

The Marginal Cost of Risk, Risk Measures, and Capital Allocation*

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Abstract

The Euler (or gradient) allocation technique defines a financial institution's marginal cost of a risk exposure via calculation of the gradient of a risk measure evaluated at the institution's current portfolio position. The technique, however, relies on an arbitrary selection of a risk measure. We reverse the sequence of this approach by calculating the marginal costs of risk exposures for a profit maximizing financial institution with risk averse counterparties, and then identifying a closed-form solution for the risk measure whose gradient delivers the correct marginal costs. We compare the properties of allocations derived in this manner to those obtained through application of the Euler technique to Expected Shortfall (ES), showing that ES generally yields economically inefficient allocations.

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1 Introduction

Practitioners have long wrestled with the problem of allocating capital to the various risks within a financial institution. And, while no particular method will work for all types of decisions (see Merton and Perold (1993)), an attractive mathematical technology for allocation appeared in an academic literature starting with Schmock and Straumann (1999), Tasche (2000), and Myers and Read (2001). Broadly speaking, these papers start with a differentiable risk measure and compute the marginal capital increase required to maintain the risk measure at a threshold value as a particular risk exposure within the portfolio is expanded, an approach referred to as “gradient” allocation or “Euler” allocation. Allocations of capital using variants of the Euler technology are now widely used in the course of pricing and performance measurement within the portfolios of financial institutions (see e.g. Society of Actuaries (2008) or McKinsey&Company (2011)).

The approach requires an arbitrarily chosen risk measure as a key input—and unfortunately, excepting highly specialized circumstances,¹ economic theory offers no guidance on the choice of the measure.² Yet the choice is not to be taken lightly: It has a profound influence on how the institution perceives risk. For example, Value-at-Risk (VaR)—perhaps the most widely used measure—is known to contain incentives for excessive risk-taking behavior (see Basak and Shapiro (2001)), and its use has been fingered by some as playing a key role in the recent financial crisis (see Nocera (2009)). Post-crisis soul-searching for alternatives to VaR has brought wider attention to the debate on the statistical properties of risk measures, with the likely outcome that measures such as Expected Shortfall (ES) will continue to gain traction among practitioners and regulators. Indeed, the ascent of ES seems well underway.³ This ascent, however, has been predicated on the technical properties of ES (e.g., coherence) and has occurred independently of any sound economic reasoning.

In this paper, we reverse the usual approach: Rather than choosing a risk measure constraint to determine the marginal cost of risk and the allocation of capital within the firm, we calculate the latter quantities as by-products of the institution’s optimization problem. We then derive the risk measure whose gradient yields the correct marginal costs and capital allocations.

The focus of our analysis is the optimal pricing behavior of a profit-maximizing financial institution with costly capital and risk-averse counterparties in the presence of a (possibly nonbinding) regulatory constraint tied to a risk measure. We find that the optimal capital allocation rule depends crucially on institutional context. As might be expected, in the scenario envisioned by Myers and Read (with fully insured counterparties), the economically optimal allocation follows from the gradient allocation principle as applied to the risk measure imposed by regulation. However, if

¹Specifically, the issue is resolved trivially if consumer preferences are defined over a particular risk measure (Zanjani (2002)), or if a particular risk measure is assumed to constitute the payoff function in a cooperative game (see Denault (2001); Powers (2007)). Similarly, if the *institutional* preferences are defined by a particular risk-averse utility function of outcomes, a particular risk measure may be implied (see e.g. Föllmer and Schied (2010) or Wächter and Mazzoni (2010)). Alternatively, Adrian and Shin (2008) find a justification for using Value-at-Risk in a model with a limited commitment and a specialized risk structure.

²Early papers in the literature, such as Myers and Read (2001), Tasche (2000), and Tasche (2004), implicitly recognized this difficulty and ultimately alluded to regulation in justifying the choice of risk measure.

³Various papers make a case for ES over VaR (see e.g. Hull (2007), Jaschke (2001)), and both regulation and practice appear to be moving in this direction as well. For instance, the International Actuarial Association (2004) recommends using ES in a risk based regulatory framework, and ES was embedded in the Swiss Solvency Test and appears to be viewed favorably by US regulators (cf. NAIC (2009)).

the counterparties are not fully insured, the optimal allocation rule is not fully determined by the regulatory constraint, even if that constraint is binding.

More specifically, when counterparties are not fully protected, the firm's marginal cost associated with the risk of a particular counterparty depends on how that risk affects the firm's other counterparties (and, thus, their willingness to pay for the firm's contracts)—so the firm must price contracts accordingly. The optimal allocation rule then ends up being a weighted average of an “external” allocation rule implied by the regulatory constraint (if it binds) and an “internal” allocation rule driven by the institution's uninsured counterparties. In the extreme case of no regulation, the allocation rule simply boils down to the “internal” rule. Intermediate cases, however, could feature marginal cost being driven mainly by the “internal” rule (if the regulatory constraint puts firm capitalization close to the level it would have chosen in the absence of regulation) or the “external” constraint (if regulation forces the firm to hold far more capital than is privately optimal).

We show further that the “internal” allocation rule can be implemented by applying the gradient allocation principle to a particular risk measure—the exponential of a weighted average of the logarithm of portfolio outcomes in states of default, with the weights being determined by the relative values placed on recoveries in the various states of default by the firm's counterparties. The weights are thus similar in concept to the “spectral” weights proposed by Acerbi (2002) as a means of capturing the “subjective risk aversion” of a financial institution, with the weights determined endogenously through the process of profit maximization. The risk measure itself is evidently a “tail” risk measure, although the functional transformations ultimately cause it to be non-convex.

We then derive closed form allocation formulas where possible, and we use numerical techniques to compare the allocations resulting from the “internal” allocation rule to those arising from the application of the gradient allocation method to ES. We show that ES-based allocations generally fail to weight default outcomes properly. Specifically, in cases where counterparties are strongly risk averse or where potential losses are large relative to counterparty wealth, ES-based allocations tend to underweight bad outcomes; in cases where counterparties are only weakly risk averse or where potential losses are relatively small, ES-based allocations tend to overweight bad outcomes. These differences flow from a fundamental difference in the basis for allocation under the “internal” rule and under ES. The starting point for evaluation of a risk's impact under ES concerns its share of the institution's *losses* in default states, whereas the starting point under the “internal” rule is the risk's share of *recoveries*—as a risk's impact on recoveries in default states is ultimately what counterparties care about.

This distinction underscores the key point of the paper: The true marginal cost of risk and the associated allocation of capital should flow from the economic context of the problem. A risk measure chosen for its technical properties such as coherence, rather than for the specific economic circumstances, will generally fail to yield correct pricing and efficient allocation of capital from the perspective of its user. In the concluding section, we consider how changing the perspective of the user—from that of the firm owner/manager, as in this paper, to, say, a regulator—changes the economic context and, hence, the appropriate measure of risk.

2 Profit Maximization and Capital Allocation

To illustrate the main ideas, we will start by considering a greatly simplified environment without securities markets and then generalize the results to the case where both the firm and its consumers have access to securities markets (see Appendix D for the generalized treatment).

The representative financial institution we have in mind in this section is an insurance company, and our language will reflect this in that we refer to the financial contracts as providing “insurance coverage” and the counterparties of the institution as “consumers.” The setup obviously fits other institutions providing similar contracts, such as reinsurance companies and private pension plan sponsors—and can be applied with little modification to institutions selling insurance-like contracts (such as credit default swaps) where the main risks emanate from risk in obligations to counterparties. The model can also be adapted to fit other institutions where capital allocation is relevant (such as commercial banks) but where the key risks emanate from the asset side of the balance sheet, by including an additional set of choice variables for investments. The key assumption of the model, however, is that the policyholders, counterparties, and/or debtholders of the institution are exposed to the failure of the institution—and their preferences for solvency drive the motivation for risk management.

Formally, we consider an insurance company that has N consumers, with consumer i facing a loss L_i modeled as a non-negative (square-integrable) random variable on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus, $L_i(\omega) > 0$ indicates that consumer i experienced a loss in the state $\omega \in \Omega$, while $L_i(\omega) = 0$ indicates that she did not.

The firm determines the optimal level of assets a for the company, as well as levels of insurance coverage for the consumers, with the coverage indemnification level for consumer i denoted as a function of the loss experienced and a parameter $q_i \in \Phi$ (where Φ is a compact choice set), as in $I_i(L_i, q_i)$, where we require $I_i(0, q_i) = 0$, $i = \{1, 2, \dots, N\}$. The latter function could take a variety of forms. For example, if indemnification promised to consumer i amounts to full reimbursement of losses subject to a policy limit q_i , the promised indemnification would be:

$$I_i = I_i(L_i, q_i) = \min \{L_i, q_i\}. \quad (1)$$

A quota share arrangement, where the insurer agrees to reimburse q_i per dollar of loss, would be represented as:

$$I_i = I_i(L_i, q_i) = q_i \times L_i. \quad (2)$$

If a consumer experiences a loss, she claims to the extent of the promised indemnification. If total claims are less than company assets, all are paid in full. If not, all claimants are paid at the same rate per dollar of coverage. The total claims submitted are:

$$I = I(L_1, L_2, \dots, L_N, q_1, q_2, \dots, q_N) = \sum_{j=1}^N I_j(L_j, q_j),$$

and we define the consumer’s recovery as:

$$R_i = \min \left\{ I_i(L_i, q_i), \frac{a}{I} I_i(L_i, q_i) \right\}.$$

Accordingly, $\{I \geq a\} = \{\omega \in \Omega \mid I(\omega) \geq a\}$ denote the states in which the company defaults whereas $\{I < a\}$ are the solvent states. The expected value of recoveries for the i -th consumer is

whence given by:

$$e_i = \mathbb{E}[R_i] = \underbrace{\mathbb{E}[R_i \mathbf{1}_{\{I < a\}}]}_{=e_i^Z} + \underbrace{\mathbb{E}[R_i \mathbf{1}_{\{I \geq a\}}]}_{=e_i^D}.$$

There is a frictional cost—including agency, taxes, and monitoring costs—associated with holding assets in the company. We represent the cost as a tax on assets:

$$\tau \times a, \quad (3)$$

although it is also possible to represent frictional costs as a tax on equity capital, as in:

$$\tau \times \left(a - \mathbb{E} \left[\sum_{i=1}^N \min \left\{ I_i(L_i, q_i), \frac{a}{I} I_i(L_i, q_i) \right\} \right] \right) \quad (4)$$

and this does not change the ensuing allocation result.

We denote the premium charged to the consumer i as p_i , and consumer utility may be expressed as:

$$v_i(a, w_i - p_i, q_1, \dots, q_N) = \mathbb{E}[U_i(w_i - p_i - L_i + R_i)], \quad (5)$$

where w_i denotes consumer i 's wealth, and we write $v_i'(\cdot) = \frac{\partial}{\partial w_i} v_i(\cdot)$.

The firm then solves:

$$\max_{a, \{q_i\}, \{p_i\}} \sum p_i - \sum e_i - \tau a, \quad (6)$$

subject to participation constraints for each consumer:

$$v_i(a, w_i - p_i, q_1, \dots, q_N) \geq \gamma_i \quad \forall i \quad (7)$$

and subject to a differentiable solvency constraint imposed by the regulator

$$s(q_1, \dots, q_N) \leq a, \quad (8)$$

where s is imagined to arise from, for example, a risk measure with a set threshold dictating the requisite capitalization for the firm.

In order to ensure differentiability of the objective function, it is necessary to impose conditions on the distributions of the loss random variables. For instance, for a quota share arrangement (2), the objective function will be continuously differentiable if the L_i are jointly continuously distributed (see Appendix A for details). For discrete loss distributions, on the other hand, if $I_i(L_i, \cdot)$ is differentiable, it is evident that the objective function is only piecewise differentiable. In this case, the analytic complications presented by the “kinks” may be overcome by considering the one-sided derivatives similarly to Zanjani (2010). In what follows, with little loss of generality, we simply assume that the solution lies in the differentiable region of the objective function. Let λ_k be the Lagrange multiplier associated with the participation constraint (7) for consumer k , and let ξ the multiplier associated with (8). The first order conditions for an interior solution are then:

$$[q_i] \quad - \sum_k \frac{\partial e_k}{\partial q_i} + \sum_k \lambda_k \frac{\partial v_k}{\partial q_i} - \frac{\partial s}{\partial q_i} \xi = 0, \quad (9)$$

$$[a] \quad - \sum_k \frac{\partial e_k}{\partial a} - \tau + \sum_k \lambda_k \frac{\partial v_k}{\partial a} + \xi = 0, \quad (10)$$

$$[p_i] \quad 1 - \lambda_i \frac{\partial v_i}{\partial w} = 0. \quad (11)$$

We show in Appendix B that a profit-maximizing firm will be able to achieve the optimum by offering each consumer a smooth and monotonic premium schedule, where consumer i is free to choose any level of q_i desired. We denote the variable premium as $p_i^*(q_i)$ and consider its construction under the assumption that each consumer is a “price taker” and ignores the impact of her own purchase at the margin on the level of recoveries in states of default. This assumption is discussed in Zanjani (2010), who followed the transportation economics literature on congestion pricing (Keeler and Small (1977)) by using the assumption when calculating the optimal consumer pricing function.⁴ With this assumption in place, the marginal price change at the optimal level of q_i must satisfy:

$$\left[\frac{\partial v_i}{\partial q_i} + \mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} U'_i \frac{a}{I^2} I_i \frac{\partial I_i}{\partial q_i} \right] \right] - \frac{\partial v_i}{\partial w} \frac{\partial p_i^*}{\partial q_i} = 0 \quad (12)$$

with $U'_i = U'_i(w_i - p_i - L_i + R_i)$. The term in brackets represents how the consumer *perceives* the marginal benefit of additional coverage, which, due to the aforementioned assumption, differs from the true impact of coverage on the utility function by $\mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} U'_i \frac{a}{I^2} I_i \frac{\partial I_i}{\partial q_i} \right]$.

Using (9) and (11), we see that (12) may be rewritten as:

$$\frac{\partial p_i^*}{\partial q_i} = \sum_k \frac{\partial e_k}{\partial q_i} + \frac{\partial s}{\partial q_i} \xi - \sum_{k \neq i} \frac{\frac{\partial v_k}{\partial q_i}}{v'_k} + \frac{\mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} U'_i \frac{a}{I^2} I_i \frac{\partial I_i}{\partial q_i} \right]}{v'_i}$$

or, simplifying and using (10):

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} \left[\sum_k \frac{\partial e_k}{\partial a} + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] + \mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U'_k}{v'_k} \frac{a}{I^2} I_k \frac{\partial I_i}{\partial q_i} \right].$$

Moving on:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} \left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] + \frac{\mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U'_k}{v'_k} \frac{1}{I^2} I_k \frac{\partial I_i}{\partial q_i} \right]}{\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k}} \times \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \times a$$

or

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} \left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] + \tilde{\phi}_i \times a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right], \quad (13)$$

where

$$\tilde{\phi}_i = \frac{\mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U'_k}{v'_k} \frac{1}{I^2} I_k \frac{\partial I_i}{\partial q_i} \right]}{\mathbb{E} \left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U'_k}{v'_k} \frac{I_k}{I} \right]}. \quad (14)$$

The last two terms of (13) imply an allocation of the marginal unit of capital to consumer that “adds up” if two conditions are met. First, we require that:

$$\frac{\partial I_i}{\partial q_i} \times q_i = I_i, \quad (15)$$

⁴The assumption is ubiquitous (although often implicit rather than stated) within many fields of economics. Its importance here is that, without it, the marginal cost associated with each consumer’s risk will feature a cost related to an implicit allocation of capital that will not “add up” across consumers to the total capital of the firm.

as would be the case under a quota share arrangement, as in (2), or under (1) if all loss distributions are binary (i.e., if a loss happens, it is of a fixed amount). If this holds, then it can be verified that:

$$a \times \sum \tilde{\phi}_i q_i = a. \quad (16)$$

Second, we require the “adding up” property on the regulatory constraint—which is satisfied under conditions previously referenced:

$$\sum \frac{\partial s}{\partial q_i} q_i = a. \quad (17)$$

When the foregoing conditions hold, the optimal marginal pricing condition (13) can be extended to fully allocate all of the firm’s costs, including the cost of capital:

$$\sum \frac{\partial p_i^*}{\partial q_i} q_i = \sum \frac{\partial e_i^Z}{\partial q_i} q_i + [\mathbb{P}(I \geq a) a + \tau a].$$

Note that the cost of capital as captured in the bracketed term breaks down as:

$$\left[\sum_k \frac{\partial e_k}{\partial a} a + \tau a \right] = \sum_i \frac{\partial s}{\partial q_i} q_i \times \left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] + \sum_i \tilde{\phi}_i q_i a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right].$$

So for any one individual consumer, their capital allocation has two components. The first derives from an “internal” marginal cost—driven by the cross-effects of consumers on each other:

$$\tilde{\phi}_i q_i a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right]$$

and the second originates from an “external” marginal cost imposed by regulators:

$$\frac{\partial s}{\partial q_i} q_i \times \left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right].$$

It is useful at this point to consider several different institutional scenarios.

Full Coverage by Deposit Insurance and Binding Regulation

If consumers are fully covered by deposit insurance, they are indifferent to the capitalization of their financial institution. Mathematically, this means that

$$\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} = 0,$$

so that (13) becomes:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} [\mathbb{P}(I \geq a) + \tau]. \quad (18)$$

Thus, the marginal cost of risk, and the attendant allocation of capital, is completely determined by the gradient of the binding regulatory constraint. This is the world of Myers and Read, Tasche, and others involved in the development of the gradient allocation principle. In this world, the marginal cost of risk is indeed completely determined by an arbitrarily chosen risk measure.

No Deposit Insurance and Non-Binding Regulation

At the opposite extreme is the case of an unregulated market with no deposit insurance. Here, $\xi = 0$, so (cf. Eq. (10)):

$$\sum_k \frac{\partial v_k}{v'_k} = [\mathbb{P}(I \geq a) + \tau],$$

meaning that (13) becomes:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \tilde{\phi}_i a \times [\mathbb{P}(I \geq a) + \tau]. \quad (19)$$

Thus, the marginal cost of risk and the attendant allocation of capital is driven completely by “internal” considerations. Specifically, (14) indicates that the allocation is driven by the time-zero value that affected consumers place on their contingent claims on recoveries in the various states of default.

General Case: Uninsured Consumers and Binding Regulation

In general, we may imagine the case where both of the considerations isolated above—an “external” regulatory constraint, and “internal” concerns driven by counterparty preferences—are influencing the marginal cost of risk. In this case, (13) remains in its original form:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} \left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\partial v_k}{v'_k} \right] + \tilde{\phi}_i \times a \times \left[\sum_k \frac{\partial v_k}{v'_k} \right], \quad (20)$$

but we are now able to see more clearly the two influences on capital allocation. When the regulatory constraint binds, we know that:

$$\mathbb{P}(I \geq a) + \tau > \sum_k \frac{\partial v_k}{v'_k},$$

with the interpretation that regulation is forcing the institution to hold assets beyond the level that would be privately efficient from the perspective of serving its counterparties. However, the extent of this distortion is key to identifying whether internal counterparty concerns or external regulatory concerns guide capital allocation. If regulation comes close to replicating the private market outcome:

$$\mathbb{P}(I \geq a) + \tau \approx \sum_k \frac{\partial v_k}{v'_k},$$

then the second term in (13) will be unimportant relative to the third term, and internal counterparty concerns will dominate. On the other hand, if regulation has the effect of pushing institutional capitalization well beyond the level that would prevail in the private market:

$$\mathbb{P}(I \geq a) + \tau \gg \sum_k \frac{\partial v_k}{v'_k},$$

then the second term in (13) will overshadow the third term, and external regulatory concerns will dominate.

3 Capital Allocation and Risk Measures

Having solved the institution's optimization problem and — as a by-product — the ensuing allocation of costs to the risks within its portfolio, we could very well terminate the discussion. However, the solution offers little obvious consolation to a practitioner faced with the problem of allocating capital in a real-world setting, where contracts may not be easily mapped to our specification and preferences may be hard to assess. It is in the hands of the practitioner where the *Euler allocation principle* gains traction, since the principle yields an implementable approach to allocation. With the Euler method, it is “only” necessary to calculate the partial derivatives of a suitable risk measure with respect to each exposure evaluated at the current portfolio.⁵

Although it is not always derived in this way, the Euler method is implied by the maximization of profits subject to a risk measure constraint. To illustrate, assume a company's profit function π depends on the parameters (volumes) q_i , $1 \leq i \leq N$, and on capital a . Then maximizing profits subject to the *risk measure constraint*

$$\rho(q_1, q_2, \dots, q_N) \chi_\rho \leq a$$

yields

$$\frac{\partial \pi}{\partial q_i} = \left(-\frac{\partial \pi}{\partial a} \right) \chi_\rho \frac{\partial \rho}{\partial q_i}$$

at the optimal values a^* , q_i^* , $1 \leq i \leq N$. Here ρ is a suitable differentiable risk measure evaluated at the aggregate claims $\sum_{j=1}^N I_j(L_j, q_j)$ and χ_ρ is an exchange rate that converts risk to capital, which is often chosen to be unity if risk is measured in monetary units. Hence, for the optimal portfolio, the risk-adjusted marginal return on marginal capital $\frac{\partial \pi / \partial q_i}{\partial \rho / \partial q_i}$ for each exposure is the same and equals the cost of a marginal unit of capital $-\frac{\partial \pi}{\partial a}$. In other words, if the marginal performance of risk i as measured by its marginal return on marginal risk capital $\chi_\rho \frac{\partial \rho}{\partial q_i}$ exceeds (respectively, falls below) the cost of a marginal unit of capital, then increasing (respectively, decreasing) the weight q_i of that exposure by a small amount improves the overall performance of the portfolio.⁶ Akin to the results of Tasche (2000) on the *suitability* of capital allocation principles, this motivates the interpretation of the marginal capital weighted by the corresponding volume $\chi_\rho \frac{\partial \rho}{\partial q_i} q_i^*$ as the amount of capital allocated to exposure i . In particular, for a homogenous risk measure, the allocations to the respective risks “add up” to the entire capital:

$$\begin{aligned} \sum_{j=1}^N \chi_\rho \frac{\partial \rho}{\partial q_j} q_j^* \left(-\frac{\partial \pi}{\partial a} \right) &= \chi_\rho \rho \left(-\frac{\partial \pi}{\partial a} \right) = a^* \left(-\frac{\partial \pi}{\partial a} \right) \\ \implies \sum_{j=1}^N \chi_\rho \frac{\partial \rho}{\partial q_j} q_j^* &= a^*. \end{aligned}$$

⁵This is not at all to say that this task is simple. In fact, the computational complexity associated with evaluating economic capital presents a serious problem for financial institutions and frequently leads them to adopt second-best calculation techniques (see e.g. Gordy and Juneja (2010) or Bauer et al. (2010)). However, the availability of a suitable model for the different risk within a company's portfolio and their interplay clearly is a necessity for the derivation for any coherent allocation of capital.

⁶Cf. Section 6.3.3 “Economic Justification of the Euler Principle” in McNeil et al. (2005).

We refer to McNeil et al. (2005) and references therein for more details on the Euler principle, and to Denault (2001), Kalkbrenner (2005), and Myers and Read (2001) for alternative derivations of the Euler principle based on cooperative game theory, formal axioms, or a contingent claim approach, respectively.

Regardless of its provenance, the Euler method's strength is the feasibility of implementation, so this raises the question of whether our allocation can be implemented using the Euler technology. That is, is it possible to identify a risk measure that would yield, after application of the Euler approach, the economically correct capital allocations identified in Section 2? In what follows, we derive this very risk measure.

Assume our institution is operating at the optimum, i.e. assets, premium schedules, and contracts are fixed at their optimal values a^* , $p_i^*(\cdot)$, q_i^* , $1 \leq i \leq N$. Consider first the case of binding regulation and full coverage by deposit insurance, i.e. the first case from Section 2. As might be expected, here the optimal allocation at the (optimal) margin can be derived by the Euler principle with profit function

$$\pi(q_1, \dots, q_N, a) = \sum_k p_k^*(q_k) - \sum_k e_k(q_1, \dots, q_N, a) - \tau a, \quad (21)$$

and risk measure constraint

$$s(q_1, q_2, \dots, q_N) \leq a,$$

which yields (cf. (18))

$$\begin{aligned} \sum_{j=1}^N \frac{\partial s}{\partial q_j} q_j^* [\mathbb{P}(I \geq a^*) + \tau] &= a^* [\mathbb{P}(I \geq a^*) + \tau] \\ \implies \sum_{j=1}^N \frac{\partial s}{\partial q_j} q_j^* &= a^*. \end{aligned}$$

For the second polar case with no deposit insurance and no binding regulation, i.e. for the second case from Section 2, we introduce the probability measure $\tilde{\mathbb{P}}$ on (Ω, \mathcal{F}) via its Radon-Nikodym derivative

$$\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}} = \frac{\sum_k \frac{U'_k}{v'_k} \frac{I_k}{I} \mathbf{1}_{\{I \geq a^*\}}}{\mathbb{E} \left[\sum_k \frac{U'_k}{v'_k} \frac{I_k}{I} \mathbf{1}_{\{I \geq a^*\}} \right]},$$

where I , U , etc. are evaluated at the concurrent (optimal) values a^* , $p_i^*(q_i^*)$, q_i^* , $1 \leq i \leq N$. Note that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to \mathbb{P} but the measures are not equivalent since under $\tilde{\mathbb{P}}$ all the probability mass is concentrated in default states. On the set of strictly positive $\tilde{\mathbb{P}}$ -square integrable random variables

$$L^2_+(\Omega, \mathcal{F}, \tilde{\mathbb{P}}) = \left\{ X \in L^2(\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \mid X > 0 \tilde{\mathbb{P}}\text{-a.s.} \right\},$$

we define the risk measure⁷

$$\tilde{\rho}(X) = \exp \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} [\log \{X\}] \right\}.$$

⁷While its functional form seems similar to that of the so-called *entropic risk measure* (which has recently gained popularity in the mathematical finance literature (see e.g. Föllmer and Schied (2002) or Detlefsen and Scandolo (2005))), note that the roles of the exponential function and the logarithm are interchanged.

Obviously $\tilde{\rho}$ is monotonic, positive homogenous, and it satisfies the constancy condition $\tilde{\rho}(c) = c$ for $c > 0$ (see Frittelli and Gianin (2002) for a discussion of properties of risk measures). However, $\tilde{\rho}$ is neither translation-invariant nor sub-additive, and is therefore *not coherent* and *not convex*. In fact, $\tilde{\rho}$ is not even a *monetary risk measure* and thus may not qualify for the use as an external risk measure.⁸ However, it is the correct risk measure for internal capital allocation based on the Euler principle.

More precisely, define $\tilde{\chi}_\rho = \frac{a^*}{\tilde{\rho}(\sum_{j=1}^N I_j(L_j, q_j^*))}$ as the “exchange rate” between units of risk and capital. Then the application of the Euler principle with profit function $\pi(q_1, \dots, q_N, a)$ as in (21) and risk measure constraint

$$\underbrace{\tilde{\rho}\left(\sum_{j=1}^N I_j(L_j, q_j)\right)}_{=\tilde{\rho}(q_1, \dots, q_N)} \tilde{\chi}_\rho \leq a,$$

yields (cf. (19))

$$\begin{aligned} \sum_{j=1}^N \tilde{\chi}_\rho \frac{\partial \tilde{\rho}}{\partial q_j} q_j^* [\mathbb{P}(I \geq a^*) + \tau] &= \sum_{j=1}^N \tilde{\chi}_\rho \mathbb{E}^{\tilde{\mathbb{P}}} \left[\frac{\partial I_j / \partial q_j}{I} \right] \tilde{\rho} q_j^* [\mathbb{P}(I \geq a^*) + \tau] \\ &= \sum_{j=1}^N \tilde{\phi}_j q_j^* a^* [\mathbb{P}(I \geq a^*) + \tau] = a^* [\mathbb{P}(I \geq a^*) + \tau] \\ \implies \sum_{j=1}^N \tilde{\chi}_\rho \frac{\partial \tilde{\rho}}{\partial q_j} q_j^* &= a^*. \end{aligned} \quad (22)$$

Hence, the correct internal capital allocation can be implemented by the Euler principle relying on the risk measure $\tilde{\rho}$ —a risk measure that, surprisingly, is neither coherent nor convex.

For intermediate cases, i.e. with binding regulation but uninsured consumers—the third and general case from Section 2—the introduced framework for the Euler principle does not immediately apply since we now look at two risk measure constraints

$$\begin{cases} s(q_1, \dots, q_N) \leq a, \\ \tilde{\rho}(q_1, \dots, q_N) \tilde{\chi}_\rho \leq a. \end{cases}$$

Thus, the allocation is determined by the gradients of both risk measures as well as the associated Lagrange multipliers for s and $\tilde{\rho}$, which at the optimum $(q_1^*, \dots, q_N^*, a^*)$ equal (cf. (20))

$$\left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} \right]$$

and

$$\left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} \right],$$

⁸While sub-additivity is subject of an ongoing debate (see e.g. Heyde et al. (2006) or Dhaene et al. (2008)), at least translation invariance is generally deemed adequate for an *external risk measure* and even “*necessary for the risk-capital interpretation [...] to make sense*” (see p. 239 in McNeil et al. (2005)).

respectively. More specifically, we have

$$\begin{aligned}
\sum_{j=1}^N \frac{\partial s}{\partial q_j} q_j^* \left[\mathbb{P}(I \geq a^*) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] + \sum_{j=1}^N \tilde{\chi}_\rho \frac{\partial \tilde{\rho}}{\partial q_j} q_j^* \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] &= a^* [\mathbb{P}(I \geq a^*) + \tau] \\
\implies \sum_{j=1}^N \left(\frac{\partial s}{\partial q_j} q_j^* (1 - \zeta) + \tilde{\chi}_\rho \frac{\partial \tilde{\rho}}{\partial q_j} q_j^* \zeta \right) &= a^* \\
\implies \sum_{j=1}^N \frac{\partial}{\partial q_j} ((1 - \zeta) s + \zeta \tilde{\chi}_\rho \rho) q_j^* &= a^*,
\end{aligned}$$

where the weight ζ is defined as

$$\zeta = \frac{\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k}}{\mathbb{P}(I \geq a) + \tau}.$$

Hence, the Euler principle still applies, but the supporting risk measure is a weighted average between the external risk measure s and the internal risk measure ρ . The weights are determined by the consumers' marginal preference for capitalization of the company relative to their marginal utility of own wealth. To determine which considerations—external or internal ones—dominate, it is therefore necessary to assess the consumers' proclivity for capitalization. However, to gain insights on how the counterparty-driven internal allocation effectively differs from the allocation based on the external risk measure, it is sufficient to study the ensuing risk measure $\tilde{\rho}$. In particular, we are interested how the economic weight assigned to various outcomes under $\tilde{\rho}$ differs from what would be obtained from the use of more popular risk measures.

To this end, we evaluate $\tilde{\rho}$ as a function of the aggregate loss $I = \sum_{j=1}^N I_j(L_j, q_j^*)$. We have

$$\begin{aligned}
\tilde{\rho}(I) &= \exp \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} [\log\{I\}] \right\} = \exp \left\{ \mathbb{E} \left[\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}} \log\{I\} \right] \right\} = \exp \left\{ \mathbb{E} \left[\mathbb{E} \left[\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}} \middle| I \right] \log\{I\} \right] \right\} \\
&= \exp \left\{ \mathbb{E} \left[\mathbb{E} \left[\frac{\sum_k \frac{U'_k}{v'_k} \frac{I_k}{I} \mathbf{1}_{\{I \geq a^*\}}}{\mathbb{E} \left[\sum_k \frac{U'_k}{v'_k} \frac{I_k}{I} \middle| I \geq a^* \right]} \middle| I \right] \log\{I\} \middle| I \geq a^* \right] \right\} \\
&= \exp \left\{ \mathbb{E} \left[\tilde{\psi}(I) \log\{I\} \middle| I \geq a^* \right] \right\}, \tag{23}
\end{aligned}$$

which illustrates that $\tilde{\rho}$ in this sense is in fact a tail risk measure, and hence is related to Expected Shortfall (ES).

Here, the weights $\tilde{\psi}(\cdot)$ perform a role similar to the *risk spectrum* within the so-called *spectral risk measures* as introduced in Acerbi (2002). According to the author, the “*subjective risk aversion of an investor can be encoded*” in this function, which may justify overweighting bad outcomes, but he does not provide guidance on how to choose an explicit form. To close this gap, Dowd et al. (2008) provide some ad-hoc examples whereas Sriboonchitta et al. (2010) and Wächter and Mazzoni (2010) attempt to establish theoretical links of the risk spectrum to the preferences of the user of the risk measure—or, depending on its application, the preferences of an external supplier such as a regulator—by relying on results from robust statistics and the dual theory of choice,

respectively. In contrast, in our setting the weights represent an adjustment to objective probabilities based on the value placed by claimants on recoveries in various states of default. Thus, the pivotal characteristics for our weights lie in the primitives of the firm's profit maximization problem (namely, the preferences of counterparties)—which ultimately determine the overall choice of capitalization as well as the values consumers place on state contingent recoveries—rather than in a subjectively specified concave preference function for the firm, which will generally fail to capture limited liability.

In the absence of weights, however, it is worth noting that the concavity of the logarithmic function will, in the course of the application of the Euler allocation methods, tend to penalize bad outcomes *less* heavily than ES. In fact, it is evident from (23) that the Euler method will effectively weight all aggregate loss outcomes in excess of the firm's capital equally, regardless of size, when $\tilde{\psi}(\cdot) \equiv 1$. The reason for this is that $\tilde{\psi}(\cdot) \equiv 1$ implies that the firm's counterparties are risk-neutral, and, thus, the value of the firm in all states of default, regardless of how extreme the default, is simply the firm's assets. At the margin, the counterparties evaluate changes in risk simply from the perspective of how the expected value of recoveries from the firm are affected, and recoveries in mild states of default are weighted no differently from severe ones. This is also the reason why $\tilde{\rho}$ is not sub-additive or translation-invariant: Adding a constant in high loss states is less precarious than in low loss states because of limited liability.

Under risk aversion, on the other hand, $\tilde{\psi}(\cdot) \neq 1$, and counterparties may well weight recoveries under severe states of default more heavily than mild ones. The answer to the question of how counterparty-driven allocation effectively differs from the allocation based ES then lies in the nature of the weights. For instance, if a specification satisfied

$$\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}} = \tilde{\psi}(I) = \text{const} \times I \times \mathbf{1}_{\{I \geq a\}}, \quad (24)$$

Equation (22) implies that the resulting counterparty-driven allocation would be identical to a gradient allocation based on the ES (for a suitably chosen level—see the next section for details). Similarly, it is conceivable that other specifications may result in qualitatively different outcomes, in either direction. The next section sheds more light on these issues by considering a variety of situations (including one which satisfies (24)).

4 Comparison of Capital Allocation Methods

In this section, we consider the practical implications of allocating capital based on the method discussed in the previous sections. In particular, we compare the resulting allocations to those obtained when applying the Euler technique to Expected Shortfall—perhaps the most widely endorsed measure within the academic and practitioner community. We then illustrate in the context of several examples.

4.1 The Case of Exponential Losses

Assume that there are N identical consumers with wealth level w in a regime with non-binding regulation that face independent, Exponentially distributed losses $L_i \sim \text{Exp}(\nu)$, $1 \leq i \leq N$.

Assume further that all consumers exhibit a constant absolute risk aversion of $\alpha < \nu$, and that their participation constraint is given by the autarky level

$$\gamma = \gamma_i = \mathbb{E} [U(w - L_i)] = -e^{-\alpha w} \frac{\nu}{\nu - \alpha}.$$

Then, the optimization problem (6)/(7)/(8) may be written as

$$\left\{ \begin{array}{l} \max_{a,q,p} \left\{ N \times p - N \times q \times \left[\frac{1}{\nu} \Gamma_{N-1,\nu} \left(\frac{a}{q} \right) - \frac{\nu^{N-1}}{(N-1)!} e^{-\nu \frac{a}{q}} \left(\frac{a}{q} \right)^{N-1} \left(\frac{1}{\nu} + \frac{1}{N} \frac{a}{q} \right) \right] \right. \\ \quad \left. - a \times \bar{\Gamma}_{N,\nu} \left(\frac{a}{q} \right) - \tau \times a \right\} \\ \text{subject to} \\ \gamma \leq e^{-\alpha(w-p)} \left\{ \frac{\nu}{\nu - (1-q)\alpha} \left[\Gamma_{N-1,\nu} \left(\frac{a}{q} \right) - \frac{e^{-\frac{a}{q}(\nu - (1-q)\alpha)} \nu^{N-1}}{((1-q)\alpha)^{N-1}} \Gamma_{N-1,(1-q)\alpha} \left(\frac{a}{q} \right) \right] \right. \\ \quad \left. + \sum_{k=0}^{\infty} \left(\frac{a}{\nu} \right)^k \frac{(N-1)!}{(N-1+k)!} e^{-\nu a} \sum_{j=0}^{N-1} \frac{(a\nu)^j}{j!} \frac{(N-j+k-1)!}{(N-j-1)!} \bar{\Gamma}_{N-j+k,\nu} \left(a \left(\frac{1-q}{q} \right) \right) \right\}, \end{array} \right. \quad (25)$$

where $\bar{\Gamma}_{m,b}(x) = 1 - \Gamma_{m,b}(x)$ and $\Gamma_{m,b}(\cdot)$ denotes the cumulative distribution function of the Gamma distribution with parameters m and b (see the Appendix C for the derivation of (25)).

For the allocation of capital to the individual consumers, we trivially obtain

$$q \tilde{\phi}_i = N^{-1}, \quad i = 1, 2, \dots, N,$$

which is the same when applying the Euler technique with *any* risk measure. More specifically,

$$q \tilde{\phi}_i \stackrel{\text{Eq. (14)}}{=} \frac{\mathbb{E} \left[\mathbf{1}_{\{qL \geq a\}} \mathbb{E} \left[\sum_{j=1}^N U' \left(w - p - L_j + a \frac{L_j}{L} \right) \frac{L_j}{L} \frac{L_i}{L} \middle| L \right] \right]}{\mathbb{E} \left[\mathbf{1}_{\{qL \geq a\}} \sum_{j=1}^N U' \left(w - p - L_j + a \frac{L_j}{L} \right) \frac{L_j}{L} \right]} = \frac{1}{N} \mathbb{E} \left[\hat{\psi}(L) \middle| qL \geq a \right],$$

where $L = \sum_j L_j$,

$$\hat{\psi}(l) = \mathbf{1}_{\{ql \geq a\}} \hat{c}_{N,\nu,\alpha,a,q} \sum_{k=0}^{\infty} \frac{(k+1)(\alpha(l-a))^k}{(N+k)!}, \quad (26)$$

and $\hat{c}_{N,\nu,\alpha,a,q}$ is a constant ensuring that $\mathbb{E} \left[\hat{\psi}(L) \middle| qL \geq a \right] = 1$.

For the risk measure $\tilde{\rho}$, we have

$$\tilde{\rho}(I) = \tilde{\rho}(qL) = \exp \left\{ \mathbb{E} \left[\tilde{\psi}(I) \log\{I\} \middle| I \geq a \right] \right\} = \exp \left\{ \mathbb{E} \left[\hat{\psi}(L) \log\{qL\} \middle| qL \geq a \right] \right\},$$

with $\hat{\psi}(\cdot)$ the corresponding weighting function.⁹ Hence, the risk measure $\tilde{\rho}$ in this case naturally accounts for risk aversion (α) as well as for diversification effects (N).

For the allocation based on the Expected Shortfall (ES) according to the *Euler principle*, it is well known that (see e.g. Dhaene et al. (2009)):¹⁰

$$\frac{a_i}{a} = \frac{q \mathbb{E} [L_i | qL \geq a]}{\mathbb{E} [I | I \geq a]} = \frac{\mathbb{E} [\mathbb{E} [L_i | L] | qL \geq a]}{\mathbb{E} [L | qL \geq a]} = \frac{1}{N} \mathbb{E} [\text{const} \times L | qL \geq a],$$

⁹The derivation of these equations, a closed form solution for $\hat{c}_{N,\nu,\alpha,a,q}$ as well as a representation of $\hat{\psi}(\cdot)$ not involving an infinite sum for implementation purposes all are provided in Appendix C.

¹⁰Here, we always assume that the confidence level is chosen corresponding to the counterparty-driven allocation, namely $\mathbb{P}(qL \geq a)$ in this case.

i.e. the Expected Shortfall can be associated with a linear weighting function of the loss states. Since $\hat{\psi}(\cdot)$ is increasing and strictly convex for all risk aversion levels $\alpha > 0$, there always exists a loss level l_0 such that the weighting function for the counterparty-driven allocation will be higher for all loss levels greater than l_0 . In this sense, the allocation based on $\tilde{\rho}$ always appears more conservative in the current setting. However, we also see that for fixed parameters,

$$\hat{\psi}(l) \longrightarrow (\mathbb{P}(qL \geq a))^{-1} > \frac{a}{\mathbb{E}[qL | qL \geq a]}, \quad N \rightarrow \infty,$$

which is the left end-point for the Expected Shortfall weighting function. Similarly, for $\alpha = 0$, we obtain $\hat{\psi}(l) \equiv (\mathbb{P}(qL \geq a))^{-1}$, i.e. a flat weighting function. Thus, for large enough companies or risk-neutral consumers, the weight on relatively low loss levels will always be higher for the counterparty-driven allocation, rendering it to appear less conservative.

Exponential Losses: Parametrizations

Nr.	N	ν	τ	α	w	a	p	q
1	5	2.0	0.050	0.25	3.0	1.4663	0.2598	0.5713
2	5	2.0	0.050	1.25	3.0	4.0036	0.7401	0.9494

Table 1: Parametrizations of the Exponential Losses model.

To further analyze this relationship, in Table 1 we present two parametrizations of the setup and the corresponding optimal parameters a , p , and q as solutions of the program (25). The properties are as expected: a , p , and q all are increasing in risk aversion. Figure 1 plots the weighting function $\hat{\psi}$ against the linear weighting function associated with the Expected Shortfall for varying risk aversion levels. We find two qualitatively different shapes:¹¹ For the high risk aversion level, $\hat{\psi}$ crosses the linear weighting function once from below; thus, in this case, relatively lower loss states are weighted more heavily for the allocation based on the Expected Shortfall, whereas the weighting is higher for the counter-party driven allocation in high loss states. For the low risk aversion level, $\hat{\psi}$ crosses the linear weighting function twice; in this case, the weighting function within the new risk measure $\tilde{\rho}$ puts more mass on low and extremely high loss states, while the weights are smaller for intermediate loss states.

Hence, when relying on the Expected Shortfall for the purpose of internal capital allocation, the loss-specific weights may be too conservative or not conservative enough, depending on, among other factors, company size or the risk aversion level. These considerations are naturally taken into account by the risk measure $\tilde{\rho}$.

¹¹Analyses with respect to other parameters such as company size N or the expected loss $1/\nu$ show similar results.

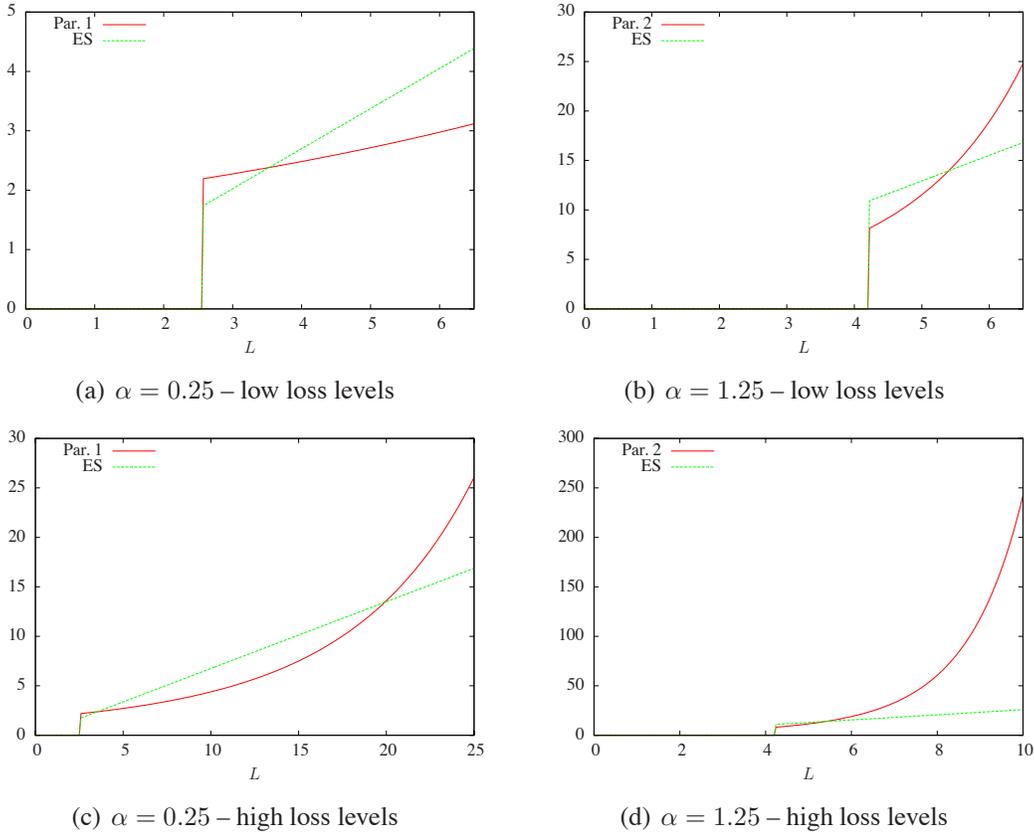


Figure 1: Weighting function $\hat{\psi}$ for varying risk aversion parameter α .

4.2 The Case of Homogenous Bernoulli Losses

Again, we consider N identical consumers with wealth level w in a regime with non-binding regulation whose preferences are given by the (same) smooth utility function $U(\cdot)$. However, in contrast to the previous section, we now assume that the consumers face Bernoulli distributed losses L_i , $1 \leq i \leq N$, with loss level l and loss probability π . Their participation constraint once again is given by the autarky level

$$\gamma = \gamma_i = \mathbb{E}[U(w - L_i)].$$

In this case, the optimization problem (6)/(7)/(8) takes the form

$$\left\{ \begin{array}{l} \max_{a,q,p} \left\{ N \times \left(p - \pi \times \sum_{k=0}^{N-1} \binom{N-1}{k} \pi^k (1-\pi)^{N-1-k} \times \right. \right. \\ \left. \left. \left[ql \mathbf{1}_{\{k < \frac{a}{ql} - 1\}} + \frac{a}{k+1} \mathbf{1}_{\{k \geq \frac{a}{ql} - 1\}} \right] \right) - \tau \times a \right\} \\ \text{subject to} \\ \gamma \leq (1-\pi) U(w-p) + \pi \times \sum_{k=0}^{N-1} \binom{N-1}{k} \pi^k (1-\pi)^{N-1-k} \times \\ \left[U(w-p - (1-q)l) \mathbf{1}_{\{k < \frac{a}{ql} - 1\}} + U\left(w-p-l + \frac{a}{k+1}\right) \mathbf{1}_{\{k \geq \frac{a}{ql} - 1\}} \right]. \end{array} \right. \quad (27)$$

For the allocation to the individual consumer, similarly to the previous section, we obtain

$$q \tilde{\phi}_i = \frac{1}{N} \mathbb{E} \left[\underbrace{\frac{1}{\mathbb{E} \left[\mathbf{1}_{\{|\Gamma| \geq \frac{a}{qt}\}} U' \left(w - p - l + \frac{a}{|\Gamma|} \right) \right]} \mathbf{1}_{\{|\Gamma| \geq \frac{a}{qt}\}} U' \left(w - p - l + \frac{a}{|\Gamma|} \right)}_{=\bar{\psi}(|\Gamma|)} \right] = \frac{1}{N},$$

where $|\Gamma|$ denotes the number of total losses. And the risk measure now takes the form

$$\tilde{\rho}(I) = \tilde{\rho}(ql|\Gamma|) = \exp \left\{ \mathbb{E} \left[\tilde{\psi}(I) \log\{I\} \mid I \geq a \right] \right\} = \exp \left\{ \mathbb{E} \left[\bar{\psi}(|\Gamma|) \log\{ql|\Gamma|\} \mid |\Gamma| \geq \frac{a}{ql} \right] \right\},$$

whereas for the Expected Shortfall based allocation, again similarly to Section 4.1,

$$\frac{a_i}{a} = \frac{q \mathbb{E} [L_i | qL \geq a]}{\mathbb{E} [I | I \geq a]} = \frac{1}{N} \mathbb{E} \left[\text{const } |\Gamma| \mid |\Gamma| \geq \frac{a}{ql} \right],$$

so again it can be associated with a linear weighting function.

Hence, in this case, the weighting function for $\tilde{\rho}$ is a composition of the marginal utility and a reciprocal function. As such, it will always be increasing if consumers are risk averse, and again we obtain a flat allocation for the risk neutral case. For $w \geq p + l$, a sufficient condition for the concavity of $\bar{\psi}$ is a level of relative prudence smaller than two (see e.g. Kimball (1990) for the concept of relative prudence). However, for high levels of relative prudence, a convex shape is possible. Thus, again, it is not immediately clear how the counterparty-driven allocation compares to the linear weighting implied by the Expected Shortfall measure.

Homogeneous Bernoulli Losses: Parametrizations

Nr.	Type	N	l	π	τ	α/γ	w	a	p	q
1	CARA	100	3.00	0.10	0.05	0.1	5.0	24.8970	0.2727	0.9762
2	CARA	100	3.00	0.10	0.05	1.0	5.0	40.4654	1.0585	0.9969
3	CRRA	10	3.25	0.45	0.05	0.5	5.0	12.9492	1.2893	0.8704
4	CRRA	10	3.25	0.45	0.05	1.0	5.0	16.5461	1.6882	0.9350
5	CRRA	10	3.25	0.45	0.05	2.0	5.0	19.5488	2.1840	0.9693

Table 2: Parametrizations of the Bernoulli model.

Table 2 now displays several parametrizations in this setup for different Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA) utility functions, where the optimal parameters a , p , and q are determined as solutions of program (27); again, they exhibit the expected relationships. Figure 2 now shows the weighting function $\bar{\psi}$ for parametrizations 1 and 2 (CARA utility) as well as the corresponding weighting function for the Expected Shortfall (ES)

and risk-neutral consumers ($\alpha = 0$). In contrast to the previous section, $\bar{\psi}$ is concave in both cases. We again we obtain two qualitatively different relationships: For the low risk aversion, $\bar{\psi}$ crosses the linear weighting function from above, implying a less conservative state-specific allocation; for the strong risk averter, on the other side, the crossing is from below, ensuing a more conservative configuration. So yet again, risk aversion appears to play an important role.

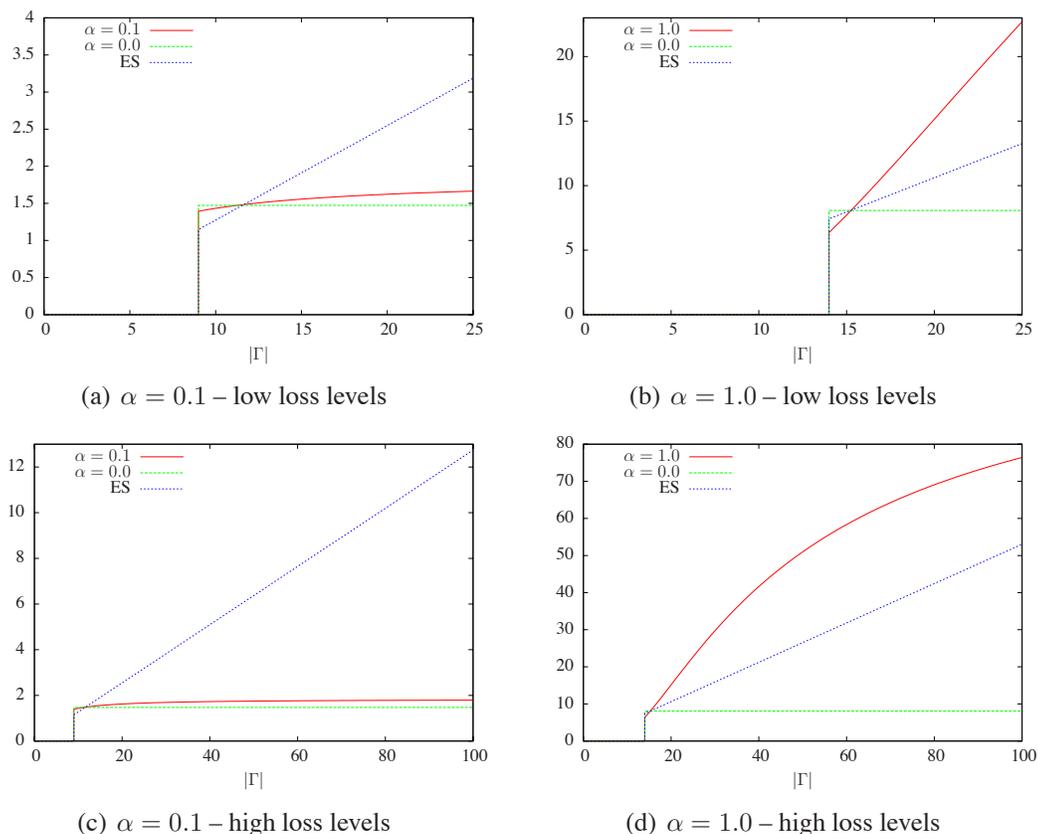
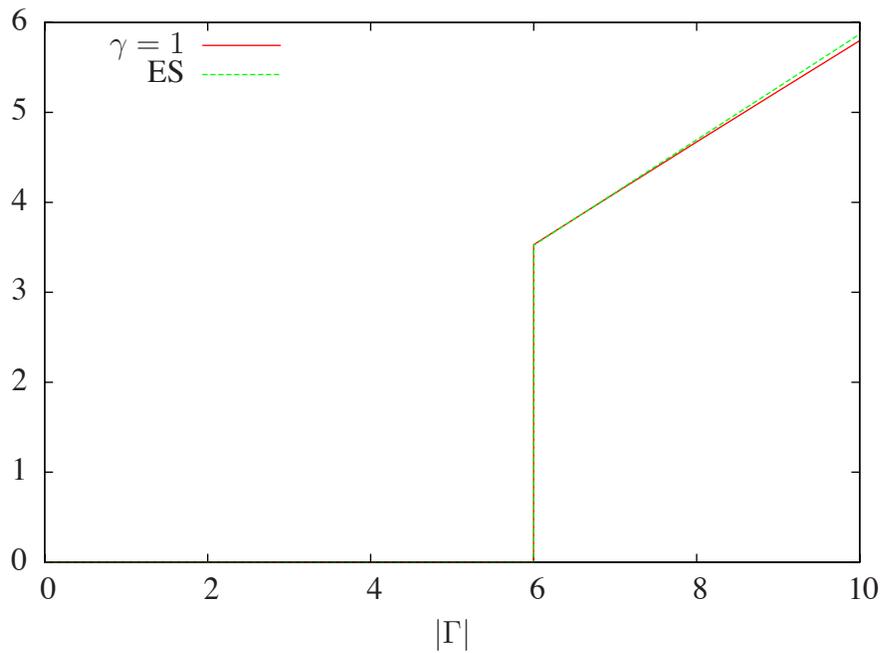


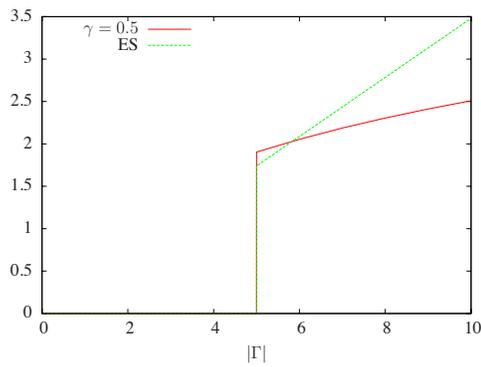
Figure 2: Weighting function $\bar{\psi}$ for varying absolute risk aversion parameter α ; parametrizations 1 and 2 (CARA utility).

For the CRRA case, the contract and firm parameters were chosen so that wealth is close to premium level plus losses, since for $w = p + l$, $\bar{\psi}$ will be concave (convex) if and only if relative prudence is smaller (greater) than two. In particular, for log-utility, where relative prudence is constant and equals two, the weighting will be linear and, therefore, identical to the Expected Shortfall weights. This can also be seen from Figure 3, where the weighting functions for different relative risk aversion levels γ are plotted: For log-utility and $w \approx p + l$, the ES weighting and $\bar{\psi}$ are roughly identical. In contrast, for the lower constant relative risk aversion—and consequently a lower level of relative prudence—the crossing is from above and the shape is concave (panel (b)). For the higher level of constant relative risk aversion and whence more prudent consumers, the crossing is from above and the shaper of $\bar{\psi}$ is convex (panel (c)).¹²

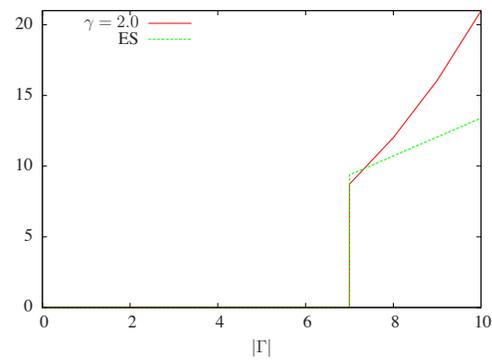
¹²The observation that for $\gamma = 1$ we obtain the Expected Shortfall is a curious analogy to the so-called *power spectral risk measures* from Dowd et al. (2008), who show that these measures degenerate into the Expected Loss for $\gamma = 1$, and that there is a qualitative difference between the case $\gamma < 1$ and $\gamma > 1$.



(a) $\gamma = 1.0$ – log utility



(b) $\gamma = 0.5$



(c) $\gamma = 2.0$

Figure 3: Weighting function $\bar{\psi}$ for varying relative risk aversion parameter γ ; parametrizations 3, 4, and 5 (CRRA utility).

Thus, we find that for the shape of the weighting function $\bar{\psi}$ and accordingly for the comparison of the resulting statewise allocation with that implied by the Expected Shortfall, other characteristics of the consumers' utility functions (in addition to risk aversion) are relevant. In particular, we can give a positive answer to the existence question raised towards the end of Section 3: Indeed, there exist special cases where the weighting is linear so that the counterparty-driven allocation and the Expected Shortfall based allocation are identical, although in general—of course—they will differ.

4.3 The Case of Heterogenous Bernoulli Losses

Similarly to the last section, we consider consumers that face Bernoulli distributed losses. However, in contrast to the previous setup, we now allow for heterogeneity in consumer preferences as well as in the losses. More specifically, we assume that there are m groups of consumers, where group i contains N_i identical consumers with wealth level w_i and utility function $U_i(\cdot)$ that face independent losses l_i occurring with a probability π_i , $i = 1, \dots, m$. The participation constraint again is given by their autarky levels:

$$\gamma_i = \mathbb{E}[U_i(w_i - L_i)] = \pi_i U_i(w_i - l_i) + (1 - \pi_i) U_i(w_i).$$

The optimization problem (6)/(7)/(8) can then be easily set up by noticing that the number of losses in the different groups follow independent Binomial(N_i, π_i) distributions.

For the counterparty-based allocation, we obtain for each group i

$$\begin{aligned} q_i \tilde{\phi}_i &= \tilde{c} \sum_{k_1}^{N_1} \dots \sum_{k_i}^{N_i} \dots \sum_{k_m}^{N_m} \binom{N_1}{k_1} \dots \binom{N_i}{k_i} \dots \binom{N_m}{k_m} \\ &\quad \times \pi_1^{k_1} \dots \pi_i^{k_i} \dots \pi_m^{k_m} (1 - \pi_1)^{N_1 - k_1} \dots (1 - \pi_i)^{N_i - k_i} \dots (1 - \pi_m)^{N_m - k_m} \\ &\quad \times \underbrace{\left[\mathbf{1}_{\{k_1 q_1 l_1 + \dots + k_m q_m l_m \geq a\}} \left\{ \sum_{j=1}^m \frac{k_j U' \left(w_j - p_j - l_j + q_j l_j \frac{a}{k_1 q_1 l_1 + \dots + k_m q_m l_m} \right) l_j}{v_j'(k_1 q_1 l_1 + \dots + k_m q_m l_m)} \right\} \right]}_{=\text{const} \times \frac{\partial \bar{\mathbb{P}}}{\partial \mathbb{P}}} \\ &\quad \times \frac{k_i q_i l_i}{k_1 q_1 l_1 + \dots + k_m q_m l_m}, \end{aligned} \tag{28}$$

where \tilde{c} is a constant such that $\sum_i q_i \tilde{\phi}_i = 1$. Thus, while the analytical form of the weights $\tilde{\psi}(I) = \mathbb{E} \left[\frac{\partial \bar{\mathbb{P}}}{\partial \mathbb{P}} \middle| I \right]$ is less transparent in this case, again we notice that they immediately depend on the marginal utilities of recoveries in various states of default. For the allocation based on the Expected Shortfall, on the other hand, we obtain

$$\begin{aligned} \frac{a_i}{a} &= \text{const} \sum_{k_1}^{N_1} \dots \sum_{k_i}^{N_i} \dots \sum_{k_m}^{N_m} \binom{N_1}{k_1} \dots \binom{N_i}{k_i} \dots \binom{N_m}{k_m} \\ &\quad \times \pi_1^{k_1} \dots \pi_i^{k_i} \dots \pi_m^{k_m} (1 - \pi_1)^{N_1 - k_1} \dots (1 - \pi_i)^{N_i - k_i} \dots (1 - \pi_m)^{N_m - k_m} \\ &\quad \mathbf{1}_{\{k_1 q_1 l_1 + \dots + k_m q_m l_m \geq a\}} (k_i q_i l_i), \end{aligned} \tag{29}$$

i.e. it is of a similar form as (28) but now 1) does not contain the adjustment based on the marginal utilities $\frac{\partial \bar{\mathbb{P}}}{\partial \mathbb{P}}$, and 2) the state-specific loss for consumer i , $(k_i q_i l_i)$, is not scaled by the aggregate

loss—which, in the counterparty-driven allocation, is a consequence of the proportional partitioning of the recoveries in states of default.

To assess the consequences of these adjustments, we consider the case of identical group sizes $N_i = N$, identical CARA preferences and wealth levels throughout the population, identical loss probabilities $\pi_i = \pi$, but differing loss levels. More specifically, in Table 3, we present model parametrizations for a setup with $m = 3$ groups, whose members face loss levels $l_1 = 1$, $l_2 = 2$, and $l_3 = 3$, respectively. For the first five parametrizations, the only difference in the assumed parameters is the group size but remarkably the influence on premiums $\{p_i\}$ and the choice parameters $\{q_i\}$ is marginal, and may be completely attributable to numerical inaccuracies. Since the participation constraint binds, it appears that the solution features an adjustment of the asset level such that the utility level is the same in all cases. In particular, since the resulting asset level is concave in the group size due to obvious diversification effects, this implies that the companies' monopoly rents increase disproportionately in relation to the firm size.

Heterogenous Bernoulli Losses: Parametrizations

Nr.	α	N	π	τ	w	a	p_1	p_2	p_3	q_1	q_2	q_3
1	1.00	2	0.1	0.01	5	5.37	0.16	0.49	1.06	0.99	0.99	1.00
2	1.00	5	0.1	0.01	5	8.84	0.16	0.49	1.06	0.99	1.00	1.00
3	1.00	10	0.1	0.01	5	13.63	0.16	0.49	1.06	0.99	1.00	1.00
4	1.00	15	0.1	0.01	5	17.86	0.16	0.49	1.06	0.99	1.00	1.00
5	1.00	25	0.1	0.01	5	25.76	0.16	0.49	1.06	0.99	1.00	1.00
6	1.25	10	0.1	0.01	5	14.07	0.18	0.60	1.31	0.99	1.00	1.00

Table 3: Parametrizations of the heterogeneous Bernoulli model with CARA preferences and loss levels $(l_1, l_2, l_3) = (1, 2, 3)$.

Figure 4 shows the counterparty-based allocation of capital to the three groups for parametrizations 1 through 5 as well as the corresponding allocations based on the Expected Shortfall (ES). At first glance, the two graphs look roughly identical. More specifically, for small group sizes, both of them entail a disproportionate allocation of capital to the high loss group ($l = 3$), whereas for an increasing group size, the allocations become more and more linear. This clearly is a consequence of diversification: For small group sizes, and individual loss in the high loss group is closely linked to a company default, whereas for extremely large group sizes, individual losses in either group become essentially independent of company default. Taking into account that positive contributions for either allocation (28) or (29) are only accrued in states where 1) the individual in view incurs a loss and 2) the company defaults, the similarity of the graphs becomes understandable. In particular, this effect appears to dominate other influences stemming from, for instance, risk aversion.

To demonstrate that such effects are present nonetheless when adjusting the focus, Figure 5 illustrates the allocation of capital to the low ($l = 1$) and high ($l = 3$) loss group for parametrization

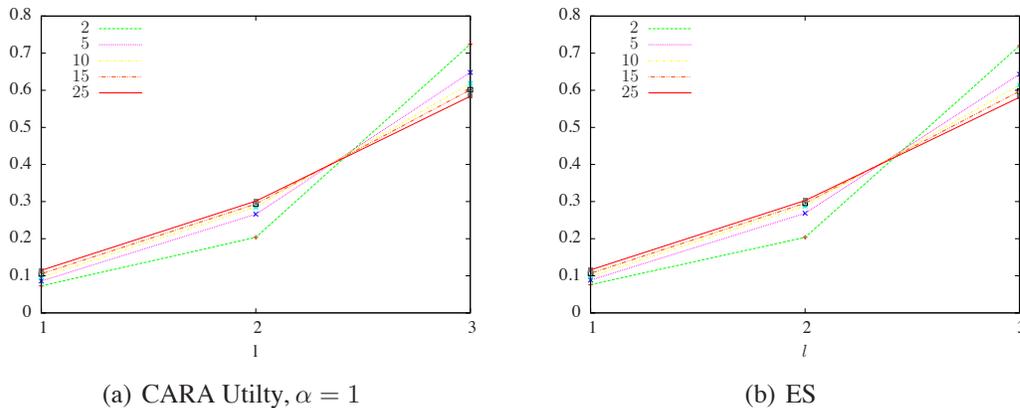


Figure 4: Allocation for varying company sizes N ; parametrizations 1-5.

6 and varying levels of absolute risk aversion. Obviously, the allocation based on the expected shortfall (ES) is independent of consumer preferences. For the counterparty-driven allocation, on the other hand, we find a decreasing and slightly concave dependence for the low loss group (panel (a)), and an increasing and slightly convex relationship for the high loss group (panel (b)). In particular, there exist risk aversion levels for which either of the allocations yield higher/lower charges for the high/low loss group, although in this case moderate levels of risk aversion cause the counterparty-driven allocation to penalize the high loss group more heavily.

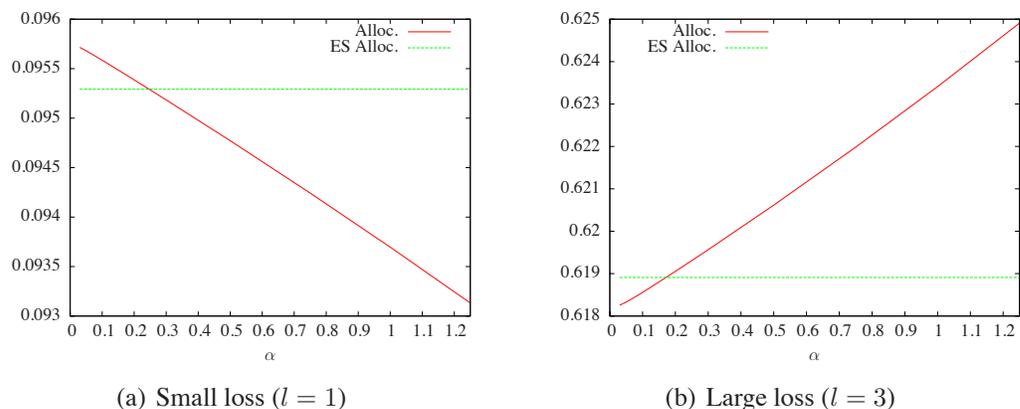


Figure 5: Allocation for varying absolute risk aversion parameter α ; parametrization 6.

5 Conclusion

The early literature on capital allocation recognized that risk measure selection was a thorny issue that could be resolved only through careful consideration of institutional context. The subsequent literature on capital allocation celebrated, refined, and justified (in mathematical terms) the technique of Euler allocation technique while ignoring the institutional context, with the consequence that the demon of arbitrary risk measure selection has never been exorcised. The demon, however, cannot and should not be ignored: Risk measure selection has a profound influence on an

organization's perception of the cost of risk.

Instead of starting with a risk measure, this paper starts with primitives and calculates the marginal cost of risk from the perspective of a profit-maximizing firm with risk averse counterparties. But, recognizing ease of implementation is paramount to various users of capital allocation methods—who will not generally have access to the blackboard parameters of a profit maximization problem—we take the additional step of identifying the risk measure whose gradient yields allocations consistent with marginal cost. To get economically correct allocations, one of course needs considerable information to choose the appropriate weighting function, yet the problem of weighting functions is generally true of spectral risk measures as well—so, from the standpoint of practice, this measure starts from an economically correct foundation for the context of a profit-maximizing firm and offers no greater level of complication than is already present.

Surprisingly, this risk measure is neither coherent nor convex; nevertheless, it is the only one that yields the appropriate allocation of capital for the profit-maximizing firm. We have shown that ES-based allocation could either be underweighting *or* overweighting severe states of default, depending on the nature of customer risk aversion, and this raises the interesting possibility that a transition away from a system of regulation based on risk measure-based solvency assessment to one based on market (counterparty) discipline will not necessarily mitigate the oft-lamented failure of financial institutions to penalize “tail” risk.

The reason for the difference in allocations derives from a fundamental difference in the source of marginal cost in an economic model based on counterparty risk aversion and one based on the imposition of ES or spectral ES. In the economic model, marginal cost derives from the impact of the expansion of a particular risk exposure on the *recoveries* of the counterparties to the firm, which is determined by the assets of the firm and the value that counterparties place on those assets in various states of default—while other risk measures such as ES tend to focus on loss outcomes themselves rather than actual recoveries. In general, we have shown that the marginal cost of risk to the firm depends on the risk preferences of its counterparties, even in cases where it faces a binding regulatory constraint—so much that the influence of the regulatory constraint on capital allocation could be dominated by the influence of counterparty risk preferences.

The calculations in this paper are done from the perspective of a profit-maximizing firm, but one could also contemplate the calculus of a regulator or social planner. In some cases, the calculus will be similar. For example, a regulator without responsibility for unpaid losses (i.e., if no deposit insurance scheme exists) but in a context where counterparties are uninformed will view risk in manner similar to the profit-maximizing firm. However, a regulator responsible for unpaid losses would have to consider the extent of that responsibility in selecting a risk measure, as well as other issues—such as bankruptcy costs not internalized by private firms and the production cost technology associated with deposit insurance—that would determine the optimal level of capitalization for financial institutions as well as the social cost of risk. These issues are of course complex and well beyond the scope of this paper, but they are intriguing areas for future research.

Appendix

A On the Differentiability of the Objective Function

To keep the presentation concise, we limit our attention to the objective function only and solely consider the case of $N = 2$ consumers under a quota share agreement; similar considerations apply for the constraints and the case of $N > 2$ consumers. Denote the corresponding losses by L_1 and L_2 with corresponding (joint) probability density function $f(x, y)$. Then the objective function (6) can be written as

$$\begin{aligned} & \sum_i p_i - \sum_i \mathbb{E} \left[\min \left\{ q_i L_i, \frac{a}{q_1 L_1 + q_2 L_2} q_i L_i \right\} \right] - \tau a \\ &= \sum_i p_i - \sum_i \int_0^\infty \int_0^\infty \min \left\{ q_i x_i, \frac{a}{q_1 x_1 + q_2 x_2} q_i x_i \right\} f(x_1, x_2) dx_1 dx_2 - \tau a. \end{aligned}$$

Obviously, this function is (continuously) differentiable with respect to p_i . For the differentiability with respect to a and q_i , problems may only arise for the summands of the second term. Without loss of generality, we focus on the first summand. Here,

$$\begin{aligned} & \int_0^\infty \int_0^\infty \min \left\{ q_1 x_1, \frac{a}{q_1 x_1 + q_2 x_2} q_1 x_1 \right\} f(x_1, x_2) dx_1 dx_2 \\ &= \int_0^\infty \int_0^{\frac{a - q_2 x_2}{q_1}} q_1 x_1 f(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_{\frac{a - q_2 x_2}{q_1}}^\infty \frac{a}{q_1 x_1 + q_2 x_2} q_1 x_1 f(x_1, x_2) dx_1 dx_2. \end{aligned}$$

By the Leibniz rule, after some simple calculus the derivatives with respect to a , q_1 , and q_2 are thus given by

$$\begin{aligned} \frac{\partial}{\partial a} &= \int_0^\infty \int_0^{\frac{a - q_2 x_2}{q_1}} \frac{q_1 x_1}{q_1 x_2 + q_2 x_2} f(x_1, x_2) dx_1 dx_2 \\ \frac{\partial}{\partial q_1} &= \int_0^\infty \int_0^{\frac{a - q_2 x_2}{q_1}} \frac{a q_2 x_1 x_2}{(q_1 x_2 + q_2 x_2)^2} f(x_1, x_2) dx_1 dx_2, \text{ and} \\ \frac{\partial}{\partial q_2} &= - \int_0^\infty \int_0^{\frac{a - q_2 x_2}{q_1}} \frac{a q_1 x_1 x_2}{(q_1 x_2 + q_2 x_2)^2} f(x_1, x_2) dx_1 dx_2, \end{aligned}$$

respectively, which are obviously continuous for $a, q_1, q_2 > 0$.

B Implementation of the Solution to (6)/(7)/(8) via a Premium Schedule

In the text we consider the solution of maximizing (6) subject to (7) and (8). We claim further that—if the consumer acts as a “price taker” with respect to the recovery rates offered by the

company within the various states of default—that the company can implement the optimum by offering a smooth and monotonically increasing premium schedule that allows each consumer to freely choose the level of coverage desired for the premium indicated by the schedule. It is subsequently shown that the marginal price increase associated with coverage must satisfy (13) when evaluated at the optimum. It remains to be shown that this premium schedule exists and can be used to implement the optimum.

A complication arises in modeling the consumer as a price-taker with free choice of coverage level. To introduce the consumer’s ignorance of his own influence on recoveries, we define price schedule described above as $p_i^*(\cdot)$ and modify the original utility function to

$$\tilde{v}_i(\mathbf{w}_i - p_i^*(q_i), q_i; \tilde{a}, \tilde{q}_1, \dots, \tilde{q}_N) = \mathbb{E} \left[U_i \left(\mathbf{w}_i - p_i^*(q_i) - L_i + \tilde{R}_i \right) \right], \quad (30)$$

where

$$\tilde{R}_i = \tilde{R}_i(q_i; \tilde{a}, \tilde{q}_1, \dots, \tilde{q}_N) = \min \left\{ I_i(L_i, q_i), \frac{\tilde{a}}{\sum_{j=1}^N I_j(L_j, \tilde{q}_j)} I_i(L_i, q_i) \right\}.$$

The idea here is to fix recovery rates by fixing the quantities \tilde{a} and $\{\tilde{q}_i\}$, leaving the consumer with the free choice of q_i —but with the caveat that this choice does not influence recovery rates.

The firm’s objective function is identical to the previous one, except that 1) the firm now specifies a price function rather than a single price point, and 2) the firm fixes the recovery rates for purposes of consumer incentive compatibility by choosing \tilde{a} and $\{\tilde{q}_i\}$ instead of the “true” levels of a and $\{q_i\}$:

$$\max_{\tilde{a}, \{p_i^*(\cdot)\}, \{\tilde{q}_i\}} \left\{ \sum p_i^*(\tilde{q}_i) - \sum e_i - \tau \tilde{a} \right\} \quad (31)$$

The firm still faces the previous constraints (7) and (8),

$$\begin{aligned} v_i(\tilde{a}, \mathbf{w}_i - p_i^*(\tilde{q}_i), \tilde{q}_1, \dots, \tilde{q}_N) &\geq \gamma_i, \\ s(\tilde{q}_1, \dots, \tilde{q}_N) &\leq a, \end{aligned}$$

and in addition the new constraint:

$$\tilde{q}_i \in \arg \max_{q_i} \tilde{v}_i(\mathbf{w}_i - p_i^*(q_i), q_i; \tilde{a}, \tilde{q}_1, \dots, \tilde{q}_N), \quad \forall i. \quad (32)$$

Equation (32) is an incentive compatibility constraint requiring the choice of coverage level to be consistent with the consumer optimizing, given her perception of own utility (which ignores her own impact on recovery rates) and the selected pricing function.

It is evident that the firm’s profits under this maximization can be no better than those achieved under the original program (maximizing (6) subject to (7) and (8)), since we have simply added another constraint and choosing the premium schedule at different points than \tilde{q}_i is immaterial to the company’s profits. It is therefore clear that, given optimal choices \hat{a} , $\{\hat{q}_i\}$, and $\{\hat{p}_i\}$ to the original program, the firm would maximize profits under the new setup if it could choose those same asset and coverage levels *and* find a pricing function $p_i^*(\cdot)$ that both satisfies $p_i^*(\hat{q}_i) = \hat{p}_i$ and induces consumers to choose the original solution:

$$\hat{q}_i \in \arg \max_{q_i} \tilde{v}_i(\mathbf{w}_i - p_i^*(q_i), q_i; \hat{a}, \hat{q}_1, \dots, \hat{q}_N), \quad \forall i.$$

The following lemma shows that this function exists.

Lemma B.1. *Suppose \hat{a} , $\{\hat{q}_i\}$, and $\{\hat{p}_i\}$ are the optimal choices maximizing (6) subject to (7) and (8). Then, for each i , there exists a smooth, monotonically increasing function $p_i^*(\cdot)$ satisfying:*

1. $p_i^*(\hat{q}_i) = \hat{p}_i$.
2. $\hat{q}_i \in \arg \max_{q_i} \tilde{v}_i(\mathbf{w}_i - p_i^*(q_i), q_i; \hat{a}, \hat{q}_1, \dots, \hat{q}_N)$.

Proof. Start by noting that it is evident that the constraints (7) all bind. Note further that the function of x

$$g(x) = \tilde{v}_i(\mathbf{w}_i - x, 0; \hat{a}, \hat{q}_1, \dots, \hat{q}_N)$$

is monotonically decreasing and, hence, invertible, so that we may uniquely define:

$$p_i^*(0) = g^{-1}(\gamma_i), \quad (33)$$

which obviously satisfies

$$\tilde{v}_i(\mathbf{w}_i - p_i^*(0), 0; \hat{a}, \hat{q}_1, \dots, \hat{q}_N) = \gamma_i.$$

Furthermore, let $p_i^*(\cdot)$ be a solution to the differential equation (initial value problem)¹³

$$\frac{\partial p_i^*(x)}{\partial x} = \frac{\frac{\partial}{\partial q_i} \tilde{v}_i(\mathbf{w}_i - p_i^*(x), x; \hat{a}, \hat{q}_1, \dots, \hat{q}_N)}{\frac{\partial}{\partial \mathbf{w}} \tilde{v}_i(\mathbf{w}_i - p_i^*(x), x; \hat{a}, \hat{q}_1, \dots, \hat{q}_N)}, \quad p_i^*(0) = g^{-1}(\gamma_i), \quad (34)$$

on the compact choice set for q_i . Due to Peano's Theorem, we are guaranteed existence of such a function and that it is smooth. Moreover, since $\frac{\partial \tilde{v}_i}{\partial \mathbf{w}}, \frac{\partial \tilde{v}_i}{\partial q_i} > 0$, we know that the function is monotonically increasing.

Moving on, by construction we know that:

$$\begin{aligned} & \tilde{v}_i(\mathbf{w}_i - p_i^*(q_i), q_i; \hat{a}, \hat{q}_1, \dots, \hat{q}_N) \\ = & \gamma_i + \int_0^{q_i} \left[\frac{\partial}{\partial q_i} \tilde{v}_i(\mathbf{w}_i - p_i^*(x), x; \hat{a}, \hat{q}_1, \dots, \hat{q}_N) - \frac{\partial}{\partial \mathbf{w}} \tilde{v}_i(\mathbf{w}_i - p_i^*(x), x; \hat{a}, \hat{q}_1, \dots, \hat{q}_N) \times \frac{\partial p_i^*(x)}{\partial x} \right] dx \\ = & \gamma_i + 0, \quad q_i > 0. \end{aligned} \quad (35)$$

In particular,

$$\tilde{v}_i(\mathbf{w}_i - p_i^*(\hat{q}_i), \hat{q}_i; \hat{a}, \hat{q}_1, \dots, \hat{q}_N) = \gamma_i,$$

which, since it is evident that the constraints (7) all bind in the original optimization, can be true if and only if

$$p_i^*(\hat{q}_i) = \hat{p}_i,$$

proving the first part of the lemma. Moreover, (35) directly implies that

$$\hat{q}_i \in \arg \max_{q_i} \tilde{v}_i(\mathbf{w}_i - p_i^*(q_i), q_i; \hat{a}, \hat{q}_1, \dots, \hat{q}_N).$$

proving the second part. □

¹³Here, $\frac{\partial \tilde{v}_i}{\partial \mathbf{w}}$ and $\frac{\partial \tilde{v}_i}{\partial q_i}$ denote the derivatives with respect to the first and the second argument of \tilde{v}_i , respectively.

C Identities in Section 4.1

Derivation of Equation (25)

For consumer N , $L_N \sim \text{Exp}(\nu)$ and the loss incurred by “the other” consumers is $L_{-N} = \sum_{i=1}^{N-1} L_i \sim \text{Gamma}(N-1, \nu)$. Then

$$e = e_N = \underbrace{\mathbb{E} \left[q L_N \mathbf{1}_{\{q(L_{-N} + L_N) < a\}} \right]}_{\text{part } i.} + a \underbrace{\mathbb{E} \left[\frac{q L_N}{q(L_{-N} + L_N)} \mathbf{1}_{\{q(L_{-N} + L_N) \geq a\}} \right]}_{\text{part } ii.}.$$

For part *ii.*, note that $\frac{L_N}{L_{-N} + L_N}$ is Beta(1, $N-1$) distributed independent of $L_{-N} + L_N \sim \text{Gamma}(N, \nu)$. Hence, part *ii.* can be written as

$$a \mathbb{P} \left(L_{-N} + L_N \geq \frac{a}{q} \right) \mathbb{E} \left[\frac{L_N}{L_{-N} + L_N} \right] = a \bar{\Gamma}_{N, \nu} \left(\frac{a}{q} \right) N^{-1}.$$

For part *i.*, we have

$$\begin{aligned} & q \mathbb{E} \left[L_N \mathbf{1}_{\{q(L_{-N} + L_N) < a\}} \right] \\ = & q \int_0^\infty \int_0^\infty \mathbf{1}_{\{i+l < a/q\}} l \nu e^{-\nu l} \frac{\nu^{N-1}}{(N-2)!} i^{N-2} e^{-\nu i} dl di \\ = & q \frac{\nu^N}{(N-2)!} \int_0^{a/q} \int_0^{a/q-i} l e^{-\nu l} dl i^{N-2} e^{-\nu i} di \\ = & q \frac{\nu^N}{(N-2)!} \int_0^{a/q} \left[\frac{1}{\nu^2} - \frac{1}{\nu} \left(\frac{a}{q} + \frac{1}{\nu} \right) e^{-\nu a/q} e^{\nu i} + i \frac{1}{\nu} e^{-\nu a/q} e^{\nu i} \right] i^{N-2} e^{-\nu i} di \\ = & q \frac{\nu^{N-2}}{(N-2)!} \int_0^{a/q} i^{N-2} e^{-\nu i} di - q \frac{\nu^{N-1}}{(N-2)!} e^{-\nu a/q} \left[\left(\frac{a}{q} + \frac{1}{\nu} \right) \int_0^{a/q} i^{N-2} di - \int_0^{a/q} i^{N-1} di \right] \\ = & \frac{q}{\nu} \Gamma_{N-1, \nu}(a/q) - q \frac{\nu^{N-1}}{(N-1)!} e^{-\nu a/q} \left(\frac{a}{q} \right)^{N-1} \left[\frac{1}{\nu} + \frac{1}{N} \frac{a}{q} \right]. \end{aligned}$$

Therefore, since all consumers are identical, the objective function (6) takes the form displayed in (25). For condition (7), on the other hand, we have

$$\begin{aligned} V &= V_N = \mathbb{E} [U(w - p - L_N + R_N)] \\ &= \underbrace{\mathbb{E} \left[U(w - p - (1-q)L_N) \mathbf{1}_{\{q(L_{-N} + L_N) < a\}} \right]}_{\text{part } i.} \\ &\quad + \underbrace{\mathbb{E} \left[U \left(w - p - L_N + a \frac{L_N}{L_{-N} + L_N} \right) \mathbf{1}_{\{q(L_{-N} + L_N) \geq a\}} \right]}_{\text{part } ii.} \end{aligned}$$

For part *i*, we obtain

$$\begin{aligned}
& \mathbb{E} \left[U \left(w - p - (1 - q) L_N \right) \mathbf{1}_{\{q(L_{-N} + L_N) < a\}} \right] \\
&= - \int_0^\infty \int_0^\infty \mathbf{1}_{\{i+l < a/q\}} e^{-\alpha(w-p-(1-q)l)} \nu e^{-\nu l} \frac{\nu^{N-1}}{(N-2)!} i^{N-2} e^{-\nu i} dl di \\
&= -e^{-\alpha(w-p)} \int_0^{a/q} \frac{1}{\nu - \alpha(1-q)} \left[1 - e^{-a/q(\nu-(1-q)\alpha)-i(1-q)\nu+iv} \right] \frac{\nu^{N-1}}{(N-2)!} i^{N-2} e^{-\nu i} di \\
&= e^{-\alpha(w-p)} \left[\frac{1}{\nu - \alpha(1-q)} \Gamma_{N-1, \nu}(a/q) - \frac{e^{-\frac{a}{q}(\nu-(1-q)\alpha)} \nu^{N-1}}{((1-q)\alpha)^{N-1}} \Gamma_{N-1, (1-q)\alpha} \left(\frac{a}{q} \right) \right].
\end{aligned}$$

For part *ii.*, note that

$$\begin{aligned}
& \mathbb{E} \left[U \left(w - p - ((L_{-N} + L_N) - a) \frac{L_N}{L_{-N} + L_N} \right) \mathbf{1}_{\{q(L_{-N} + L_N) \geq a\}} \right] \\
&= \int_0^1 \int_0^\infty e^{-\alpha(w-p-(l-a)y)} \mathbf{1}_{\{l \geq a/q\}} \frac{\nu^N}{(N-1)!} l^{N-1} e^{-\nu l} (N-1)(1-y)^{N-2} dl dy \\
&= -e^{-\alpha(w-p)} \int_{a/q}^\infty \frac{\nu^N}{(N-1)!} e^{-\nu l} l^{N-1} \underbrace{\int_0^1 e^{\alpha(l-a)y} (N-1)(1-y)^{N-2} dy}_{\text{mgf}_{\text{Beta}(1, N-1)}(\alpha(l-a))} dl \\
&= -e^{-\alpha(w-p)} \int_{a/q}^\infty \frac{\nu^N}{(N-1)!} e^{-\nu l} l^{N-1} \sum_{k=0}^\infty \frac{(\alpha(l-a))^k}{(N-1+k)!} dl \\
&= -e^{-\alpha(w-p)} \sum_{k=0}^\infty \sum_{j=0}^{N-1} \binom{N-1}{j} a^j e^{-\nu a} \frac{\alpha^k}{(N-1+k)!} \frac{(N+k-j-1)!}{\nu^{-j+k}} \\
&\quad \times \int_{a/q}^\infty \frac{\nu^{N+k-j}}{(N+k-j-1)!} e^{-\nu(l-a)} (l-a)^{N+k-j-1} dl \\
&= -e^{-\alpha(w-p)} \sum_{k=0}^\infty \sum_{j=0}^{N-1} \frac{(N-1)!}{(N-1+k)!} \frac{(a\nu)^j}{j!} e^{-\nu a} \left(\frac{\alpha}{\nu} \right)^k \frac{(N+k-j-1)!}{(N-1-j)!} \bar{\Gamma}_{N-j+k, \nu} \left(a \left(\frac{1-q}{q} \right) \right).
\end{aligned}$$

Derivation of Equation (26)

Similar to the previous part, for consumer N with $L = \sum_{i=1}^N L_i$:

$$\begin{aligned}
& \mathbb{E} \left[\sum_{j=1}^N U' \left(w - p - L_j + a \frac{L_j}{L} \right) \frac{L_j}{L} \frac{L_N}{L} \middle| L \right] \\
&= \sum_{j=1}^{N-1} \underbrace{\mathbb{E} \left[U' \left(w - p - (L-a) \frac{L_j}{L} \right) \frac{L_j}{L} \frac{L_N}{L} \middle| L \right]}_{\text{part } i.} + \underbrace{\mathbb{E} \left[U' \left(w - p - (L-a) \frac{L_N}{L} \right) \left(\frac{L_N}{L} \right)^2 \middle| L \right]}_{\text{part } ii.}.
\end{aligned}$$

Note that $\frac{L_j}{L}, \frac{L_N}{L} \sim \text{Beta}(1, N-1)$ and for the joint distribution

$$f_{\frac{L_j}{L}, \frac{L_N}{L}}(x, y) = (1-x-y)^{N-3} (N-2)(N-1) \mathbf{1}_{\{x, y \geq 0, x+y \leq 1\}}, \quad j \neq N.$$

Whence, for part *i.*,

$$\begin{aligned}
& \mathbb{E} \left[U' \left(w - p - (L - a) \frac{L_{N-1}}{L} \right) \frac{L_{N-1}}{L} \frac{L_N}{L} \middle| L \right] \\
&= \alpha e^{-\alpha(w-p)} \int_0^1 \int_0^{1-x} e^{\alpha(L-a)x} x y (N-1)(N-2)(1-x-y)^{N-3} dy dx \\
&= \alpha e^{-\alpha(w-p)} \int_0^1 e^{\alpha(L-a)x} x \underbrace{\int_0^{1-x} y (N-1)(N-2)(1-x-y)^{N-3} dy}_{=(1-x)^{N-1}} dx \\
&= \alpha e^{-\alpha(w-p)} \beta(2, N) \underbrace{\int_0^1 e^{\alpha(L-a)x} \frac{1}{\beta(2, N)} x (1-x)^{N-1} dx}_{=\text{mgf}_{\text{Beta}(2, N)}(\alpha(L-a))} \\
&= \alpha e^{-\alpha(w-p)} \frac{1}{N(N+1)} (N+1)! \sum_{k=0}^{\infty} \frac{(k+1) (\alpha(L-a))^k}{(N+k+1)!},
\end{aligned}$$

whereas for part *ii.*,

$$\begin{aligned}
& \mathbb{E} \left[U' \left(w - p - (L - a) \frac{L_N}{L} \right) \left(\frac{L_N}{L} \right)^2 \middle| L \right] \\
&= \alpha e^{-\alpha(w-p)} \mathbb{E} \left[\underbrace{\exp \left\{ \alpha(L-a) \frac{L_N}{L} \right\} \left(\frac{L_N}{L} \right)^2}_{=\frac{\partial^2}{\partial t^2} \text{mgf}_{\text{Beta}(1, N-1)}(t) \Big|_{t=\alpha(L-a)}} \middle| L \right] \\
&= \alpha e^{-\alpha(w-p)} (N-1)! \sum_{k=0}^{\infty} \frac{(k+1)(k+2) (\alpha(L-a))^k}{(N+k+1)!},
\end{aligned}$$

so that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{j=1}^N u' \left(w - p - L_j + a \frac{L_j}{L} \right) \frac{L_j}{L} \frac{L_N}{L} \middle| L \right] \\
&= \alpha e^{-\alpha(w-p)} (N-1)! \sum_{k=0}^{\infty} \frac{(N-1)(k+1) (\alpha(L-a))^k + (k+1)(k+2) (\alpha(L-a))^k}{(N+k+1)!} \\
&= \alpha e^{-\alpha(w-p)} (N-1)! \sum_{k=0}^{\infty} \frac{(k+1)(N+1+k) (\alpha(L-a))^k}{(N+k+1)!} \\
&= \alpha e^{-\alpha(w-p)} (N-1)! \sum_{k=0}^{\infty} \frac{(k+1) (\alpha(L-a))^k}{(N+k)!}.
\end{aligned}$$

For the denominator,

$$\begin{aligned}
& \mathbb{E} \left[\mathbf{1}_{\{qL \geq a\}} \sum_{j=1}^N U' \left(w - p - L_j - a \frac{L_j}{L} \right) \frac{L_j}{L} \right] \\
&= \mathbb{E} \left[\mathbf{1}_{\{qL \geq a\}} N \alpha e^{-\alpha(w-p)} (N-1)! \sum_{k=0}^{\infty} \frac{(k+1) (\alpha(L-a))^k}{(N+k)!} \right] \\
&= N! \alpha e^{-\alpha(w-p)} \sum_{k=0}^{\infty} \frac{k+1}{(N+k)!} \alpha^k \int_{a/q}^{\infty} \frac{\nu^N}{(N-1)!} (l-a)^k l^{N-1} e^{-\nu l} dl \\
&= N! \alpha e^{-\alpha(w-p)} \sum_{k=0}^{\infty} \left(\frac{\alpha}{\nu} \right)^k \frac{k+1}{(N+k)!} e^{-\nu a} \sum_{j=0}^{N-1} \binom{N-1}{j} (a\nu)^j \frac{(N+k-j-1)!}{(N-1)!} \\
&\quad \times \int_{a/q}^{\infty} \frac{\nu^{N+k-j}}{(N+k-j-1)!} e^{-\nu(l-a)} (l-a)^{N+k-j-1} dl \\
&= N! \alpha e^{-\alpha(w-p)} \sum_{k=0}^{\infty} \left(\frac{\alpha}{\nu} \right)^k \frac{k+1}{(N+k)!} e^{-\nu a} \sum_{j=0}^{N-1} \frac{(a\nu)^j (N+k-j-1)!}{j! (N-j-1)!} \bar{\Gamma}_{N+k-j, \nu} \left(a \left(\frac{1-q}{q} \right) \right).
\end{aligned}$$

Hence,

$$q \tilde{\phi}_i = \frac{1}{N} \mathbb{E} \left[\mathbf{1}_{\{qL \geq a\}} \frac{\sum_{k=0}^{\infty} \frac{(k+1) (\alpha(L-a))^k}{(N+k)!}}{\underbrace{\sum_{k=0}^{\infty} \left(\frac{\alpha}{\nu} \right)^k \frac{k+1}{(N+k)!} e^{-\nu a} \sum_{j=0}^{N-1} \frac{(a\nu)^j (N+k-j-1)!}{j! (N-j-1)!} \bar{\Gamma}_{N+k-j, \nu} \left(a \left(\frac{1-q}{q} \right) \right)}_{=\hat{\psi}(L)}} \right].$$

For implementation purposes, the numerator can be expressed as

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(k+1) t^k}{(N+k)!} \Big|_{t=\alpha(L-a)} &= \frac{\partial}{\partial t} \left[\sum_{k=0}^{\infty} \frac{t^{k+1}}{(N+k)!} \right] \Big|_{t=\alpha(L-a)} = \frac{\partial}{\partial t} \left[t^{-(N-1)} \sum_{k=N}^{\infty} \frac{t^k}{k!} \right] \Big|_{t=\alpha(L-a)} \\
&= \frac{\partial}{\partial t} \left[t^{-(N-1)} \left(e^t - \sum_{k=0}^{N-1} \frac{t^k}{k!} \right) \right] \Big|_{t=\alpha(L-a)} \\
&= -(N-1) t^{-N} \left(e^t - \sum_{k=0}^{N-1} \frac{t^k}{k!} \right) + t^{-(N-1)} \left(e^t - \sum_{k=0}^{N-1} \frac{k t^{k-1}}{k!} \right) \Big|_{t=\alpha(L-a)} \\
&= \left(e^t - \sum_{k=0}^{N-2} \frac{t^k}{k!} \right) \times (t^{-N+1} - t^{-N} (N-1)) + \frac{(N-1)}{(N-1)!} t^{-1} \Big|_{t=\alpha(L-a)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\tilde{\psi}(I) &= \tilde{\psi}(qL) = \mathbb{E} \left[\frac{\partial \tilde{\mathbb{P}}}{\mathbb{P}} \Big| L \right] \\
&= \text{const} \mathbb{E} \left[\sum_{j=1}^N \alpha \exp \left\{ -\alpha \left(w - p - L_j + a \frac{L_j}{L} \right) \right\} \frac{L_j}{L} \Big| L \right] \\
&= \text{const} N \mathbb{E} \left[e^{\alpha(L-a)L_j/L} \frac{L_j}{L} \Big| L \right] \\
&= \text{const} \frac{\partial}{\partial x} \text{mgf}_{\text{Beta}(1, N-1)}(x) \Big|_{x=\alpha(L-a)} = \hat{\psi}(L),
\end{aligned}$$

since $\mathbb{E} \left[\tilde{\psi}(I) \right] = 1$.

D Allocation in a Security Market Equilibrium

To keep the setup as simple as possible, we limit our considerations to a one-period market with a finite number of securities (M), each security with potentially distinct payoffs in X states and assume that the risk-free rate is zero. More specifically, let $\Omega^{(S)} = \{\omega_1^{(S)}, \dots, \omega_X^{(S)}\}$ be the set of these states with associated sigma-algebra $\mathcal{F}^{(S)}$ given by its power set and let $p_j^{(S)} = \mathbb{P}(\{\omega_j^{(S)}\})$ denote the associated physical probabilities. Let then D be the $M \times X$ matrix with D_{ij} describing the payoff of the i^{th} security in state $\omega_j^{(S)}$, where we assume

$$\text{span}(D) = \mathbb{R}^X.$$

This condition allows us to define state prices, consistent with the absence of arbitrage within the securities market, denoted by π_j , $j = 1, \dots, X$. Thus, any arbitrary menu of securities-market-sub-state-contingent consumption can be purchased at time zero. However, it would be misleading to characterize markets as complete, since $\Omega^{(S)}$ does not provide a complete description of the “states of the world”. Instead, we characterize the full probability space as $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, with

$$\begin{aligned}
\bar{\Omega} &= \Omega^{(S)} \times \Omega = \{ \bar{\omega} = (\omega^{(S)}, \omega) \mid \omega^{(S)} \in \Omega^{(S)}, \omega \in \Omega \}, \\
\bar{\mathcal{F}} &= \mathcal{F}^{(S)} \vee \mathcal{F}, \text{ and} \\
\bar{\mathbb{P}}(\bar{A}) &= \sum_{j \in \Upsilon_A} p_j^{(S)} \times \mathbb{P}(A_j \mid \{\omega_j^{(S)}\})
\end{aligned}$$

for $\bar{A} = \bigcup_{j \in \Upsilon_A} \{\omega_j^{(S)}\} \times A_j \in \bar{F}$ with $A_j \in \mathcal{F}$, $j = 1, 2, \dots, |\Upsilon_A|$.

Our problem now, however, is that the market is no longer complete so that we need a notion of what insurance liabilities are “worth” when they cannot be hedged completely. We make the assumption that the insurance market is “small” relative to the securities market and, for purposes of valuing insurance liabilities, define the probability measure

$$\mathbb{Q}(\bar{A}) = \sum_{j \in \Upsilon_A} \pi_j \times \mathbb{P}(A_j \mid \{\omega_j^{(S)}\}), \quad \bar{A} \subseteq \bar{\Omega},$$

i.e. it given by the Radon-Nikodym derivative $\frac{\partial \mathbb{Q}}{\partial \mathbb{P}}((\omega_j^{(S)}, \omega)) = \frac{\pi_j}{p_j^{(S)}}$.

Consumer utility now depends on the individual's chosen security market allocation and may be expressed as

$$v_i = \mathbb{E}^{\mathbb{P}} [U_i (W_i - p_i - L_i + R_i)] \text{ with } v'_i = \mathbb{E}^{\mathbb{P}} [U'_i (W_i - p_i - L_i + R_i)],$$

where W_i is $\mathcal{F}^{(S)}$ -measurable with $w_{ij} = W_i(\omega_j^{(S)})$ and $\sum_j \pi_j w_{ij} = w_i$ whereas L_i —as before—is \mathcal{F} -measurable. The recovery R_i , on the other hand, now depends both on insurance loss activity as well as portfolio decisions made within the insurance company. To elaborate on this, the budget constraint of the insurance company may be expressed as

$$a = \sum_j \pi_j K_j a \Rightarrow 1 = \sum_j \pi_j K_j,$$

where $K_j a$ reflects consumption purchased in the securities market state $\omega_j^{(S)}$ or—more precisely—in the states of the world $\bar{\Omega}_j = \{ \bar{\omega} = (\omega^{(S)}, \omega) \mid \omega^{(S)} = \omega_j^{(S)} \}$. We write K to denote the corresponding $\mathcal{F}^{(S)}$ -measurable random variable. Consumer i 's recovery can then be expressed as

$$R_i = \min \left\{ I_i, \frac{K a}{I} I_i \right\}$$

and the fair valuation of claims is thus

$$e_i = \mathbb{E}^{\mathbb{Q}} [R_i] = \underbrace{\mathbb{E}^{\mathbb{Q}} [R_i \mathbf{1}_{\{I < K a\}}]}_{=e_i^Z} + \underbrace{\mathbb{E}^{\mathbb{Q}} [R_i \mathbf{1}_{\{I \geq K a\}}]}_{=e_i^D}.$$

Hence, the firm's problem becomes

$$\max_{a, \{q_i\}, \{p_i\}, \{K_j\}, \{w_{ij}\}} \sum p_i - \sum e_i - \tau a,$$

subject to

$$\begin{aligned} v_i &\geq \gamma_i, \\ s(q_1, \dots, q_N) &\leq a, \\ \sum_j \pi_j K_j &= 1, \\ \sum_j \pi_j w_{ij} &= w_i. \end{aligned}$$

In addition to a new set of optimality conditions connected with $\{K_j\}$ and $\{w_{ij}\}$, we have the same set of first order conditions (as before, we sacrifice technical rigor by assuming a solution in a smooth part of the function):

$$\begin{aligned} [q_i] \quad & - \sum_k \frac{\partial e_k}{\partial q_i} + \sum_k \lambda_k \frac{\partial v_k}{\partial q_i} - \frac{\partial s}{\partial q_i} \xi = 0, \\ [a] \quad & - \sum_k \frac{\partial e_k}{\partial a} - \tau + \sum_k \lambda_k \frac{\partial v_k}{\partial a} + \xi = 0, \\ [p_i] \quad & 1 - \lambda_i \frac{\partial v_i}{\partial w} = 0. \end{aligned}$$

The first order condition for $\{w_{ij}\}$ is:

$$\begin{aligned} [w_{ij}] \quad \lambda_i \frac{\partial v_i}{\partial w_{ij}} - \eta_i \pi_j &= 0 \\ \Leftrightarrow \lambda_i p_j^{(S)} \mathbb{E}^{\mathbb{P}} \left[U'_i(W_i - p_i - L_i + R_i) | \omega_j^{(S)} \right] - \eta_i \pi_j &= 0, \end{aligned} \quad (36)$$

where $\{\eta_i\}$ are the Lagrange multipliers for the individual wealth constraints. Since

$$\begin{aligned} 0 &= \sum_j \left(\lambda_i p_j^{(S)} \mathbb{E}^{\mathbb{P}} \left[U'_i(W_i - p_i - L_i + R_i) | \omega_j^{(S)} \right] - \eta_i \pi_j \right) \\ &= \lambda_i \frac{\partial v_i}{\partial w} - \eta_i, \end{aligned}$$

with $[p_i]$ we obtain $\eta_i \equiv 1$.

As before, we seek a pricing function satisfying:

$$\left[\frac{\partial v_i}{\partial q_i} + \mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_{\{I \geq K a\}} U'_i \frac{K a}{I^2} I_i \frac{\partial I_i}{\partial q_i} \right] \right] - \frac{\partial v_i}{\partial w} \frac{\partial p_i^*}{\partial q_i} = 0.$$

Proceeding analogously to Section 2, we arrive at the marginal pricing condition associated with a decentralized implementation:

$$\frac{\partial p_i^*}{\partial q_i} = \sum_k \frac{\partial e_k}{\partial q_i} + \frac{\partial s}{\partial q_i} \xi - \sum_{k \neq i} \frac{\frac{\partial v_k}{\partial q_i}}{v'_k} + \frac{\mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_{\{I \geq K a\}} U'_i \frac{K a}{I^2} I_i \frac{\partial I_i}{\partial q_i} \right]}{v'_i}.$$

Simplifying, we obtain

$$\begin{aligned} \frac{\partial p_i^*}{\partial q_i} &= \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} \left[\sum_k \frac{\partial e_k}{\partial a} + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] + \mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_{\{I \geq K a\}} \sum_k \frac{U'_k}{v'_k} \frac{K a}{I^2} I_k \frac{\partial I_i}{\partial q_i} \right] \\ &= \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} \left[\mathbb{E}^{\mathbb{Q}} [K \mathbf{1}_{\{I \geq K a\}}] + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] + \frac{\mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_{\{I \geq K a\}} \sum_k \frac{U'_k}{v'_k} \frac{K}{I^2} I_k \frac{\partial I_i}{\partial q_i} \right]}{\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k}} \times \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \times a \\ &= \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} \left[\mathbb{E}^{\mathbb{Q}} [K \mathbf{1}_{\{I \geq K a\}}] + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right] + \tilde{\phi}_i \times a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v'_k} \right], \end{aligned}$$

where

$$\tilde{\phi}_i = \frac{\mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_{\{I \geq K a\}} \sum_k \frac{U'_k}{v'_k} \frac{K}{I^2} I_k \frac{\partial I_i}{\partial q_i} \right]}{\mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_{\{I \geq K a\}} \sum_k \frac{U'_k}{v'_k} \frac{I_k}{I} \right]}.$$

Hence, we essentially obtain the same result as before—with the only differences worth noting being that 1) the marginal cost of capital, $\mathbb{E}^{\mathbb{Q}} [K \mathbf{1}_{\{I \geq K a\}}] + \tau$, as well as the component of marginal cost deriving from claimant recoveries in solvent company states, $\frac{\partial e_i^Z}{\partial q_i}$, now reflect state prices derived from the security markets, and 2) the “weights” used in the internal capital allocation formula now include a factor K , reflecting the fact that available company resources may vary

across default states due to company asset allocation. Importantly, note that the capital allocation weights are still determined by customer marginal utility, rather than state prices—although the latter will be connected to the former due to customer asset allocation choices.

For instance, in the limiting case of a complete market, L_i and R_i are $\mathcal{F}^{(S)}$ -measurable so that we can write $l_{ij} = L_i(\omega_j^{(S)})$ and $r_{ij} = R_i(\omega_j^{(S)})$, $l_{ij}, r_{ij} \in \mathbb{R}$, and (36) becomes

$$\lambda_i p_j^{(S)} U_i'(w_{ij} - p_i - l_{ij} + r_{ij}) = \pi_j.$$

Thus, with $[p_i]$:

$$\begin{aligned} \tilde{\phi}_i &= \frac{\mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{I \geq K a\}} \sum_k \frac{K}{I^2} I_k \frac{\partial L_i}{\partial q_i} \right]}{\mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{I \geq K a\}} \right]} \\ &= \mathbb{E}^{\mathbb{Q}} \left[K \frac{\partial L_i}{\partial q_i} \middle| I \geq K a \right], \end{aligned}$$

i.e. $q_i \times \tilde{\phi}_i \times \mathbb{Q}(I \geq K a) \times a$ is the fair price of the recovery.

References

- Acerbi, C. (2002). “Spectral Measures of Risk: A Coherent Representation of Subjective Risk Aversion.” *Journal of Banking and Finance* 26, 1505-1518.
- Adrian, T., and Shin, H. (2008). “Financial Intermediary Leverage and Value-at-Risk.” *Federal Reserve Bank of New York Staff Report* 338.
- Basak, S., and Shapiro, A. (2001). “Value-at-risk-based Risk Management: Optimal Policies and Asset Prices.” *Review of Financial Studies* 14, 371-405.
- Bauer, D., Bergmann, D., and Reuss, A. (2010). “On the Calculation of the Solvency Capital Requirement based on Nested Simulations.” *Working paper, Georgia State University*.
- Denault, M. (2001). “Coherent Allocation of Risk Capital.” *Journal of Risk* 4, 7-21.
- Detlefsen, K., and Scandolo, G. (2005). “Conditional and dynamic convex risk measures.” *Finance and Stochastics* 9, 539-561.
- Dhaene, J., Laeven, R.J.A., Vanduffel, S., Darkiewicz, G., and Goovaerts, M. J. (2008). “Can a Coherent Risk Measure be too Subadditive?” *Journal of Risk and Insurance* 75, 365-386.
- Dhaene, J., Tsanakas, A., Valdez, E., and Vanduffel, S. (2009). “Optimal Capital Allocation Principles.” *Journal of Risk and Insurance*, forthcoming.
- Dowd, K., Cotter, J., Sorwar, G. (2008). “Spectral Risk Measures: Properties and Limitations.” *Journal of Financial Services Research* 34, 61-75.
- Föllmer, H., and Schied, A. (2002). “Convex measures of risk and trading constraints.” *Finance and Stochastics* 6, 429-447.

- Föllmer, H., and Schied, A. (2010). "Coherent and Convex Risk Measures." In *Encyclopedia of Quantitative Finance*, Cont, R. (Ed.), Wiley, 355-363.
- Frittelli, M., Gianin, E.R. (2002). "Putting Order in Risk Measures." *Journal of Banking and Finance* 26, 1473-1486.
- Froot, K.A., and Stein, J. (1998). "Risk Management, Capital Budgeting, and Capital Structure Policy for Financial Institutions: An Integrated Approach." *Journal of Financial Economics* 47, 55-82.
- Gordy, M.B., and Juneja, S. (2010). "Nested Simulation in Portfolio Risk Measurement." *Management Science* 56, 1833-1848.
- Heyde, C.C., Kou, S.G., Peng, X.H. (2006). "What Is a Good External Risk Measure: Bridging the Gaps between Robustness, Subadditivity, and Insurance Risk Measures." *Working paper, Columbia University*.
- Hull, J. (2007) "VAR versus Expected Shortfall." *Risk Magazine*, March 2007.
- International Actuarial Association (2004). "A Global Framework for Insurer Solvency Assessment." IAA Insurer Solvency Assessment Working Party Research Report. Available at http://www.actuaries.org/LIBRARY/Papers/Global_Framework_Insurer_Solvency_Assessment-public.pdf.
- Jaschke, S.R. (2001). "Quantile-VaR is the Wrong Measure to Quantify Market Risk for Regulatory Purposes." *EFA 2002 Berlin Meeting Discussion Paper*.
- Kalkbrener, M. (2005). "An Axiomatic Approach to Capital Allocation." *Mathematical Finance* 15, 425-37.
- Keeler, T.E., and Small, K.A. (1977). "Optimal Peak Load Pricing, Investment, and Service Levels on Urban Expressways." *Journal of Political Economy* 85, 1-25.
- Kimball, M.S. (1990). "Precautionary Savings in the Small and in the Large." *Econometrica* 58, 53-73.
- McKinsey&Company (2011). "The use of economic capital in performance management for banks: A perspective." Prepared by T. Baer, A. Mehta, and H. Samandari. *McKinsey Working Papers on Risk*, Number 24. Available at http://www.mckinsey.com/client-service/risk/pdf/risk_working_papers/24_The_Use_of_Economic_Capital.pdf.
- McNeil, A.J., Frey, R., and Embrechts, P. (2005). "Quantitative Risk Management: Concepts, Techniques, and Tools." Princeton: Princeton University Press.
- Merton, R.C., and Perold, A.F. (1993). "Theory of Risk Capital in Financial Firms." *Journal of Applied Corporate Finance* 6, 16-32.
- Mildenhall, S.J. (2004). "A Note on Myers and Read Capital Allocation Formula." *North American Actuarial Journal* 8, 32-44.

- Mildenhall, S.J. (2006). "Actuarial Geometry." Paper presented at the 2006 Risk Theory Seminar.
- Myers, S.C., and Read, J.A. (2001). "Capital Allocation for Insurance Companies." *Journal of Risk and Insurance* 68, 545-80.
- National Association of Insurance Commissioners (NAIC) (2009). "Consultation Paper on Regulatory Capital Requirements and Overarching Accounting/Valuation Issues for the Solvency Modernization Initiative." Available at http://www.naic.org/documents/committees_ex_isftf_iswg_capital_requirements.doc.
- Nocera, J. (2009). "Risk Mismanagement." *The New York Times Magazine*, January 2009.
- Perold, A.F. (2005). "Capital Allocation in Financial Firms." *Journal of Applied Corporate Finance* 17, 110-18.
- Powers, M.R. (2007). "Using Aumann-Shapley Values to Allocate Insurance Risk: The Case of Inhomogeneous Losses." *North American Actuarial Journal* 11, 113-27.
- Schmock, U., and Straumann, D. (1999). "Allocation of Risk Capital and Performance Measurement". Presentation at the Risk Day 1999, Zürich. Available at <http://www.fam.tuwien.ac.at/~schmock/slides/AllocationSlidesOct1999.pdf>.
- Society of Actuaries (2008). "Economic Capital for Life Insurance Companies." Research report prepared by I. Farr, H. Mueller, M. Scanlon, and S. Stronkhorst. Available at <http://www.soa.org/files/pdf/research-ec-report.pdf>.
- Sriboonchitta, S., Nguyen, H.T., Kreinovich, V. (2010). "How to Relate Spectral Risk Measures and Utilities." *International Journal of Intelligent Technologies and Applied Statistics* 3, 141-158.
- Stoughton, N.M., and Zechner, J. (2007). "Optimal Capital Allocation Using RAROC and EVA." *Journal of Financial Intermediation* 16, 312-342.
- Tasche, D. (2000). "Risk Contributions and Performance Measurement." *Working paper*.
- Tasche, D. (2004). "Allocating Portfolio Economic Capital to Sub-portfolios." In: *Economic Capital: A Practitioner Guide*, A. Dev (ed.). Risk Books, 2004, 275-302.
- Wächter, H.P., Mazzoni, T. (2010). "Consistent Modeling of Risk Averse Behavior with Spectral Risk Measures." *Working paper, Fernuniversität Hagen*.
- Zanjani, G. (2002). "Pricing and Capital Allocation in Catastrophe Insurance." *Journal of Financial Economics* 65, 283-305.
- Zanjani, G. (2010). "An Economic Approach to Capital Allocation." *Journal of Risk and Insurance* 77, 523-549.