A Dynamic Politico-Economic Model
of Intergenerational Contracts*

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Abstract

This paper investigates the conditions for the emergence of implicit intergenerational contracts without assuming reputation mechanisms, commitment technology and altruism. We present a tractable dynamic politico-economic model in OLG environment where politicians play Markovian strategies in a probabilistic voting environment, setting multidimensional political agenda. Both backward and forward intergenerational transfers, respectively in the form of pension benefits and education investments, are simultaneously considered in an endogenous human capital setting with labor income taxation. On one hand, social security sustains investment in public education; on the other hand investment in education creates a dynamic linkage across periods through both human and physical capital driving the economy toward different Welfare State Regimes. Embedding a repeated-voting setup of electoral competition, we find that in a dynamic efficient economy both forward and backward intergenerational transfers simultaneously arise. The equilibrium allocation is education efficient, but, due to political overrepresentation of elderly agents, the electoral competition process induces overtaxation compared with a Benevolent Government solution with balanced welfare weights.

JEL Classification: C61, D71, E62, H11

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“Why should I care about future generations? What have they done for me?”
(Groucho Marx)

1 Introduction

The implementation and sustainability of public intergenerational redistributive programs are crucial issues in the current political debate. On one hand, the shift of the age-balance in favor of the elderly alters the economic nature underlying the enforcement of redistributive welfare programs. On the other hand, the increasing age of median voter makes the political choices more responsive to the need of the crucial group of voters - the old cohort.\(^1\) The demographic changes and the political power reassessments suggest that past intergenerational transfers promises, especially in the form of pension benefits, become more and more unaffordable over time.

![Figure 1: Source: Comparative Welfare State (2004)](image)

Although in the last decades political debates have often concerned reforms of current pension systems based on retributive schemes in favor of private contributive schemes; nonetheless in most developed countries the bulk of retirement income comes mostly from the public sector, around 60%. As shown in Figure 1, the share of GDP per-capita devoted to finance social security in OECD countries has increased over time.

One of the main implications of this trend is that the intergenerational disagreement over the allocation of public resources turns to be a battleground, pitting young against old and taxpayers against recipients, especially when balanced budget conditions are required to be met. For this reason, it becomes critical to explore the conditions under which intergenerational transfers, as an outcome of a political voting game, can be implemented and why the welfare system developed so far has became a stable institution of modern society. The main objective of this

\(^1\)As reported by OECD (2007), between 1975 and 2005 the average age of median voter increased three times faster than it had in the preceding 30 years. At the same time, the dependency ratio - the ratio of workers to pensioners - has steadily deteriorated in all rich countries. It is expected to shift from 4 points in 1970 to around 2 in 2050.
work is to provide a tractable dynamic politico-economic theory to analyze how intergenerational conflicts affect, through the political mechanism in the form of democratic vote, the size and composition of public expenditure in a context of population aging.

Figure 2 reports the fiscal policy pattern for the USA in 2001. The plot suggests that the intergenerational redistribution scheme matches the self-enforcing requirements when the working age-class collects taxes and both elderly and young enjoy benefits in the form of backward and forward transfers, respectively. Building on Boldrin and Montes (2005), who provide normative prescriptions for the optimal intergenerational redistribution, we show that multidimensional efficient intergenerational transfer schemes in the form of PAYG and public education transfers can be supported as a politico-economic equilibrium of an intergenerational game.

We consider a three period OLG economy populated by ideological heterogeneous agents who participate to repeated political elections. When young, agents acquire skills if education transfers are publicly provided without having access to private credit markets. When adult, the individuals offer inelastically labor and pay taxes; while when old, agents retire and receive unproductive transfers to sustain their consumption. The presence of a political system is justified by the need to finance the provision of the public spending under credit market constraint. The electoral competition takes place in a majoritarian probabilistic environment, where politicians compete by proposing multidimensional fiscal platforms under uncertainty (i.e. before the realization of ideological shock) subject to intra-period balanced budget. We assume away the provision of public goods - a key element in the political economy of fiscal policy - to pick out the impact of political institutions on intergenerational transfers. We introduce two restrictions to the neoclassical growth environment in order to isolate the basic mechanism at work. We rule out altruism and we consider a small open economy where the wage per efficient unit and the return on saving are fixed.3

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This issue is well investigated by Tabellini (1991), Lizzeri and Persico (2001), Hassler et al. (2005) and Bassetto (2008).

3Focusing on a small open economy we avoid the crowding out effect on prices, then the conflict between age-classes arise not only because of different life span but also for the difference in ownership of productive factors as well as in the source of income.
Technically, this paper highlights two main features concerning fiscal policies. Firstly, several political choices have to be set simultaneously, so the political space cannot be reduced to a mere unidimensional problem. Secondly, political decisions and private intertemporal choices are mutually affected over time. Forward-looking selfish agents internalize how current political choices influence the evolution of the economy and the implementation of future policies. To fully describe the endogenous feedback effects, we embody the minor causes should have minor effect principle to implement differentiable stationary Markov perfect equilibria, where the policies are conditioned on the two payoff-relevant state variables of the economy: Physical and human capital. We characterize the policy rules as an equilibrium outcome in a finite horizon environment when time goes to infinity. As a result, we overcome the main limit of earlier literature, related to the adoption of trigger strategies equilibria, which are not robust to finite horizon refinement.

We argue that selfish adults buy insurance for their future old age by paying productive education transfers to their children in order to raise the labor productivity of the next period. When old, they partially grab the bigger output, in the form of PAYG transfers, by exerting political power in a probabilistic voting environment. The redistributive scheme works only if the cost of providing a productive transfer is low with respect to the value of receiving a pork-barrel transfer when old. Therefore, if a PAYG pension scheme is introduced, its future beneficiaries may become supportive of higher funding in public education via taxes. In other words, the existence of a retributive social security system provides incentives to invest optimally in human capital and, as a consequence, it becomes efficient-enhancing for the economic system. Thus, the two age-specific redistributive programs may self-sustain reaching optimality.

Despite the simple structure of the game, we reach several interesting results, consistent with the empirical correlations. We find an analytical solution for the intergenerational contract, which is robust to finite horizon restrictions. Both physical capital and human capital play a relevant role in shaping the politico-economic equilibrium and in affecting the transition dynamics. Due to politicians’ opportunistic behavior, strategic persistency drives the setting of the income tax. The equilibrium predicts that the higher the physical capital, the lower the income tax rate, consistently with previous literature. Furthermore, human capital plays a crucial role in two different ways. On one hand it mitigates the politicians’ strategic behavior. Precisely, the higher the level of human capital, the flatter the equilibrium policy function and the lower responsiveness of taxation policy decisions on the level of private savings, weakening the strategic channel through which politicians can increase the probability to win elections. On

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4PAYG pension schemes have often been criticized as being detrimental to growth. According to Feldstein (1974) unfunded pension schemes have a negative effect on capital accumulation, since they discourage private saving and, unlike funded pension system, the payments into the PAYG scheme do not contribute to national saving. Moreover, the implicit rate of return on contributions to a PAYG scheme typically falls short of the interest rate. According to this line of analysis, PAYG pension systems reduce per capita income. This standard argument is focused on physical capital accumulation and fails to take notice of the effect of PAYG pension systems have on the accumulation of human capital, particularly through public education. Primary and secondary education is now overwhelmingly publicly financed in all OECD countries, and universities also receive substantial funding from public sources.

the other hand, human capital perturbs the political choice concerning the size of government, driving the economy towards different welfare state regimes and long-run convergence dynamics. Specifically, the presence of endogenous human capital formation generates two welfare state regimes. A complementarity regime, in which higher the human capital and, consequently, larger the tax base, higher the tax rate, and a substitutability regime, in which the opposite relation holds.

The sustainability conditions of the intergenerational contract strongly rely on both the dynamic efficiency condition and the ideological heterogeneity of voters. The dynamic efficiency requirement is necessary for the simultaneous existence of PAYG and public education programs. As long as the implicit net return of pensions is higher than the capital rental price, incentives in spending simultaneously on both sides of the redistribution programs arise. As soon as the interest rate falls below the economic growth rate, than no incentives for the adults to reduce current consumption in order to sustain forward transfers would emerge. As a result, in equilibrium only backward transfer would be provided. The ideological heterogeneity is necessary for the sustainability of the contract. Suppose the agents are ideologically homogeneous and decisions are taken through a deterministic majoritarian voting system. If adults detain power at each time, then forward transfers cannot be sustained as a Markov perfect politico-economic equilibrium of the intergenerational game. If agents anticipate that when old they will be prevented to extract rent by being in power and to benefit from higher return to physical capital, they never reduce their current consumption by transferring resources forward. Whereas, they still decide to optimally self-sustain backward transfers under dynamic inefficiency. Equivalently, if the political power is assigned to elderly, then old would act myopically by expropriating adults and by transferring the collected resources backward.

To conclude the existence of PAYG social security programs supports public investment in education even in absence of altruism and general equilibrium price effects. At the same time in equilibrium the impact of education spending on the social security transfers is always positive. By supporting a current higher education cost, the adults internalize that it will generate a higher taxable income, guaranteeing a higher level of pension benefits when old. Furthermore, demographic aging increases the equilibrium per-capita level of forward transfers, whereas the political aging has a positive impact on taxation.

Compared with a benevolent government solution with balanced welfare weights, the equilibrium allocation is education efficient, but, due to political overrepresentation of elderly agents, the electoral competition process induces overtaxation. These distortions come from the politicians’ strategic behavior. In setting taxation rules, short-lived politicians take into account that future representatives will compensate the fiscal cost of current adults by paying the pensions in their old age. Thus, the expected policy response of future politicians reduces the current cost of transferring resources to the elderly and leads to overspending, unless the adult enjoy an unusually large political power. Consequently, by transferring too many resources to old, the

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5 The sustainability problem of redistributive programs in favor of the elderly appears exacerbated by the dynamically efficient growth path experienced by developed countries since the Second World War (Abel et al., 1989).
politicians fail to provide the optimal income tax rate policy.

The paper is organized as follows. Section 2 reviews the literature. Section 3 presents the model characterizing the economic and political environment. Section 4 describes the politico-economic equilibrium and the maximization problem. Section 5 fully characterizes the equilibrium in transition and in the long run. Section 6 introduces the government problem without commitment, comparing the results with the decentralized one. Section 7 concludes. All proofs are contained in the Appendix.

2 Literature Review

This paper relies on the dynamic political-economy literature that incorporates forward-looking decision makers in a multidimensional policy space (Krusell et al., 1997). Recent works (Forni, 2005, and Gonzales-Eiras and Niepelt, 2008) present models on social security in a repeated voting environment. We depart from these studies by supporting and giving new theoretic fundamentals to the existing literature, which recognizes the link between productive and redistributive public spending. From a purely economic point of view, Boldrin and Montes (2005) formalize public education and PAYG system as two parts of an intergenerational contract where public pension is the return on the investment into the human capital of the next generation. The authors show how an interconnected pension and public education system can replicate the allocation achieved by a complete market. Allowing issue-by-issue voting, Rangel (2003) studies the ability of non-market institutions to optimally invest in forward and backward intergenerational goods. Bellettini and Berti Ceroni (1999) incorporate politics in an economic model to analyze how societies might sustain public investments (i.e. education) even if the interests of those benefitting from the investment are not represented in the political process. As main shortcoming the earlier studies have assumed that voters play trigger strategies. Although trigger strategies may be analytically convenient, they lead to multiplicity of equilibria. Furthermore, they require coordination among agents and costly enforcement of a punishment technology, which may not work when agents are not sufficiently patient. Finally, they are not robust to refinement such as backward induction in a finite horizon economy when time tends to infinity. Departing from the existing literature, instead of emphasizing that the complementarity between education and pension payments is mainly sustained because of reputation mechanisms, our model adopts a different perspective. We focus on the resolution of the intergenerational conflicts over the determination of the amount of the two public spending components by adopting Markovian strategies. Closer in the spirit of our work is Gonzales-Eiras and Niepelt (2011). The authors study how demographic ageing affects the per-capita long run endogenous growth in a two period OLG economy. Differently from them, our theory put emphasis on the analyses of the intergenerational fiscal sustainability during the transition path, in a context in which positive spillover from general equilibrium price effects and endogenous growth are ruled out.

Many recent studies have identified the public good provision as the critical variable that allows the emergence of an intergenerational redistributive scheme. Bassetto (2008) studies how
intergenerational conflicts shape government policies when taxes, transfers and public good are jointly determined. Without public good provision the government would be running a purely redistribution scheme, to which any household that is a net loser would object: Hence, the only possible equilibrium would entail no taxes and no transfers. In a simpler underlying economic environment of majority voting Hassler et al. (2007) develop an OLG model of welfare state where tax revenues are used to finance public goods and age-dependent transfers. Studying a linear quadratic economy, they provide analytical solution, but the voting strategies equilibrium turns out to be either constant or independent of fundamentals. Unlike these models we exclude public good provision.

Finally, some studies have analyzed the interaction between education and social security by adopting an altruistic motive. Kaganovich and Zilcha (1999) study a model where altruistically-motivated parents invest in the human capital of their children. When parents retire, the labor income of their children is taxed to finance their social security benefits. The link between the human capital of children and the parents’ retirement benefits is disregarded in each parental educational decision, but it is captured by the government. In Ehrlich and Lui (1998) children provide support to parents in old age, so that parents have an interest in the education of their children due to pure altruistic motives. Despite the usefulness of these studies, they adopt a centralized point of view to justify the simultaneous investment in both redistributive programs. In contrast to these studies, as a device to bring out incentives to pay tax, we allow for productive and long-lasting impact transfers to the future generation of workers. The dynamic intertemporal linkage occurs by affecting the benefits of adults, which expect to grab future output by exerting political power when old.

3 The Model

Consider an OLG economy populated by an infinite number of ideologically heterogeneous agents, living up to three-periods: Young, Adult and Old. $i \in \{1, 2\}$ denotes the adult and elderly cohorts, respectively. Time is discrete, indexed by $t$, and runs from zero to infinity. Population grows at a constant rate $n > -1$, thus the mass of a generation born at time $\tau$ and living at time $t$ is equal to $N_\tau^t = N_0 (1 + n)^t$. At each time two infinite living parties, denoted by $i \in \{A_t, B_t\}$, compete proposing an electoral platform in order to maximize their probability of winning election. All variables are expressed in per-adult terms.

3.1 Households

An agent $j$ born at time $t - 1$ and living at time $t$ evaluates consumption according to the following expected intertemporal non altruistic utility function:

$$ U_{t-1}^j = u(c_{1t}^j) + \sigma_{jt}^1 + \varphi_t + \beta E_t \left[ u(c_{2t+1}^j) + \sigma_{jt+1}^2 + \varphi_{t+1} \right] $$

(1)

where $\beta \in (0, 1)$ is the individual discount factor. $\sigma_{jt}^i$ represents the idiosyncratic ideological shock uniformly distributed over the interval $[-\frac{1}{2\sigma^1}, \frac{1}{2\sigma^1}]$, whose distribution is cohort specific.
It measures voter $j$’s individual preferences toward party $B$. $\varphi_t$ describes the aggregate shock uniformly distributed over the interval $[-\frac{1}{2\pi}, \frac{1}{2\pi}]$ and measures the average relative popularity of candidates from party $B$ relative to those from party $A$. $c^1_{1t}$ represents the consumption at time $t$ when adult and $c^2_{t+1}$ denotes the consumption at time $t+1$ when old. In the first period of life (i.e. childhood), the individual does not consume. The function $u(\cdot)$ is concave, twice continuously differentiable and satisfies the usual Inada condition. We assume preferences exhibit logarithmic form, i.e. $u(c) = \log c$.

When young, agents spend all their time endowment in acquiring skills if education transfers, $e_{et}$, are publicly provided without having access to private credit markets. When adult, the individuals supply inelastically labor and consume their labor income per unit of time, $w_t$, net of the proportional labor income taxes, $z_{it}$, and individual savings, $s_t$. When old, the individuals retire and consume their total income, equal to the sum of pension benefits that their children pass to them in the form of PAYG transfers, $p_{it+1}$, and the capitalized savings at a fixed interest rate $R$. Then, the individual budget constraints for the adults and old are, respectively:

\[
\begin{align*}
    c^1_{1t} &\leq w_t (1 - z_{it}) - s_t \\
    c^2_{t+1} &\leq Rs_t + (1 + n) p_{it+1}
\end{align*}
\]

The net present value at time $t$ of the lifetime wealth of an agent born at time $t-1$ is equal to:

\[
I_t = w_t (1 - z_{it}) + \left(\frac{1 + n}{R}\right) p_{it+1}
\]

### 3.2 Production

At each time $t$ the economy produces a single homogenous private good, $Y_t$, combining labor, $L_t$, and physical capital, $K_t$, according to a linear technology. The linearity of the production function can be derived as an equilibrium outcome in a context of perfect international capital mobility and factor price equalization in the presence of goods trade. It emphasizes the inter-generational conflicts arising by the divergent economic interests between the two productive classes: workers (i.e. adults) and capitalists (i.e. old). Then, the production function at time $t$ is as follows:

\[
Y_t = w_t L_t + RK_t
\]

where $Y_t \equiv y_t N^t_{1t-1}$, $L_t \equiv N^t_{1t-1}$, and $K_t \equiv k_t N^t_{1t-1}$. The wage rate, $w_t = \omega (1 + h_t)$, and the gross rental price to capital, $R$, are determined by the marginal productivity conditions for factor prices. The wage per efficiency unit, $\omega$, is augmented by the level of human capital acquired the period before, $h_t$. Without loss of generality we normalize $\omega = 1$.

The human capital of an agent born at time $t$ is an increasing function of the public expenditure on education and the parental education. Public education transfers are supplied in an

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7 For a recent discussion on the missing credit markets to finance education, see Kehoe and Levine (2000).

8 The importance and the empirical relevance of both the public spending in schooling and the parental education input in the formation of the human capital of the young people has been explored theoretically, as well as empirically. For a comprehensive survey of the related literature see Becker and Tomes (1986).
egalitarian way. The following Cobb-Douglas human capital technology, \( h_{t+1} = H(h_t, e_{tt}) \), is adopted:

\[
(1 + n) h_{t+1} = (\alpha h_t + (1 - \alpha) \theta)^\theta e_{tt}^{1-\theta}
\]  

(5)

where \( \theta \in [0, 1] \) represents the public education productivity. \( h_t \) is the dynasty’s human capital at time \( t \) and \( \theta \) is the constant society endowment of human capital.\(^9\) If \( \alpha \in (0, 1) \), then the externality generated by \( \theta \) takes the form of positive spillover from the society endowment of human capital to the formation of future skills. If parties decide to not invest in public education, then \( H(h_t, 0) = 0 \). As a consequence, the next period economy will be reverted to an endowment economy.

Physical capital fully depreciates each period. Consequently, the level of saving determines the dynamics of per-capita physical capital accumulation. The capital market clears when:

\[
(1 + n) k_{t+1} = s_t
\]

(6)

### 3.3 Fiscal Constitution

In order to provide intergenerational transfers, the agents in the economy have to devise a politician. In each period, the politician raises revenues through labor income taxes and uses the proceeds to purchase consumption goods to pay transfers to the young and old generations. We assume that the politician is prevented from borrowing, then the public balance must hold in every period. This implies that total benefits paid to young and elderly equalize total contributions collected from the adult generations. The balanced budget constraint requires:

\[
w_t z_{tt} = e_{tt} + p_{tt}
\]

(7)

Condition (7) reduces the multidimensionality of political platform to a bidimensional plan \( f_{tt} \equiv \{z(e_{tt}, p_{tt}), e_{tt}, p_{tt}\} \) where \( z(\cdot) \in [0, \bar{z}] \), \( p_{tt} \in [0, \bar{p}_t] \) and \( e_{tt} \in [0, \bar{e}_t] \) at each time \( t \), with \( \bar{p}_t = \bar{e}_t \equiv w_t \bar{z} \) equal to the maximum feasible level of redistributive transfers.

### 4 Maximization and Equilibrium

As in Krusell et al. (1997), we characterize the equilibrium of the economy as a dynamic politico-economic equilibrium. To fully describe the mechanism underlying the resolution strategy, we provide a detailed description of the timing of the game. At each time \( t \):

i. a new generation of young people is born;

ii. before the realization of the ideological shocks among voters, parties compete proposing their political fiscal platforms;

\(^9\)The constant society endowment, \( \theta \), is the country-specific human capital endowment (or civic capital), which imperfectly substitutes the dynasty’s human capital. It is introduced to appropriately perform cross-countries analyses and for analytically convenience. By simultaneously adopting the final good technology as in Eq. (4) and the human capital technology as in Eq. (5), we rule out the effects generated by endogenous growth, at the same time enabling the economy to reach a stable steady state different from autarchy.
iii. after the realization of the ideological shocks among voters, agents vote for their preferred candidates;

iv. the firms hire workers and rent capital;

v. agents take economic decision on saving;

vi. the older generation dies; the young and adult generations age and become adult and old, respectively.

Due to the sequential nature of the timing of the game we solve backward. First, the agents determine the individual level of saving and firms produce the homogenous final good given the fiscal stance (Competitive Economic Equilibrium). Second, elected short-lived office-seeking politicians determine both the level of taxation and the amount of backward and forward transfers in order to win elections (Politico-Economic Equilibrium).

4.1 Competitive Economic Equilibrium

In a competitive economic equilibrium, each individual $j$ when adult chooses her lifetime consumption taking fiscal and redistributive policies as given. Maximizing Eq. (1) subject to the individual budget constraints (2) and (3), and feasibility constraints $c_{it}^1 > 0$ and $c_{it+1}^2 > 0$, the following first order condition for interior solutions must hold:

$$0 = \eta(\cdot) \equiv u_c(c_{it}^1) - R\beta E_t \left[ u_c(c_{it+1}^2) + \sigma^2_{jt+1} + \varphi_{t+1} \right]$$  \hspace{1cm} (8)

In equilibrium by implicit function theorem a unique saving function, $s_t = K(h_t, e_t, p_t, p_{t+1})$, which satisfies Eq. (8) exists. Thus, using Eq. (6) we obtain:

$$(1 + n) k_{t+1} = K(h_t, e_t, p_t, p_{t+1})$$  \hspace{1cm} (9)

Definition 1 (Competitive Economic Equilibrium) Given the initial conditions $\{h_0, k_0\}$ and the sequence of policies $\{f_t\}_{t=0}^{\infty}$, a competitive economic equilibrium is a sequence of allocations $\{c_{it}^1, c_{it+1}^2, h_{t+1}, k_{t+1}\}_{t=0}^{\infty}$ that solves the maximization problem of the adults, i.e. Eq. (9) is satisfied, and the market for production inputs clears, i.e. (5) and (6) hold at each time.

After plugging Eq. (9) into Eq. (2) and (3) the following individual consumption levels are attained:

$$c_{it}^1 \equiv c_{it}^1(f_{it}, h_t, k_{t+1}) = w_t - e_t - p_t - (1 + n) k_{t+1}$$  \hspace{1cm} (10)

$$c_{it+1}^2 \equiv c_{it+1}^2(f_{it+1}, k_{t+1}) = (1 + n)(Rk_{t+1} + p_{t+1})$$  \hspace{1cm} (11)

The indirect intertemporal utility of an individual $j$ belonging to the adult cohort living at time $t$, up to the current realization of the ideological bias, is equal to:

$$W_{jt}^1 = u(C_{jt}^1) + \beta E_t \left[ u(C_{jt+1}^2) + \sigma^2_{jt+1} + \varphi_{t+1} \right]$$  \hspace{1cm} (12)
For an old individual born a time $t-2$ the indirect utility, up to the realization of the ideological bias, is equal to:

$$\mathcal{W}^2_{jt} \equiv u \left( C^2_{it} \right) \quad (13)$$

We call *autarky* indirect utility, $\mathcal{W}_{jt}$, the lifetime utility of an agent $j$, when taxation and public spending are precluded:

$$\mathcal{W}_{jt} \equiv \max_{s_t} \left\{ u^j_{t-1} \mid I_t = 1 \right\}$$

Suppose there is no government that has the authority to levy taxes. As a consequence, the adults keep the entirety of their labor income to purchase final good and to save. Capital earns a gross return of $R$, used by old to buy consumption goods. Clearly, the economy converges to the unique steady state in one period, where $h = 0$, $k = \frac{\beta}{(1+r)(1+\eta)}$, $c^1_t = \frac{1}{1+\beta}$, $c^2_t = \frac{\beta}{1+\beta} R$ and $w = 1$.

**Definition 2 (Equilibrium Feasible Allocation)** An equilibrium feasible allocation is a sequence of competitive economic equilibrium allocations $\left\{ c^1_t, c^2_{t+1}, h_{t+1}, k_{t+1} \right\}_{t=0}^{\infty}$ and policies $\{ f_t \}_{t=0}^{\infty}$ that satisfy, the balanced budget constraint, Eq. (7), and the fiscal feasibility conditions at each time $t$.

### 4.2 Political Competition

In this section we describe how parties interact in the electoral competition. Public policies are chosen through a repeated voting system according to majority rule. Young have no political power. To describe the behavior of politicians we consider a probabilistic voting setting. Suppose there are two parties, $\{A_t, B_t\}$, that compete to attain political power, with no ability to extract individual rent from election. As a consequence, their objective is the maximization of the probability of winning elections at each time in order to implement the proposed political platform, $f_t$, with no ability to commit to future policies. At each time, first parties propose their political platforms, second any individual votes. Agents decide to vote for

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10 We reflect the fact that young people show a much lower turnout rate at elections in comparison to adults and old. As Galasso and Profeta (2004), report in some countries the elderly have a higher rate at elections than the youth. In the USA turnout rates among those aged 60-69 years is twice as high as among the young (19-29 years). Again in France it is almost 50% higher.

11 Due to the multidimensionality of the political platform Condorcet winner generally fails to exist. Consequently the median voter theorem does not hold (Plot, 1967). In the literature there are three main influential approaches to making predictions when the policy space is multi-dimensional. The first is the implementation of structure-induced equilibria. By following Shepsle (1979), agents vote simultaneously, yet separately (i.e. issue by issue), on the issues at stake. Votes are then aggregated over each issue by the median voter. See Condez-Ruiz and Galasso (2005) for a detailed discussion of this approach. The second is the legislative bargaining approach, which stems from the seminal work of Baron and Ferejohn (1989) and is further developed by Battaglini and Coate (2006). This approach applies when legislators’ first loyalty is to their constituents and when legislative coalitions are fluid across time and issues. The latter concerns the adoption of probabilistic voting rule. While this model of voting dates back to the 1970s, its resurgence in popularity stemmed from Lindbeck and Weibull (1987). It applies to political environments where party discipline is strong and the winning political party simply implements its platform. See Persson and Tabellini (2000) for a survey of this framework.
party $B_t$ as long as the following inequality holds:

$$\sigma^i_{jt} \geq \sigma^j (k_t, h_t) \equiv W^i_{jA_t} - W^j_{jB_t} - \varphi_t$$  \hspace{1cm} (14)$$

Formally, $\sigma^j (k_t, h_t)$ represents the swing voter in cohort $i$. The share of voters belonging to cohort $i$ for party $B_t$ is equal to:

$$\lambda^i_t \equiv \frac{1}{2} - \sigma^i (W^i_{jA_t} - W^j_{jB_t} - \varphi_t)$$  \hspace{1cm} (15)$$

Under majoritarian rule, party $B_t$ wins the election if and only if it obtains the largest share of votes, which implies that $\varphi_t$ must be larger than the threshold level $\varphi (k_t, h_t)$. As a consequence, the objective function of party $B_t$ simplifies to:

$$\max_{f_{B_t}} \Pr (\varphi_t \geq \varphi (k_t, h_t)) = 1 - \varphi_\varphi (k_t, h_t)$$  \hspace{1cm} (16)$$

and equivalently for party $A_t$:

$$\max_{f_{A_t}} \Pr (\varphi_t \leq \varphi (k_t, h_t)) = \frac{1}{2} + \varphi_\varphi (k_t, h_t)$$  \hspace{1cm} (17)$$

4.3 Politico-Economic Equilibrium

To characterize the politico-economic equilibrium, we need to consider the dynamic aspects of the political process that take place in the economy. We restrict the notion of politico-economic equilibrium to the differentiable Markov perfect concept as equilibrium refinement of subgame perfect equilibria.\footnote{The Markov-perfect concept implies that outcomes are history-dependent only in the fundamental state variables. The stationary part is introduced to focus only on the current value of the payoff relevant state variable. Consequently the vector of equilibrium policy decision rules is not indexed by time, i.e. the structural relation among payoff-relevant state variables and political controls is not time variant. The differentiable part is a convenient requirement to avoid multiplicity of equilibrium outcomes and in order to give clear positive political predictions.} The payoff-relevant state variables of our economy are the assets held by the adults and old, i.e. human and physical capital.

At each time the implementation of a particular political platform generates dynamic linkages of policies across periods. Due to the non-negligible impact of current political actions on future equilibria, rational agents internalize this dynamic feedback. In our framework the dynamic linkages generated by physical and human capital arise both directly, affecting asset accumulation decision, and indirectly, affecting future political choices. Focusing on Markov strategies, agents are able to fully internalize the overall direct and indirect impact of taxation and redistribution through the evolution of the asset variables.

We denote by $e_{it} = E_{it} (h_t, k_t)$ and $p_{it} = P_{it} (h_t, k_t)$ the intergenerational equilibrium policies rules, and by $\mathcal{F}_{it} = \{ E_{it}, P_{it} \}$ their collection. Using condition (7), $z_{it} = Z_{it} (h_t, k_t)$ represents the taxation equilibrium policies rule. In a perfect forward-looking environment, where parties play Markov strategies, the following Definition of politico-economic equilibrium is adopted.
**Definition 3 (Politico-Economic Equilibrium)** A Markov perfect politico-economic equilibrium is a sequence of competitive economic equilibrium allocations \( \{c^1_t, c^2_t, h_{t+1}, k_{t+1}\}^\infty_{t=0} \) and policies \( \{f_t\}^\infty_{t=0} \), such that the functional vector of differentiable policy decision rules, \( F_t = \{E_t, P_t\} \), where \( E_t : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, \bar{e}] \) and \( P_t : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, \bar{p}] \), satisfies the following points:

i. Parties \( B_t \) and \( A_t \) solve the maximization program described in (16) and (17), respectively;

ii. The fixed point condition holds, i.e. the policies are fixed points of the mappings \( E_t \) into \( E^e_t (h_t, k_t) \) and \( P_t \) into \( P^e_t (h_t, k_t) \), where the apex \( e \) stands for expected.

The first equilibrium condition requires the political control variables, \( f_t \), to be chosen in order to maximize the party’s objective function, taking account that future redistribution and taxation depend on the current policy choices via both the equilibrium private decisions and future equilibrium policy rules. The second equilibrium condition requires that, if the equilibrium exists, it must satisfy the fixed point requirement. From a technical point of view, we are looking for two differentiable policies which obey the recursive rules given by the functions \( p_t = P_t (h_t, k_t) \) and \( e_t = E_t (h_t, k_t) \), where \( P_t \) and \( E_t \) are infinite dimensional objects and the key endogenous variables of the problem. The second fundamental element we are looking for is a function, \( K (h_t, k_t, f_t) \), which describes the private sector response to a one-shot deviation of the government, when agents expect future policies to be set by politicians according to \( F_t \), as a function of current state and political control variables. From Eq. (9) the function \( K (\cdot) \) fully describes the equilibrium behavior of private agents as a function of current state and both current and future policies. If differentiable functions, \( E_t \) and \( P_t \), which describe the policy behavior followed by politicians in equilibrium, exist, these rules can be internalized by fully rational private agents. It follows that:

\[
k_{t+1} = K (h_t, e_t, p_t, P_{t+1} (h_{t+1}, k_{t+1})) \tag{18}
\]

Plugging Eq. (5) into Eq. (18) and rearranging the terms, \( k_{t+1} = K (h_t, e_t, p_t) \) is obtained.\(^{13}\)

For notational purpose, we denote by \( \phi \equiv \frac{\sigma^2}{\delta} \in [0, \infty) \) a synthetic measure of the ideological bias among voters, which also represents the relative political weight of the voters belonging to old cohort. If \( 0 < \phi < 1 \), then on average the old cohort cares less about ideology and has more swing-voters than the adults one. For \( \phi > 1 \) the opposite holds, where the elderly represent the majority in the political debate. Finally, when \( \phi = 1 \), all voters are equally represented.

**Proposition 1** If a Markov perfect politico-economic equilibrium of the intergenerational voting game exists, then the parties maximize the following objective function with respect to \( \{e_t, p_t, h_{t+1}, k_{t+1}\} \):

\[
(1 + n) \left( u \left( C^1_t (e_t, p_t, h_t, k_{t+1}) \right) + \beta u \left( C^2_{t+1} (P_{t+1}, k_{t+1}) \right) \right) + \phi u \left( C^2_t (p_t, h_t, k_{t+1}) \right) \tag{19}
\]

\(^{13}\)Under full depreciation of physical capital and exogenous prices, the function \( K \) is not affected by the current level of physical capital.
subject to:

\[ u_c (C^1_t) - \beta Ru_c (C^2_{t+1}) = 0 \]  \hspace{1cm} (20)

As a consequence, parties \( A \) and \( B \) propose the same political platform, i.e. \( f_A t = f_B t = f t \).

**Proof.** (See appendix).

At each time \( t \) the political objective function, Eq. (19), has to be simultaneously maximized with respect to its arguments, i.e. the pair \( \{p_t, e_t\} \), subject to the balanced budget constraint, Eq. (7) and the Euler condition of the economic optimization problem, Eq. (20). The following first order conditions are attained for \( p_t \) and \( e_t \), respectively:

\[ p_t : 0 = \phi u_{c2}^2 - u_{c1}^2 + \beta (1 + n) u_{c2}^{t+1} \frac{dP_{t+1}}{dp_t} \]  \hspace{1cm} (21)

\[ e_t : 0 = -u_{c1}^2 + \beta (1 + n) u_{c2}^{t+1} \frac{dP_{t+1}}{de_t} \]  \hspace{1cm} (22)

Let us first refer to Eq. (21). At each time an interior solution for the pension transfers is simply determined as the outcome of a weighted bargaining between current elderly and adults, who reap benefits and sustain costs by a variation in the current pension scheme. The first term represents the elderly’s marginal benefits due to the increase in PAYGO social security rate. Since tax levying on labor income makes adults sustain the whole tax burden, the second term captures the adults’ marginal cost caused by a positive variation on the backward dimension. Finally, the third term measures the expected marginal impact of current variation in the pension benefits on the utility of future elderly. Similarly, an increase in public education transfers have a double-edged sword impact. On one hand, it makes current adult sustain direct costs due to an increase in the total fiscal burden, represented by the first part of Eq. (22). On the other hand, future old enjoy direct benefits from expected returns of productive investment in human capital, whose effects are captured by the second part.

Eq. (21) and (22) internalize the strategic effects, capturing how politicians can affect future policies through their current choices of \( f_t \). If \( \frac{dP_{t+1}}{dp_t} > 0 \) (\(< 0\)) and \( \frac{dP_{t+1}}{de_t} > 0 \) (\(< 0\)) agents expect that larger pension and education transfers lead to a higher (lower) level of pension in the future. Thus, representatives may strategically increase (reduce) \( p_t \) and \( e_t \) in order to affect the pension rate tomorrow.

### 4.3.1 Intergenerational Conflicts

In this section we discuss how the introduction of human capital as an additional state variable in an environment characterized by uncertainty over the political competition outcome modifies the incentives to sustain intergenerational contracts. Only when these two elements are jointly considered agents may credible coordinate their expectation over a growth-enhancing state variable and, in turns, achieve efficiency. To make clear the analysis, we consider the following four cases, separately.
Case I: Physical capital, adult median voter and no human capital, i.e. \( k_t > 0, \phi = 0 \) and \( h_t = 0 \). In this environment agents are precluded to access to a growth-enhancing technology. From Eq. (21), the first order condition to be satisfied requires \( 1 - (1 + \frac{n}{R}) \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} K_{p_t} = 0 \). Since \( \hat{K}_{p_t} = \frac{1}{1 + n} < 0 \), in equilibrium agents sustain positive transfers to the elderly by coordinating their expectations on the level of aggregate physical capital. The lower the level of savings generated by the income taxes paid when adult, the higher the level of future benefits when old, i.e. \( \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} < 0 \). As a consequence, the intergenerational contract where only pensions emerge is sustainable if and only if the economy experiences a dynamic inefficiency growth path, i.e. \( 1 + n > R \). If the rental price of capital were larger than the economic growth rate, then agents would not have incentive to substitute private savings with public one.

Case II: Physical capital, adult median voter and human capital, i.e. \( k_t > 0, \phi = 0 \) and \( h_t > 0 \). When the productive channel is at disposal, agents might coordinate on an additional payoff relevant state variable, i.e. human capital. From Eq. (21) and (22), the equilibrium condition requires that \( 1 - (1 + e_t) \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} K_{p_t} = 0 \) and \( \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} (K_{p_t} - K_{e_t}) - \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} H_{e_t} = 0 \) have to be simultaneously satisfied. Since \( K_{e_t} = -\frac{1}{1 + n} - \frac{1}{R} \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} H_{e_t} \) and \( K_{p_t} = -\frac{1}{1 + n} \), then the feasible equilibrium relation is \( \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} = 0 \). Adults have no incentives to credible coordinate on the human capital because they perfectly anticipate that the future generation has incentive to hold up. As a consequence, in equilibrium public education investment is set to zero and the economy reverses to a bad equilibrium, where only backward intergenerational transfers emerge. To induce agents to sustain forward transfers a credible coordination device has to be introduced in the economy, that we identify in the political uncertainty.

Case III: No physical capital, political uncertainty and human capital, i.e. \( k_t = 0, \phi > 0 \) and \( h_t > 0 \). Since elderly actively participate to the political debate, they always gain a rent in terms of backward transfers by exerting political power. As a result, agents credibly coordinate on human capital. At each time an agreement on intergenerational contract between different generations always emerge. From Eq. (21) and (22), the equilibrium condition requires that \( u_{c_{t+1}} = \phi u_{c_{t+1}} = 0 \) and \( (1 + n) \beta u_{c_{t+2}} \frac{\partial \hat{P}_{t+2}}{\partial \hat{h}_{t+2}} H_{e_t} - u_{c_{t+1}} = 0 \) have to be simultaneously satisfied. The presence of \( \phi \) makes the pension benefits emerge as a bargaining outcome between current adults and old without the requirement of infinite repetition of the game. As a consequence, the equilibrium consumption of the elderly is a constant share of total production, i.e. \( C_t^2 = \frac{\phi}{1 + \phi} Y_t \). Since \( H_{e_t} > 0 \), in equilibrium agents sustain positive transfers to young by generating a credible coordination on the level of aggregate human capital.

---


15 We distinguish three possible cases, i.e. \( \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} \geq 0 \). Suppose \( \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} > (\leq) 0 \), than from Eq. (22) the condition \( (K_{p_t} - K_{e_t}) < (>) 0 \) should be satisfied. This is not an equilibrium because the relational structure between policies and saving choice requires \( K_{e_t} < (>) K_{p_t} \). As a consequence, the unique equilibrium condition is \( \frac{\partial \hat{P}_{t+1}}{\partial \hat{h}_{t+1}} = 0 \) and \( (K_{p_t} - K_{e_t}) = 0 \).

16 To make forward transfers emerge previous literature (Bellettini and Berti Ceroni, 1999; Rangel, 2003) has focussed on reputational mechanisms, which works only if the game is infinitely repeated. Contrarily by introducing political uncertainty in the electoral process intergenerational contract emerge in finite horizon and drive the economy toward good equilibrium.
**Case IV:** Physical capital, political uncertainty and human capital, i.e. \( k_t > 0, \phi > 0 \) and \( h_t > 0 \). Differently from Case III, by introducing physical capital as an additional payoff relevant state variable, intergenerational conflicts arise due to the divergent economic interests between the different age-cohorts: workers (i.e. adults) and capitalists (i.e. old). When the share of physical capital per GNP is large enough, elderly should partially transfer their resources to adults in order to subsidize their consumption. However they always have incentives to walk out from the contract breaking intergenerational cooperation. In this circumstance adults decide to invest in human capital only if when old the relative share of physical capital will be reduced and they will enjoy backward transfers from the future generations. Even if intergenerational cooperation might be challenged by the agents’ divergent economic interests, the credible coordination over the evolution of human capital enables agents to sustain better equilibria compared to an adult median voter scenario (Case II). By using Eq. (21) and (22) the following Proposition holds.

**Proposition 2** If an intergenerational contract emerges as Markov perfect politico-economic equilibrium, then
\[
\frac{dp_{t+1}}{de_t} > 0 \quad \text{and} \quad \frac{dp_{t+1}}{dp_t} > \frac{dp_{t+1}}{dp_t}.
\]

**Proof.** (See appendix). □

Proposition (2) describes the intertemporal structural relations among policies to make forward and backward transfers simultaneously emerge as an intergenerational contract. The adults sustain the education spending only if they expect higher tax revenue that, in turns, generates a positive variation in the level of future pension benefits. As long as the return of public education investment is equal to the return of private savings, then the incentives to support the formation of skills arise. Furthermore, the marginal impact of current education investment on future pension benefits has to be larger than the marginal impact of current pension transfers on future one. By contradiction, if \( \frac{dp_{t+1}}{de_t} \) were smaller than \( \frac{dp_{t+1}}{dp_t} \), in order to maximize expected public returns it would be sufficient to support intergenerational cooperation over backward transfers without investing in public education and Case II arises.

## 5 Equilibrium Characterization

When voters coordinate their expectation on the level of human capital and political uncertainty characterizes the political environment, intergenerational contract emerges in a finite horizon. Therefore, the Markov perfect politico-economic equilibrium can be obtained as the limit of a finite-horizon equilibrium, whose characteristics do not significantly depend on the time horizon, as long as the time horizon is long enough. Once a recursive structure is identifiable, we obtain the equilibrium policy rules as the fixed point of the recursive problem in multidimensional environment.

**Lemma 1** Let \( m : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), such that \( m\left(\psi_{(j)}\right) := \left(\frac{\alpha \theta}{R} \psi_{(j)} + \frac{1-\theta}{R}\right)^{\frac{1}{q}} \). A stable fixed point of \( \psi_{(j+1)} = m \left(\psi_{(j)}\right) \) exists and is equal to \( \psi \equiv \psi\left(\alpha, \theta, R\right) \), where \( \psi_\alpha \geq 0, \psi_\theta \leq 0 \) and \( \psi_R \leq 0 \).

**Proof.** (See appendix). □
For notational purposes we denote by \( \Omega \equiv \frac{\phi}{\sigma + (1+n)(1+\beta)} \) and its complement to one the relative political bargaining power of the elderly and adults, respectively. The more the population ages (i.e. \( n \) decreases and \( \phi \) increases), the smaller the relative political weight of adults and larger the relative political weight of old. Furthermore, let \( \rho \equiv \frac{R}{R-(1+n)} \) be a measurement of the economy dynamic efficiency. The following Proposition characterizes the equilibrium outcomes of public choices in a fully rational environment when Markov strategies are implemented.

**Proposition 3** Under Lemma (1) and dynamic efficiency, a unique Markov perfect politico-economic equilibrium exists. The feasible rational policies, \( f_t = \{e_t, p_t\} \), which can be supported as Markov perfect politico-economic equilibrium, have the following functional forms:

i. \[
E_t = \epsilon_1 h_t + \epsilon_0
\] where \( \epsilon_0 \equiv (1 - \alpha) \theta \psi \), \( \epsilon_1 \equiv \alpha \psi \);

ii. \[
P(h_t, k_t) = \max \{0, -\pi_2 k_t + \pi_1 h_t + \pi_0\}
\] where \( \pi_2 \equiv R(1-\Omega) \), \( \pi_1 \equiv \left(1 + \frac{\theta}{1-\gamma} \epsilon_1 \right) \Omega \) and \( \pi_0 \equiv (1 + \rho) \left(1 + \frac{\theta}{1-\gamma} \epsilon_0 \right) \Omega \).

By balanced budget constraint the taxation equilibrium policy rule is:

iii. \[
Z(h_t, k_t) = \min \left\{1, -\zeta_3 \frac{k_t}{1 + h_t} + \zeta_2 \frac{h_t}{1 + h_t} + \zeta_1 \frac{1}{1 + h_t} + \zeta_0 \right\}
\] where \( \zeta_3 \equiv R(1-\Omega) \), \( \zeta_2 \equiv \epsilon_1 \left(1 + \frac{\theta}{1-\gamma} \Omega \right) \), \( \zeta_1 \equiv \epsilon_0 \left(1 + (1 + \rho) \frac{\theta}{1-\gamma} \Omega \right) + \rho \Omega \) and \( \zeta_0 \equiv \Omega \).

**Proof.** (See appendix). \( \blacksquare \)

From a structural point of view, the policy rules associated with the backward and forward transfers are linear in the level of the relevant payoff state variables. While education investment is positively affected only by the human capital production, the amount of pension benefits depends also negatively on the level of physical capital. As a consequence, the fiscal policy rule is a linear and negative function in the physical capital but not in the human capital level. Depending on the political bargaining intensity between adults and old, embedded in the coefficient \( \Omega \), the marginal impact of human capital on taxation decision can be either positive or negative.

A necessary condition to sustain an intergenerational contract characterized by the pair (23) and (24) is dynamic efficiency, i.e. \( R > (1+n) \). By contraction, suppose the opposite relation holds. Then adults would have incentives to transfer all private savings into public one and invest in public education. As a consequence future generations will coordinate their expectations only on human capital, by destroying the physical capital coordination channel and supporting good equilibria.\(^{17}\)

\(^{17}\)See case III of Section 4.3.1
5.1 Dynamics

We now discuss the transition dynamics of the economy during the adjustment towards the steady state.

**Definition 4 (Law of Motion)** The laws of motion of the collection \( \{e_t, p_t, h_t, k_t\}_{t=0}^{\infty} \) are definite as the mappings:

\[
\begin{align*}
    h_{t+1} &= H (E (h_t), h_t), \\
    e_{t+1} &= E (H (E (h_t), h_t)), \\
    k_{t+1} &= K (E (h_t), P (h_t, k_t), h_t), \\
    p_{t+1} &= P (K (E (h_t), p_t, h_t), H (E (h_t), h_t))
\end{align*}
\]

The economy dynamics is basically driven by the human capital evolution which affects both the education transfers’ law of motion and the transition dynamics of taxation policy. While the former is directly influenced by human capital, the latter is affected by human capital both directly and indirectly through physical capital. This implies that convergence conditions in the state-space are also sufficient for the stable convergence of the policy rules evolution. The following Lemma states the conditions for economy’s convergence stability.

**Lemma 2** The laws of motion of the collection \( \{e_t, p_t, h_t, k_t\}_{t=0}^{\infty} \) are equal to:\(^{18}\)

\[
\begin{align*}
    h_{t+1} &= \kappa_1 h_t + \kappa_0, \\
    e_{t+1} &= \xi_1 h_t + \xi_0, \\
    k_{t+1} &= \varrho_2 k_t + \varrho_1 h_t + \varrho_0, \\
    p_{t+1} &= \chi_2 p_t + \chi_1 h_t + \chi_0
\end{align*}
\]

Given any feasible initial condition \((h_0, k_0)\), if \(\kappa_1 < 1\), \(\varrho_2 < 1\), \(\varrho_1 h_t + \varrho_0 > 0\) and \(\chi_1 h_t + \chi_0 > 0\) then the sequence \(\{e_t, p_t, h_t, k_t\}_{t=0}^{\infty}\) is characterized by stable convergence.

**Proof.** (See appendix). ■

Given the differentiability of the policy functions and Lemma 2, the following Proposition holds:

**Proposition 4** A feasible steady state \((e^*, p^*, h^*, k^*)\) exists and is unique.

**Proof.** (See appendix). ■

Thus, depending on the initial condition, \(\{h_0, k_0\}\), and the level of the human capital society endowment, \(\vartheta\), the control and the state variables converge monotonically to the unique feasible steady state.

5.2 Welfare State Regimes

Depending on both the political bargaining intensity between adults and old and the level of physical capital, the marginal impact of human capital on taxation decisions can be either positive or negative driving the economy toward different welfare state regimes. We denote by

\(^{18}\)The parameters are fully described in the appendix.
$k \equiv \frac{\zeta_1 - \zeta_2}{x_3}$ the physical capital threshold level. The following Corollary fully characterizes the conditions for the emergence of different regimes.\footnote{Formally, we define $\bar{\Omega} \equiv \frac{(1-\kappa_1)(1-\kappa_2)}{\pi_1 - (1-\rho_1)(1-\kappa_2)(1-\kappa_1)}$. If elderly have weak political bargaining power, i.e. $\Omega \leq \bar{\Omega}$, then $\zeta_1 \leq \zeta_2$. Contrarily, if elderly are endowed with strong political power, i.e. $\Omega > \bar{\Omega}$, then $\zeta_1 > \zeta_2$.}

**Corollary 1** Given the stationary equilibrium policy rules $E(h_t)$ and $P(h_t, k_t)$:

i. if $\zeta_1 \leq \zeta_2$, then the Politico Complementarity Regime (PCR) arises, i.e. $\frac{dZ_t}{dh_t} \geq 0$;

ii. if $\zeta_1 > \zeta_2$ and $k_t \geq k$, then the Politico-Economic Complementarity Regime (PECR) arises, i.e. $\frac{dZ_t}{dh_t} \geq 0$;

iii. if $\zeta_1 > \zeta_2$ and $k_t < k$, then the Politico-Economic Substitutability Regime (PESR) arises, i.e. $\frac{dZ_t}{dh_t} < 0$;

**Proof.** (See appendix).

Corollary (1) states that when elderly are harmed by a weak level of bargaining power in the political process, i.e. $\zeta_1 \leq \zeta_2$, the economy experiences a PCR, for any level of $k_t$. Contrarily, if $\zeta_1 > \zeta_2$, then PECR characterizes a high-capitalized economy, whereas PESR emerges in low-capitalized environment.

Fig. 3 reports the equilibrium fiscal policy rule described in Eq. (25), providing a representation of the welfare state regimes.

Panel (a) describes the structural relation between the equilibrium tax rate and the level of human capital. The intercept, $Z(k_t, 0)$, is a decreasing function in physical capital. As long as $k_t < k$, the larger the human capital, the higher the opportunity cost to tax levy, i.e. $\frac{dZ_t}{dh_t} < 0$. Otherwise if $k_t \geq k$, incentives to increase the income tax rate arise, i.e. $\frac{dZ_t}{dh_t} \geq 0$. Panel (b) illustrates the structural relation between the equilibrium tax rate and the level of physical capital. The equilibrium predicts for any value of $k_t$ that the higher the physical capital, the...
lower the income tax rate, consistently with previous literature.\textsuperscript{20} The intuition for the fiscal policy function to be non-increasing in the capital stock is the following. By contradiction, if $Z(h_t, k_t)$ were increasing in $k_t$, current adults would have incentives to save in order to provide the next generation with a higher level of capital and, therefore, receive a higher pension. This cannot be an equilibrium, since the higher amount of backward transfer reduces the level of saving that workers are able to make.

In our environment human capital plays a crucial role in two different ways. On one hand, it mitigates the politicians’ strategic behavior. Precisely, the higher the level of human capital, the flatter the equilibrium policy function and the lower the elasticity of $Z(h_t, k_t)$ with respect to physical capital. The lower responsiveness of taxation policy decisions on the level of private savings weakens the strategic channel through which politicians can increase the probability to win elections. On the other hand, human capital perturbs the political choice concerning the size of government, driving the economy towards different welfare state regimes.

**Corollary 2** At each time $t$, there exists a positive threshold level $\tilde{h}$, such that for any $h_t > \tilde{h}$ the pension benefits in PESR are lower than in PCR and larger than in PECR, i.e. $p_t^{PESR} < p_t^{PECR} < p_t^{PCR}$.

**Proof.** (See appendix). \hfill \blacksquare

The emergence of different welfare state regimes has also implications in terms of pension benefits. In the regime supported by adults, the equilibrium pension benefits reach the highest feasible level. Intuitively, in equilibrium a higher level of current income tax rate determines a decrease of future physical capital stock and, in turns, an increase of future tax rate. In the PCR, adults anticipate that, if they invest in current education, an increase in future human capital will determine a further positive variation in the level of income tax rate tomorrow. Given the increase in both the future tax rate and taxable income, which maximize adult intertemporal utility, PCR emerges as the only sustainable welfare state regime when adult bargaining power prevails.

### 6 Benevolent Government Allocation

We consider the benevolent government allocation as benchmark for the politico-economic equilibrium studied in the previous sections. As in the political game, we exclude private agents’ default on the implemented fiscal plane within the period. Furthermore, under balanced budget constraint, the government platform is characterized by the vector $f_{gt} = \{z(e_{gt}, p_{gt}), e_{gt}, p_{gt}\}$, where the apex $g$ stands for government. Given the initial conditions $\{h_0, k_0\}$, we define the government optimization program in sequential version, as follows:

$$\max_{\{f_{gt}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (1 + n)^t \delta^t \left((1 + n) \delta u (C_t^1) + \beta u (C_t^2)\right)$$

where the equilibrium policies become non-trivially dependent on fundamental asset variables. Let us consider the restriction \( \delta < \delta_0 \equiv \frac{1}{1+n} \). Differently from
the relative political bargaining power of the adults and old, the infinite-horizon government
takes account of both the relative welfare weight gap between current and future pensioners,
\( \Omega_g = \frac{\beta(1-\delta(1+n))}{\beta+\delta(1+n)} \), and the relative welfare weight of the representative agent, \( 1 - \Omega_g \).

As in Klein et al. (2008),\(^{21}\) we rewrite the sequential government program in a recursive way
to derive the Generalized Euler Equations (GEEs). They capture the government optimal trade-
offs between taxation and redistribution wedges over time. Due to stationarity, we omit the time
subscript, denoting by the prime symbols next-period values. Eq. (8), requires \( \eta \left( f_g, f'_g, h, k' \right) = \right. \)
In equilibrium, by using the implicit function theorem, a unique \( k' = K \left( f_g, f'_g, h \right) \) satisfying
\( \eta \left( f_g, f'_g, h, K \left( \cdot \right) \right) = 0 \) exists. If the equilibrium policy rules exist \( \mathcal{E}_g \left( h, k \right) \) and \( \mathcal{P}_g \left( h, k \right) \), then by
using \( h' = H \left( c_g, h \right) \) we derive the recursive formulation of \( K \left( \cdot \right) \), whose functional form is equal
to \( k' = K \left( f_g, h \right) \). The recursive economic first order condition becomes \( \eta \left( f_g, h, K \left( f_g, h \right) \right) = 0 \).
Derivating the function \( \eta \left( \cdot \right) \) with respect to its arguments we obtain:
\( K_{f_g} = -\frac{\eta_{f_g}}{\eta_{K}} \) and \( K_h = -\frac{\eta_h}{\eta_{K}} \),
which gives a measure of the variation in the amount of savings due to a change in either policies
or human capital.

After some manipulations, Eq. (26) can be reformulated in terms of Bellman equation, as
follows:
\[
V \left( h, k \right) = \max_{\left\{ f_g, h', k' \right\}} \left[ \left( 1 + n \right) \delta u \left( C^1 \right) + \beta u \left( C^2 \right) \right] + \left( 1 + n \right) \delta V \left( h', k' \right)
\]  
\( (27) \)
We now provide the formal Definition of the government equilibrium allocation.

**Definition 5 (Benevolent Government Allocation)** A perfect foresight Markov perfect equi-
librium of the benevolent government problem is a sequence of competitive economic equilibrium
allocations \( \{ c^1, c^2, h', k' \} \) and policies \( f_g \), such that the functional vector of differentiable economic policy
decision rules, \( \mathcal{F}_g = \{ \mathcal{E}_g, \mathcal{P}_g \} \), where \( \mathcal{E}_g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, \bar{c}_i] \) and \( \mathcal{P}_g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, \bar{p}_i] \), satisfies
the following conditions:

i. \( \{ \mathcal{E}_g \left( h, k \right), \mathcal{P}_g \left( h, k \right) \} = \arg \max_{\left\{ \mathcal{E}_g, \mathcal{P}_g \right\}} \left[ \left( 1 + n \right) \delta u \left( C^1 \right) + \beta u \left( C^2 \right) \right] + \left( 1 + n \right) \delta V \left( h', k' \right) \)

ii. \( V \left( h, k \right) = \mathcal{M} \left( V \right) \left( h, k \right) \)

where the functional \( \mathcal{M} : \mathcal{C}^{\infty} \left( \mathbb{R}^2 \right) \rightarrow \mathcal{C}^{\infty} \left( \mathbb{R}^2 \right) \) is defined as follows:
\[
\mathcal{M} \left( V \right) \left( h, k \right) = \max_{\left\{ p_g, c_g \right\}} \left[ \left( 1 + n \right) \delta u \left( C^1 \right) + \beta u \left( C^2 \right) \right] + \left( 1 + n \right) \delta V \left( h', k' \right)
\]

\(^{21}\)Recent studies have extended the dynamic politico-economic modelling to the infinite-horizon Gvt problem
(Klein et al., 2008, and Azzimonti et al., 2009). These models differ from ours in that the policy space is one-
dimensional and the dynamic linkages are not long-run persistent due to the full depreciation of the relevant-payoff
state variables. Departing from earlier literature, we find analytical solutions in a multi-dimensional state space
where the equilibrium policies become non-trivially dependent on fundamental asset variables.
The first condition requires the political variables, $e_g$ and $p_g$, have to be chosen by the government in order to maximize the utilitarian social welfare, internalizing the equilibrium private saving decision and all the direct and indirect feedback effects. The second requirement is the fix point condition, given the mapping $\mathcal{M}(V)$.

6.1 The Government SMPE

To solve the government optimization problem, we guess a bidimensional policy, structurally equivalent to Eq. (23) and (24), which verifies the government Euler conditions.\footnote{For details on the government Euler conditions see Appendix.} Then the next Proposition characterizes the optimal feasible policy rules.

**Proposition 5** Under dynamic efficiency, the feasible rational policies, $f_g = \{p_g, e_g\}$, which can be supported as a Markov perfect equilibrium of the benevolent government problem, has the following functional form:

i. \[ E_g(h) = \epsilon_{g1}h + \epsilon_{g0} \] (28)

where $\epsilon_{g1} = \epsilon_1$ and $\epsilon_{g0} = \epsilon_0$;

ii. \[ P_g(h, k) = \pi_{g2}k + \pi_{g1}h + \pi_{g0} \] (29)

where $\pi_{g2} = R(1 - \Omega_g)$, $\pi_{g1} = \left(\frac{R\psi(1-\epsilon_{g0})}{R\psi - \epsilon_{g1}}(1 - \epsilon_{g1})\right)\Omega_g$ and $\pi_{g0} = \left(\frac{R\psi(1-\epsilon_{g0}) - (\epsilon_{g1} - \epsilon_{g0})}{R\psi - \epsilon_{g1}}\right)\rho\Omega_g$.

**Proof.** (See appendix). $\blacksquare$

The two equilibrium concepts described in Definition 3 and 5 lead to the implementation of the same education program. Specifically, in equilibrium both the government and the office-seeking politicians set the same amount of forward transfers, inducing education-efficient political fiscal planes, i.e. $E_g(h) = E(h)$ for any level of human capital. The main difference concerns their quantitative predictions on the taxation policy dimension, which are fully captured by the policy parameters. In the following subsection we discuss in details how a political equilibrium divergences from the government optimal allocation.

6.2 Are the political choices on pensions and education optimal?

Both the politicians and the government have incentives to provide intergenerational transfers. Moreover, their equilibrium policies share similar structural properties. Even if politicians only take care of welfare of current living voters, they fully internalize the infinite spillover effects generated by human capital. Consequently, they succeed in providing efficient forward transfers. Parties anticipate that the next-period generation will internalize the expectation of the future generations on coordination over the level of human capital. Due to the internalization of the higher order beliefs on intergenerational cooperation, in equilibrium politicians set education...
transfers as a benevolent government, whose objective function correspond to the maximization of dynastic welfare. On the dimension of backward transfers the political choice quantitatively differs from the benevolent government allocation. In order to obtain clear predictions, we assign \( \delta = \frac{\beta}{p(1+n)+\phi} \). Consequently we write the relative welfare weights, in terms of political weights, making the two solutions comparable.

**Proposition 6** For any \( \theta > \bar{\theta} \) the Markov perfect politico-economic equilibrium induces over-redistribution of backward transfers with respect to the Government solution, i.e. \( P(h_t, k_t) > P_g(h_t, k_t) \).

**Proof.** (See appendix).

According to Proposition 6, if the government adopts a politically equivalent system of welfare weights, the level of pension benefits is always lower than in the political case, i.e. \( P(h_t, k_t) > P_g(h_t, k_t) \), for any value of human and physical capital. We conclude that politicians involved in a Markov game among successive generations of players deliver the government allocation if they reduce the political weight they assign to the elderly agents. These distortions come from the politicians’ strategic behavior. In determining taxation rules, short-lived politicians take into account that future politicians will compensate the fiscal cost of current adults by paying the pensions in their old age. This stems from the fact that higher taxes on today environment lead to a lower private wealth in old age, i.e. to a lower state variable in the following period, thereby triggering more transfers by the future politicians. Thus, the policy response of the future politicians reduces the current cost of transferring resources to the elderly and leads to overspending, unless the adults enjoy an unusually large political power. Consequently, by transferring too much resources to old age due to both the overrepresenting of current elderly agents and the policy response of the future politicians, the politicians fail to provide the optimal backward transfers.

### 7 Conclusions

In this paper we investigate the conditions for the emergence of implicit intergenerational contracts without assuming reputation mechanisms, commitment technology and altruism, when agents coordinate their expectation on the evolution of human capital and the political environment is characterized by uncertainty. We present a tractable dynamic politico-economic model in OLG environment where political representatives compete proposing multidimensional fiscal platforms. Both backward and forward intergenerational transfers, respectively in the form of pension benefits and education investments, are simultaneously considered in an endogenous human capital setting with income taxation when agents play Markovian strategies. The infinite horizon government solution without commitment is used as benchmark to evaluate the efficiency of politically determined rules.

23 Quantitatively, the internalization of higher order beliefs on intergenerational cooperation is captured by the fractal series \( \psi_{\{j+1\}} = m \left( \psi_{\{j\}} \right) \) in Lemma 1.
The dynamic mechanism driving our results is intuitive. Political uncertainty, as a commitment device, induces agents to credibly coordinate their expectation on the evolution of human capital. Consequently an intergenerational contract characterized by both forward and backward is sustained as a Markov perfect politico-economic equilibrium of the voting game. Social security system sustains investment in public education, that, in turns, creates a dynamic linkage across periods through both human and physical capital driving the economy towards different welfare state regimes.

We show that intergenerational contracts may be politically sustained uniquely as long as the economy is in dynamic efficiency, i.e. the rental gross price of capital is larger than the economic growth rate. Our economic environment is in line with empirical findings on the dynamic efficiency status of most developed countries, especially after the demographic transition. By endogenizing human capital formation through public education investments, backward and forward redistributive programs may optimally self-sustain each other even in the absence of a benevolent government. In equilibrium political decisions are education efficient.

Relatively to the prediction about the transition towards the steady state, we find three different welfare state regimes may emerge depending on both the relative political bargaining power between adults and old and the endogenous capital asset accumulation. The emergence of different regimes leads the economy towards different dynamic paths and persistence degrees of politically distortionary redistribution. In the regime supported by adults, the equilibrium pension benefits reach the highest feasible level.

Finally, due to the distortions generated by the repeated political competition process and to the political overrepresentation of elderly agents, political equilibrium is characterized by overtaxation compared with the Gvt solution.

Our analysis leaves some natural directions for future research. We have assumed only adults and old compete in the political debate. Using the developed methodology, a change in the voting rule, which enables also the young to vote, would generate different equilibrium allocations both in terms of education transfers and government size. Another direction for future research concerns the introduction of a dynamic electoral stage by endogenizing the probability of re-election, which would introduce a new source of distortion.
8 Appendix A

Proposition (1). Let us consider a finite horizon economy, which ends at time \( T = 2 \). In the last period the political maximization program for each party \( i_t \) is equal to:

\[
\max_{f_{i_t}} \frac{\sigma^2}{\sigma_T} u(C^2_{i_t}) + (1 + n) u(C^1_{i_t})
\]

Since the parties’ objective function is simply the probability of winning elections, then in equilibrium they propose the same platform, \( f_{A_2} = f_{B_2} \). Using Eq. (15), it follows that \( \varphi(k_2, h_2) = 0 \) and \( \Pr(\varphi_2 > \varphi(k_2, h_2)) = \Pr(\varphi_2 < \varphi(k_2, h_2)) = \frac{1}{2} \).

At time \( T = 1 \) the maximization program for party \( i_1 \) is:

\[
\max_{f_{i_1}} \frac{\sigma^2}{\sigma_1} u(C^2_{i_1}) + (1 + n) (u(C^1_{i_1}) + \beta E(\mathcal{P}_{B_1}(\mathcal{W}^2_{i_1} + \theta_2 + \psi^2_{j_2}) + \mathcal{P}_{A_1}\mathcal{W}^2_{i_1}))
\]

where \( \mathcal{P}_{B_1} \equiv \frac{1}{2} - \varphi \varphi(k_2, h_2) \) and \( \mathcal{P}_{A_1} = \frac{1}{2} + \varphi \varphi(k_2, h_2) \). Since \( \varphi(k_2, h_2) \) is equal to 0 independently from the incumbent party, then the conditional probability of being elected turns out to be equal to:

\[
\mathcal{P}_{B_1} = \mathcal{P}_{A_1} = \frac{1}{2}
\]

As a result, the maximization program at time \( T = 1 \) reduces to:

\[
\max_{f_{i_1}} \frac{\sigma^2}{\sigma_1} u(C^2_{i_1}) + (1 + n) (u(C^1_{i_1}) + \beta u(C^2_{i_1}))
\]

Replicating the same argument of time \( T = 2 \), we have \( f_{A_1} = f_{B_1} \). Hence the two candidates’ platform converges in equilibrium to the same fiscal policy that maximizes a weighted utility of current adults and old:

\[
\max_{f_t} \frac{\sigma^2}{\sigma_1} u(C^2(p_t, k_t)) + (1 + n) (u(C^1(e_t, p_t, h_t, k_{t+1})) + \beta u(C^2(\mathcal{P}_{t+1}, k_{t+1})))
\]

under the economic constraint \( u_c(C^1_c) - \beta Ru_c(C^2_{t+1}) = 0 \).

Proposition (2). By using Eq. (20) and (22), an interior solution for the forward transfer policy requires \( \frac{dp_{t+1}}{de_t} = \frac{R}{1+n} > 0 \). It follows that Eq. (21) can be rewritten as \( \frac{\phi u_{c^2}}{u_{c^1}} = \frac{dp_{t+1}}{dp_t} \). Substituting the equilibrium level of \( \frac{dp_{t+1}}{dp_t} \), we obtain \( \frac{dp_{t+1}}{dp_t} = \frac{R}{1+n} \Lambda \) where \( \Lambda = \frac{1 - \phi u_{c^2}}{u_{c^1}} \) is a measure of how current adults internalize the impact of current policies on future one. Since \( \Lambda \) is always less than one, then \( \frac{dp_{t+1}}{dp_t} > \frac{dp_{t+1}}{dp_t} \).

Lemma (1). The resolution strategy consists in computing the first order conditions starting from a time \( t < \infty \) large enough and solving backward for each time \( t-j \) with \( j = 1, 2, \ldots \), subject to: 1) the economic Euler condition, Eq. (8), 2) the balanced budget constraint, Eq. (7), and 3) the equilibrium policy rules of the following periods. We recursively determine the conditions for the existence of fixed points taking the limit for \( j \to \infty \).
In the last period $t < \infty$, the adults have one period temporal-horizon. Thus, the political objective function is as follows:

$$\phi u \left(C_t^1 \right) + (1 + n) u \left(C_t^2 \right)$$

where $C_t^1 \equiv w_t - p_t - e_t$ and $C_t^2 \equiv (1 + n) (Rk_t + p_t)$. No incentives to invest in education arise, i.e. $e_t = 0$. Let $\Omega_t \equiv \frac{\phi}{1 + \frac{n}{\alpha}}$. Under logarithmic utility, the equilibrium policy rules, $\mathcal{F}_t = \{P_t, E_t\}$, are equal to:

$$\mathcal{F}_t : \begin{cases} E_t = 0 \\ P_t = -b_{1(0)}k_t + b_{0(0)}h_t + b_{0(0)} \end{cases}$$

(1A)

where $b_{1(0)} \equiv R (1 - \Omega_t)$ and $b_{0(0)} \equiv \Omega_t$. In the brackets we report the number of iterations.

At time $t - 1$ the adults born at time $t - 2$ live up three periods. The political objective function is as follows:

$$(1 + n) \left( u \left(C_{t-1}^1 \right) + \beta u \left(C_{t-1}^2 \right) \right) + \phi u \left(C_{t-2}^2 \right)$$

(2A)

where $C_{t-1}^1 \equiv w_{t-1} - p_{t-1} - e_{t-1} - (1 + n) k_t$ and $C_{t-1}^2 \equiv (1 + n) (Rk_{t-1} + p_{t-1})$. After plugging the equilibrium policy rules of period $t$, Eq. (1A), into Eq. (2A), we maximize with respect to $f_{t-1} = \{p_{t-1}, e_{t-1}\}$. Let us denote $\psi(1) \equiv \left(\frac{1 - \theta}{\alpha}\right)^{\frac{1}{2}}$ and $\gamma(1) \equiv \frac{1 + n}{\theta}$. Furthermore, $\Omega_{t-1} \equiv \Omega \equiv \frac{\phi}{\phi + (1 + n) (1 + \beta)}$ is the indexes of the elderly relative political power in an economy that lasts more than one period. By applying the envelope theorem, after some manipulations we obtain the following equilibrium policy rules:

$$\mathcal{F}_{t-1} : \begin{cases} E_{t-1} = a_{1(1)} h_{t-1} + a_{0(1)} \\ P_{t-1} = -b_{4(1)} k_{t-1} + (b_{9(1)} + b_{0(1)} - a_{1(1)}) h_{t-1} + (b_{2(1)} + b_{1(1)} + b_{0(1)} - a_{0(1)}) \end{cases}$$

(3A)

where the coefficients associated to the forward transfers are $a_{0(1)} \equiv (1 - \alpha) \phi \psi(1)$ and $a_{1(1)} \equiv \alpha \psi(1)$, whereas those associated to the backward transfers are $b_{0(1)} \equiv \Omega$, $b_{1(1)} \equiv \gamma(1) \Omega$, $b_{2(1)} \equiv (1 - \alpha) (1 - \Omega) \psi(1)$, $b_{3(1)} \equiv \alpha \psi(1) \left(1 + \frac{\theta}{1 - \theta} \Omega \right)$ and $b_{4(1)} \equiv R (1 - \Omega)$.

Finally let us consider time $t - 2$ when all the direct dynamic feedbacks are internalized. The political objective function is equivalent to Eq. (2A). The recursive problem is now subject to the equilibrium policy rules of the next two periods, (1A) and (3A). Let us now denote with $\psi(2) \equiv \left(\frac{\alpha \theta}{\alpha} \left(\frac{1 - \theta}{\alpha}\right)^{\frac{1}{2}} + \frac{1 - \theta}{\alpha}\right)^{\frac{1}{2}}$, $\gamma(2) \equiv \frac{1 + n}{\theta} + \left(\frac{1 + n}{\theta}\right)^2$ and $g(2) \equiv \frac{1 + n}{\theta} \psi(1) + \psi(2)$. Maximizing the political objective function with respect to $f_{t-2} = \{e_{t-2}, p_{t-2}\}$ and rearranging the terms we obtain the following pair of equilibrium policy rules at time $t - 2$:

$$\mathcal{F}_{t-2} : \begin{cases} E_{t-2} = a_{1(2)} h_{t-2} + a_{0(2)} \\ P_{t-2} = -b_{4(2)} k_{t-1} + (b_{9(2)} + b_{0(2)} - a_{1(2)}) h_{t-1} + (b_{2(2)} + b_{1(2)} + b_{0(2)} - a_{0(2)}) \end{cases}$$

(3A)

where the coefficients associated to the forward transfers are $a_{0(2)} \equiv (1 - \alpha) \phi \psi(2)$ and $a_{1(2)} \equiv \alpha \psi(2)$, whereas those associated to the backward transfers are $b_{0(2)} \equiv b_{0(1)}$, $b_{1(2)} \equiv \gamma(2) \Omega$, $b_{2(2)} \equiv b_{2(1)} + b_{0(1)} - a_{0(1)}$, $b_{3(2)} \equiv (1 - \alpha) (1 - \Omega) \psi(2)$ and $b_{4(2)} \equiv R (1 - \Omega)$.
where \( b_{2(2)} = (1 - \alpha) \left( \left( 1 + \frac{\theta}{1 - \theta} \Omega \right) \psi(2) + \frac{\theta}{1 - \theta} \Omega g(2) \right) \), \( b_{3(2)} = \alpha \psi(2) (1 + \frac{\theta}{1 - \theta} \Omega) \) and \( b_{4(2)} = b_{4(1)} \). It is straightforward to show that \( \psi(2) \) can be derived as a differentiable monotonic transformation, \( m(\cdot) \), of \( \psi(1) \). It is characterized by \( m(0) > 0 \), \( m_\psi > 0 \), and \( m_{\psi \psi} > 0 \). In particular \( m(\psi(1)) := \left( \frac{\alpha}{\pi} \psi(1) + \frac{1 - \theta}{\pi} \right)^{\frac{1}{\theta}} \). The argument can be repeated for each time \( j > 0 \) such that:

\[
\psi_{(j+1)} = m(\psi_{(j)})
\]

(4A)

For each \( j \) the following series can be derived:

\[
\gamma(j) = \sum_{i=1}^{j} \left( 1 + \frac{n}{R} \right)^{i}, \quad \text{and} \quad g(j) = \left( 1 + \frac{n}{R} \right)^{j-1} \psi(1) + \left( 1 + \frac{n}{R} \right)^{j-2} \psi(2) + ... + \psi(j)
\]

Using the above notation, starting from \( t - 3 \) we finally derive the recursive structure, which characterizes the political problem:

\[
\mathcal{F}_{t-j}:
\begin{align*}
\mathcal{E}_{t-j} &= a_1(j) \psi_{t-j} + a_0(j) \\
\mathcal{P}_{t-2} &= -b_4(j) \psi_{t-1} + (b_3(j) + b_0(j) - a_1(j)) \psi_{t-1} + (b_2(j) \psi + b_1(j) + b_0(j) - a_0(j))
\end{align*}
\]

(5A)

where \( a_0(j) = (1 - \alpha) \psi(j) \), \( a_1(j) = \alpha \psi(j) \), \( b_0(j) = b_0(1), b_1(j) = \gamma(j) \Omega, b_3(j) = \alpha \psi(j) \left( 1 + \frac{\theta}{1 - \theta} \Omega \right), b_4(j) = b_4(1) \).

If a political SMPE exists, then the limits for \( j \to \infty \) of the set of time-variant parameters \( \{a_0(j), a_1(j), b_0(j), b_1(j), b_2(j), b_3(j), b_4(j)\} \) exist and are finite. The determination of the fixed points for the two stationary policy rules depends on the existence of the fixed point for the forward transfer and, in final instance, on the determination of the limit for \( \psi_{(j)} \). The computation consists in solving the non-linear difference equation (4A). The \( \lim_{j \to \infty} \psi_{(j)} \) is equivalent to the solution, if any, of such difference equation given \( \psi(1) \) as initial condition. Let us denote with \( \hat{\psi} \) the value of \( \psi_{(j)} \) such that \( m_{\psi(1)} \psi_{(j)} = \hat{\psi} = 1 \). We yield respectively zero, one or two fixed points as solution of the difference equation iff \( m(\hat{\psi}) > 0 \). Thus, \( \hat{\psi} \) is equal to \( \hat{\psi} = \left( \frac{R}{\alpha} \right)^{\frac{1}{\theta}} \frac{1 - \theta}{\alpha \theta} \). Given that \( R \geq \alpha \theta \) in all the parametric space, then one stable fixed point exists and is equal to the zero of the non linear equation \( \psi - \left( \frac{\alpha \theta}{\pi} \psi + \frac{1 - \theta}{\pi} \right)^{\frac{1}{\theta}} = 0 \), which is equal to the function
ψ := ψ(α, θ, R).

Recall that \( m(\psi_{j}) = \psi \leq 1 \) and \( \alpha \psi < 1 \), then by using implicit function, the following relations hold:
\[
ψ_\alpha \geq 0 \\
ψ_\theta \leq 0 \\
ψ_R \leq 0
\]

Under the condition \( R > (1 + n) \) the \( \lim_{j \to \infty} \gamma(\psi_j) < \infty \) is equal to \( \rho \equiv \frac{R}{R - (1 + n)} \). Consequently starting from \( t \) large enough, the \( \lim_{j \to \infty} g(\psi_j) = \lim_{j \to \infty} \psi \sum_{i=t}^{j} (\frac{1 + n}{R})^i < \infty \) is equal to \( \psi \rho \). Under the convergence conditions of \( \gamma(\psi_j) \) and \( g(\psi_j) \) the fixed points are characterized.

**Proposition (3).** Under Lemma 1, using the convergence conditions, the following equilibrium policy rules are attained as Markov perfect politico-economic equilibrium of the intergenerational voting game:
\[
\mathcal{E}(h_t) = \epsilon_1 h_t + \epsilon_0 \\
\mathcal{P}(h_t, k_t) = \max \{0, -\pi_2 k_t + \pi_1 h_t + \pi_0\}
\]
where \( \epsilon_0 \equiv (1 - \alpha) \partial \psi, \epsilon_1 \equiv \alpha \psi \) and \( \pi_2 \equiv R (1 - \Omega), \pi_1 \equiv \left(1 + \frac{\theta}{1 - \theta} \right) \Omega \) and \( \pi_0 \equiv (1 + \rho) \left(1 + \frac{\theta}{1 - \theta} \epsilon_0\right) \Omega \).

The taxation equilibrium policy rule is obtained by balanced budget constraint and is equal to:
\[
\mathcal{Z}(h_t, k_t) = -\zeta_3 \frac{k_t}{1 + h_t} + \zeta_2 \frac{h_t}{1 + h_t} + \zeta_1 \frac{1}{1 + h_t} + \zeta_0
\]
where \( \zeta_3 \equiv R (1 - \Omega), \zeta_2 \equiv \epsilon_1 \left(1 + \frac{\theta}{1 - \theta} \Omega\right), \zeta_1 \equiv \frac{\alpha}{\alpha} \zeta_2 + \left(1 + \frac{\theta}{1 - \theta} \epsilon_0\right) \rho \Omega \) and \( \zeta_0 \equiv \Omega \).

**Lemma (2).** Let us first consider the transition dynamics of \( h_t \) and \( e_t \). Plugging the equilibrium education transfers, Eq. (23), into the human capital production, Eq. (5), we obtain the law of motion \( h_{t+1} = H^d(h_t) \), which is equal to:
\[
h_{t+1} = \kappa_1 h_t + \kappa_0 \quad (6A)
\]
where \( \kappa_0 \equiv \frac{(1-\alpha) \psi^{1-\theta}}{1+n} \) and \( \kappa_1 \equiv \frac{\alpha \psi^{1-\theta}}{1+n} \). Since \( \kappa_1 \geq 0 \), the serial correlation between current and future level of human capital is always positive. To determine the law of motion of the redistributive policy we plug Eq. (5) into the equilibrium education policy rule at time \( t+1 \). The law of motion \( e_{t+1} = E^d (h_t) \) is as follows:

\[
e_{t+1} = \xi_1 h_t + \xi_0
\]  

(7A)

where \( \xi_0 \equiv \left( \frac{\alpha}{(1+n) \psi^{1-\theta}} + \psi \right) (1 - \alpha) \) \( \psi \) and \( \xi_1 \equiv \frac{\alpha^2}{(1+n) \psi^{1-\theta}} \). If the dynamics of \( h_t \) is characterized by stable convergence, i.e. \( \kappa_1 < 1 \), then also the dynamics of \( e_t \) converge toward the steady state. Thus, using the expression of \( \kappa_1 \), the sufficient condition for the convergence stability of both \( h_t \) and \( e_t \) requires \( n > n_1 \equiv \alpha \psi^{1-\theta} - 1 \). Due to linearity, both \( h_t \) and \( e_t \) converge monotonically toward the steady states.

Let us now analyze the transition dynamics of \( k_t \) and \( p_t \). First, consider the following recursive formulation for the equilibrium saving under log-utility, \( k_{t+1} = K(e_t, p_t, h_t) \), which is obtained plugging the human capital production, Eq. (5), and the expected equilibrium policies \( e_{t+1} \) and \( p_{t+1} \) according to Eq. (23) and (24). The saving function can then be rewritten as follows:

\[
k_{t+1} = \frac{\beta R ((1 + h_t) - e_t - p_t)}{(1+\beta) - \pi_2} - \frac{\pi_1 H (e_t, h_t) + \pi_0}{R(1+\beta) - \pi_2}
\]  

(8A)

Plugging the equilibrium policy rules, Eq. (23) and Eq. (24), into Eq. (8A), we obtain the law of motion \( k_{t+1} = K^d (h_t, k_t) \):

\[
k_{t+1} = \varrho_2 k_t + \varrho_1 h_t + \varrho_0
\]  

(9A)

where:

\[
\varrho_2 \equiv \frac{\beta R \pi_2}{(1+n) (R(1+\beta) - \pi_2)}
\]

\[
\varrho_1 \equiv \frac{1}{(R(1+\beta) - \pi_2)} \left( \frac{\beta R (1 - e_1 - \pi_1)}{1+n} - \pi_1 \kappa_1 \right)
\]

\[
\varrho_0 \equiv \frac{1}{(R(1+\beta) - \pi_2)} \left( \frac{\beta R (1 - e_0 - \pi_0)}{1+n} - \pi_1 \kappa_0 - \pi_0 \right)
\]

Current and future level of physical capital are positively interrelated each other, \( \varrho_2 \geq 0 \), on the contrary the way \( h_t \) perturbs \( k_{t+1} \) depends on the welfare state regime intensity embedded in \( \pi_1 \).

The dynamics of physical capital is characterized by stable convergence if \( \varrho_2 < 1 \), i.e. \( \phi > \phi \equiv \beta (R - (1+n)) \), and \( \varrho_1 h_t + \varrho_0 > 0 \). Plugging Eq. (6A), (7A) and (8A) into the equilibrium pension transfer policy at time \( t+1 \), after some manipulations, we attain the law of motion \( p_{t+1} = P^d (p_t, h_t) \), as follows:

\[
p_{t+1} = \chi_2 p_t + \chi_1 h_t + \chi_0
\]  

(10A)
where:

\[
\begin{align*}
\chi_2 & = \frac{\beta R \pi_2}{(1 + n) (R(1 + \beta) - \pi_2)} \\
\chi_1 & = \frac{R ((1 + \beta)(1 + n)\pi_1 \kappa_1 - \beta \pi_2 (1 - \xi_1))}{(1 + n) (R(1 + \beta) - \pi_2)} \\
\chi_0 & = \frac{R(1 + n)(1 + \beta)\pi_1 \kappa_0 + ((1 + n)\pi_0 - \beta R (1 - \xi_0))}{(1 + n) (R(1 + \beta) - \pi_2)}
\end{align*}
\]

The dynamics of pension transfers is characterized by stable convergence if \( \chi_2 < 1 \) and \( \chi_1 h_t + \chi_0 > 0 \). Furthermore the convergence for \( p_t \) basically depends on the welfare state regime characterizing the economy, resumed in the coefficient \( \chi_1 \). Specifically if \( \frac{\chi_1}{\pi_2} > (\leq) \frac{\beta}{(1 + \beta)(1 + n) \frac{1 - \xi_1}{\kappa_1}} \) then \( \chi_1 > (\leq) 0 \) the speed of convergence toward the steady state is lower (higher) than in the opposite case.

**Proposition (4).** Under Lemma 2, due to linearity of the laws of motion, Eq. (6A), (7A), (9A) and (10A), there exists a unique steady state \( \{e^*, p^*, h^*, k^*\} \). Equating \( h_{t+1} = h_t = h^* \) in Eq. (6A) and \( k_{t+1} = k_t = k^* \) in Eq. (9A), the following steady state levels for the state variables are obtained:

\[
h^* = \frac{(1 - \alpha) \vartheta \psi^{1-\theta}}{(1 + n) - \alpha \psi^{1-\theta}}
\]

\[
k^* = \frac{\beta R (1 - \epsilon_1 - \pi_1) - (1 + n)\pi_1 \kappa_1 h^* + \beta R (1 - \epsilon_0 - \pi_0) - (1 + n) (\pi_1 \kappa_0 + \pi_0)}{(1 + n) (R(1 + \beta) - \pi_2) - \beta R \pi_2}
\]

(11A)

(12A)

Plugging Eq. (11A) and (12A) into the equilibrium policy rules described in Theorem 1, we obtain the following the steady states levels for the political control variables:

\[
e^* = \frac{(1 - \alpha) (1 + n) \vartheta \psi}{(1 + n) - \alpha \psi^{1-\theta}}
\]

and:

\[
p^* = \frac{(R(1 + \beta) - (1 - \kappa_1) \pi_2) (1 + n)\pi_1 - \beta R (1 - \epsilon_1) \pi_2 h^*}{(1 + n) (R(1 + \beta) - \pi_2) - \beta R \pi_2}
\]

\[
+ \frac{R(1 + n)(1 + \beta)\pi_0 - \beta R \pi_2 (1 - \epsilon_0) + (1 + n)\pi_2 \pi_1 \kappa_0}{(1 + n) (R(1 + \beta) - \pi_2) - \beta R \pi_2}
\]

By balanced budget constraint the pension steady state level is:

\[
z^* = \frac{p^* + e^*}{1 + h^*}
\]

**Corollary (1).** The marginal impact of \( h_t \) on the equilibrium tax rate, Eq. (25), is equal to:

\[
\frac{dz_t}{dh_t} = \zeta_3 \frac{k_t}{(1 + h_t)^2} + (\zeta_2 - \zeta_1) \frac{1}{(1 + h_t)^2}
\]

(13A)
For any level of $k_t$, if $\zeta_1 \leq \zeta_2$, then $\frac{dz_t}{dh_t} \geq 0$. It implies that $\Omega \leq \hat{\Omega}$ where $\hat{\Omega} \equiv \frac{(1-\theta)(\zeta_2-\zeta_1)}{\theta(1-\rho)}$. Otherwise, if $\zeta_1 > \zeta_2$, then the sign of Eq. (13A) depends on the value of $k_t$. When $k_t < k$, where $k \equiv \frac{\zeta_1 - \zeta_2}{\zeta_3}$, the income tax rate is a decreasing function of $h_t$, i.e. $\frac{dz_t}{dh_t} < 0$. The opposite holds for $k_t \geq k$.

**Corollary (2).** From Proposition (3) recall that $\pi_2 \equiv \zeta_3$, $\pi_1 (\zeta_2) \equiv \zeta_2 + \zeta_0 - \epsilon_1$ and $\pi_0 (\zeta_1) \equiv \zeta_1 + \zeta_0 - \epsilon_1$. We first consider the case of Politico-Economic welfare state regime, i.e. $\zeta_4^{PE} > \zeta_2^{PE}$. In PECR $k_t^C > k$ and in PESR $k_t^S < k$, then the level of pension benefits turns out to be $p_t^{PECR} \equiv \pi_2 k_t^C + \pi_1 (\zeta_2^{PE}) h_t + \pi_0 (\zeta_1^{PE}) < p_t^{PESR} \equiv \pi_2 k_t^S + \pi_1 (\zeta_2^{PE}) h_t + \pi_0 (\zeta_1^{PE})$ for any level of $h_t$. In the case of Politico welfare state regime, i.e. $\zeta_4^P \leq \zeta_2^P$, the level of backward transfer sustained in equilibrium is equal to $p_t^{PCR} \equiv \pi_2 k_t + \pi_1 (\zeta_2^P) h_t + \pi_0 (\zeta_1^P) > p_t^{PESR}$ for any $h_t > \bar{h} \equiv \frac{\pi_2 (k_t^P - k_t) + \pi_0 (\zeta_2^P) - \pi_0 (\zeta_1^P)}{\pi_1 (\zeta_2^P) - \pi_1 (\zeta_3^P)}$. ■
9 Appendix B

9.1 Derivation of Generalized Euler Equation

We derive the recursive formulation of the Gvt program starting from its sequential version:

\[ V_0(h_0, k_0) = \max_{\{f_g, h_{t+1}, k_{t+1}\}} \sum_{t=0}^{\infty} (1 + n)^t \delta^t U(f_g, h_t, k_t, k_{t+1}) \]

where \((h_0, k_0)\) are the initial conditions of the payoff-relevant state variables of the dynamic optimization program and \(U(f_g, h_t, k_t, k_{t+1}) = (1 + n) \delta u(C^1_t(e_g, p_g, h_t, k_{t+1})) + \beta u(C^2_t(p_g, k_t))\).

Equivalently we rewrite the above value function in the following terms:

\[ V_0(h_0, k_0) = \max_{\{f_{g_0}, k_1\}} U(f_{g_0}, h_0, k_0, k_1) + (1 + n) \delta \max_{\{f_{g_0}, h_{t+1}, k_{t+1}\}} \sum_{t=0}^{\infty} (1 + n)^t \delta^t U(f_g, h_t, k_t, k_{t+1}) \]  

By definition, we have:

\[ V_1(h_1, k_1) = \max_{\{f_{g_0}, h_{t+1}, k_{t+1}\}} \sum_{t=0}^{\infty} (1 + n)^t \delta^t U(f_g, h_t, k_t, k_{t+1}) \]  

Due to stationarity condition, the indirect utility function satisfies \(V_0(\cdot) \equiv V_1(\cdot) \equiv \ldots \equiv V_t(\cdot)\).

We omit time indexes and denote by prime symbol next period variables. Plugging Eq. (2B) into Eq. (1B) we yield the following Bellman equation:

\[ V(h, k) = \max_{\{f_g, h', k'\}} U(f_g, h, k, k') + (1 + n) \delta V(h', k') \]

subject to the constraints \(k' = K(f_g, h)\) and \(h' = H(e_g, h)\). We rewrite the Bellman equation as follows:

\[ V(h, k) = \max_{f_g} U(f_g, h, k, K(f_g, h)) + (1 + n) \delta V(H(e_g, h), K(f_g, h)) \]  

The GEE are obtained as the FOC of the government optimization plan. The derivation follows the method proposed by Klein et al. (2008) and it is extended to OLG case with two political controls in bidimensional state-space. For simplicity of notation we will omit the apex \(g\). The political first order conditions of Eq. (3B) with respect to \(f = \{e, p\}\) are equal to:

\[ 0 = U_e + U_p + (1 + n) \delta (V_{h'}H_e + V_{k'}K_e) \]

\[ 0 = U_p + U_p + (1 + n) \delta V_{k'}K_p \]
Using Benveniste-Scheinkman formula we obtain the following expression for \( V_h \) and \( V_k \):

\[
V_h = U_h + U_k K_h + (1 + n) \delta (V_h H_h + V_k K_h) \tag{6B}
\]

\[
V_k = U_k \tag{7B}
\]

From Eq. (4B) and (5B) we obtain the expression for \( V_{h'} \) and \( V_{k'} \):

\[
V_{h'} = \frac{1}{(1 + n) \delta H_e} \left( \frac{U_p K_e - U_r K_p}{K_p} \right) \tag{8B}
\]

\[
V_{k'} = -\frac{U_p + U_k K_p}{(1 + n) \delta K_p} \tag{9B}
\]

Plugging Eq. (8B) and (9B) into (6B) we get the final expression for \( V_h \):

\[
V_h = U_h + \frac{U_p K_e - U_r K_p}{K_p} H_h - \frac{U_r K_h}{K_p} \tag{10B}
\]

Using stationarity condition and plugging Eq. (7B) and (10B) into (4B) and (5B), we obtain the GEEs of the government problem respectively for \( e \) and \( z \):

\[
0 = U_e + U_{k'} K_e + (1 + n) \delta \left( \left( U'_{h'} + \frac{U'_p K'_{e'} - U'_r K'_{p'}}{K'_p} H'_h - \frac{U'_r K'_h}{K'_p} H'_e \right) H_e + U'_{k'} K_e \right) \tag{11B}
\]

\[
0 = U_p + (U_{k'} + (1 + n) \delta U'_p) K_p \tag{12B}
\]

From definition of \( U \), we have \( U_e = -\delta (1 + n) u_{c1} \), \( U_p = -(1 + n) (\delta u_{c1} - \beta u_{c2}) \), \( U_h = (1 + n) \delta u_{c1} \), \( U_k = \beta R (1 + n) u_{c2} \) and \( U_{k'} = -\delta (1 + n)^2 u_{c1} \).

**Proposition (5).** Let us guess as equilibrium policy functions for the Benevolent Government solution the following functional form for \( e \) and \( p \), respectively:

\[
e_g = e_{g1} h + e_{g0} \tag{13B}
\]

\[
p_g = p_{g2} k + p_{g1} h + p_{g0} \tag{14B}
\]

If Eq. (13B) and (14B) are the equilibrium of the government problem, then they must satisfy simultaneously the GEEs given by conditions (11B) and (12B). Let us manipulate the GEEs, plugging the expressions for each partial derivative and omitting the apex \( g \). We obtain for \( e \) and \( p \), respectively:

\[
0 = -u_{c1} + \left( (1 + n) (\delta u_{c1} - \beta u_{c2}) \left( \frac{K'_{h'}}{K'_{p'}} - \frac{K'_{e'}}{K'_{p'}} H'_h - \frac{H'_e}{H'_e} \right) \right) H_e + \delta u_{c1} \left( 1 + \frac{H'_{h'}}{H'_{e'}} \right) \tag{15B}
\]

\[
0 = \delta u_{c1} - \beta u_{c2} \tag{16B}
\]
Using the equation of $H(\cdot)$, the following expressions result:

$$H_e = \frac{1 - \theta}{1 + n} \left( \frac{\alpha h + (1 - \alpha) \vartheta}{e} \right)^\theta$$

$$H_h = \frac{\theta}{1 + n} \left( \frac{\alpha e}{\alpha h + (1 - \alpha) \vartheta} \right)^{1-\theta}$$

then it follows that:

$$\frac{H_{h'}}{H_{e'}} = \frac{\theta}{1 - \theta \alpha h' + (1 - \alpha) \vartheta} \alpha e'$$

Under logarithmic utility and linear production function, plugging the guess given by Eq. (13B) and (14B) into the saving function, we obtain the following recursive function for saving choice:

$$k' = \frac{\beta R}{(1 + n)(R(1 + \beta) + \pi_0)} ((1 + h) - e - p) - \frac{1}{R(1 + \beta) + \pi_2} (\pi_{g1} H(e, h) + \pi_{g0}) \quad (17B)$$

Using Eq. (16B) and simplifying, we get:

$$\frac{K'_{h'}}{K'_{e'}} - \frac{K'_{e'}}{K'_{h'}} H'_{h'} = -1 + \frac{\theta}{1 - \theta \alpha h' + (1 - \alpha) \vartheta} \alpha e'$$

Finally rearranging all the terms, Eq. (15B) becomes as follows:

$$-u_{c1} + (1 + n) \delta u_{c1'} \left( 1 + \frac{\theta}{1 - \theta \alpha h' + (1 - \alpha) \vartheta} \alpha e' \right) H_e \quad (18B)$$

Using the political Euler condition $\beta u_{c2} - \delta u_{c1} = 0$ and the economic one $u_{c1} - R\beta u_{c2} = 0$, Eq. (18B) simplifies to:

$$1 = \frac{(1 + n) R}{1 + \theta \alpha h' + (1 - \alpha) \vartheta} \alpha e' H_e \quad (19B)$$

Eq. (19B) is equivalent to:

$$e = \left( \left( (1 - \theta) + \frac{\alpha e'}{\alpha h' + (1 - \alpha) \vartheta} \right) \frac{1}{R} \right)^\frac{1}{\theta} \left( (1 \alpha h + (1 - \alpha) \vartheta) \right) \quad (20B)$$

Let us now make a further assumption on the guess on $e$, considering the following variant of Eq. (13B):

$$e_g = e_{g1} (\psi_g) h + e_{g0} (\psi_g) \vartheta$$

such that $e_{g1} (\psi_g) = \alpha \psi_g$ and $e_{g0} (\psi_g) = (1 - \alpha) \psi_g$, i.e. we guess the policy $e$ as a linear convex combination between parental human capital $h$ and human capital society endowment $\vartheta$ scaled by a constant which has to be determined, $\psi_g$. Then Eq. (20B) can be rewritten as follows:

$$e_g = \alpha \psi_g h + (1 - \alpha) \psi_g \vartheta$$
where \( \tilde{\psi}_g \equiv \left( (1 - \theta) + \theta \frac{\alpha \epsilon'}{\alpha h_t (1 - \alpha \theta)} \right) \frac{1}{R} \). Plugging the guess of \( e_g \) given by Eq. (21B) into the expression of \( \tilde{\psi}_g \) and simplifying we get:

\[
\tilde{\psi}_g \equiv \left( \frac{\alpha \theta}{R} \psi_g + \frac{1 - \theta}{R} \right) \frac{1}{R}
\]

By fixed-point condition \( \tilde{\psi}_g = \psi_g \) which is equal to the fixed point achieved in Proof of Lemma 1. Let us denote \( \epsilon_{g1} \equiv \alpha \psi_g \) and \( \epsilon_{g0} \equiv (1 - \alpha) \psi_g \theta \) then the optimal public education investment is equal to:

\[
e_g = \epsilon_{g1} h + \epsilon_{g0}
\] (22B)

Then the human capital law of motion is:

\[
h_{t+1} = \left( \frac{\psi_g}{1 + n} \right)^{1-\theta} (\alpha h_t + (1 - \alpha) \theta)
\]

After plugging Eq. (14B), Eq. (22B) and the recursive saving function, Eq. (17B), into Eq. (16B), the GEE for the policy \( p \) is as follows:

\[
\frac{1}{(1 + n) R k + (1 + n) p} = \frac{1}{\delta (1 + h) - \epsilon_1 h - \epsilon_0 - p - (1 + n) K (e, p, h)}
\]

After some algebraic manipulations we obtain the following two solutions for \( p \):

\[
p_g^1 = -\pi_{g2}^1 k + \pi_{g1}^1 h + \pi_{g0}^1
\] (23B)

where, under \( \Omega_g \equiv \frac{\beta (1 - \delta (1 + n))}{\beta + \delta (1 + n)} \):

\[
\begin{align*}
\pi_{g2}^1 &= R (1 - \Omega_g) \\
\pi_{g1}^1 &= \left( \frac{R \psi^\theta}{R \psi^\theta - \epsilon_{g1}} (1 - \epsilon_{g1}) \right) \Omega_g \\
\pi_{g0}^1 &= \left( \frac{R \psi^\theta (1 - \epsilon_{g0}) - (\epsilon_{g1} - \epsilon_{g0})}{R \psi^\theta - \epsilon_{g1}} \right) \rho \Omega_g
\end{align*}
\]

and

\[
p_g^1 = -\pi_{g2}^1 k + \pi_{g1}^1 h + \pi_{g0}^1
\] (24B)

where \( \pi_{g2}^1 = R, \pi_{g1}^1 = 0 \) and \( \pi_{g0}^1 = 0 \).

Note that the Eq. (23B) is equivalent to Eq. (24B) under the condition \( \Omega_g = 0 \), which implies \( \delta = \frac{1}{1 + n} \). Recall that, for the existence of the fixed point, the condition \( \delta < \frac{1}{1 + n} \), which induces \( \Omega_g \) to be strictly greater than zero, is required. Consequently the Eq. (24B) is not feasible. ■

**Proposition (6).** Let us first consider the following normalization of the relative welfare weights after assigning \( \delta \equiv \frac{\beta}{\beta (1 + n) + \phi} \). Comparing the parameters of the policy rules of Eq. (24)
and Eq. (29) we obtain for any $\theta > \theta_0$. Then we conclude $\mathcal{P}(h_t, k_t) > \mathcal{P}_g(h_t, k_t)$. ■
References


