How does the Bond Market Perceive Government Interventions?

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Abstract
The investor believes in Ricardian equivalence but fears that it might not hold in reality. The investor is exposed to two types of government uncertainties. First, uncertainty about whether Ricardian equivalence holds. Second, uncertainty about the expected effectiveness of government interventions. The estimation reveals that 80% of variations in the latter premium are explainable by variations in GDP growth and inflation. The results confirm results in Christiano, Eichenbaum and Rebelo (2011) which suggest that uncertainty about the effectiveness of government interventions is lowest at the zero lower bound. Uncertainty premiums for government interventions make the real and nominal yield curve slope upwards.

Keywords:

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1. Introduction

Do government policies affect aggregate consumption and real interest rates? The answer to that fundamental economic question is ambiguous. A large fraction of economists believe that Ricardian equivalence is a good workhorse model for that question. That theory says that government policies do not affect aggregate consumption and real interest rates because investors anticipate that they have to re-pay the additional government stimulus and reduce private consumption to save for the anticipated future tax increase. The corresponding government multiplier would be zero. Empirical support comes from Barro and Redlick (2011) who estimate a multiplier of approximately 0.4.

On the other hand, the U.S. government has responded to the most current financial crisis by issuing a gigantic fiscal stimulus plan, the American Recovery and Reinvestment Act, worth approximately 2.5% of GDP in 2009 and 2010. The government issued that stimulus plan because it believes to stimulate aggregate consumption. That belief is supported by findings in Romer and Bernstein (2009) and Nakamura and Steinsson (2011) who both argue that the fiscal multiplier is around 1.5. This seemingly contradictory evidence shows that investors are confronted with substantial uncertainty about whether government policies affect future consumption growth.

The research community has made considerable progress in understanding how uncertainties about government policies affect investments. The effect on asset prices is less well understood. Pastor and Veronesi (2010) is the first account of an equilibrium model that analyzes the effect of policy uncertainty on the stock market. The effect of government policy uncertainty on bond markets has not been analyzed.

In this paper, I provide a theoretical and empirical bond pricing model

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2 Julio and Yook (2008), Yonce (2009), Hassett and Metcalf (1999) and Rodrick (1999) analyze the effects of policy uncertainty on investment.

3 They show in a Bayesian learning equilibrium model where government policies affect the expected profitability of firms that if a quasi-benevolent government announces to follow an untested new policy stock prices tend to fall. This happens because uncertainty about the effectiveness of the policy increases the firm’s discount rate more than expected future cash flow growth rises.
which accounts for different types of government policy uncertainties. I follow neo-classical economic theory and assume that the representative investor believes in Ricardian equivalence. This implies that he believes that the government cannot systematically affect aggregate consumption growth. I deviate only slightly from the neo-classical view in the sense that my agent does not fully trust in Ricardian equivalence. This exposes the investor to two types of Knight (1921) uncertainty. First, the investor faces uncertainty about whether or not Ricardian equivalence holds. This type of uncertainty is a "known unknown" in the model because the investor can quantify it. I call it also policy intervention uncertainty, because it captures uncertainty about whether government policies try to systematically change aggregate consumption growth. Second, the investor and the government face uncertainty about the effectiveness of each policy intervention model on future consumption. The model treats this uncertainty as truly unobservable, which is consistent with the notion of an "unknown unknown". I call this also multiplier uncertainty, because the macro research focuses on the multiplier to measure the expected effectiveness of government policies.

In the model, the investor is confronted with a set of multiple data generating processes for consumption growth. His most trustworthy model assumes Ricardian equivalence. Other models in the set differ in terms of the anticipated effect that government policy interventions have on consumption growth. The representative agent takes government policy uncertainties into account when pricing assets.

The research questions of the paper are as follows. First, is the agent’s marginal utility affected by both types of policy uncertainties? Second, despite the assumption that the investor believes in Ricardian equivalence, do both types of uncertainty affect real interest rates? Third, if real interest rates were affected, would this raise or lower these? Fourth, if one takes the model to data, can we learn something about the empirical properties of both policy uncertainties? Are there any macro variables that seem to

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4The analogy to "known unknown" and "unknown unknown" goes back to D.H. Rumsfeld who mentioned in a news briefing: "As we know, There are known knowns. There are things we know we know. We also know There are known unknowns. That is to say We know there are some things We do not know. But there are also unknown unknowns, The ones we don’t know We don’t know."
be good proxies for both uncertainties? How do the empirical results relate to Christiano et al. (2011) who argue that government interventions are especially effective in states where the nominal short rate is close to the zero lower bound?

With regard to the first research question, uncertainty about policy interventions and multiplier uncertainty affect marginal utility of the investor in an otherwise standard Lucas (1978) type endowment economy with a log-utility agent. The intuition for this result is that the robust inference about future growth of an ambiguity responsive agent is contaminated by the investor’s fear that Ricardian equivalence does not hold in the data and that policy interventions might have a different degree of effectiveness than anticipated. Second, expected consumption growth and both policy uncertainties have first-order importance for the yield curve of real bonds. Given the first result, this is intuitive because real bond yields are monotone transformations of expected growth rates of marginal utility.

Moreover, the model is flexible in the sense that both types of uncertainty can either increase or lower real interest rates. Uncertainty about Ricardian equivalence increases real interest rates if the investor perceives government policy interventions as benevolent (multiplier bigger than zero). In analogy to the ambiguity aversion, robust control literature, which assumes that malevolent nature might choose the worst-case consumption model from the set of models, I explicitly allow nature, which in my model is the government, to be perceived as benevolent or malevolent. The empirical estimation will recover which of these two perceptions is the proper one. Real interest rates also increase in states of increased multiplier uncertainty, because potential government interventions could be even more effective in such states, than

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what observable proxies indicate. The latter is consistent with the recent American Recovery and Reinvestment Act, which the government anticipates a "best-case" multiplier of 1.6, while most other research studies predict a substantially smaller degree of effectiveness. Under these two perceptions, the investor is willing to pay a premium for obtaining positive exposure to both types of government uncertainty. This is intuitive, because if both perceptions were indeed true, the expected life-time utility of the agent would be higher than under the most trustworthy model which postulates Ricardian equivalence.

While these two ambiguity perceptions make government uncertainties to be good news for the expected life-time utility of the agent, it is bad news for bond investments. As any other consumption-based asset pricing model, my model implies that real bonds fall in value when expected consumption growth increases. This makes bonds to be a recession hedge. That implies that in non-Ricardian models, where the government can systematically increase consumption growth, benevolent government policies lower the probability and severity of recessions. This takes away expected profit opportunities from bond investors. The bond market stays only in zero-net supply if a bond investment offers an additional policy uncertainty premium for expected foregone profit opportunities that arise if the government could indeed systematically increase consumption growth.

These uncertainty premiums for government policies make the real yield curve slope upwards. The model therefore shows that accounting for uncertainty about government policies overcomes the short-comings in Ang et al. (2008a), Piazzesi and Schneider (2006), Piazzesi and Schneider (2010) and Ulrich (2010) whose upward sloping nominal yield curve comes at the cost of a downward sloping real yield curve. This is especially important given that the U.S. real yield curve, measured by U.S. Treasury-Inflation-Protected-Securities (TIPS), is upward sloping. Liquid TIPS data from 2003 onwards suggests that the average ten-year minus five-year spread has been 45 basis points.

\footnote{Compare Barro and Redlick (2011), Woodford (2011), Cogan et al. (2010), among others.}
I evaluate the model with U.S. macro and bond data. I use a forward-looking Taylor (1993) rule and cross-sectional forecast data on the three-month T-bill, next quarter inflation and GDP growth to extract an uncertainty measure for anticipated Fed interventions in the Federal funds market. The forecast data is from the Survey of Professional Forecasters and the uncertainty measure coincides with the variance of the cross-sectional Taylor rule error. The Federal Reserve is the monetary branch of the U.S. government and policy uncertainty that is revealed by Fed actions are a lower bound on government uncertainty that consists of monetary and fiscal uncertainties.

Estimating the model with bond data allows me to recover the time-series of the "unknown unknown" factor that drives uncertainty about the expected effectiveness of policy interventions and to judge how the bond market perceives government policy interventions. This factor is highly correlated with aggregate macro variables. I regress this factor (normalized) on normalized expected inflation, expected GDP growth and several normalized measures of uncertainty and find that it correlates strongly with expected inflation and expected GDP growth. More precisely, 80% of variations in multiplier uncertainty is related to changes in expected nominal growth. For the normalized factors I find that a one percent increase in expected inflation (expected GDP growth) leads to a 0.93% (0.14%) increase in uncertainty about the expected effectiveness of policy interventions. The more the economy expands, the more uncertainty does the investor and government have about the expected effectiveness of new policy interventions. This also implies that if expected inflation approaches zero, multiplier uncertainty goes to zero as well; meaning the investor and the government are very certain about the effectiveness of policy interventions at the zero lower bound. This finding is consistent with Christiano et al. (2011) who argue that the effectiveness of government spending is particularly obvious in periods of low expected inflation because these are periods in which the zero-lower bound on the short-term nominal interest rate binds.

\footnote{The normalization makes all variables to have a zero mean and unit variance.}
2. Government Policies and the Business Cycle

2.1. Reference belief

The reference belief describes the investor’s most trustworthy approximation of reality. Based on the work of Lucas (1978) and Barro (1989) the investor’s most trustworthy model for his endowment is that government policies have no effect on aggregate consumption growth. I model this by assuming that consumption good $c$ follows

$$d \ln c_t = (c_0 + z_t)dt + \sigma_c dW_t^c,$$

(1)

where $c_0 > 0$ is the unconditional growth rate of consumption growth, $\sigma_c > 0$ is the conditional volatility of consumption growth and the process $z$ is the predictable business cycle component. The business cycle component $z$ follows a simple mean-reverting Ornstein-Uhlenbeck process

$$dz_t = k_z z_t dt + \sigma_z dW_t^z,$$

(2)

with $-\kappa_z, \sigma_z \in \mathcal{R}_+.$

The complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, Q^0)$ describes mathematically the reference belief and $E[\cdot]$ denotes expectations under that belief. The agent’s expected life time utility is denoted as $U(c_t; z_t)$, i.e.

$$U(c_t; z_t) := E_t \left[ \int_t^\infty e^{-\rho(s-t)} \ln c_s ds \right],$$

(3)

where $\rho > 0$ is the subjective time discount factor.

Real yield curve. As is known from the literature, the above assumptions for the reference belief endogenize a one factor Vasiček (1977) type real yield curve. Yields of all maturities depend only on the most current realization of $z_t$. Holding real bonds would not pay any risk premiums because the agent has log utility and $W^c$ is orthogonal to $W^z$. The volatility of real yields would be driven by volatility of expected consumption growth. Government policy interventions have no effect on aggregate growth or real interest rates.
2.2. Uncertainty about Ricardian Equivalence

The absence of government policies in the reference model coincides with assuming that the government spending multiplier is zero, meaning that every dollar spent by the government reduces a dollar spent by the private agent. So the net effect on the aggregate consumption process is zero. Accounting for measurement errors one can decompose the business cycle shock $W^z$ into two orthogonal components. One captures noise coming from policy interventions, while the other is a pure RBC-type technology shock. Thus,

\[ dW^z_t = \hat{\rho} dW^z_t + \sqrt{1 - \hat{\rho}^2} dW^G_t, \]

where $dW^z_t$ is the technology shock and $dW^G_t$ is a government policy shock. The zero-mean of the policy shock captures Ricardian equivalence and a government multiplier of zero because it states that the expected impact of a policy intervention on the business cycle is zero.

The shock $W^G_t$ captures net effects of government policies, which are assumed to be zero in expectation. These are net effects because the agent anticipates already that an increase in government spending must be financed by higher taxes in the future or a tax cut today will be offset by lower government spending in the future. So if the multiplier is zero, the net effect on growth will be zero as well. Government spending includes fiscal policy as well as monetary policy. Examples for fiscal policy are Keynesian-type anti-cyclical government spending or public investments in infrastructure, military and education. Examples for monetary policy include money supply policies or Quantitative Easing policies that have the goal to affect firm’s investment decisions through changing the firms real cost of capital or through changing real labor costs.

**Uncertainty about Richardian Equivalence.** My investor does not fully trust the Ricardian equivalence model. He is aware of several empirical findings that argue that Ricardian equivalence might not hold and that government interventions $W^G_t$ might indeed have a systematic impact on $z$.$^8$ Compared to the Ricardian reference belief this means that government interventions, net of the investor’s adjusting behavior, might add a non-zero drift

$^8$Romer and Bernstein (2009) and Nakamura and Steinsson (2011) find a multiplier of around 1.5 which implies a systematic and positive impact on $z$. 8
to the business cycle component of aggregate consumption growth. These systematic net effects can be positive or negative. The net effect is positive if the multiplier is positive.

I incorporate Ricardian equivalence uncertainty by introducing Knightian (1921) uncertainty about the government policy shock $dW^G$. This means that the investor knows that policy models in which Ricardian equivalence does not hold impose a non-zero conditional expected growth rate on $dW^G$, i.e.

$$dW^G_t = dW^{G,h}_{t} + h^G_t dt,$$

where $h^G_t \neq 0$ quantifies the amount of instantaneous expected deviations from Ricardian equivalence. I call $dW^{G,h}_{t}$ to be a zero-mean government policy shock under the distorted belief $Q^{h^G}$. Following Hansen and Sargent (2008) and Chen and Epstein (2002), $h^G_t$ quantifies the expected instantaneous distortion in beliefs that arises from uncertainty. Note that the reference belief is a special distorted belief, because it implies $h^G_t \equiv 0$.

In summary, the reference model postulates Ricardian equivalence ($h^G_t \equiv 0$), while the ”true” government policy model might be characterized by non-zero stochastic drift distortions $h^G_t$.

From the view of the data generating process, distorted models imply

$$dz_t = (\sigma_z \sqrt{1 - \bar{\rho}^2} h^G_t + \kappa z_t) dt + \sigma_z \left(\bar{\rho} dW^z_t + \sqrt{1 - \bar{\rho}^2} dW^{G,h}_{t}\right),$$

which are more complex processes compared to the CIR process of the Ricardian model. If $h^G_t$ is sufficiently small, an econometrician, who does not know the true DGP of $z$, cannot distinguish between realized $z$’s being generated by the Ricardian model (2) or by a distorted model (6) where Ricardian equivalence does not hold.

This lack of knowledge exposes the investor to Knight (1921) uncertainty that Ricardian equivalence might not hold in reality, despite the fact that the investor’s most trustworthy model assumes it holds. Indeed, it could be that the Ricardian model, i.e. $h^G_t \equiv 0$ is wrong, and that government policy interventions have a systematic, but potentially hard to measure, impact on the drift of $z$. 
Government’s decision problem. The expected net effect of Government policy interventions can be non-zero in a non-Ricardian world. The investor assumes that if the true model of the world was non-Ricardian, the government would implement policies with the goal of setting \( h_t^G \) such that the agent’s expected life-time utility is maximized. This coincides with perceiving government as benevolent. The government is constrained by an exogenously given set of policy models that it can choose from. Mathematically, this set is modeled via an entropy growth constraint between the Ricardian model and each non-Ricardian model.

Formally the government solves the following dynamic problem

\[
\max_{\{h_t^G\}} E_t \left[ \int_t^{\infty} e^{-\rho s} \ln c_s ds \right] \quad (7)
\]

\[s.t. (1), (6) \quad (8)\]

\[s.t. \frac{1}{2} (h_t^G)^2 dt \leq A (\eta_t)^2 dt, \quad (9)\]

where \( A > 0 \) and where the left hand side of the last equation coincides with \( E_t^{h_t^G} \left[ d \ln \frac{dq^{h_t^G}}{dq^0} \right] \). The right hand side of the last equation characterizes the observed size of the set of non-Ricardian policy intervention models. The term \( (\eta_t)^2 dt \) measures time-variations in the size of the set. If \( A (\eta_t)^2 = 0 \), the set of non-Ricardian models equals zero, which means that Ricardian equivalence is expected to hold under all models.

Solution to government’s decision problem. For the moment I assume that the government knows the net effect that different intervention models have on the business cycle. I will relax that assumption later, allowing for uncertainty about the expected effectiveness of policy intervention models.

The solution methodology to the government’s dynamic problem follows Chen and Epstein (2002), Sbuelz and Trojani (2002b), Sbuelz and Trojani (2008) and Ulrich (2010). In order to derive an analytically convenient solution I assume that the government observes \( \eta_t^2 \) and knows for sure that \( \eta_t \) follows a Feller (1951) process,

\[
d\eta_t = (a_\eta + \kappa_\eta \eta_t) dt + \sigma_\eta \sqrt{\eta_t} dW_\eta, \quad (10)
\]
where \( a_\eta > 0, \kappa_\eta < 0 \) and \( \sigma_\eta > 0 \).

The appendix shows that the optimal policy intervention is given by:

\[
h^G_t dt = \sqrt{2A_\eta} dt \equiv m^G_\eta dt,
\]

where I defined \( \sqrt{2A} \equiv m^G \). The optimal policy function implies the following. First, the expected net effect of policy interventions on the business cycle is time-varying. The expected net effect in a non-Ricardian world is particularly large in periods where the set of policy models is large. Second, in a non-Ricardian world, a benevolent government increases the instantaneous expected consumption growth rate by \( \sigma z \sqrt{1 - \bar{\rho}^2 m^G_\eta} dt > 0 \). Third, if the government was malevolent, minimizing the expected life-time utility of the agent, the optimal policy function would be the same with \( m^G = -\sqrt{2A} \).

**Quantifying the trust in Ricardian equivalence.** I assume that the government is able to communicate its optimization problem and set of models from which it can choose from to the agent. This allows the investor to anticipate the optimization problem of the government and its solution. I leave it to future research to add complications that could arise from miss-communications.

Remember that the investor’s most trustworthy model for \( z \) (reference belief) assumes a government multiplier of zero, or equivalently Ricardian equivalence, i.e. \( h^G_t = 0 \). Being exposed to Ricardian equivalence uncertainty and knowing the optimal policy of the government makes the investor sensitive to the argument that the reference model for \( z \) might be miss-specified and that the ”true” instantaneous expected growth rate of \( z \) equals \( \sigma z \sqrt{1 - \bar{\rho}^2 m^G_\eta} + k_z z_t \). The investor applies likelihood ratio tests on realized values of \( z \) to determine whether \( h^G \) equals zero (his reference belief) or whether \( h^G \) equals \( m^G_\eta \).

Let \( a^G_t \) denote the ratio of the empirical likelihood that \( z_t \) was generated by the non-Ricardian model (distorted model) \( Q^h^G \), compared to that it was generated by the Ricardian model (reference model) \( Q^0 \). In my model that

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9The term \( \eta_t \) is to be understood as \( \sqrt{\eta_t^2} \) in order to ensure positivity of \( \eta_t \).
likelihood ratio has a convenient analytical characterization

\[ a_T^G := \frac{dQ_T^{h_t^G}}{dQ_T^T} = \exp \left( -\frac{1}{2} \int_0^T (h_t^G)^2 dt + \int_0^T h_t^G dW_t^G \right), \quad a_0 \equiv 1. \] (12)

This likelihood ratio is a martingale under the reference belief, i.e.

\[ \frac{da_t^G}{a_t^G} = h_t^G dW_t^G. \] (13)

The last equation has three practically relevant implications. First, the investor does not expect to reject his Ricardian view of the world. This holds because the expectation of the last equation is zero. Said differently, the investor expects his reference model to stay the most trustworthy model. Second, observing policy interventions in the form of \( dW_t^G \), changes the investor’s amount of Ricardian equivalence uncertainty. This is very intuitive because since the investor is not sure that the world is Ricardian, he cannot be sure that an observed policy intervention is noise and not the result of a systematic policy intervention.

The following example illustrate this. Imagine the business cycle component in consumption growth increases because of a positive government intervention \( \sigma_z \sqrt{1 - \bar{\rho}^2} (dW_t^{G,H} + h_t^G dt) > 0 \). Since the agent faces Ricardian equivalence uncertainty, he does not know with certainty that \( h_t^G \) is zero. If the agent believes that the government is benevolent, it could indeed be that \( h_t^G \) is positive and that this caused the observed increase in consumption growth. Applying a likelihood ratio test will indeed tell the agent that it is more likely that the world is non-Ricardian. Mathematically this means \( \frac{da_t^G}{a_t^G} > 0 \) puts more empirical weight on the hypothesis that the non-Ricardian (distorted) model, and not the most trustworthy Ricardian model, generated \( z \). In summary, observing a positive policy intervention increases the agent’s fear that the reference model of Ricardian equivalence is misspecified.

Third, there is a non-zero probability that a sequence of positive government intervention shocks "fools" the empirical likelihood function in the sense that it argues that this positive sequence arose because Ricardian equivalence does not hold, \( h_t^G > 0 \), although Ricardian equivalence might indeed be the true data generating process for \( z \). This implies that there are states of the world where the empirical likelihood ratio can move towards zero, making
the investor trust fully in Ricardian equivalence; or move towards infinity, making the investor totally mistrust that assumption.

I call an investor to be ambiguity responsive, if the government’s decision problem in (7) persuades the investor to evaluate his expected life-time utility under the non-Ricardian (distorted) measure $Q^{hG}$, i.e.$^{10}$

$$E_0^{hG} \left[ \int_0^{\infty} e^{-\rho t} \ln c_t dt \right].$$  \hspace{1cm} (14)

As explained in Hansen and Sargent (2008), from the perspective of the reference model, the investor’s expected life-time utility depends on a multiplicative preference shock which quantifies the trustworthiness of the Ricardian equivalence assumption, i.e.

$$U^{hG} (c_0, z_0, \eta_0) := E_0 \left[ \int_0^{\infty} a_t^G \cdot e^{-\rho t} \ln c_t dt \right].$$  \hspace{1cm} (15)

The last equation shows that the risk attitude is characterized by $\ln c_t$, whereas the uncertainty attitude about Ricardian equivalence is quantified by the exponential martingale $a_t^G$.

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$^{10}$Ghirardato et al. (2004) show that the subjective measure of the agent can be distorted towards the worst-case, best-case or any convex combination of both. They summarize this as $\alpha$-maxmin expected utility with multiple priors.
2.3. Multiplier Uncertainty

Already in 1967, Brainard (1967) argued that the government itself faces uncertainty about the effectiveness of its interventions. Recent empirical and theoretical research points out that even half a century later the government is exposed to a substantial amount of uncertainty about the expected effectiveness of policy interventions.\footnote{Empirical evidence for a multiplier of 0.4 magnitude comes from Barro and Redlick (2011). Woodford (2011) shows that neo-classical models support this view. Cogan et al. (2010) use a New Keynesian model to support the view that the multiplier is smaller than 0.5. On the contrary, Nakamura and Steinsson (2011) estimate a multiplier of 1.5. This is supported by the model of Romer and Bernstein (2009). Woodford (2011) and Christiano et al. (2011) show that realistically calibrated New Keynesian models produce only a multiplier of bigger than one in states where the nominal short rate is at the zero lower bound.} I refer to this uncertainty as "multiplier uncertainty". An increase in multiplier uncertainty indicates that it becomes more uncertain whether a policy intervention creates a positive or a negative net effect on the business cycle.

**Modeling multiplier uncertainty.** Equation (10) assumed that the government knows the statistical model that describes size of the set of models. This is equivalent to saying the government can form "correct" expectations about the future effectiveness of its chosen intervention policy. To relax this restrictive assumption, I specify that the "true" but unknown data generating process for $\eta$ can follow a much more complex distribution than the simple Ornstein-Uhlenbeck process in equation (10). This means that the "true" but unknown size of the set of government intervention models solves

$$d\eta_t = (a_\eta + \kappa_\eta \eta_t)dt + \sigma_\eta \sqrt{\eta_t} \left( dW^{\eta,h}_t + h^{\eta}_t dt \right)$$

(16)

where $h^{\eta}_t dt$ is an instantaneous stochastic perturbation that characterizes multiplier uncertainty.

From the government's perspective this models uncertainty about the expected effectiveness of different policy intervention models. From the investor's perspective this uncertainty comes on top of Ricardian equivalence uncertainty. In short, it coincides with the investor's uncertainty about his
Ricardian equivalence uncertainty.

**Government decision problem on choosing the multiplier.** The above discussion has shown that the U.S. government launched the American Recovery and Reinvestment Act with the belief that the government spending multiplier is 1.6, although most estimates argue it is lower than 0.5. Considering the vast amount of different multiplier estimates, how did the government choose to work with a multiplier that is on the higher end of the spectrum?

The government has several possibilities to respond in a robust way to multiplier uncertainty. It can believe that potential interventions are more effective compared to what the dynamic of $\eta$ in equation (10) suggests. Evidence from the American Recovery and Reinvestment Act suggests that the Obama administration chose to believe that the true multiplier is actually larger than what most models and empirical studies suggest.

I assume that multiplier uncertainty is a true "unknown unknown", meaning it characterizes the amount of Knightian (1921) uncertainty about policy interventions that the investor and the government cannot quantify. In the model, I capture this by assuming that the expected relative entropy growth rate between the reference model for $\eta$ and its distorted dynamic is driven by an unobserved factor, which I call $L$. The letter $L$ stands for latent or unobserved. If this factor was observed it would not capture "unknown unknowns".

In summary, the government must find the anticipated effectiveness of its chosen policy. Thus,

$$\max_{h_t^\eta} U^{h^G}(c_0; z_0, \eta_0)$$

subject to:

$$\frac{1}{2}(h_t^\eta)^2 dt \leq \frac{1}{2\eta_t}L_t^2 dt$$

I assume that $L$ follows a Feller (1951) process, which implies

$$dL_t = (a_L + \kappa_L L_t)dt + \sigma_L \sqrt{L_t} dW_t^L,$$

with $a_L > 0$, $\kappa_L < 0$ and $\sigma_L > 0$. 

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Solution to government decision problem on choosing the multiplier. The appendix shows that the solution to problem (17) is given by the following policy function

\[ h^\eta_t = L_t \in \mathcal{R}_+ dt. \]  

(20)

The solution implies that in periods where multiplier uncertainty is high \((L^2)\) is high), the government enlarges the set of policy models that it considers by \(\sigma^\eta L_t dt > 0\).

Alternatively, if the government believed that chosen policy will be less effective compared to the approximation in (10), it would run a minimization problem instead of the maximization in (17). The structure of the solution would be \(h^\eta_t = L_t \in \mathcal{R}_- dt\). In the empirical study, I use data to back out the investor’s perception about how the government copes with multiplier uncertainty. For the reader’s convenience and because of the empirical findings, I will continue to assume that the government solves (17).

The robust utility function of the investor, evaluated under the Ricardian equivalence belief, is given by

\[ U^h(c_0; z_0, \eta_0, L_0) := E_0 \left[ \int_0^\infty a_t^G \cdot a^\eta_t \cdot e^{-\rho t} \ln c_t dt \right], \]  

(21)

where the pricing kernel for multiplier uncertainty, \(a^\eta_t\), is given by

\[ \frac{da^\eta_t}{da^\eta_t} = h^\eta_t dW^\eta_t. \]  

(22)

Relation to the ambiguity literature. The investor’s robust inference of future consumption growth will rise in periods where the empirical likelihood favors the non-Ricardian (distorted) model over the Ricardian (reference) model. Moreover, it will also rise in periods where the government faces more uncertainty about the expected effectiveness of its policy models. In the language of Klibanoff et al. (2005), the investor ”likes” being exposed to uncertainty about Ricardian equivalence and about the size of the multiplier. Compared to Ghirardato et al. (2004) the investor has a constant ”loving attitude” towards Ricardian uncertainty, while the investor and the government have a constant ”loving attitude” towards multiplier uncertainty.
While the ambiguity attitudes are constant, the amount of Ricardian equivalence uncertainty and multiplier uncertainty are stochastic. In the empirical section I use bond yields to evaluate whether the assumed ambiguity attitudes are supported in the data.

Ellison and Sargent (2010) assume that the FOMC, which in my model is part of the government, faces model uncertainty with regard to the output and inflation dynamics in the economy. They assume it has a constant ambiguity aversion attitude towards using too optimistic inflation and output forecasts.\footnote{The robust forecasts of the FOMC are therefore different from the staff forecasts who themselves forecast under the reference model. The authors argue this explains why both forecasts differ, as was first noted by Romer and Romer (2008).}

\subsection*{2.4. Government Policy and Marginal Utility}

How can it be that marginal utility is affected by policy interventions, although the investor believes in Ricardian equivalence? Doubts about the correctness of this belief are the reason. The investor knows that the world might be non-Ricardian, despite his belief that Ricardian equivalence is the most trustworthy workhorse model. Marginal utility is affected by both types of policy uncertainty.

\textbf{Uncertainty kernel and risk kernel.} I denote the agent’s marginal utility as $M$. It accounts for the risk-free rate, the market price of risk and both market prices of uncertainty. There are two market prices of uncertainty, one for Ricardian equivalence uncertainty and another one for multiplier uncertainty. In detail, $M$ is defined as the growth rate in marginal utility, i.e.

\begin{equation}
\frac{M_s}{M_t} := e^{-\rho(s-t)} \frac{U^h_c(c_s; z_s, \eta_s, L_s)}{U^h_c(c_t; z_t, \eta_t, L_t)}, \quad s > t
\end{equation}

and coincides with the real stochastic discount factor (sdf). In my log-utility and government uncertainty set-up, it reduces to the product of four eco-
nomically meaningful terms:

\[
\frac{M_s}{M_t} = e^{-\rho(s-t)} \left( \frac{c_s}{c_t} \right)^{-1} \underbrace{\left( \frac{a_s^G}{a_t^G}, \frac{a_s^G}{a_t^G} \right)}_{\text{RiskKernel}} \overbrace{\left( \frac{a_s^G}{a_t^G}, \frac{a_s^G}{a_t^G} \right)}^{\text{UncertaintyKernel}}
\]

(24)

Risk kernel. The risk kernel is the standard log-utility consumption-based risk kernel. It captures expected and unexpected changes in consumption growth. Expected changes in consumption growth feed into the risk-free rate, while unexpected changes, \(\sigma_c dW^c\), characterize the market price and source of priced consumption risk. One can refer to consumption risk \(W^c\) as "known" consumption risk, because the concept of risk assumes the model is known. In the absence of policy uncertainty, the sdf collapses to the risk kernel.

Uncertainty kernel. The uncertainty kernel consists of two components, each being an exponential martingale under the reference belief. Since these are exponential martingales, they do not contribute to the risk-free rate, but they contribute to the market price of uncertainty. This already implies that long-term real interest rates might be affected by policy uncertainty.

The first component, \(\frac{a_s^G}{a_t^G}\), characterizes the amount of trust that the investor has in his Ricardian equivalence model. Said differently, it measures the trust into the assumption that the government does not systematically intervene in the business cycle. The term \(\frac{a_s^G}{a_t^G}\) rises if the empirical likelihood ratio favors the non-Ricardian model over the Ricardian equivalence model. Assuming the agent perceives the government as benevolent, the expected life-time utility of the investor is higher in a non-Ricardian model. The agent likes the non-Ricardian model and is willing to pay a premium of \(-h_t^G < 0\) to get a positive exposure to that kind of uncertainty.

The term \(\frac{a_s^G}{a_t^G}\) quantifies the "known unknown", because I assume that the agent is able to quantify the amount of government intervention uncertainty. This means the agent can infer the level of \(\eta^2\).

The second component, \(\frac{a_s^G}{a_t^G}\), characterizes the amount of uncertainty
about the expected effectiveness of different government intervention models. It captures the amount of multiplier uncertainty that the government itself faces. The term, $\eta^{G}_{G}$, quantifies the "unknown unknown" because the investor has no observable information about the amount of uncertainty that the government faces.

If the agent believes that the government increases the set of models in states of increased uncertainty about the expected effectiveness of policy interventions, the investor’s expected life-time utility increases in such states. The agent will be willing to pay a premium of $-h^{\eta} dt < 0$ for a unit of exposure to uncertainty about the effectiveness of policy interventions.

The transition density of the sdf summarizes this as follows

$$-\frac{dM_t}{M_t} = (\rho + c_0 - \sigma_c^2 + z_t)dt + \sigma_c dW^c + (-h^G_t) dW^G_t + (-h^{\eta}_t) dW^\eta_t \quad (25)$$

where the drift term coincides with the equilibrium instantaneous real risk-free rate, $r$. The term $\sigma_c \in \mathcal{R}_+$ coincides with the market price of consumption risk, $-h^G_t \in \mathcal{R}_-$ is the market price of Ricardian equivalence uncertainty and $-h^{\eta}_t \in \mathcal{R}_-$ coincides with the market price of multiplier uncertainty. The only priced source of risk is $W^c$, while the priced sources of policy uncertainty are $W^G$ and $W^\eta$. These three shocks are priced because they drive unpredictable changes of the agent’s marginal utility.

**Difference to government interventions being certain.** The reader might wonder how my set-up differs from a model where the investor knows for sure that the government affects the drift of consumption growth and where the government itself has full information about the effectiveness of its interventions. The short answer is that such a model would coincide with a standard consumption based asset pricing model where expected consumption growth and the instantaneous risk-free rate were higher because of the government factor in $z$.

In more detail, in such a model, $z$ would contain an additional government intervention factor. The structure of the instantaneous risk-free rate would be the same, $r = \rho + c_0 - \sigma_c^2 + z_t$, but it would be higher if the government factor in $z$ was positive. The market price of risk would be the same,
and there would not be any market prices of policy uncertainty, because by assumption there was no uncertainty.

In summary, one can obtain a meaningful dynamic of marginal utility even in the log-utility case, as in equation (24), if one accounts for the complex uncertainty issues that arise in reality because of potential government business cycle interventions.
3. Real Bonds as a Mirror Image of Marginal Utility

Zero-coupon bonds are in zero net supply if they are priced according to a standard Euler equation. Thus,

\[ B_t(\tau) = E_t \left[ \frac{M_{t+\tau}}{M_t} \right], \quad \tau > 0 \quad (26) \]

where \( B_t(\tau) \) is the price of a real zero-coupon bond that matures in \( \tau \) periods.

Marginal utility and real bonds. Observing these bond prices is equivalent to observing the conditional expected growth rates of marginal utility of the representative investor. A good model for marginal utility translates into low pricing errors when the bond model is taken to data. This has the implication that we can use observed bond prices to learn how the investor copes with both types of government policy uncertainty.

Intuitively, both types of policy uncertainty affect the price of real bonds, because they affect the investor’s robust inference of future consumption growth. If there was were no policy uncertainties, government would not have any impact on marginal utility and real interest rates.

But, if the investor faces Ricardian equivalence uncertainty this will increase his robust forecast of expected future growth. Similarly, if the investor thinks the government faces multiplier uncertainty, this will affect his robust forecast about the aggressiveness with which the benevolent government might try to systematically affect the path of future consumption growth. An ambiguity responsive investor quantifies how these two policy uncertainty channels might affect future growth. The price and yield of real bonds allows to back out these concerns.

The appendix shows that \( B_t(\tau) \) is an exponentially affine function in the instantaneous expected consumption growth rate \( z_t \), as predicted by the reference model, as well as exponentially affine in the two uncertainty factors.
that characterize the time-varying amount of trust that the investor has in that model. Thus,

\[ B_t(\tau) = \exp (A(\tau) + B_z(\tau)z_t + B_\eta(\tau)\eta_t + B_L(\tau)L_t), \]  

(28)

where the factor loadings are deterministic functions of the underlying parameters of the economy.

Continuously compounded real yields, \( y_t(\tau) \), are affine in \( z_t, \eta_t, L_t \), i.e.

\[ y_t(\tau) := -\frac{\ln B_t(\tau)}{\tau} = -\frac{1}{\tau} (A(\tau) + B_z(\tau)z_t + B_\eta(\tau)\eta_t + B_L(\tau)L_t). \]  

(29)

The previous equation shows that expected consumption growth, the "known unknown" uncertainty about Ricardian equivalence and the "unknown unknown" uncertainty about the expected effectiveness of policy interventions are all of first-order importance for explaining the term structure of real bond yields. The factor loadings reflect the relative importance of these three macro factors for real yields of different maturities.

3.1. Government Can Take Over the Bond Market

Expected business cycle growth. The instantaneous real interest rate depends only on the factor \( z_t \), which makes \( z_t \) the natural level factor for the real yield curve. If \( z_t \) goes up, real interest rates rise to offset the agent’s temptation to reduce his savings because of the expected increase in future endowments. A rise in real rates is equivalent to falling bond prices. As the bond loadings \( B_z(\tau) \) show, the magnitude by which long-term real bond prices fall depends only on the persistence of \( z \) itself:

\[ B_z(\tau) = \frac{1 - e^{\kappa_z \tau}}{\kappa_z} \in \mathcal{R}_-, \quad B_z(0) = 0. \]  

(30)

This shows that \( B_z(\tau) \) starts in zero and decays exponentially towards its steady-state value of \( B_z(\infty) = \frac{1}{\kappa_z} < 0 \).Buying and holding a real bond provides the investor with negative exposure to \( z \), as is well known in the literature and evident from (30). It is the negative \( z \) exposure that makes real bonds to be recession hedges. Below, I will show explicitly that this negative exposure makes investors expect to earn a positive uncertainty premium for
benevolent but uncertain government interventions into the business cycle.

Uncertainty about Ricardian equivalence. The investor observes \( \eta \), which provides him with an estimate for the amount of uncertainty about whether the government can systematically increase consumption growth. The loading \( B_\eta(\tau) \) quantifies how the corresponding uncertainty premium affects real bonds of different maturities. The appendix verifies that this loading is given by

\[
B_\eta(\tau) = \frac{m^G \sqrt{1 - \bar{\rho}^2 \sigma_z}}{-\kappa_z \kappa_\eta} \left( 1 - e^{\kappa_\eta \tau} + \frac{\kappa_\eta (e^{\kappa_z \tau} - e^{\kappa_\eta \tau})}{\kappa_z - \kappa_\eta} \right). \tag{31}
\]

Studying this loading is instructive for understanding how the bond market reacts to a temporary increase in Ricardian equivalence uncertainty. First, the uncertainty premium is paid because uncertainty about government interventions in the business cycle introduces a potentially systematic error into the investor’s forecast of business cycle growth that comes out of his Ricardian (reference) model. Compared to the Ricardian (reference) model, the instantaneous expected growth rate of \( z \) is higher under the non-Ricardian (distorted) model, namely in steady-state by factor

\[
m^G \sqrt{1 - \bar{\rho}^2 \sigma_z}/(-\kappa_z) > 0.
\]

Second, given that a bond has negative \( z \) exposure, it pays out well if business cycle growth is weak and it pays out poorly if business cycle growth is strong. If the non-Ricardian business cycle model was indeed the correct DGP for \( z \), the economy would grow stronger than forecasted by the agent’s Ricardian model and being long a bond would turn out to pay out less than forecasted by the Ricardian model.

In summary, if the government could indeed increase systematically \( z \) over the business cycle, this would lower the probability and severity of recessions, but take profit opportunities from negative \( z \) exposure assets such as bonds. Since the investor is uncertain about such a systematic government impact, he requires an uncertainty premium for expected losses, in case the non-Ricardian model is indeed true. Otherwise the bond market would not remain in zero-net supply.

Fundamentals vs. Ricardian equivalence uncertainty. What affects the long-term expected growth rate in marginal utility, or equivalently

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long-term real bond yields stronger? A one percent shock to expected consumption growth or a one percent shock to uncertainty about whether Ricardian equivalence holds? Straightforward intuition would argue that a one percent change in expected consumption growth has a bigger impact than a one percent change in government intervention uncertainty. This argument relies on the idea that government intervention uncertainty affects marginal utility through its impact on expected consumption growth, and has therefore a form of second-order importance for marginal utility and real bond yields.

The appendix shows by comparing $B_\eta(\infty)$ with $B_z(\infty)$, that this intuition is absolutely right under one of the following two conditions. First, the set of non-Ricardian models is sufficiently small, which implies that the potential instantaneous forecast error when working with the wrong model is sufficiently small. Second, persistence in Ricardian equivalence uncertainty is sufficiently small, implying that long-term forecasts for different models are sufficiently close to each other. Both of these conditions ensure that the long-term forecast under the Ricardian (reference) model and the "best-case" non-Ricardian model are sufficiently close to each other. Whether these conditions are fulfilled depends on the empirical properties of $\eta$ and $z$ and is not under the control of the modeler.

Interpreted from another angle, it can indeed happen that uncertainty about government interventions in the business cycle dominates long-term marginal utility and long-term real yields if $\eta$ is extremely persistent. This can happen even if different models within the set of models produce very similar short-term forecasts.

**Uncertainty about the government multiplier.** The factor loading $B_L$ quantifies the impact that the government's uncertainty about the expected effectiveness of its interventions has on the investor's expected marginal utility and the real yield curve in the economy. The appendix
verifies that
\[ B_L(\tau) = \frac{m^G \sqrt{1 - \rho^2 \sigma_z \sigma_\eta}}{-\kappa_z} \cdot g(\tau) \] (32)

\[ g(\tau) := \left( \frac{1 - e^{\kappa_L \tau}}{-\kappa_L \kappa_\eta} + \frac{e^{\kappa_\eta \tau} - e^{\kappa_L \tau}}{\kappa_\eta (\kappa_L - \kappa_\eta)} + \frac{e^{\kappa_z \tau} - e^{\kappa_L \tau}}{(\kappa_z - \kappa_L)(\kappa_z - \kappa_\eta)} - \frac{e^{\kappa_\eta \tau} - e^{\kappa_L \tau}}{(\kappa_\eta - \kappa_L)(\kappa_z - \kappa_\eta)} \right). \] (33)

The first part of the loading confirms that multiplier uncertainty has an "uncertainty about uncertainty" feature. If the agent thinks the government solves (17), he anticipates that the government will expand the set of potential intervention models in periods where it is more uncertain about the multiplier. If in addition, he believes that government is benevolent, his "best-case" non-Ricardian business cycle model anticipates an even more aggressive government intervention in the future. The first term of the last equation captures this. The nominator \( m^G \sqrt{1 - \rho^2 \sigma_z \sigma_\eta} > 0 \) quantifies the amount by which a one unit increase in multiplier uncertainty increases the business cycle growth forecast of the "best-case" model. Scaling it by the positive denominator relates it to the increase in steady-state.

This "uncertainty about uncertainty" name indicates that multiplier uncertainty might be of third-order importance when it comes to expected changes in long-term marginal utility and long-term real bond yields. The appendix shows that indeed \( B_L(\infty) = B_\eta(\infty) \cdot \sigma_\eta/(-\kappa_L) < 0 \). The economic interpretation is that multiplier uncertainty is of third-order importance if it is not "too" persistent. This means as long as its persistence is sufficiently low, i.e. \(-\kappa_L > \sigma_\eta\), Ricardian equivalence uncertainty will dominate long-term yields over multiplier uncertainty.

Viewed from another angle, if multiplier uncertainty, \( L \), is sufficiently persistent, meaning \(-\kappa_L < \sigma_\eta\), multiplier uncertainty has a second-order impact on long-term real interest rates. If in addition \( \eta \) was sufficiently persistent, multiplier uncertainty could even become the main driver for long-term real interest rates and long-term growth in marginal utility. The persistence of both sources of policy uncertainty is determined by their empirical time-series.

In summary, the model is flexible enough to explain that it can happen that long-term real yields fluctuate not because of news about the funda-
mental growth process $z$, but instead because of news about government interventions and the expected effectiveness of interventions.

4. Policy Uncertainties and Real Bond Premiums

The investor requires a premium for holding real bonds if the holding return correlates with changes in marginal utility. The premium is positive if real bonds pay out poorly in periods where the investor needs income most (high marginal utility). On the other hand, the premium is negative if real bonds pay out well in periods of high marginal utility. These statements hold for all equilibrium asset pricing models. My model is distinct from existing asset pricing models because uncertainty about whether Ricardian equivalence holds and uncertainty about the government multiplier affect marginal utility of the investor.

**Real bond premium.** I define $RP_t^B(\tau)$ to denote the instantaneous risk premium on a $\tau$-maturity real bond. It coincides with

$$RP_t^B(\tau)dt = -\frac{dM_t}{M_t} \frac{dB_t(\tau)}{B_t(\tau)} = -B_z(\tau)\sigma_z \sqrt{1 - \bar{\rho}^2} m^G \eta_t dt - B_{\eta}(\tau)\sigma_\eta(L_t) dt. \quad (34)$$

The premium for real bonds has two components. The first component captures a premium for exposure to Ricardian uncertainty. The second premium exists in equilibrium because of multiplier uncertainty.\(^{13}\)

The first premium is positive if the investor believes that the world might indeed be non-Ricardian and that a benevolent government exploits this by trying to systematically intervene in the business cycle. The intuition for this premium is as follows. A real bond does not hedge an increase in uncertainty about government interventions. To see that, assume that there is a positive shock to $W^G$ which increases $z$. The investor looses trust into his Ricardian reference model for $z$ because the empirical likelihood puts more weight on

\(^{13}\)The structure of the bond premium is consistent with Gagliardini et al. (2009a, 2009b) in the sense that factors that are uncorrelated with aggregate consumption are the main driver for the bond premium.
the hypothesis that the true DGP for \( z \) follows a non-Ricardian model. As a result uncertainty about Ricardian equivalence increases which leads to an increase in marginal utility. At the same time, the bond investment produces an unexpected loss because bonds have negative \( z \) exposure. This implies that the bond payoff is low in periods of high marginal utility. It is consistent with standard asset pricing intuition that such an asset must pay a positive premium.

The second bond premium is positive if the investor thinks that the government expands the set of potential multipliers in periods of increased multiplier uncertainty.

The economic intuition for the premium on multiplier uncertainty is as follows. The bond does not hedge this uncertainty. Assume there is a positive shock to \( W^\eta \), which increases the amount of models that the government can choose from. The empirical likelihood tells the investor that the likelihood that the reference model for \( \eta \) is correct has fallen, because the distorted model predicted that the \( W^\eta \) shock will be positive. The investor and government are more uncertain about the true magnitude of the multiplier which means that the uncertainty about the size of the set of policy models goes up. Marginal utility increases because "uncertainty about uncertainty" has increased.

Being long a bond gives the investor negative exposure to \( \eta \). This holds because the robust forecast for future growth \( z \) goes up if \( \eta \) goes up. A bond has negative \( z \) exposure so will fall as a response to an increase in \( \eta \). Hence, a positive shock to \( W^\eta \) results in an unpredictable loss in the bond investment, which happens in a period where marginal utility has increased.

**Variance of the bond premium.** Compared to the Ricardian equivalence (reference) model, the bond premium exhibits excess and stochastic volatility, arising from the stochastic amount of Ricardian equivalence uncertainty and multiplier uncertainty. Formally,

\[
< dR_P^B(\tau), dR_P^B(\tau) > = B^2_z(\tau) \sigma^2_z (1 - \bar{\rho}^2)(m_G^c)^2 \sigma^2_\eta \eta dt + B^2_\eta(\tau) \sigma^2_\eta \sigma^2_L L dt,
\]

where \( < dR_P^B(\tau), dR_P^B(\tau) > \) stands for the instantaneous unpredictable variation of the bond risk premium. This stochastic variation is driven by the
"known unknown" $\eta_t$ and the "unknown unknown" $L_t$. Periods of increased uncertainty coincide with higher volatility in bond yields and bond premiums.

5. Estimation Strategy: Theoretical Considerations

In order to identify empirically how the bond market perceives government interventions I have to bring the model to data. Ideally, the model would be tested with a large panel of real yields, $y_t(\tau)$. An ideal estimation would in addition use observable variables for the business cycle component in expected consumption growth, for the observable amount of government uncertainty and for consumption growth. The unobserved uncertainty about the expected effectiveness of government interventions could be inferred from the rich panel of real yields. Nominal yields would not be necessary for answering the research question because the effect on the real yield curve carries over to the nominal yield curve. Even if government interventions were correlated with inflation, this would only add a second-order effect to the nominal yield curve and is therefore not of first-order importance for nominal yields. It would not have implications for marginal utility and real yields.

**Data availability problem: real yields.** Data for real yields are available for the U.S. only since the U.S. Treasury started to issue Treasury-Inflation-Protected Securities (TIPS) in the late 1990’s. D’Amico et al. (2007) show that TIPS yields contained a significant liquidity premium until the early 2000. This shortens the panel of observable real interest rates dramatically. This is not the first project that encounters such a problem. There are three strategies to overcome this problem.

First, Gagliardini et al. (2009) derive a model for real yields and tests the model with a rich panel of nominal Treasury bond yields. This strategy assumes that real yields have exactly the same characteristics than nominal yields.

Second, the Fisher Hypothesis argues that the spread between nominal yields and real yields coincides with inflation expectations. That means that if one subtracts inflation expectations from nominal yields, one has an unbiased estimate of the corresponding real yield.
Third, the disadvantage of the second strategy is that it might overstate the importance of the government uncertainty premium. Ang et al. (2008a) show that real yields can be recovered from nominal yields and inflation if one properly subtracts inflation compensation from the nominal yield curve. Inflation compensation consists of inflation expectations (Fisher Hypothesis) plus an inflation premium. Although empirical evidence on the magnitude, sign and economic channel for an inflation premium are mixed, it seems that a conservative estimation strategy should take such a premium into account.\(^{14}\)

Taking all these suggestions into account, I will subtract expected inflation and an inflation premium from nominal yields in order to get a proxy for the implied real yield curve. While Ang et al. (2008a) is a statistical model, I follow the suggestion in the equilibrium model of Ulrich (2010). Importantly for my analysis, this paper shows that incorporating an inflation ambiguity premium in a log-utility economy recovers the upward sloping inflation premium in nominal bond yields. In addition, his results show that the real yield curve is practically not affected by the inclusion of inflation ambiguity. This happens because as the name suggests, the inflation ambiguity premium is a nominal phenomenon which leaves the real yield curve mainly unaffected. This is important for my research study because it allows me to use nominal yields and a nominal inflation premium without distorting the implications for the effects of government uncertainty on the real yield curve.

**Data availability solution: real yields.** Summarizing, I will follow Ulrich (2010) and add an inflation process which introduces an inflation ambiguity premium to the yield curve of nominal bonds. The positive inflation premium ensures that the inclusion of nominal yields into the estimation does not overstate the premium for government uncertainty. Moreover, the implications for the real yield curve are unaffected by the inflation ambiguity premium, as shown in Ulrich (2010).

The appendix shows in detail that if one adds the following inflation

\(^{14}\)Hördahl and Tristani (2007), Joslin et al. (2010) and Grishchenko and Huang (2010) find that the inflation premium is small, whereas Buraschi and Jiltsov (2005) and Ang et al. (2008a) argue that nominal yields contain a substantial inflation premium.
process, $d\ln p_t$, to the economy

$$d\ln p_t = (p_0 + w_t)dt + \sigma_p \left( \rho_{pc} dW^c_t + \sqrt{1 - \rho_{pc}^2} dW^p_t \right), \quad (37)$$

with $p_0 > 0$ being the unconditional mean of inflation and $\rho_{pc}$ being the correlation between consumption growth and inflation; and the predictable process for expected inflation, $w_t$, being

$$dw_t = \kappa_w w_t dt + \sigma_w dW^w_t, \quad (38)$$

where expected inflation is correlated with expected consumption growth, i.e. $E_t[dW^w_t dW^z_t] = \rho_{wz} dt$, and subject to model uncertainty with $(\eta^w_t)^2$ characterizing the amount of inflation uncertainty, where $\eta^w_t$ itself is assumed to follow:

$$d\eta^w_t = (a_{\eta^w} + \kappa_{\eta^w} \eta^w_t) dt + \sigma_{\eta^w} \sqrt{\eta^w_t} dW^\eta^w_t, \quad (39)$$

with $a_{\eta^w}, \sigma_{\eta^w}, -\kappa_{\eta^w} > 0$. Under these assumptions one can derive a nominal yield curve which is the sum of real yields plus expected inflation and an inflation ambiguity premium.

The appendix shows in detail that nominal yields are given by

$$y^s_t(\tau) = -\frac{1}{\tau} \left( A^s(\tau) + B_z(\tau) \alpha_t + B_\eta(\tau) \eta_t + B_L(\tau) L_t + B^s_w(\tau) w_t + B^s_{\eta^w}(\tau) \eta^w_t \right) \quad (40)$$

where the loading without the $\$ index coincide with the loadings for the real yield curve in equation (29). This implies that government intervention uncertainty and multiplier uncertainty affect nominal yields in the same way they affect real yields. The theoretical section explained in detail how real yields are affected; all of these explanations carry over to nominal yields. The assumed orthogonality between the inflation dynamics and the government intervention dynamics ensures that government uncertainty affects the nominal bond market in the same way as it affects the real bond market. I can therefore use nominal bond yields in the estimation without distorting the interpretations for real interest rates. The loadings $B^s_w(\tau)$ and $B^s_{\eta^w}$ summarize the time-varying effect of expected inflation and of the inflation uncertainty premium. All loadings are specified in the appendix.
The inflation compensation (expected inflation and inflation premium) in the nominal yield curve, \( B^S_w(\tau)w_t + B^S_{\eta^w}\eta^w_t \), ensures that the estimated implications for government interventions are not distorted by using nominal yields in the estimation. In addition, if \( \rho_w \) is sufficiently small, there will be no measurable difference between the real yield curve in (29) and an extended real yield curve that accounts for the inflation uncertainty premium in the real yield curve. I denote such an extended real yield curve as \( \bar{y}_t(\tau) \). The appendix shows that it is given by

\[ \bar{y}_t(\tau) = y_t(\tau) + \bar{a}(\tau) + b_{\eta^w}(\tau)\eta^w_t \] (41)

where \( y_t(\tau) \) is given in equation (29) and \( \bar{a}(\tau) \to 0 \) and \( b_{\eta^w}(\tau) \to 0 \) for \( \rho_w \to 0 \). Intuitively, this means that adding an inflation uncertainty premium to nominal yields comes at the cost of introducing such a premium into real yields. But if the inflation non-neutrality, which is introduced through \( \rho_w \), is small, the inflation uncertainty premium has no effect on the real yield curve, but it accounts for the inflation premium in nominal bond yields.

**Data availability problem: Uncertainty about Ricardian equivalence.** The model assumes that the investor can infer the amount of government intervention uncertainty. In order to match the model with data it is advantageous if one proxies the amount of this uncertainty by an observable and government related intervention process; instead of working with a latent factor approach. This is of even more interest given that the model assumes that uncertainty about the expected effectiveness of government policies is truly unobservable.

I use the following observation to derive an empirical proxy for the amount of observable uncertainty about policy interventions in the business cycle. That measure builds on observable data about the Federal Reserve which is the monetary branch of the U.S. government. At a 2010 conference, former Federal Reserve Chairman Alan Greenspan responded to the question of whether he thinks that Fed interventions created a ”Greenspan put” during his Chairman tenure:¹⁵

¹⁵The term ”Greenspan put” describes the stock market perception that the Fed, under former Chairman Greenspan, used its discretionary but mandate consistent interventions to ”bail out” the stock market. Compare minutes 44 to 50 of the recorded panel discussion: [http://www.frbatlanta.org/news/conferences/10jekyll_webcast.cfm](http://www.frbatlanta.org/news/conferences/10jekyll_webcast.cfm).
We are responding to the economy, not to the markets, not to interest rates. We are responding essentially to what our job is, namely to stabilize the system. Now, if in effect the Greenspan put is a notion which says you are stabilizing the system, I say well I hope so.  

Taylor (1993), Woodford (2003), Ang et al. (2008b), Clarida and Gertler (1997) and Clarida et al. (2000) assume that the investor has incomplete information about how the Fed sets short-term interest rates. The state-of-the-art approach is to assume that a trustworthy approximation for government actions is to assume that they set short-term interest rates according to a rule that is linear in expected GDP growth and expected inflation. While there is consensus that this is a good approximation, the true government model is unknown and different from that Taylor-type rule. The spread between the interest rate that the government sets and the Taylor-rule implied interest rate is called government intervention. I use Survey of Professional Forecasters (SPF) data on the three-month T-bill, inflation, and GDP growth to run in each quarter a cross-sectional Taylor-rule regression among forecasters

\[ R_i^t = c + a z_i^t + b w_i^t + \varepsilon_i^t, \]

where index \(i\) stands for a particular forecaster, \(R\) coincides with the forecast of the next quarter 3-month T-bill, \(z\) stands for the forecast on GDP growth over the next quarter and \(w\) stands for the inflation forecast over the next quarter and \(\varepsilon_t\) stands for the anticipated government intervention. These regressions assume that all forecasters have had access to the same information about the aggressiveness with which the Fed targets output growth and inflation in a particular quarter. This means \(c, a, b\) are assumed to be constant within a quarter. Rather than using \(\varepsilon_i^t\), I use the cross-sectional variance of this error term to proxy for \(\eta_t^2\).

In the model \(\eta^2\) proxies for uncertainty about fiscal and monetary interventions, while empirically, I can only identify a proxy for uncertainty about the monetary branch of government interventions. The empirical \(\eta^2\)

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16 Compare minute 47 of the above-mentioned recorded panel discussion.

is therefore a lower bound on government intervention uncertainty. This observable measure has several advantages. First, an increase in $\eta_2^t$ indicates that through the lenses of a Taylor rule, SPF forecasters are exposed to an increased amount of uncertainty about government interventions in the money market. Second, to use the cross-sectional variance in survey data as a proxy for the amount of model uncertainty is well established and follows Anderson et al. (2009), Ulrich (2010) and Patton and Timmermann (2010). Third, monetary and empirical asset pricing research, such as Taylor (1993), Woodford (2003) and Ang et al. (2008b), interpret deviations of the Federal funds rate from a Taylor rule as an intervention of the Fed. My measure therefore captures the anticipated uncertainty about next quarter’s policy intervention of the monetary government.

Data availability problem: business cycle component. The model requires the investor to observe the business cycle component in expected consumption growth, $z$. Empirically, there is no data on such a business cycle component in consumption growth. Bansal and Yaron (2004) argue that such a component exists and has important implications for asset prices. I follow Ulrich (2010) and assume that the (demeaned) median forecast for next quarter GDP growth coincides with $z$. For consistency reasons, I will work with the time-series for GDP growth instead of consumption growth, since $z$ measures the time-varying component in expected GDP growth.


Data. I use quarterly data from 1981.III to 2009.II. I start in 1981.III because this is the earliest date at which the data allows me to identify an observable proxy for government intervention uncertainty, $\eta^2$. As explained above, $\eta^2$ is identified as the cross-sectional variance of anticipated Fed interventions in the money market. Following Ulrich (2010, JME), I identify the amount of inflation uncertainty, $(\eta^w)^2$, with the cross-sectional dispersion in one-quarter ahead SPF inflation forecasts.

State variables $z$ and $w$ coincide with the demeaned one quarter ahead median forecast of GDP growth and inflation, respectively, as published by the SPF. I use $c_0$ and $p_0$ to demean the respective median forecast, where
$c_0$ is matched with the sample mean of quarterly GDP growth and $p_0$ is set to coincide with the sample mean of quarterly inflation. Realized GDP growth and realized inflation are matched with the quarterly continuously compounded GDP growth rate and the quarterly continuously compounded GDP implicit price deflator growth rate, respectively. Realized GDP growth and realized inflation is published by the FRED database of the St. Louis Fed.\textsuperscript{18}

The unobservable amount of uncertainty about the expected effectiveness of government interventions will be inferred from the yield curve. I use the Chen and Scott (1998) methodology and invert for each parameter vector the ten-year nominal yield to get a time-series estimate for $L_t$. As explained in the theoretical section, $h_t^\eta = L_t \in \mathcal{R}_+$ if the government enlarges the set of policy intervention models in periods of increased uncertainty about the expected effectiveness of its policy models. On the contrary, $h_t^\eta = L_t \in \mathcal{R}_-$ if it shrinks the set in such states of the world. The sign of the inverted $L_t$ reveals which strategy the government followed.

The bond data consists of the Federal Funds rate, continuously compounded yields on nominal U.S. Treasury bonds with maturity 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 years, continuously compounded yields on CPI-indexed U.S. Treasury bonds of maturity 5, 6, 7, 8, 9, 10 years. The latter yields, also called TIPS yields, are direct observations of U.S. real government bond yields. D’Amico et al. (2007) point out that TIPS yields contained a large liquidity premium during the late 1990’s and early 2000. I therefore use TIPS yields from 2003.\textsuperscript{19} I onwards. The yield data is published by the Board of Governors of the Federal Reserve System. The Federal Funds rate is from the FRED database of the St. Louis Fed.

\textbf{ML.} The model is estimated with a one-step ML method. The model has five conditionally Gaussian state variables, $(z, \eta, L, w, \eta^w)$ and 19 measurement equations. The measurement errors for the yields and the Federal Funds Rate are assumed to be Gaussian and orthogonal to each other and to the other processes in the economy.\textsuperscript{19}

\textsuperscript{18}Compare: http://research.stlouisfed.org/fred2/.
\textsuperscript{19}Ang and Piazzesi (2003) is only one example of papers who use such an assumption.
7. Empirical Results

Model Implied Real and Nominal Yield curve.
The model explains the macro and yield curve dynamics well and explains why the U.S. TIPS yield curve is upward sloping. The estimation ensures that the macro variables, GDP growth and inflation, are perfectly matched. The annualized mean absolute pricing error for nominal yields is 20 basis points and 14 basis points for real yields. The low pricing errors for real bond yields show that policy intervention uncertainty and uncertainty about the expected effectiveness of policy interventions can explain the term structure of real bond yields. This is especially satisfying from a financial economic point of view because real yields coincide with a logarithmic transformation of the representative investor’s expected growth rate of marginal utility.

Real yields and marginal utility.
Probably the single most important concept of asset pricing is the pricing kernel, or equivalently marginal utility of the investor. Observing the term structure of real bond yields provides an image of the expected growth rates of marginal utility. The yield curve of real bonds is upward sloping in the data. The ten-year minus five-year spread is 45 basis points in TIPS yields and 80 basis points in nominal Treasury yields. This poses a challenge to the real yield curve models in Ang et al. (2008a) and the long-run risk bond models of Bansal and Shaliastovich (2010) and Piazzesi and Schneider (2006) or the inflation ambiguity model of Ulrich (2010), because their models imply that the real yield curve in the U.S. is downward sloping.

For comparison, the U.K. yield curve of real bonds is downward sloping in that period. The most likely reason for this counterintuitive term structure is that the U.K. Pension Acts from 1995 and 2004 provided strong incentives for pension funds to hold long-term inflation-indexed U.K. bonds, pressuring...
long-term real yields in the U.K. bond market to drop significantly.

The bond model allows us to study the yield curve of real and nominal bonds over the entire sample of 1981 to 2009. The two solid lines in figure (1) show the model implied real and nominal U.S. yield curve. The former is upward sloping with a real term spread of 90 basis points, while the latter is upward sloping with a 1.6% nominal term spread. In more detail, the average one-year TIPS yield is 3.6%, whereas the average ten-year TIPS yield is 4.5%. The average one-year nominal Treasury yield is around 5.5%, while it is roughly 7% for the ten-year nominal Treasury yield.

Figure (1) also shows that the positive term premium in real bonds arises because of uncertainty about government interventions. Namely, setting $m^G$ to zero results in a flat real yield curve of 3%. Said differently, the real yield curve would be flat at 3% in a world without uncertainty about potential government interventions. The dotted line corresponds to the implied nominal yield curve that consists of real yield curve plus inflation expectations. That curve is upward sloping despite a slightly downward sloping term structure of inflation expectations, because government uncertainty makes the real yield curve slope upwards. The spread between this dotted line and the solid green line that corresponds to the estimated nominal yield curve coincides with the inflation ambiguity premium in the nominal term spread.

In summary, figure (1) reveals that roughly half of the nominal term spread in U.S. Treasury bonds comes from the government uncertainty premiums that are inherent in the real term spread. The magnitude is consistent with the data implied real and nominal term spread. The 1.6% nominal term spread contains the 90 basis points real term spread plus a one percent term spread for the inflation ambiguity premium. The missing negative 30 basis points arises from a slightly downward sloping term structure of inflation expectations. The magnitude of the inflation ambiguity premium is consistent with Ulrich (2010) and ensures that the estimation does not overstate the importance of the government uncertainty premiums.\textsuperscript{20}

\textsuperscript{20}In contrast, Ang et al. (2008a) and Piazzesi and Schneider (2010) are recent accounts where the inflation premium in the nominal term spread is captured by an inflation risk premium.
The finding that the nominal term spread inherits a significant part of its shape from the real term spread supports findings in Hördahl and Tristani (2007), Joslin et al. (2010) and Grishchenko and Huang (2010). Consistent with my findings, Hördahl and Tristani (2007) find that the nominal term spread in the Euro area is driven mainly by variables that also affect the real term spread. Joslin et al. (2010) and Grishchenko and Huang (2010) find in independent work and with different set-ups that the inflation premium in U.S. bond markets is rather smaller, implying that a significant component of the nominal bond premium must arise from bond premiums that real bonds must pay. Other results, such as Buraschi and Jiltsov (2005), Kim and Wright (2005), Piazzesi and Schneider (2006), Ang et al. (2008a) argue that the inflation premium in the nominal bond spread is positive and of an approximate magnitude of 1% or more for the ten-year horizon. My inflation ambiguity premium captures this inflation premium. My results therefore combine both strands of findings. First, the ten-year inflation premium is roughly one percent. Second, roughly half of the nominal term spread is driven by the real term spread.

**Reason for upward sloping real yield curve.**

Figure (2) presents the model implied factor loadings that drive the real yield curve. Consistent with findings in Ulrich (2010), the real bond loading on inflation ambiguity is essentially zero, which implies that inflation ambiguity does not affect real yields but it affects nominal yields. My model for real bonds is essentially a three factor model, as shown in equation (29) and (41). The upper panel in figure (2) shows that expected consumption growth, $z$, generates a positive and downward sloping real yield curve. The steepness of the slope is entirely governed by the empirical persistence of the $z$ time-series.

The second panel of figure (2) shows that uncertainty about systematic government policy interventions increases yields of all maturities and generates a negative slope of the yield curve. The loading on $\eta$ reveals that the investor likes the prospect of having a government that might decide to systematically increase business cycle growth $z$. The negative $z$-exposure of

---

21 The figure for the real bond loading on inflation ambiguity is available upon request.
bonds makes him require an additional yield for holding bonds. The longer
the duration of the bond the smaller the required premium for uncertainty
about policy interventions. This is true because first, uncertainty about pol-
icy interventions mean-reverts quickly, with an estimated \( \kappa_\eta \) of \(-0.46\). This
implies that the expected distortion in long-term growth forecasts is smaller
than for short-term forecasts. Mathematically, this corresponds to \((1 - e^{\kappa_\eta \tau})\)
in the parametric expression of \( B_\eta(\tau) \) in equation (31). Second, the expected
growth spread that arises in the model with government policy interventions
compared to the model without government policy interventions is fairly
small. The term \( e^{\kappa_\eta \tau} - e^{\kappa_\eta \tau} \) in equation (31) is close to zero because the
estimated persistence of \( z \) and \( \eta \) is of similar magnitude.

The third panel of figure (2) reveals that uncertainty about the future
effectiveness of potential policy interventions results in an upward sloping
real yield curve. It is instructive to analyze the corresponding factor load-
ing in order to derive more intuition about why the real yield curve slopes
upwards. The slope of the real yield curve is governed by \(-g(\tau)\) in (32).
It is \(-g(\tau)\) because \( B_L(\tau) \) is the loading for bond prices and \(-B_L(\tau)\) is the
loading for bond yields. The first term in \(-g(\tau)\) is downward sloping because
\( L \) is a mean-reverting factor. The positive slope in \(-\frac{B_L(\tau)}{\tau}\) arises because the
second and fourth component in \(-g(\tau)\) is upward sloping. Both components
quantify how uncertainty about the expected effectiveness of a potential pol-
icy intervention amplifies the spread in expected long-term growth rates in
non-Ricardian models with an uncertain multiplier versus a certain multi-
plier. The investor thinks that the current expected policy intervention in
the business cycle could generate higher growth in the future than what is
suggested by the current observation of \( \eta^2_t \). This means it could turn out
to have a higher multiplier than suggested by \( \eta^2 \). The high persistence in
multiplier uncertainty makes long-term bond investments less attractive than
short-term bond investments. As a result, long-term real yields have a higher
yield than short-term real yields.

**Macro variables drive uncertainty.**
Table (3) summarizes unconditional correlations between the macro factors.
The estimated process for multiplier uncertainty correlates with the observed
measure for policy intervention uncertainty, expected inflation and inflation
uncertainty. The estimation finds that tracking expected inflation provides
a good proxy for the amount of multiplier uncertainty. Roughly 88% of variations in $L$ are related to variations in $w$. This implies that multiplier uncertainty was highest during the monetary policy experimentation of the early 1980s, a period where fiscal as well as monetary interventions were actively pursued.

The estimated measure of multiplier uncertainty correlates also with the observable measures of policy intervention uncertainty and inflation uncertainty. The unconditional correlations are 66% and 43%, respectively. That means that although $L$ quantifies the “unknown unknown” in the model, it correlates highly with different observable measures of aggregate uncertainty.

In run a regression to understand which macro variables explain variations in $L$. I normalize all variables to have a zero mean and unit variance. The regression output is

$$
\bar{L}_t = 0.9276 \bar{w}_t + 0.1384 \bar{z}_t + 0.0183 \bar{\eta}_t - 0.0535 \bar{\eta}^w_t + \varepsilon_t, \quad R^2 = 78%
$$

where the “bar” indicates that these variables are normalized. Only the t-stats for $\bar{w}_t$ and $\bar{z}_t$ are bigger than two, 12.90 and 2.92, respectively. This says that the “unknown unknown” or "uncertainty about uncertainty" is strongly related to variations in aggregate macro variables. 80% of variations in multiplier uncertainty is related to expected changes in nominal growth.

The estimates of the last regression state that a one percent increase in the normalized value of expected inflation (expected GDP growth) leads to a 0.93% (0.14%) increase in the normalized value of multiplier uncertainty. The more the economy expands, the more uncertain is the investor about the effectiveness of new policy interventions. This also implies that if expected inflation approaches zero, uncertainty about the effectiveness of new policy interventions goes to zero as well; meaning the investor is very certain about the effectiveness of new policy interventions.

This finding is consistent with Christiano et al. (2011) who argue that the effectiveness of government policies is particularly high in periods of low expected inflation because these are periods in which the zero-lower bound on the short-term nominal interest rate rate binds. It happens in their model that if expected inflation rises, the nominal short rate escapes the zero lower
bound and the effectiveness of government interventions gets questionable. My last regression indicates that the uncertainty about the effectiveness of government interventions shoots up in periods where expected inflation, and therefore the nominal short-term interest rate, goes up.

**Uncertainty premiums in bond returns.** Table (2) decomposes the expected instantaneous return of a nominal bond into the expected instantaneous return of a real bond and the inflation ambiguity premium. The former consists of a premium for intervention uncertainty and uncertainty about the future effectiveness of a potential intervention. The expected bond premium for a one-year real bond is 0.79% and 1.06% for a ten-year real bond. The nearly the entire real bond premium is attributable to the government’s uncertainty about the future effectiveness of potential interventions. This is remarkable because the premium loadings \( B_z \) and \( B_\eta \) indicate that a bond exposes the investor stronger to a potential misspecification of \( z \) than to a potential misspecification of \( \eta \). But on the other hand, the investor is highly concerned about the "unknown unknown" future effectiveness of potential interventions. This is reflected in the market price of multiplier uncertainty, \( h_\eta \), which is larger than the market price for policy interventions, \( h_G \).

In summary, a bond investment exposes the investor relatively weakly to uncertainty about the effectiveness of policy interventions, but the bond investor dislikes that uncertainty very strongly. He therefore requires a substantial bond premium for that type of uncertainty.

The last two columns of Table (2) shows that the expected return on a nominal bond exceeds the expected return on a real bond by an upward sloping inflation ambiguity premium. The average ten-year inflation ambiguity premium is 11.6 basis points. This is consistent with Joslin et al. (2010) who find that the expected bond return on inflation is time-varying and rather small in terms of absolute magnitude.

**Impulse responses.**

How does a one standard deviation increase in both types of uncertainty affect the bond market? I answer this question by analyzing the impulse response function for both types of uncertainty.

The upper three pictures in Figure (3) present how the one-year (real
and nominal) bond yield, the slope of the (real and nominal) yield curve and the bond premium of the one-year (real and nominal) bond reacts to a one standard deviation increase in $L_t$. The one-year yield goes up by 1.4%, the slope of the yield curve increases by 1.2%. The steep increase in the slope coincides with a 0.6% increase in the expected bond premium for short-term bonds. The estimated high persistence in uncertainty about the future effectiveness of potential government interventions gives the "unknown unknown" uncertainty a long-lasting effect on the real and nominal bond market. In summary, uncertainty about the effectiveness of policy interventions has economically and quantitatively meaningful effects on the bond market. This is even more remarkable, if one takes into consideration that the investor believes in Ricardian equivalence.

The lower three pictures in Figure (3) show that a one standard deviation increase in intervention uncertainty has a weaker impact on the bond market. This has to do with the substantially less volatile observable process for intervention uncertainty. In more detail, the instantaneous reaction on the yield curve is an increase in the level of the yield curve and a flattening of the term structure. More precisely, the one-year yield goes up by 0.1 basis points, while the yield curve flattens by the same amount.

8. Conclusion

This paper finds that bond investors request a positive premium for potential government interventions in the business cycle. This premium gives rise to a time-varying bond premium and an upward sloping yield curve for real and nominal bonds. The premium reveals two perceptions about government intervention policies. First, government interventions are perceived as benevolent policies that try to amplify consumption growth. Second, if the government is confronted with an increased uncertainty about the expected effectiveness of policy interventions, it might launch new policies with an increased optimistic expectation about the effectiveness. This can potentially explain why the government launched the American Recovery and Reinvestment Act with an anticipated multiplier of 1.6, while many empirical and theoretical studies predict a multiplier of smaller than 0.5.
References


Table 1: PARAMETER ESTIMATES (Standard Errors)

**Panel A: State Variables**

<table>
<thead>
<tr>
<th></th>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>drift</td>
<td>-0.2305</td>
<td>0.0079</td>
<td>0.0 (fixed)</td>
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<tr>
<td>$z$</td>
<td>volatility</td>
<td>-1.2270</td>
<td>0.0061</td>
<td>0.0 (fixed)</td>
</tr>
<tr>
<td>$\eta$</td>
<td></td>
<td>-0.4554</td>
<td>0.0032</td>
<td>0.0005 ()</td>
</tr>
<tr>
<td>$\eta^w$</td>
<td></td>
<td>-0.2002</td>
<td>0.1751</td>
<td>0.0016 ()</td>
</tr>
<tr>
<td>$L$</td>
<td></td>
<td>-0.0098</td>
<td>0.5734</td>
<td>0.4064 ()</td>
</tr>
</tbody>
</table>

**Panel B: Growth and Inflation**

<table>
<thead>
<tr>
<th></th>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_0$</td>
<td></td>
<td>0.0068</td>
<td>(fixed)</td>
<td></td>
</tr>
<tr>
<td>$p_0$</td>
<td></td>
<td>0.0066</td>
<td>(fixed)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td></td>
<td>0.0055</td>
<td>()</td>
<td></td>
</tr>
<tr>
<td>$\sigma_p$</td>
<td></td>
<td>0.04864</td>
<td>()</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td>0.001</td>
<td>(fixed)</td>
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<tr>
<td>$\bar{\rho}$</td>
<td></td>
<td>0.9554</td>
<td>()</td>
<td></td>
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<tr>
<td>$\rho_{pc}$</td>
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<tr>
<td>$\rho_w$</td>
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<td>-7.9900e-05</td>
<td>()</td>
<td></td>
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<tr>
<td>$m^G$</td>
<td></td>
<td>2.5743</td>
<td>()</td>
<td></td>
</tr>
<tr>
<td>$m^w$</td>
<td></td>
<td>14.2252</td>
<td>()</td>
<td></td>
</tr>
</tbody>
</table>

Note: The table presents ML parameter estimates and their standard error (in parenthesis). The asymptotic standard errors are determined based on the score of the log likelihood. The ML estimation uses bond yield and macro data from 1981.III to 2009.II.
Table 2: Instantaneous Expected Bond Return

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Intervention P.</th>
<th>Multiplier P.</th>
<th>Total Real P.</th>
<th>Inflation P.</th>
<th>Total Nominal P.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0008</td>
<td>0.7904</td>
<td>0.7913</td>
<td>0.0699</td>
<td>0.8612</td>
</tr>
<tr>
<td>2</td>
<td>0.0009</td>
<td>1.0139</td>
<td>1.0147</td>
<td>0.0977</td>
<td>1.1125</td>
</tr>
<tr>
<td>3</td>
<td>0.0009</td>
<td>1.0502</td>
<td>1.0511</td>
<td>0.1088</td>
<td>1.1599</td>
</tr>
<tr>
<td>4</td>
<td>0.0009</td>
<td>1.0573</td>
<td>1.0581</td>
<td>0.1132</td>
<td>1.1713</td>
</tr>
<tr>
<td>5</td>
<td>0.0009</td>
<td>1.0568</td>
<td>1.0576</td>
<td>0.1149</td>
<td>1.1726</td>
</tr>
<tr>
<td>6</td>
<td>0.0009</td>
<td>1.0593</td>
<td>1.0601</td>
<td>0.1156</td>
<td>1.1758</td>
</tr>
<tr>
<td>7</td>
<td>0.0009</td>
<td>1.0593</td>
<td>1.0601</td>
<td>0.1159</td>
<td>1.1761</td>
</tr>
<tr>
<td>8</td>
<td>0.0009</td>
<td>1.0573</td>
<td>1.0581</td>
<td>0.1160</td>
<td>1.1741</td>
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<tr>
<td>9</td>
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<td>1.0578</td>
<td>1.0586</td>
<td>0.1161</td>
<td>1.1747</td>
</tr>
<tr>
<td>10</td>
<td>0.0009</td>
<td>1.0593</td>
<td>1.0601</td>
<td>0.1161</td>
<td>1.1762</td>
</tr>
</tbody>
</table>

The table presents the instantaneous expected return for real and nominal bonds of different maturities. The columns from left to right stand for: bond maturity in years, premium for uncertainty about government interventions, premium for uncertainty about the future effectiveness of potential interventions, total bond premium on real bonds, inflation uncertainty premium inherent in nominal bonds, and total bond premium on nominal bonds. The sample is 1981.III to 2009.II and sample averages are used to compute the sample mean of the bond premiums.

Table 3: Correlations of Macro Factors

<table>
<thead>
<tr>
<th></th>
<th>w</th>
<th>z</th>
<th>η</th>
<th>ηₘ</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>1.0000</td>
<td>-0.2680</td>
<td>0.7653</td>
<td>0.5564</td>
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<tr>
<td>z</td>
<td>-0.2680</td>
<td>1.0000</td>
<td>-0.2845</td>
<td>-0.3085</td>
<td>-0.0989</td>
</tr>
<tr>
<td>η</td>
<td>0.7653</td>
<td>-0.2845</td>
<td>1.0000</td>
<td>0.5280</td>
<td>0.6606</td>
</tr>
<tr>
<td>ηₘ</td>
<td>0.5564</td>
<td>-0.3085</td>
<td>0.5280</td>
<td>1.0000</td>
<td>0.4296</td>
</tr>
<tr>
<td>L</td>
<td>0.8766</td>
<td>-0.0989</td>
<td>0.6606</td>
<td>0.4296</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

The table presents unconditional correlations of the macro variables; w is expected inflation, z is expected GDP growth, η captures uncertainty about policy interventions, ηₘ captures uncertainty about inflation, L captures uncertainty about the expected effectiveness of policy interventions. The sample is 1981.III to 2009.II.
Table 4: Absolute Pricing Error of Bond Yields, in %, annualized

**Panel A: Nominal Yields, 1981-2009**

<table>
<thead>
<tr>
<th>maturity</th>
<th>$R$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
<th>8</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.10</td>
<td>0.33</td>
<td>0.05</td>
<td>0.21</td>
<td>0.26</td>
<td>0.35</td>
<td>0.28</td>
<td>0.23</td>
<td>0.17</td>
<td>0.13</td>
<td>0.04</td>
</tr>
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</table>

**Panel B: Real Yields, 2003-2009**

<table>
<thead>
<tr>
<th>maturity</th>
<th>$y^r$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.26</td>
</tr>
</tbody>
</table>

Note: Panel A shows the mean absolute pricing error for nominal bond yield and real bond yields. The ML estimation uses bond yields and macro data from 1981.III to 2009.II.
Figure 1: **Government Interventions in the Yield Curve, 1981.III-2009.II**

This figure presents premium components of the model implied term structure for real and nominal Treasury bonds for the sample period 1981.III to 2009.II. The estimated real yield curve is shown in the blue $-$ > graph. The implied real yield curve without the government uncertainty premiums is shown in the red $>$ graph. The green solid curve represents the model implied nominal yield curve. The purple dotted curve corresponds to the nominal yield curve without an inflation ambiguity premium. The ML estimation uses bond yield and macro data from 1981.III to 2009.II.
Figure 2: **Factor Loadings of Real Bond Yields**

This figure presents the estimated factor loadings for expected consumption growth, uncertainty about government interventions and multiplier uncertainty. These factors affect the real and nominal yield curve alike. The ML estimation uses bond yield and macro data from 1981.III to 2009.II.
Figure 3: Impulse Response: Government Uncertainty and the Bond Market

This figure presents impulse responses for a one standard deviation shock to both forms of government uncertainty. The upper three graphs, from left to right, analyze how a one standard deviation shock in multiplier uncertainty ($L$) affects the one year (real and nominal) interest rate, the slope of the (real and nominal) yield curve and the one year bond premium of real and nominal bonds. The lower three graphs, from left to right, analyze how a one standard deviation shock in intervention uncertainty ($\eta$) affects the one year (real and nominal) interest rate, the slope of the (real and nominal) yield curve and the one year bond premium of real and nominal bonds. The ML estimation uses bond yield and macro data from 1981.III to 2009.II.
Online Appendix to Paper: How Does the Bond Market Perceive Government Interventions?

1 Equilibrium

The government solves its two decision problems in one step. Consistent with the paper, the government wants to intervene in the business cycle to maximize the expected life-time utility of the investor. It faces uncertainty about the expected effectiveness of different intervention models. The model assumes that the government extends the set of potential models in periods of increased multiplier uncertainty.

The value function $J$ of the problem is affine in expected consumption and the three state variables, i.e.

$$ J = J(c_t; z_t, \eta_t, L_t) $$

$$ J = \delta_0 + \delta_z z_t + \delta_\eta \eta_t + \delta_L L_t + \frac{\ln c_t}{\rho} $$

where $\delta_0, \delta_z, \delta_\eta, \delta_L$ are deterministic functions of the underlying economy.

The government solves the HJB

$$ \rho J = \max_{h_t^G, h_t^\eta} \ln c_t + $$

$$ + \theta_t^G \left( \frac{1}{2} (h_t^G)^2 - A \eta_t^2 \right) + $$

$$ + \theta_t^\eta \left( \frac{1}{2} (h_t^\eta)^2 - \frac{L_t^2}{2 \eta_t} \right) + $$

$$ + A^H J $$

where $\theta^G$ and $\theta^\eta$ are the Lagrange multipliers for the corresponding relative entropy constraint. $A^H J$ is the second-order differential operator under the robust probability measure, applied to the value function, i.e.

$$ A^H J = \rho^{-1} (c_0 + z_t) + \delta_z (\kappa_z z_t + \sigma_z \sqrt{1 - \rho^2 h_t^G}) + $$

$$ + \delta_\eta (a_\eta + \kappa_\eta \eta_t + \sigma_\eta \sqrt{\eta_t h_t^\eta}) + \delta_L (a_L + \kappa_L L_t) $$

1
First-order conditions reveal:

\[
h_t^G : \quad \theta_t^G h_t^G = -\delta_z \sigma_z \sqrt{1 - \bar{\rho}^2} \quad (5)
\]
\[
\theta_t^G : \quad h_t^G = \pm \sqrt{2A\eta_t} \quad (6)
\]

The HJB is maximized at

\[
h_t^G = \sqrt{2A\eta_t} \equiv m^G \eta_t \quad (7)
\]
\[
\theta_t^G = -\frac{\delta_z \sigma_z \sqrt{1 - \bar{\rho}^2}}{\sqrt{2A\eta_t}} \quad (8)
\]

The second set of first-order conditions reveals

\[
h_t^\eta : \quad \theta_t^\eta h_t^\eta = -\delta_\eta \sigma_\eta \sqrt{\eta_t} \quad (9)
\]
\[
\theta_t^\eta : \quad h_t^\eta = \pm \frac{L_t \sqrt{\eta_t}}{\eta_t} \quad (10)
\]

The HJB is maximized at

\[
h_t^\eta = \frac{L_t}{\sqrt{\eta_t}} \quad (11)
\]
\[
\theta_t^\eta = -\frac{\delta_\eta \sigma_\eta \eta_t}{L_t} \quad (12)
\]

Verifying that guess for value function is correct

\[
\rho (\delta_0 + \delta_z z_t + \delta_\eta \eta_t + \delta_L L_t) + \ln c_t
\]
\[
= \ln c_t + \rho^{-1}(c_0 + z_t) + \delta_0 + \delta_z (\kappa_z z_t + \sigma_z \sqrt{1 - \bar{\rho}^2} m^G \eta_t) +
\]
\[
+ \delta_\eta (a_\eta + \kappa_\eta \eta_t + \sigma_\eta L_t) + \delta_L (a_L + \kappa_L L_t) \quad (13)
\]

Matching coefficients reveals:

\[
z_t : \delta_z = \frac{\rho^{-1}}{\rho - \kappa_z} \quad (14)
\]
\[
\eta_t : \delta_\eta = \frac{\delta_z \sigma_z \sqrt{1 - \bar{\rho}^2} m^G}{\rho - \kappa_\eta} \quad (15)
\]
\[
L_t : \delta_L = \frac{\delta_\eta \sigma_\eta}{\rho - \kappa_L} \quad (16)
\]
The distorted dynamics in the real economy are:

\[ dz_t = \left( \kappa z_t + \sigma_z \sqrt{1 - \overline{\rho}^2} mG \eta_t \right) dt + \sigma_z \left( \tilde{\rho} dW_t^z + \sqrt{1 - \overline{\rho}^2} dW_t^{G,h} \right) \] (17)

\[ d\eta_t = (a_\eta + \kappa_\eta \eta_t + \sigma_\eta L_t) dt + \sigma_\eta dW_t^{\eta,h} \] (18)

\[ dL_t = (a_L + \kappa_L L_t) dt + \sigma_L dW_t^L. \] (19)

2 Term Structure of Real Bonds

I guess that the price of a real bond is exponentially affine in the state variables, i.e.

\[ F = F_\tau(\tau) = \exp \left( A(\tau) + B(\tau)X_t \right), \quad X_t = (z_t, \eta_t, L_t). \] (20)

\( F \) solves

\[ rF = A^H F - F_\tau \] (21)

where \( r \) is the real risk-free rate and \( F_\tau \) is \( \frac{\partial F}{\partial \tau} \).

The model implies:

\[ r = \rho + c_0 - \sigma_z^2 + z_t \] (22)

\[ \frac{F_\tau}{F} = \hat{A}(\tau) + \hat{B}(\tau)X_t \] (23)

\[ \frac{A^H F}{F} = B_z(\tau) \left( \kappa z_t + \sigma_z \sqrt{1 - \overline{\rho}^2} mG \eta_t \right) + \frac{1}{2} B_z^2(\tau) \sigma_z^2 + \\
+ B_\eta(\tau) (\kappa_\eta \eta_t + a_\eta + \sigma_\eta L_t) + \frac{1}{2} B_\eta^2(\tau) \sigma_\eta^2 \eta_t + \\
+ B_L(\tau) (a_L + \kappa_L L_t) + \frac{1}{2} B_L^2(\tau) \sigma_L^2 L_t. \] (24)

The solution to the pricing pde is unique. A closed-form solution can be obtained if one evaluates the second-order pre-cautionary savings effects under the unconditional expectation. This implies that the small pre-cautionary risk terms will affect the constant loading \( A(\tau) \) and not the time-varying loadings \( B(\tau) \). This is an innocuous approximation because (i) pre-cautionary savings are only of second order, (ii) it is known that their economic magnitude is small, (iii) the approximation accounts for the level of the precautionary
savings effect, and (iv) a numerical approximation would create a discretization error and make an estimation infeasible.

This means that I solve the following PDE:

\[ r = \rho + c_0 - \sigma_z^2 + z_t \]

\[ \frac{F_t}{F} = \dot{A}(\tau) + \dot{B}(\tau) X_t \]

\[ \frac{A_H F}{F} = B_z(\tau) \left( \kappa_z z_t + \sigma_z \sqrt{1 - \rho^2 m^G \eta_t} \right) + \frac{1}{2} B_z^2(\tau) \sigma_z^2 + \]

\[ + B_\eta(\tau) (\kappa_\eta \eta_t + a_\eta + \sigma_\eta L_t) + \frac{1}{2} B_\eta^2(\tau) \sigma_\eta^2 E[\eta_t] + \]

\[ + B_L(\tau) (a_L + \kappa_L L_t) + \frac{1}{2} B_L^2(\tau) \sigma_L^2 E[L_t], \]

where \( E[L_t] = \frac{a_L}{\kappa_L} \) and \( E[\eta_t] = \frac{a_\eta}{\kappa_\eta} \).

Matching coefficients reveals

\[ B_z(\tau) = \frac{1 - e^{\kappa_z \tau}}{\kappa_z} \]

\[ B_\eta(\tau) = \frac{m G \sqrt{1 - \rho^2} \sigma_z}{-\kappa_z \kappa_\eta} \left( 1 - e^{\kappa_\eta \tau} + \frac{\kappa_\eta (e^{\kappa_z \tau} - e^{\kappa_\eta \tau})}{\kappa_z - \kappa_\eta} \right) \]

\[ B_L(\tau) = \frac{m G \sqrt{1 - \rho^2} \sigma_\eta \sigma_L}{-\kappa_\eta} \cdot g(\tau) \]

\[ g(\tau) := \left( \frac{1 - e^{\kappa_L \tau}}{-\kappa_L \kappa_\eta} + \frac{e^{\kappa_\eta \tau} - e^{\kappa_L \tau}}{\kappa_\eta (\kappa_L - \kappa_\eta)} + \frac{e^{\kappa_z \tau} - e^{\kappa_L \tau}}{(\kappa_z - \kappa_L)(\kappa_z - \kappa_\eta)} - \frac{e^{\kappa_\eta \tau} - e^{\kappa_L \tau}}{(\kappa_\eta - \kappa_L)(\kappa_z - \kappa_\eta)} \right). \]

The constant \( A(\tau) \) is obtained in closed-form by integrating

\[ A(\tau) = - (\rho + c_0 - \sigma_z^2) \tau + \frac{1}{2} \sigma_z^2 \int_0^\tau B_z^2(u) du + \frac{1}{2} \sigma_\eta^2 E[\eta_t] \int_0^\tau B_\eta^2(u) du + \]

\[ + \frac{1}{2} \sigma_L^2 E[L_t] \int_0^\tau B_L^2(u) du + a_\eta \int_0^\tau B_\eta(u) du + a_L \int_0^\tau B_L(u) du \]

The loadings, evaluated at steady-state reveal:

\[ B_\eta(\infty) = B_z(\infty) \left( -\frac{m G}{\kappa_\eta} \right) \sqrt{1 - \rho^2 \sigma_z}. \]
Hence, \( B_\eta(\infty) > B_\infty \) if

\[
\frac{m^G}{-\kappa_\eta} > \frac{1}{\sigma_z \sqrt{1 - \bar{\rho}^2}} \tag{34}
\]

which means if either the set of government intervention models is sufficiently large or the
dynamic of the set of intervention models is sufficiently persistent.

Moreover,

\[ B_L(\infty) = B_\eta(\infty) \frac{\sigma_\eta}{\sigma_L} \tag{35} \]

### 3 Estimation

#### 3.1 Accounting for Nominal Yields

In order to use nominal bond yields in the estimation and to not over predict the government
premium I add an inflation ambiguity premium as in Ulrich (2010).

\[
\begin{align*}
\min_{\hat{h}^w} & \quad E \left[ \int_0^\infty a_t^G a_t^w e^{-\rho t} \ln c_t \, dt \right] \\
\text{s.t.} & \quad \frac{1}{2} (\hat{h}^w)^2 \leq A \eta_w^2 \\
& \quad dz_t = \left( \kappa_z z_t + \sigma_z \sqrt{1 - \bar{\rho}^2} m^G_t \bar{n}_t \right) \, dt + \rho \sigma_z h_t^w \, dt + \\
& \quad \frac{1}{2} \left( \kappa_z z_t + \sigma_z \sqrt{1 - \bar{\rho}^2} m^G_t \bar{n}_t \right) \, dt + \rho \sigma_z h_t^w \, dt + \\
& \quad dh_t^w = (a_{\eta^w} + \kappa_{\eta^w} n_t^w) \, dt + \sigma_{\eta^w} \sqrt{\eta_t^{w^2}} \, dW_t^G \tag{39}
\end{align*}
\]

Following Ulrich (2010), the the robust distortion in inflation is

\[ h_t^w = \sqrt{2 A \eta_t^w} > 0. \tag{41} \]

I define \( m^w \equiv \sqrt{2 A} \).
Nominal SDF $M^\$$. 

\[
- \frac{dM^\$}{M^\$} = - \frac{d(M/p)}{M/p} = \left( (\rho + c_0 - \sigma_c^2 + p_0 - \sigma_c \sigma_p \rho_{pc} - \sigma_p^2) + \omega_t + \zeta_t \right) dt \\
+ \left( \sigma_c + \sigma_p \rho_{pc} \right) dW^s + \sigma_p \sqrt{1 - \rho_{pc}^2} dW^p - h^G dW^G - h^\eta dW^\eta
\] (42)

Note: The drift term coincides with the nominal short rate $R_t$.

Nominal Bond Yields. The price of a nominal bond solves $F_t = F_t(\tau) = e^{A^t(\tau) + B^t(\tau) X_t}$, $X_t = (z_t, \omega_t, \eta_t, \eta^w_t, L_t)$. The nominal bond price solves 

\[
R_t F_t(\tau) = A^t F_t(\tau) - F_\tau. 
\] (45)

Plugging in $R_t$, the partial derivatives, together with the same innocuous approximation for second-order precautionary savings terms, as explained above in the section on Real Bond Yields, reveals

\[
B^s_z(\tau) = B_z(\tau) 
\] (46)

\[
B^s_\eta(\tau) = B_\eta(\tau) 
\] (47)

\[
B^s_L(\tau) = B_L(\tau) 
\] (48)

\[
B^s_\omega(\tau) = \frac{1 - e^{\kappa^w \tau}}{\kappa^w} 
\] (49)

\[
B^s_{\eta^w}(\tau) = - \frac{\bar{\rho}_w \sigma^w_z}{\kappa^w} \left( \frac{1 - e^{\kappa^w \eta^w \tau}}{\kappa^w \eta^w + \kappa^w - \kappa^\eta^w} \right) \\
- \frac{\sigma^w_z}{\kappa^w} \left( \frac{1 - e^{\kappa^w \eta^w \tau}}{\kappa^w \eta^w + \kappa^w - \kappa^\eta^w} \right) 
\] (50)

$A^s(\tau)$ is given in closed-form as the solution of simple integrals

\[
A^s(\tau) = - \left( \rho + c_0 + p_0 - (\sigma_c^2 + \sigma_p^2 + \rho_{pc} \sigma_c \sigma_p) \right) \tau + \frac{1}{2} \sigma^2_z \int_0^\tau B^2_z(u) du + \frac{1}{2} \sigma^2_\eta E[\eta^w] \int_0^\tau B^2_\eta(u) du + \\
\frac{1}{2} \sigma^2_L E[L_t] \int_0^\tau B^2_L(u) du + a_\eta \int_0^\tau B_\eta(u) du + a_L \int_0^\tau B_L(u) du + \\
\frac{1}{2} \sigma^2_w \int_0^\tau (B^w(u))^2 du + \frac{1}{2} \sigma^2_{\eta^w} E[\eta^w_t] \int_0^\tau (B^w_{\eta^w}(u))^2 du + \\
a_{\eta^w} \int_0^\tau B^w_{\eta^w}(u) du
\] (51)
Extended Real Yield Curve.

Adding inflation ambiguity changes the real yield curve

\[ F = F_t(\tau) = \exp \left( A(\tau) + B(\tau) X_t \right), \quad X_t = (z_t, \eta_t, \eta^w_t, L_t). \]  

(52)

\( F \) solves

\[ rF = A^H F - F_{\tau} \]  

(53)

where \( r \) is the real risk-free rate and \( F_{\tau} \) is \( \frac{\partial F}{\partial \tau} \).

The model implies:

\[ r = \rho + c_0 - \sigma^2_z + z_t \]  

(54)

\[ \frac{F_{\tau}}{F} = \dot{A}(\tau) + \dot{B}(\tau) X_t \]  

(55)

\[ \frac{A^H F}{F} = B_z(\tau) \left( \kappa_z z_t + \sigma_z \sqrt{1 - \rho^2 m^G \eta_t + \rho \rho_w \sigma_z m^w \eta^w_t} \right) + \]  

\[ + \frac{1}{2} B^2_z(\tau) \sigma^2_z + \]  

\[ + B_{\eta}(\tau) (\kappa_\eta \eta_t + a_\eta + \sigma_\eta L_t) + \frac{1}{2} B^2_{\eta}(\tau) \sigma^2_\eta \eta_t + \]  

\[ + B_L(\tau) (a_L + \kappa_L L_t) + \frac{1}{2} B^2_L(\tau) \sigma^2_L L_t + \]  

\[ + B_{\eta^w}(\tau) (\kappa_{\eta^w} \eta^w_t + a_{\eta^w}) + \frac{1}{2} \sigma^2_{\eta^w} \eta^w_t B^2_{\eta^w}(\tau) \]  

(56)

The solution to the pricing pde is unique. A closed-form solution can be obtained if one evaluates the second-order pre-cautionary savings effects under the unconditional expectation. This implies that the small pre-cautionary risk terms will affect the constant loading \( A(\tau) \) and not the time-varying loadings \( B(\tau) \). This is an innocuous approximation because (i) pre-cautionary savings are only of second order, (ii) it is known that their economic magnitude is small, (iii) the approximation accounts for the level of the precautionary savings effect, and (iv) a numerical approximation would create a discretization error and make an estimation infeasible. The resulting solution is
\[ B_z(\tau) = \frac{1 - e^{\kappa_z \tau}}{\kappa_z} \] (57)

\[ B_\eta(\tau) = \frac{n^G \sqrt{1 - \bar{\rho}^2 \sigma_z}}{-\kappa_z \kappa_\eta} \left( 1 - e^{\kappa_\eta \tau} + \frac{\kappa_\eta (e^{\kappa_z \tau} - e^{\kappa_\eta \tau})}{\kappa_z - \kappa_\eta} \right) \] (58)

\[ B_L(\tau) = \frac{n^G \sqrt{1 - \bar{\rho}^2 \sigma_z \sigma_\eta} \cdot g(\tau)}{-\kappa_z} \] (59)

\[ g(\tau) := \left( 1 - e^{\kappa_L \tau} - e^{\kappa_L \tau} \right) \frac{\kappa_\eta (\kappa_L - \kappa_\eta)}{(\kappa_z - \kappa_L)(\kappa_z - \kappa_\eta)} - \frac{e^{\kappa_\eta \tau} - e^{\kappa_L \tau}}{\kappa_\eta - \kappa_L}(\kappa_z - \kappa_\eta) \] (60)

\[ B_{\eta^w}(\tau) = -\frac{\bar{\rho} \rho w \sigma_z m^w}{\kappa_z} \left( 1 - e^{\kappa_{\eta^w} \tau} + \frac{\kappa_{\eta^w} - e^{\kappa_{\eta^w} \tau}}{\kappa_z - \kappa_{\eta^w}} \right). \] (61)

The inflation ambiguity loading goes to zero for \( \rho_w \to 0 \). This implies that if the inflation non-neutrality is small, inflation ambiguity does not distort the real yield curve.

The constant \( A(\tau) \) is obtained in closed-form by integrating

\[ A(\tau) = -(\rho + c_0 - \sigma_z^2) \tau + \frac{1}{2} \sigma_z^2 \int_0^\tau B_z^2(u)du + \frac{1}{2} \sigma_\eta^2 E[\eta_t] \int_0^\tau B_\eta^2(u)du + \frac{1}{2} \sigma_L^2 E[L_t] \int_0^\tau B_L^2(u)du + a_\eta \int_0^\tau B_\eta(u)du + a_L \int_0^\tau B_L(u)du + \frac{1}{2} \sigma_{\eta^w}^2 E[\eta_{\eta^w}^w] \int_0^\tau B_{\eta^w}^2(u)du + a_{\eta^w} \int_0^\tau B_{\eta^w}(u)du. \] (62)

The last two integrals go to zero for \( \rho_w \to 0 \).