</td>
<td>1.718***</td>
<td>1.454***</td>
<td>13475</td>
</tr>
<tr>
<td></td>
<td>(0.543)</td>
<td>(0.165)</td>
<td>(0.526)</td>
<td></td>
</tr>
</tbody>
</table>

Notes - This table reports risk exposure of ‘Young’ and ‘Other’ firms. In each sample period, a firm is classified as ‘Young’ if it belongs to the set of the 25% youngest firms; otherwise it is classified in the group ‘Other’. All estimates are based on the second-stage regression (22):

$$\Delta \ln a_{ijt} = \begin{cases} 
\xi_{0i} + \phi_Y \Delta \ln \bar{A}_t + \tilde{\varepsilon}_{ijt} & i \in \text{Young} \\
\xi_{0i} + \phi_O \Delta \ln \bar{A}_t + \tilde{\varepsilon}_{ijt} & \text{otherwise.}
\end{cases}$$

OMY refers to $\phi_O - \phi_Y$. Regression (1) is obtained using the whole sample. To control for exit bias, in regression (2) we add the Inverse Mills Ratio (IMR). In regression (3) we exclude the years with negative aggregate productivity growth. All the estimation details are reported in Appendix B. Numbers in parentheses are standard errors. We use *, **, and *** to indicate p-values smaller than 0.10, 0.05, and 0.01, respectively.

Our model the same way as the US NIPA account. In particular, we treat investment in intangible capital, $J_t$ as expenses rather than investment when we compute measured output, $Y_{M,t}$ from the model. We focus on a long sample of US annual data ranging from 1929 to 2003. Data on consumption ($C_t$) and tangible investment ($I_t$) are from the Bureau of Economic Analysis (BEA). Aggregate intangible investment ($J_t$) is measured in the same way as in Corrado, Hulten, and Sichel (2006). The details of the construction of the $J_t$ series can be found in Appendix A.3. All variables are real and per-capita. See Appendix C for more details. Annual data on asset returns are from the Fama-French dataset.\(^8\) We use the Fama-French $HML$ factor as a measure of the spread between tangible and intangible capital. The model is calibrated at an annual frequency, and all moments are annual.

\(^8\)We thank Kenneth French for making the data available online: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/RF/F-F_Research_Data_Factors.zip.
Table 4: Main Components of Our Economy

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vintage capital</td>
<td>Yes ($\phi_0 = 0$)</td>
<td>No ($\phi_0 = 1$)</td>
<td>No ($\phi_0 = 1$)</td>
<td>No ($\phi_0 = 1$)</td>
</tr>
<tr>
<td>Long-run productivity risk</td>
<td>Yes ($\sigma_x \neq 0$)</td>
<td>Yes ($\sigma_x \neq 0$)</td>
<td>No ($\sigma_x = 0$)</td>
<td>No ($\sigma_x = 0$)</td>
</tr>
<tr>
<td>Intangible capital</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Recursive preferences</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Notes - This table summarizes the main components active in each of our four models. All parameter values are reported in Table 5.

A Parameter Values

Our model has three major components: heterogeneous productivity of vintage capital, long-run productivity risk, and intangible capital. In order to understand the importance of each component, we compare four different calibrations. The Benchmark model has all three components and is our preferred calibration. Model 1 does not have heterogeneous productivity of vintage capital but retains the other features of the Benchmark model, namely, long-run productivity risk and intangible capital. In Model 2, we further exclude fluctuations in long-run productivity growth (by setting $\sigma_x = 0$); this model incorporates intangible capital but requires the production units of all generations to have the same productivity (by setting $\phi_0 = 1$). Finally, we consider the case without intangible capital in Model 3. Essentially, Model 3 is the neoclassical growth model with recursive preferences and i.i.d. productivity growth rates. The details of the four models are summarized in Table 4.

The parameters of the models can be divided into two groups. The first group includes risk aversion, $\gamma$; intertemporal elasticity of substitution, $\psi$; capital share, $\alpha$; depreciation rates, $\delta_K$ and $\delta_S$; average growth rate of the economy, $\mu$; and the first-order autocorrelation of the predictable component in productivity growth, $\rho$. These parameters are identical across all four calibrations. We choose the risk aversion and the intertemporal elasticity of substitution in line with the long-run risk literature. Specifically, we set $\gamma = 10$ and $\psi = 2$. We set the capital share $\alpha = 0.3$ and the annual depreciation rate of physical capital $\delta_K = 10\%$, consistent with the RBC literature.
(Kydland and Prescott (1982)). We choose the same rate of depreciation for intangible capital: \( \delta_S = 10\% \) per year.\(^9\) We calibrate \( \mu = 2\% \) per year, consistent with the average annual real growth rate of the US economy. We set \( \rho = 0.93 \), which is the point estimate obtained in Croce (2008).

The second group of parameters includes the discount factor, \( \beta \); the standard deviation of the persistent component of productivity growth, \( \sigma_x \); the short-run shock volatility, \( \sigma_a \); and parameters of the aggregator \( G(I,S) \). In all calibrations, we set the discount factor \( \beta \) to match the level of the risk-free interest rate in the data if possible. An exception is Model 3, which lacks sufficient parameters to match both the level of the risk-free rate and the consumption–tangible investment ratio. We therefore choose \( \beta \) in Model 3 to match the consumption–tangible investment ratio but not the level of the risk-free rate. We set \( \sigma_a \) and \( \sigma_x \) in both the Benchmark model and Model 1 to approximately match the standard deviation and the first-order autocorrelation of the annual growth rate of measured output. In both models 2 and 3, we impose \( \sigma_x = 0 \) and set \( \sigma_a \) to match the standard deviation of the annual growth rate of measured output.

We choose the aggregator \( G(I,S) \) to be of the CES form

\[
G(I,S) = \left( \nu I^{1-\frac{1}{\eta}} + (1-\nu) S^{1-\frac{1}{\eta}} \right)^{\frac{1}{1-\frac{1}{\eta}}}.
\]  

\(^{(23)}\)

The aggregator \( G \) is described by two parameters, \( \nu \) and \( \eta \), chosen to approximately match the

---

\(^9\)Note there are two ways to measure the depreciation rate of intangible capital in our model. We use the term depreciation rate for the death rate of unexercised growth options, \( \delta_S \). Alternatively, one can count the amount of exercised options, \( G(I,S) \), as depreciation as well, since they are not stored for the next period. Measured this way, the steady state depreciation rate of intangible capital in our model is about 60%. Empirical estimates of the depreciation rate of intangible is sparse, and most of them do not distinguish between the two interpretations. Corrado, Hulten, and Sichel (2006) use depreciation rate of intangible capital ranging from 20% to 60% for different categories of intangible capital.

We experimented with alternative calibrations of our model to match some target level of depreciation rate of intangible capital. This substantially affect the steady state stock of intangible capital without changing significantly the basic asset pricing implications of the model.

\(^{10}\)Ideally, we would like to choose the parameters of the \( G \) function by calibrating the density \( f(\theta) \) to micro-evidence on the distribution of the quality of growth options. However, the quality of growth options are hard to measure in the data. Here we choose \( G(I,S) \) to be of the CES form and calibrate the parameters of \( G \) to match macro-moments. As shown in Ai (2009b), for any smooth \( G \) function that is concave and homogeneous of degree 1, there is a well-defined density function \( f(\theta) \) such that \( G \) is the aggregator of the option exercise problem described in (4) and (5).
Table 5: Calibrated Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Benchmark</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preference parameters</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discount factor</td>
<td>$\beta$</td>
<td>0.97</td>
<td>0.97</td>
<td>0.98</td>
</tr>
<tr>
<td>Risk aversion</td>
<td>$\gamma$</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Intertemporal elasticity of substitution</td>
<td>$\psi$</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Production function/Aggregator parameters</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capital share</td>
<td>$\alpha$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Depreciation rate of physical capital</td>
<td>$\delta_K$</td>
<td>10%</td>
<td>10%</td>
<td>10%</td>
</tr>
<tr>
<td>Depreciation rate of intangible capital</td>
<td>$\delta_S$</td>
<td>10%</td>
<td>10%</td>
<td>10%</td>
</tr>
<tr>
<td>Weight on physical investment</td>
<td>$\nu$</td>
<td>0.84</td>
<td>0.79</td>
<td>0.815</td>
</tr>
<tr>
<td>Elasticity of substitution</td>
<td>$\eta$</td>
<td>1.40</td>
<td>1.40</td>
<td>1.75</td>
</tr>
<tr>
<td>TFP parameters</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average growth rate</td>
<td>$\mu$</td>
<td>2.0%</td>
<td>2.0%</td>
<td>2.0%</td>
</tr>
<tr>
<td>Volatility of short-run risk</td>
<td>$\sigma_a$</td>
<td>5.00%</td>
<td>6.30%</td>
<td>7.00%</td>
</tr>
<tr>
<td>Volatility of long-run risk</td>
<td>$\sigma_x$</td>
<td>0.85%</td>
<td>0.85%</td>
<td>–</td>
</tr>
<tr>
<td>Autocorrelation of expected growth</td>
<td>$\rho$</td>
<td>0.93</td>
<td>0.93</td>
<td>–</td>
</tr>
<tr>
<td>Risk exposure of new investment</td>
<td>$\phi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Notes - This table reports the parameter values used for our calibrations. The following parameters are common across all models: risk aversion, $\gamma$; intertemporal elasticity of substitution, $\psi$; capital share, $\alpha$; depreciation rates, $\delta_K$ and $\delta_S$; and the average productivity growth rate, $\mu$. We choose the rest of the parameters to match the moments reported in Table 6 whenever possible. All models are calibrated at an annual frequency.

We solve the model using a second-order local approximation around the stochastic steady state.\footnote{Model 3 does not have intangible capital, so $E[I/J]$ is not defined. In Model 2, the parameter $\eta$ has only minor effects on the stochastic steady state; therefore, it is not possible to match both $E[C/I]$ and $E[I/J]$ simultaneously. In Model 2, we follow the RBC literature and set $\nu$ to match the consumption–physical investment ratio observed in the data.} We also solved the models numerically using a finite element-based global approximation method to check the accuracy of the local approximation method. Overall, the two numerical
Table 6: Moments Used for Model Calibration

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Benchmark</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[C/I]$</td>
<td>5.62</td>
<td>5.60</td>
<td>5.62</td>
<td>5.62</td>
<td>5.64</td>
</tr>
<tr>
<td>$E[I/J]$</td>
<td>1.00</td>
<td>1.01</td>
<td>0.98</td>
<td>(0.80)</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma[\Delta \ln Y_M]$</td>
<td>3.49</td>
<td>3.49</td>
<td>3.49</td>
<td>3.49</td>
<td>3.50</td>
</tr>
<tr>
<td>$AC1[\Delta \ln Y_M]$</td>
<td>0.45</td>
<td>0.45</td>
<td>(0.49)</td>
<td>(0.30)</td>
<td>(0.27)</td>
</tr>
<tr>
<td>$E[\gamma]$</td>
<td>0.86</td>
<td>0.86</td>
<td>0.87</td>
<td>0.86</td>
<td>(12.65)</td>
</tr>
</tbody>
</table>

Notes - This table reports the moments used to calibrate the parameters of the models evaluated in this paper. Our database refers to US annual data from 1930 to 2003 (see Appendix C). All moments that cannot be matched are in parentheses. In Model 1, the autocorrelation of measured output, $Y_M \equiv C + I$, cannot be matched, as we keep the calibration of the long-run risk component constant across models. In Model 2, the parameter $\nu$ is set to match the $C/I$ ratio, even though the implied $I/J$ ratio is lower than in the data. Neither Model 2 nor Model 3 is able to match the persistence of output growth, since they have no long-run risk. In Model 3, the discount factor $\beta$ is chosen to match the steady-state consumption-investment ratio, even though this choice makes the risk-free interest rate too high.

B Quantity Dynamics

In this section, we show that all four models produce largely similar macroeconomic quantity dynamics and that our Benchmark model improves slightly upon the RBC model (Model 3) along several dimensions. In this sense, our model inherits the success of the RBC models on the quantity side of the economy.

The quantity dynamics produced by our calibrations are shown in the first panel of Table 7. All four calibrations produce a small volatility of consumption growth and a high volatility of tangible investment growth, consistent with the data. Recall that Model 3 is essentially the standard RBC model with recursive preferences. We know from Tallarini (2000) that the risk aversion parameter of the recursive preference has little effect on the quantity dynamics. Therefore, on the quantity side, the model behaves just like the standard RBC model with CRRA preferences where $\gamma = \frac{1}{\psi} = 0.5$. The second moments generated by Model 3 are consistent with those in Kydland and Prescott (1982). In particular, this model produces a small standard deviation of consumption ($2.47\%$ per year), and a standard deviation of investment about five times larger ($12.61\%$ per year).
Comparing Models 2 and 3, we see that the addition of intangible capital to the standard RBC model reduces the volatility of physical investment growth. This is because the concavity of the aggregator $G$ implies decreasing marginal production of physical investment and affects the volatility of physical investment similarly to adjustment cost functions in neoclassical models. In order to generate a high volatility of tangible investment, therefore, the curvature of $G(I, S)$ needs to be low, or, equivalently, the elasticity of substitution between $I$ and $S$, $\eta$, needs to be sufficiently high. All of our calibrated models with intangible capital have this feature.

Long-run productivity risk provides an additional source of variation in the expected return of physical capital. Comparing Models 1 and 2, we see that long-run risk increases the volatility of the growth rate of physical investment. A similar positive effect is also produced by the introduction of different vintages of capital. As a result, our Benchmark model produces a 12.28% annual volatility of investment growth, similar to that of Model 3.

The persistence of the growth rates of macroeconomic quantities produced by our model is similar to that in the data. In Models 2 and 3, both output and consumption are autocorrelated, even if productivity growth is not. This result is generated by the persistent fluctuations of our endogenous state variables, $K$ and $S$ (as in Kaltenbrunner and Lochstoer (2010)). The persistence generated in these two models, however, is smaller than that in the data. The addition of long-run productivity risk increases the autocorrelation of consumption and output growth rates (Croce (2008)). Since both the Benchmark model and Model 1 feature long-run productivity shocks, they produce a higher autocorrelation in output growth (Table 6) and consumption growth (Table 7) than Models 2 and 3. The introduction of long-run productivity shock, therefore, brings our model closer to the data.

The correlation between consumption and physical investment growth rates in our calibration is consistent with its empirical pattern. A well-known feature of standard RBC models is that they produce large correlations of consumption and investment growth. As shown in Table 7, Models 2 and 3 share this feature. This result is driven by the existence of only one source of exogenous
shocks, the short-run productivity shock. Since both consumption and investment co-move with this shock, the correlation of their growth rates is quite high. In the data, however, the correlation of consumption and investment growth during the sample period 1929–2003 is 39%, a much lower value than that produced by Models 1 and 2. Both our Benchmark model and Model 1 are consistent with this feature of the data: the correlation between consumption and investment growth is 20% in the Benchmark model and 62% in Model 1. These results are driven by the introduction of long-run productivity risk. Realizations of long-run shocks, carry news about future productivity shocks but have little effect on the total output in the current period. Because of the resource constraint, consumption and total investment must move in opposite directions in response to these shocks, reducing their unconditional correlation.

A dimension of the data that our model does not match well is the volatility of investment in intangible capital. The volatility of $J_t$ in our benchmark model is similar to that of physical investment, 15% per year. The volatility of intangible investment in the data is much smaller, 5% per year. This, however, speaks in favor of rather than against our model, as many components of intangible investment is estimated by interpolation methods. While this procedure may have little effect on the first moment of intangible investment, it tends to understate volatility by construction.

C Asset Price Dynamics

In this section we examine the asset pricing implications of our model. While the quantity dynamics of the Benchmark model inherits the basic features of the standard RBC model, thanks to the lagged risk exposure of new vintage capital, asset returns in our model respond to long-run risks similarly to endowment based long-run risk models, for example, Bansal and Yaron (2004). More importantly, our model is able to produce a large spread between the expected return on tangible and intangible capital.

Campbell (2000) summarizes the challenge to general equilibrium asset pricing models as three puzzles: the equity premium puzzle (Mehra and Prescott (1985)), the stock market volatility puzzle
Table 7: Quantities and Prices

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Benchmark</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma (\Delta \ln C)$</td>
<td>02.53 (00.56)</td>
<td>02.45</td>
<td>02.95</td>
<td>02.83</td>
<td>02.47</td>
</tr>
<tr>
<td>$\sigma (\Delta \ln I)$</td>
<td>16.40 (03.24)</td>
<td>12.28</td>
<td>09.20</td>
<td>08.87</td>
<td>12.61</td>
</tr>
<tr>
<td>$\rho (\Delta \ln C, \Delta \ln I)$</td>
<td>00.39 (00.29)</td>
<td>00.20</td>
<td>00.62</td>
<td>00.76</td>
<td>00.82</td>
</tr>
<tr>
<td>$AC_1 (\Delta \ln C)$</td>
<td>00.49 (00.15)</td>
<td>00.45</td>
<td>00.50</td>
<td>00.46</td>
<td>00.34</td>
</tr>
<tr>
<td>$\sigma [SDF]$</td>
<td>–</td>
<td>93.09</td>
<td>98.21</td>
<td>68.08</td>
<td>39.01</td>
</tr>
<tr>
<td>$E[r_K - r_f]$</td>
<td>–</td>
<td>02.00</td>
<td>00.80</td>
<td>00.72</td>
<td>00.28</td>
</tr>
<tr>
<td>$\sigma [r_K]$</td>
<td>–</td>
<td>02.15</td>
<td>01.46</td>
<td>01.37</td>
<td>00.98</td>
</tr>
<tr>
<td>$E[r_S - r_f]$</td>
<td>–</td>
<td>00.55</td>
<td>00.65</td>
<td>00.44</td>
<td>–</td>
</tr>
<tr>
<td>$\sigma [r_S]$</td>
<td>–</td>
<td>01.19</td>
<td>01.12</td>
<td>00.89</td>
<td>–</td>
</tr>
<tr>
<td>$E[r_f]$</td>
<td>00.86 (00.31)</td>
<td>00.86</td>
<td>00.86</td>
<td>00.86</td>
<td>12.65</td>
</tr>
<tr>
<td>$\sigma [r_f]$</td>
<td>00.97 (00.31)</td>
<td>01.18</td>
<td>01.10</td>
<td>00.75</td>
<td>00.68</td>
</tr>
<tr>
<td>$E[r_L^M - r_f]$</td>
<td>05.71 (02.25)</td>
<td>05.66</td>
<td>02.22</td>
<td>02.16</td>
<td>00.85</td>
</tr>
<tr>
<td>$\hat{\sigma}_M$</td>
<td>05.24 (—.—)</td>
<td>04.81</td>
<td>02.98</td>
<td>03.14</td>
<td>02.27</td>
</tr>
<tr>
<td>$E[r_L^K - r_{L,S}]$</td>
<td>04.32 (01.39)</td>
<td>04.08</td>
<td>00.44</td>
<td>01.11</td>
<td>–</td>
</tr>
</tbody>
</table>

Notes - All figures are multiplied by 100, except contemporaneous correlations (denoted by $\rho$) and first-order autocorrelations (denoted by $AC_1$). Empirical moments are computed using US annual data from 1930 to 2003. Numbers in parentheses are GMM Newey-West adjusted standard errors. $E[r_K - r_f]$ measures the average difference between the levered returns of tangible and intangible capital. We use the $HML$ Fama-French factor as empirical counterpart of $r_K^L - r_S^L$. $r_L^M$ indicates levered market returns; $\hat{\sigma}_M$ measures the volatility of levered market returns explained by productivity. Returns are in log units. In the models, the levered premiums are three times greater than the unlevered ones. All the parameters are calibrated as in Table 5. The entries for the models are obtained by repetitions of small sample simulations. (Campbell (1999)), and the risk-free rate puzzle (Weil (1989)). These puzzles are even more difficult to solve in production economies, as the model must (1) not only generate a pricing kernel that is volatile enough, but also endogenously produce a high risk exposure of the stock market returns, and (2) be consistent with the empirical evidence from the quantity side of the economy. The literature has relied primarily on adjustment cost or other forms of rigidity in investment to generate the variation in the price of physical capital. In the next sections we show that it is difficult to reconcile the high risk exposure of the market returns and the high volatility of tangible investment when relying on rigidity in investment as the only means by which to generate variations in the price of physical capital. In our Benchmark model, however, thanks to heterogeneous productivity of capital of different vintages, we can simultaneously produce volatile stock market returns and aggregate


physical investment. The adoption of recursive preferences with high intertemporal elasticity of substitution also allows us to solve the risk-free rate puzzle.

The empirical evidence on the value premium imposes a strong discipline on general equilibrium asset pricing models with intangible capital. Stocks with high book-to-market ratios earn higher returns than stocks with low book-to-market ratios, and the difference between market value and book value can be attributed to the value of intangible capital owned by the firm. This evidence suggests that intangible capital earns a lower average return than physical capital. Qualitatively, the Benchmark model and Models 1 and 2 are consistent with intangible capital being less risky than physical capital (Ai (2009b)). Quantitatively, however, only the Benchmark model is capable of producing a significant value premium. The interaction between lagged risk exposure of new vintage capital and long-run productivity risk is the main driver of this result.

In the following subsections, we first discuss the common features of all four calibrations (Section C.1), then examine the models’ implications for the volatility of returns $r_K$ and $r_S$ (Section C.2). Finally, we study the models’ implications for the value premium (Section C.3). The asset pricing implications of all four calibrations are summarized in Table 7.

C.1 Common Features

All calibrations except Model 1 are able to generate a low and relatively smooth risk-free interest rate.\textsuperscript{13} The volatilities of the risk-free interest rates are low because we adopt an intertemporal elasticity of substitution greater than one: since agents are very willing to substitute consumption across time, fluctuations in the expected consumption growth rate produce only small variations in the equilibrium interest rate.

All four models produce a fairly high volatility of the stochastic discount factor. Since the representative agent is endowed with recursive preferences, fluctuations in expected consumption

\textsuperscript{13}In standard RBC models, there is always a tension in simultaneously producing a high consumption-physical investment ratio and a low level of the risk-free rate through the subjective discount factor $\beta$. This explains why in Model 3 we are not able to match the level of the risk-free rate, as we set $\beta$ to reproduce the consumption-investment ratio observed in the data.
growth (long-run risk, in the language of Bansal and Yaron (2004)) strongly affect marginal utility. Models 2 and 3 feature predictability in consumption growth because of the endogenous fluctuations in $K$ and $S$. The introduction of long-run productivity shocks almost doubles the volatility of the stochastic discount factor. As explained earlier, in both the Benchmark model and Model 1, long-run productivity risk produces substantial variations in the predictable component of consumption growth, which further increases the market price of risk.
C.2 Returns Dynamics

As shown in Kaltenbrunner and Lochstoer (2010) and Croce (2008), an important challenge to the long-run risk–based asset pricing model with production is to account for the high volatility of investment and stock returns simultaneously. Although recursive preferences generate a high volatility of the stochastic discount factor, the return to physical capital is typically very smooth, unless one is willing to assume a large adjustment cost. High levels of adjustment cost, however, are typically associated with counterfactually low levels of volatility in investment growth.

This tension is present in Models 1, 2, and 3, but it is resolved in our Benchmark model, where the annual volatility of the unlevered returns on physical capital is 2.15% and investment is as volatile as in an RBC model. To explain our results, we plot in Figures 1 and 2 the impulse response functions of quantities and prices, respectively, generated by both short-run shocks and long-run shocks in the Benchmark model and Model 1.

The left panels of Figures 1 and 2 show that the introduction of heterogeneous productivity of vintage capitals does not significantly alter the model’s response to short-run shocks. This result has two important implications. First, since the quantity dynamics in the Benchmark model are mostly driven by short-run shocks, they inherit the success of standard RBC models with i.i.d. productivity growth (Model 3). Second, the risk premia associated with short-run shocks are small in both models. Therefore, in order to understand the success of our Benchmark model for both equity and value premiums, we must focus on the interaction between long-run shocks and heterogeneous productivity of vintage capitals.

As shown in the right hand panels of Figures 1 and 2, the impulse responses to long-run shocks are significantly different across Model 1 and the Benchmark model. With a one-standard-deviation change in the long-run productivity shock, the return on physical capital, $r_K$, in the Benchmark model increases by about 1.5%, whereas the change in $r_K$ in Model 1 is barely visible. This implies that the exposure to the long-run productivity risk of physical capital is very small in Model 1, whereas that in the Benchmark model is larger by several orders of magnitude.
To explain the different behavior of \( r_K \) across the Benchmark model and Model 1, we focus our attention on the ex-dividend price of physical capital, \( q_{K,t} \) (see Figure 2, fourth panel, right column). Using the definition of \( q_{K,t} \) in (15), and iterating equation (3) forward, we can express \( q_{K,t} \) as the present value of the infinite sum of all future payoffs:

\[
q_{K,t} = \sum_{j=1}^{\infty} (1 - \delta_K)^j E_t \left[ A_{t,t+j} \alpha K_{t+j}^{\alpha-1} (A_{t+j})^{1-\alpha} \right],
\]

Equation (24) implies that the price of physical capital, \( q_{K,t} \), is the present value of the marginal product of physical capital in all future periods. This equation holds in Model 1 as well. A positive innovation in the long-run productivity component \( x_t \) has two effects on the future marginal product of physical capital. The first is a direct effect: keeping everything else constant, an increase in \( x_t \) raises the marginal product of physical capital by increasing all future \( A_{t+j} \) for \( j = 1, 2, \cdots \). The second effect comes from the general equilibrium. An increase in the marginal productivity of capital also triggers more investment, which augments \( K_{t+j} \) in all future periods. Due to decreasing returns to scale (\( \alpha < 1 \)), an increase in \( K_{t+j} \) mitigates the direct effect.

In Model 1, the elasticity of substitution between physical investment and intangible capital, \( \eta \), is set to 1.4. This implies that the supply of physical investment is quite elastic. Consequently, the return on physical capital responds very little to long-run shocks. To see this point more clearly, note that without overlapping generations of vintage capital, we have \( \varpi_t = 1 \ \forall t \), and equation (12) can be written as

\[
q_{K,t} - (1 - \delta_S) = \frac{1}{G_I(I_t, S_t)} = \frac{1}{\nu} \left( \frac{I_t}{G(I_t, S_t)} \right)^{\frac{1}{\eta}}.
\]

By equation (25), as \( \eta \) increases, \( I_t \) becomes more sensitive to changes in \( q_{K,t} \). Equation (24) implies that if investment adjusts elastically to productivity shocks, then the effect of the long-run productivity shock on \( q_{K,t} \) is small, due to decreasing return to scale of physical capital. This intuition is confirmed by our impulse response functions. Innovations in the long-run productivity
Fig. 2 – Impulse Response Functions for Prices

Notes - This figure shows annual log-deviations from the steady state. All the parameters are calibrated to the values reported in Table 5. Returns are not levered. The solid lines refer to Model 1; the dashed lines refer to the Benchmark model.

component are accompanied by a nearly permanent increase in the I/S ratio (Figure 1, third panel, right column, solid line). As a result, the changes in $q_K$ after a long-run productivity shock are almost negligible (Figure 2, fourth panel, right column). To summarize, in Model 1 the return on physical capital responds little to long-run productivity shocks because the direct effect on the price of physical capital is mostly offset by movements in investment (the general equilibrium effect). As with standard adjustment cost models, it is difficult to simultaneously produce a high volatility of investment growth and returns on physical capital in Model 1.
In the Benchmark model, however, after a long-run productivity shock, investment rises, but after a substantial delay, whereas the return on physical capital increases immediately and sharply. Figure 1 shows that the $I/S$ ratio initially drops and then starts to rise, always staying below the level obtained in Model 1 (fourth panel, right column). The last panel in the right column plots the impulse response of physical capital stock normalized by productivity ($k_t = K_t/A_t$) after a long-run shock. Because of the lagged response of investment, the level of physical capital in the Benchmark model stays nearly permanently behind that obtained in Model 1. Since the marginal product of capital, $\alpha k_t^{-(1-\alpha)}$, is a decreasing function of normalized capital stock, in the Benchmark model the marginal product of physical capital stays almost permanently above that observed in Model 1, producing a strong increase in $q_{K,t}$.

In this case, both the direct and the general equilibrium effects of long-run productivity shocks affect $q_{K,t}$ in the same way, thereby reinforcing each other. The marginal product of capital increases both because a positive shock in $x_t$ increases $A_{t+j}$ in all future periods and because the sluggish response of investment to long-run shocks results in a nearly permanent reduction of physical capital stock relative to that in Model 1.

To understand the lagged response of investment to long-run news in the Benchmark model, note that a long-run shock increases the productivity of all existing vintages of capital almost permanently, but it affects the productivity of the new production units only after a delay. This generates an incentive to postpone the exercise of new growth options. As a result, a long-run productivity shock immediately produces a strong income effect (the agent anticipates a persistent increase in the productivity of all existing vintages of capital and prefers to consume more) without generating a significant substitution effect (the return on new physical investment is unaffected by long-run productivity shock for an extended period of time). At time 1, when a positive long-run shock materializes, the net effect is an immediate increase in consumption and a decrease in investment, exactly the opposite of what happens in Model 1, in which the substitution effect dominates the income effect and investment increases.
Notes - This figure shows the correlation of one-period market excess returns with investment growth leads \((j > 0)\) and lags \((j < 0)\). The left panel refers to one-year investment growth, whereas the right panel considers five-year investment growth. The thin solid line represents the point estimate of the correlations computed using US data from 1930 to 2003. The dotted lines mark the 95% confidence interval for the correlations. The solid line with circles represents the correlations obtained in the Benchmark model. The diamond-shaped markers refer to Model 1. All the parameters are calibrated to the values reported in Table 5. The entries from the models are obtained through repetitions of small sample simulations.

In the Benchmark model, positive long-run shocks, although small, have quite significant and prolonged negative effects on physical investment. This sluggish response of investment is generated by the persistence of the long-run shocks: after positive long-run news, the relative productivity of new investment remains behind that of existing vintages for an extended period of time, thereby discouraging a fast and full recovery of investment.

The economic mechanism in the Benchmark model discussed above is consistent with the empirical evidence on the correlation between investment growth and market excess returns. In Figure 3, we depict the cross-correlations between one-period market excess returns, \(r_{m,t+1}^{ex}\), and leads \((j > 0)\) and lags \((j < 0)\) of one-period (left panel) and five-period investment growth (right panel). In the data, the contemporaneous correlation between one-year investment growth and excess returns is close to zero. Market excess returns lead investment growth, while there is no significant
relationship with lagged investment growth rates. The Benchmark model is consistent with this pattern in the data. Model 1, in contrast, produces a contemporaneous correlation that is too high, and it exaggerates the cross-correlations on the leads side (j > 0). Because the key implications of the Benchmark model rely on its long-run dynamics, in the right panel of Figure 3 we also plot this cross-correlation structure using long-horizon investment growth rates. The Benchmark model is again consistent with the pattern in the data, whereas Model 1 fails to reproduce the correlations of market returns with the lagged investment growth rates (j < 0).

In addition, we compare the volatility of the stock market returns in our economy and in the data. Our model explains the link between productivity shocks and stock market fluctuations but does not feature dividend-specific shocks. In the data, however, it is well known that dividend-specific shocks explain a substantial share of the volatility of the market returns. Consequently, our model cannot and should not account for the total volatility of the stock market returns. To isolate and better quantify productivity-related stock market fluctuations, under the guidance of our model we project returns onto the space of long-run and short-run productivity shocks. For any return $r_t$, therefore, we define its productivity-related volatility as follows:

\[
\hat{\sigma} \equiv \text{St.Dev.}(E[r_t | \Delta x_t, \Delta \ln A_t])
\] (26)

In practice, we calculate the conditional expectation in (26) by the following regression: \[ r_{M,t} = \xi_0 + \xi_1 \Delta x_t + \xi_2 \Delta \ln A_t + \varepsilon_t. \] (27)

A potential difficulty of calculating $\hat{r}_t$ is that $\Delta x_t$ is unobservable in the data, and thus we must substitute it with an estimated proxy, $\Delta \hat{x}_t$. A merit of our approach is that we do not need to measure the latent variable $\Delta x_t$ accurately in order to accurately compute $\hat{\sigma}$. Since we are only interested in the conditional expectation, as long as the couple $(\Delta \hat{x}_t, \Delta \ln A_t)$ spans the same space

\[ \text{Here we are assuming that the conditional expectation is linear in the short-run and long-run shocks. It turns out that this is a very good approximation under the null of the model.} \]
as \((\Delta x_t, \Delta \ln A_t)\), we can obtain a perfect measure of \(\hat{\sigma}\). The details of the construction of \(\hat{\sigma}\) in the data and in the model can be found in Appendix C.\(^{15}\)

According to our procedure, only 25% of the total observed volatility of the market returns can be attributed to productivity shocks. Therefore, even though in our sample the total annual volatility of the returns is 19.42%, the volatility directly related to productivity is about 5.25%. After leveraging the returns, our model accounts for almost all of the productivity-related volatility of the stock market, as shown in the last row of Table 7.

We conclude this section by pointing out that the volatility puzzle is more severe in economies with decreasing returns to physical capital (see also Kaltenbrunner and Lochstoer (2010) and Croce (2008)). In AK models, such as the one studied in Ai (2009a), the response of investment to productivity shocks does not dampen the direct effect of productivity shocks on returns; hence, generating a volatile return on capital is easier. Although AK models are useful in studying economic growth in the long run, empirical studies (see, for example, Benhabib and Jovanovic (1991), Benhabib and Spiegel (1994), Romer (1990), and King and Levine (1994)) emphasize the importance of decreasing returns to physical capital in understanding macroeconomic quantities. In light of this empirical evidence, our work suggests that different exposure of capital vintages to productivity risk is important in explaining productivity-related volatility of market returns.

**C.3 Value Premium**

To understand the difference in the expected return of tangible and intangible capital, we can use the functional form of \(G(I, S)\) in equation (23) and write the return of intangible capital in equation (19) as

\[
 r_{S,t+1} = \frac{1 - \nu}{\nu} \left( \frac{I_{t+1}}{S_{t+1}} \right)^{\frac{1}{\eta}} + (1 - \delta_{S}) .
\]

\(^{15}\)Our empirical investigation confirms that our calculation of the productivity-related volatility of the market returns is robust to various methods of estimation in both the data and the model simulations.
As explained in Section II.C, the term $\frac{1-\nu}{\nu} \left( \frac{I_{t+1}}{S_{t+1}} \right)^{1/\gamma}$ can be interpreted as the expected payoff of in-the-money options in period $t+1$. Because $S_{t+1}$ is determined in period $t$, innovations in the return on intangible capital responds positively to innovations in $I_{t+1}$. The intuition for this result is that an increase in the $\frac{I}{S}$ ratio lowers the option exercise threshold $\theta^* (t) = G_I (I_t, S_t)$ and raises the probability of option exercise, thereby enhancing the payoff of growth options. As shown in Figures 1 and 2, in our Benchmark model, $\frac{I}{S}$ responds negatively to long-run productivity shocks. Therefore, our model is able to account for the empirical fact that growth stocks are less exposed to long-run economic risks as documented in Bansal, Dittmar, and Lundblad (2005); Hansen, Heaton, and Li (2008); Kiku (2006).

To understand the lower exposure of growth options with respect to long-run productivity shocks compared to assets in place, note that the payoff of growth options can be replicated by long positions in assets in place and short positions in the cost of the strike asset. Exercising growth options at time $t$ costs $\frac{1}{\theta}$ unit of investment goods and produces a generation $t$ production unit, which is equivalent to $\omega_t$ initial generation production units. Since installed physical capital in this economy is measured in terms of generation-0 production unit equivalents, effectively, the expected cost of creating an additional unit of $K_t$ on date $t$ is $E_t \left[ \omega_{t+1}^{-1/\gamma} \right] = \frac{1}{\theta} e^{\frac{1-\alpha}{\alpha} (x_t + \frac{1}{2} \sigma_x^2)}$. Growth options have low exposure to long-run risk, $x_t$ because the cost of option exercise, $E_t \left[ \omega_{t+1}^{-1/\gamma} \right]$, covaries positively with $x_t$ and acts as a hedge. From the planner’s perspective, since the cost of option exercise is high when $x_t$ increases, she optimally choose to reduce the total amount of option exercised by cutting the supply of the scarce resource needed for option exercise, $I_t$. This implies that physical investment and the payoff to growth option boths respond negatively to long-run productivity shocks.

The implications of our model for the value premium are summarized in the last panel of Table 7. We make the following observations. First, all models with intangible capital yield a higher return for physical capital than for intangible capital. This is true because the "out-of-the-money" component of option price accounts for a quantitively large fraction of the return on intangible
capital and is risk-free due to Proposition 1.

Second, despite the introduction of long-run risk, Model 1 produces a lower spread between physical and intangible capital than does Model 2. In Model 1, intangible capital is more exposed to long-run risk than is tangible capital. Specifically, without heterogeneous productivity of vintage capital, after a positive long-run productivity shock, physical investment increases sharply, but \( q_{K,t} \) remains almost flat (Figure 2). At the same time, the increase in \( I/S \) ratio is associated with a drop in the option exercise threshold, \( \theta^*(t) \) and a positive innovation in the return on intangible capital. As a result, as shown in Table 7, simply adding long-run productivity shocks to Model 2 increases the market risk premium only slightly and eliminates most of the spread in the expected return on physical and intangible capital.

Finally, compared to Model 1, our Benchmark model produces both a larger risk premium on physical capital and a smaller one on intangible capital, thus simultaneously improving on equity and value premiums. The heterogeneous productivity of vintage capital is responsible for both improvements because it causes the \( I/S \) ratio to drop after good long-run news. This feature of the model produces a sharp increase in the return on tangible capital, \( r_K \), and a fall in the intangible capital return, \( r_S \). It increases the riskiness of physical capital and makes growth options an insurance device against long-run risk.

Overall, the Benchmark model produces a market risk premium more than two times larger than that in Model 1, and a spread between tangible and intangible capital larger by an order of magnitude. When leverage is taken into account, the Benchmark model produces a log value premium of 4.08% per year, more than 90% of the observed value premium in the data.
V Extensions and Sensitivity Analysis

A Implications for Labor

Focusing on the US economy, Barsky and Sims (2010) and Kurmann and Otrok (2010) show that positive news about future productivity shocks are associated with increases in consumption and immediate drops in investment, hours worked, and total output. Here we show that our model is consistent with these empirical findings once we endogenize labor supply in the model. The quantitative asset pricing implications of our model, on the other hand, are unaffected by endogenous labor supply.

For the sake of simplicity, we adopt the well known Cobb-Douglas aggregator of consumption goods, $C_t$, and leisure, $1 - N_t$, as follows:

$$u_t = C_t^o (A_{t-1}(1 - N_t))^{1-o}.$$ 

This formulation implies that the aggregator is homogeneous of degree one in $A_{t-1}$, and, therefore, guarantees balanced growth. In economic terms, $A_{t-1}$ is often interpreted as a simple measure of the standards of living. As commonly done in the literature, we set the parameter $o$ so that the average number of hours worked is equal to $1/3$ of the total number of workable hours. The intratemporal first order condition for labor is:

$$\frac{1 - o}{o} \cdot \frac{C_t}{1 - N_t} = (1 - \alpha) \frac{Y_t}{N_t},$$

and the stochastic discount factor with leisure becomes:

$$A_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-1} \left( \frac{u_{t+1}}{u_t} \right)^{1 - \frac{1}{\psi}} \left[ \frac{V_{t+1}}{E_t \left[ V_{t+1}^{\frac{1}{1-\gamma}} \right]^{\frac{1}{\psi}}} \right]^{\frac{1}{\psi} - \gamma}.$$
We plot the impulse response function of consumption, investment and labor (denoted as ‘n’) in figure 4. We notice three things. First, as in standard RBC models, labor co-moves with consumption and investment after short-run shocks. Second, consistent with the empirical literature on macroeconomic news, hours worked drop after good news for long-run productivity shocks due to income effect. Third, the dynamics of consumption and investment preserve all the main properties discussed in the previous section. Table 8 shows that our model is also able to replicate key moments of the joint distribution of consumption and labor in the US while simultaneously retaining its success on the asset pricing side.
Table 8 Key Moments with Endogenous Labor Supply

<table>
<thead>
<tr>
<th></th>
<th>$\sigma(\Delta \ln C)$</th>
<th>$AC_1(\Delta \ln C)$</th>
<th>$\sigma(\Delta \ln n)$</th>
<th>$\rho(\Delta \ln C, \Delta \ln n)$</th>
<th>$E[r^M - rf]$</th>
<th>$E[r^K - r_S^L]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Labor</td>
<td>02.45</td>
<td>00.45</td>
<td>–</td>
<td>–</td>
<td>05.66</td>
<td>04.08</td>
</tr>
<tr>
<td>Labor</td>
<td>02.47</td>
<td>00.58</td>
<td>01.77</td>
<td>00.07</td>
<td>05.55</td>
<td>04.14</td>
</tr>
<tr>
<td>DATA</td>
<td>02.53</td>
<td>00.49</td>
<td>02.07</td>
<td>00.28</td>
<td>05.71</td>
<td>04.32</td>
</tr>
<tr>
<td></td>
<td>(00.56)</td>
<td>(00.15)</td>
<td>(00.21)</td>
<td>(00.07)</td>
<td>(02.25)</td>
<td>(01.39)</td>
</tr>
</tbody>
</table>

Notes - All figures are multiplied by 100, except contemporaneous correlations (denoted by $\rho$) and first-order autocorrelations (denoted by $AC_1$). Empirical moments are computed using US annual data from 1930 to 2003. Numbers in parentheses are GMM Newey-West adjusted standard errors. $E[r^M - rf]$ measures the average difference between the levered returns of tangible and intangible capital. We use the $HML$ Fama-French factor as empirical counterpart of $r^K - r_S^L$. $r^M$ indicates levered market returns; returns are in log units. In the models, the levered premiums are three times greater than the unlevered ones. All the parameters are calibrated as in Table 5; the parameter $o$ is set to match an average share of hours worked of 1/3. The entries for the models are obtained by repetitions of small sample simulations.

VI Conclusion

We present a general equilibrium asset pricing model with intangible capital to account for some of the salient features of macroeconomic quantity and asset price dynamics. The incorporation of intangible capital presents additional challenges to general equilibrium asset pricing models with production. Because of the well-known difficulty in generating a high equity premium in production economies, one might be tempted to assume that intangible capital is much riskier than physical capital and propose this as a resolution of the equity premium puzzle. The empirical evidence on the value premium, however, suggests the exact opposite. In the US, the portfolios of firms with low book-to-market ratios pay substantially lower returns than those of firms with high book-to-market ratios. This suggests that intangible capital earns a much lower risk premium than tangible capital, making it even harder to account for the overall market equity premium.

We document novel empirical evidence that is the key to understanding the empirical facts on equity returns, especially the spread between tangible and intangible capital. We show that in the data, new investment is less exposed to aggregate productivity shocks than is the capital of older vintages. We build a general equilibrium model with intangible capital based on Ai (2009b) and incorporate long-run productivity risk as in Croce (2008). We show that the lower exposure of
new investment is quantitatively important in accounting for the high equity premium, the high volatility of the stock market return, and the large spread between book-to-market–sorted portfolios in the data.

Several remarks are in order. First, our model is silent on the reason why young firms are less exposed to aggregate productivity shocks than older firms. A fully satisfactory answer to this question would requires a theory of endogenous productivity shocks. We leave this for further research. Second, we have assumed that shocks to the quality of blueprints are \( i.i.d. \). While unrealistic, this assumption simplifies our aggregation results making our model very tractable. In our economy, indeed, we avoid to keep track of the cross-sectional distribution of option quality and calibrate the aggregator \( G(I, S) \) to match only aggregate moments. Allowing for more general shock processes to the quality of the options will allow us to calibrate the model to micro-evidence as well. Third, extending our model to a setting with heterogeneous firms will allow us to implement portfolio sorting exercises in the context of our current model, to study firms’ transition among value and growth portfolios, and to confront the model with a wider set of empirical evidence at the portfolio level, as done by Ai and Kiku (2009). Based on their insights, we are optimistic that the basic intuition in this paper will remain valid even with heterogeneous firms. Finally, we believe that our model provides a valuable general equilibrium framework for the measurement of intangible capital by exploring the information from both the quantity and pricing sides of the economy.
References


Appendices

Appendix A: Aggregation of Production Units

Lemma 1 Suppose there are \( m \) types of firms. For \( i = 1, 2, 3, \ldots m \), the productivity of the type \( i \) firm is denoted by \( A(i) \), and the total measure of the type \( i \) firm is denoted by \( K(i) \). The production technology of the type \( i \) firm is given by
\[
y(i) = [A(i) n(i)]^{1-\alpha},
\]
where \( n(i) \) denotes the labor hired at firm \( i \). The total labor supply in the economy is \( N \). Then the aggregate production function is given by
\[
Y = \left[ \sum_{i=1}^{m} K(i) \left[ \frac{A(i)}{A(1)} \right]^{\frac{1-\alpha}{\alpha}} \right]^{\alpha} [A(1) N]^{1-\alpha}
\]

Proof. Without loss of generality, we assume at the optimal production plan that firms of the same type employ the same amount of labor. The total production in the economy is given by
\[
Y = \max \sum_{i=1}^{m} K(i) A(i)^{1-\alpha} n(i)^{1-\alpha} \tag{A.1}
\]
subject to \( \sum_{i=1}^{m} K(i) n(i) = N \)

The first-order condition of the above optimization problem implies that for all \( i \),
\[
\frac{n(i)}{n(1)} = \left( \frac{A(i)}{A(1)} \right)^{\frac{1-\alpha}{\alpha}}
\]

Using the resource constraint, we determine the labor employed in firm 1:
\[
\sum_{i=1}^{m} K(i) \left( \frac{A(i)}{A(1)} \right)^{\frac{1-\alpha}{\alpha}} n(1) = N
\]
This implies that
\[
n(1) = \left[ \sum_{i=1}^{m} K(i) \left[ \frac{A(i)}{A(1)} \right]^{\frac{1-\alpha}{\alpha}} \right]^{-1} N \tag{A.2}
\]
Therefore, the total production is given by

\[ Y = \sum_{i=1}^{m} K(i) A(i)^{1-\alpha} \left[ \left( \frac{A(i)}{A(1)} \right)^{\frac{1-\alpha}{\alpha}} n(1) \right]^{1-\alpha} \]

\[ = \left[ A(1)^{-\frac{1-\alpha}{\alpha}} n(1) \right]^{1-\alpha} \sum_{i=1}^{m} K(i) A(i)^{1-\alpha} \]

\[ = \left[ \sum_{i=1}^{m} K(i) A(i)^{1-\alpha} \right]^{\alpha} N^{1-\alpha} \]

\[ = A(1) \left[ \sum_{i=1}^{m} K(i) \left( \frac{A(i)}{A(1)} \right)^{\frac{1}{\alpha}} \right]^{\alpha} N^{1-\alpha} \]

Plugging in the expression for \( n(1) \) in equation (A.2), we have

\[ Y = \left[ (1-\delta_K)^t K_0 + \sum_{j=0}^{t-1} (1-\delta_K)^{t-j-1} E_j \left( \frac{A_j}{A_t} \right)^{\frac{1-\alpha}{\alpha}} \right]^{\alpha} N^{1-\alpha} \]

as needed.

Proof of Proposition 1:

At time \( t \), there are \( t+1 \) types of operating production units in the economy, namely, production units of generation \(-1, 0, 1, \cdots, t-1\). The measures of these production units are \((1-\delta_K)^t K_0, (1-\delta_K)^{t-1} E_0, (1-\delta_K)^{t-2} E_1, \cdots, E_{t-1}\). Using the above lemma, at date \( t \), the total production in the economy is given by

\[ Y_t = A_t \left[ (1-\delta_K)^t K_0 + \sum_{j=0}^{t-1} (1-\delta_K)^{t-j-1} E_j \left( \frac{A_j}{A_t} \right)^{\frac{1-\alpha}{\alpha}} \right]^{\alpha} N_t^{1-\alpha} \]

Clearly, if we define the \( \{K_t\}_{t=0}^{\infty} \) according to (??), the aggregate production function can be summarized as in (3).
Table B.1: Exposure to Aggregate Risk by Firm Age

<table>
<thead>
<tr>
<th>Regression</th>
<th>Δ ln A</th>
<th>AGE</th>
<th>AGE ∗ Δ ln A</th>
<th>Obs.</th>
<th>Firms</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0.086</td>
<td>-0.001***</td>
<td>0.011***</td>
<td>74138</td>
<td>7373</td>
</tr>
<tr>
<td></td>
<td>(0.216)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>1.522***</td>
<td>-0.002*</td>
<td>0.015**</td>
<td>22513</td>
<td>4033</td>
</tr>
<tr>
<td></td>
<td>(0.522)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>0.482</td>
<td>-0.001***</td>
<td>0.016***</td>
<td>62109</td>
<td>7270</td>
</tr>
<tr>
<td></td>
<td>(0.320)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes - This table reports firms’ risk exposure by age. All estimates are based on the following second-stage regression: \( Δ \ln a_{ijt} = ξ_0i + ξ_1 \Delta \ln A_t + ξ_2 AGE_{i,j,t} + ξ_3 AGE_{i,j,t} ∗ Δ \ln A_t + ε_{ijt} \). Regression (1) is obtained using the whole sample. To control for exit bias, in regression (2) we use the Inverse Mills Ratio (IMR) as an additional explanatory variable. In regression (3) we exclude the years with negative aggregate productivity growth. All the estimation details are reported in Appendix B. Numbers in parentheses are standard errors. We use *, **, and *** to indicate p-values smaller than 0.105, 0.05, and 0.01, respectively.

Appendix B: Robustness Analysis of Firms’ Risk Exposure

**Inverse Mills Ratio.** The exit of firms from the COMPUSTAT database universe is not a random process; hence, there is a sample selection issue that may bias our estimation results. Regressions (2) and (3) of Tables 2 and 3 address the sample selection bias issue and show that our results are very robust. While the implementation of regressions (1) and (3) is straightforward, regression (2) deserves more attention. We implement the Heckman (1979) two-stage procedure as follows. First, we project exit on the Altman (2000) Score, size (measured by total assets in real terms), size squared, R&D expenditure-sales ratio, earnings over sales, capital-labor ratio, and aggregate productivity growth. Second, we compute the implied Inverse Mills Ratio (IMR) (see Greene (2002)) and include it as an additional independent variable in regression (21).

**Endogeneity and production function.** An alternative way to estimate the production function avoiding endogeneity issues is to work with the following regression:

\[
\ln y_{i,j,t} = v_j + z_{i,j} + w_{j,t} + α_{1,j} \ln k_{i,j,t} + α_{2,j} \ln n_{i,j,t} + u_{i,j,t}. \tag{B.1}
\]

The parameters \( v_j, z_{i,j}, \) and \( w_{j,t} \) indicate an industry dummy, a firm fixed effect, and an industry-specific time dummy, respectively. The residual from the regression is denoted by \( u_{i,j,t} \). Using the production function in (20), the log productivity for each firm can be calculated as

\[
\ln a_{i,j,t} = v_j + z_{i,j} + w_{j,t} + u_{i,j,t}. \tag{B.2}
\]

Given this estimation of firms’ productivity, we proceed as before in estimating equation (21). The results are summarized in Table B.1 and are consistent with those reported in Table 2.
Table B.2: Exposure to Aggregate Risk by Firm Age & B/M

<table>
<thead>
<tr>
<th>Regression</th>
<th>$\Delta \ln A$</th>
<th>$AGE$</th>
<th>$AGE * \Delta \ln A$</th>
<th>B/M</th>
<th>Obs.</th>
<th>Firms</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>-0.034</td>
<td>-0.001***</td>
<td>0.008***</td>
<td>-0.138***</td>
<td>68450</td>
<td>7303</td>
</tr>
<tr>
<td></td>
<td>(0.222)</td>
<td>0.000</td>
<td>(0.003)</td>
<td>(0.008)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>1.235***</td>
<td>-0.000</td>
<td>0.019***</td>
<td>-0.171***</td>
<td>21253</td>
<td>3956</td>
</tr>
<tr>
<td></td>
<td>(0.527)</td>
<td>(0.001)</td>
<td>(0.007)</td>
<td>(0.017)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>0.043</td>
<td>-0.001***</td>
<td>0.017***</td>
<td>-0.149***</td>
<td>57151</td>
<td>7197</td>
</tr>
<tr>
<td></td>
<td>(0.340)</td>
<td>(0.000)</td>
<td>(0.004)</td>
<td>(0.009)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes - This table reports firms’ risk exposure by age. All estimates are based on the following second-stage regression: $\Delta \ln a_{ijt} = \xi_0 + \xi_1 \Delta \ln A_t + \xi_2 AGE_{i,j,t} + \xi_3 AGE_{i,j,t} * \Delta \ln A_t + \xi_4 B/M_{i,j,t} + \varepsilon_{ijt}$. Regression (1) is obtained using the whole sample. To control for exit bias, in regression (2) we use the Inverse Mills Ratio (IMR) as an additional explanatory variable. In regression (3) we exclude the years with negative aggregate productivity growth. All the estimation details are reported in Appendix B. Numbers in parentheses are standard errors. We use *, **, and *** to indicate p-values smaller than 0.10, 0.05, and 0.01, respectively.

Controlling for B/M. In the data, the assets of a firm are a combination of young and old vintages of installed physical capital and growth options. We use B/M to proxy for growth option intensity and add it to equation (21) as an additional regressor. Table B.2 shows that after controlling for B/M our results are still in line with those reported in table 2. We notice also that when we control for B/M in equation (22), we still find that: (1) the estimated exposure of young firms, $\phi_Y$, is small and statistically not different from zero; (2) the exposure of the other older firms is positive and statistically significant; and (3) $OMY = \phi_O - \phi_Y$ is positive, statistically different from zero and close to its model counterpart, ie, 1.12. Results available upon request.
Appendix C: Measurement of Quantities and Prices in the Data and in the Model

1. **Consumption** \( (C_t) \). Per capita consumption data are from the National Income and Product Accounts (NIPA) annual data reported by the Bureau of Economic Analysis (BEA). The data are constructed as the sum of consumption expenditures on nondurable goods and services (Table 1.1.5, lines 5 and 6) deflated by corresponding price deflators (Table 1.1.9, lines 5 and 6).

2. **Physical Investment** \( (I_t) \). Physical investment data are also from the NIPA tables. We measure physical investment by fixed investment (Table 1.1.5, line 8) minus information processing equipment and software (Table 5.5.5, line 3) deflated by its price deflator (Table 1.1.9, line 8). Information processing equipment and software is interpreted as investment in intangible capital and is therefore subtracted from fixed investment.

3. **Measured Output** \( (Y_{M,t}) \). It is the sum of total consumption and physical investment, that is, \( C_t + I_t \). We exclude government expenditure and net export because not explicitly modelled in our economy. Notice also that the NIPA tables do not account for \( J_t \) over the sample 1929–2003.

4. **Intangible Investment** \( (J_t) \). We follow the procedure in Corrado, Hulten, and Sichel (2006) to construct a measure of intangible investment from 1953 to 2003. Prior to 1953 data are not available.

5. **Share of hours worked** \( (N_t) \). We divide total private hours worked by civilian labor force. Data are annual from the Bureau of Labor and Statistics (BLS). Prior to 1948 data are not available.

6. **Asset returns.** Both the market returns and the HML factor are from the Fama-French dataset available online on K. French’s webpage.

7. **Risk-free rate.** The nominal risk-free rate is measured by the annual 3-month T-Bill return. The real risk-free rate is computed by subtracting realized inflation from the nominal risk-free rate.

8. **Projected Volatility of Market Returns.** We use the multifactor productivity index for the private nonfarm business sector from the Bureau of Labor Statistics (BLS) as a proxy for total factor productivity (denoted by \( A_t \) in the model). We estimate the long-run component of productivity \( x_t \) using a three year moving average. At this point we are able to estimate equation (27), construct the projection of the market return, and measure the amount of volatility explained by productivity shocks. We also repeated this procedure using other different kinds of moving averages of productivity growth to proxy for the long-run component. We got similar results. We adopt this procedure both in the data and in the model simulations.