1 Introduction

Social networks are an important feature of labor markets.\textsuperscript{1} Approximately half of all American workers report learning about their job through their social network (friends, acquaintances, relatives etc).\textsuperscript{2} On the other side of the labor market, half of all employers report using the social networks of their current employees when hiring.\textsuperscript{3} Finally, referrals are ubiquitous in labor markets in that they occur in large numbers in all occupations, industries and educational levels. However, social networks are usually not included in models of the labor market.

\textsuperscript{1}See Ioannides and Loury (2006) and Topa (2010) for useful recent surveys of the topic.

\textsuperscript{2}The exact proportion differs across data sets but it is substantial across the board. For instance, Corcoran et al (1980) report that, according to the PSID, 50\% of all workers find their job through their social network; the NLSY puts that proportion to 30\% while 24 studies surveyed by Bewley (1999) range from 30\% to 60\%.

\textsuperscript{3}Again, the exact proportions differ across studies but are everywhere substantial (e.g. 53\% according to EOPP as reported by Holzer (1987) and 60\% according to the interviews in Bewley (1999)).
In this paper an endogenous social network is introduced in a fairly standard model of the labor market. The model has two stages: in the first stage workers form their social network in a one-shot game and in the second stage they participate in an infinite-horizon labor market. Workers have either high or low productivity, are risk-neutral and maximize expected discounted utility. There is free entry of firms who can employ one worker, are risk-neutral and maximize expected discounted profits.

During network formation a worker announces the measure of links that he would like to have with workers of each type and these links are created if there are available counterparties. Attention is restricted to equilibria in symmetric strategies for the network formation stage. The labor market is subject to search and informational frictions. Search frictions are captured by the usual matching function, while informational frictions are introduced by assuming that all unemployed workers search in the same market regardless of their productivity level. Vacancy creation occurs in two different ways: a new firm enters and starts searching for workers in the market; an existing firm expands and asks its current employee for a referral which leads one of the unemployed links of that worker to be contacted for the job. The new position is then sold off to keep with one-worker firms.

It is shown that given an arbitrary symmetric network there exists an equilibrium in the labor market and such equilibria are characterized. When the network is separated across types (i.e. high types only have high-type friends and similarly for low types), it is shown that the labor market equilibrium is unique. Last, there exists a unique network equilibrium when the network is separated by type.

Attention is focused on the equilibrium where the network is separate by type. This is the most empirically plausible equilibrium out of the model’s stable equilibria. This equilibrium yields a number of interesting predictions. First, a match created through a referral is more likely to be of high productivity than a match created through the market. The reason is that high-productivity workers are more likely to be employed at any point in time which makes them refer their (high-productivity) links more often. This prediction is consistent with both

A second prediction is that the productivity premium of referrals is only applicable to the occupations where unobserved worker heterogeneity is an important feature. If there is a single productivity type, then referrals will still be used to alleviate search frictions but they will yield no productivity advantage. This prediction may help rationalize the evidence in Pistaferri (1999) and Pellizzari (2010) who find that, after controlling for observable worker characteristics, finding a job though referrals has an insignificant or even negative effect on wages.\footnote{The model, as it currently stands, cannot rationalize lower wages for referred workers.}

A final prediction concerns the effect of social networks on business cycle dynamics of the labor market. The aggregate matching function (including both market and referrals) is procyclical in that a given labor market tightness yields more matches when the unemployment rate is low than when it is high. Consistent with this prediction, Cheremukhin and Restrepo (2009) present evidence that the efficiency of the matching function is pro-cyclical.

## 2 The Model

The model has two distinct stages: a one-shot network formation game among workers and a subsequent infinite-horizon search-production stage.

Workers have either high or low productivity, are risk-neutral and maximize expected discounted utility. The two productivity types are of equal measure which is normalized to one: $\mu_H = \mu_L = 1$. There is free entry of firms who can employ one worker, are risk-neutral and maximize expected discounted profits.
2.1 Network Formation

Network formation is modeled as a large, one-shot, non-cooperative game among workers. A worker’s action is to announce his desired measure of links with workers of either type, taking as given everyone else’s simultaneous announcement. The action’s outcome is the realization of these links which depends on the announcement of other workers. Payoffs depend on the action, which is costly, and the outcome, which affects the worker’s labor market. Attention is restricted to Nash equilibria of the network formation game where workers follow symmetric strategies.\(^5\)

**Actions:** The action of worker \(j\) of type \(i\) is a pair of numbers, \(\hat{n}_{ji} = (\hat{n}^H_{ji}, \hat{n}^L_{ji})\), where \(\hat{n}^k_{ji}\) is the measure of links that worker \(j\) wants to create with workers of type \(k\) (symmetry implies that every type-\(k\) worker is identical). The symmetric (aggregate) action of every other worker is denoted by \(\hat{N}^k_i\). Let \(\hat{N} = (\hat{N}^H_H, \hat{N}^L_H, \hat{N}^H_L, \hat{N}^L_L)\) denote the aggregate announcement.

**Outcomes:** A worker’s realized links are given by \(n_{ji} = (n^H_{ji}, n^L_{ji})\) and they depend on his own announcement and on the aggregates in the following way:

\[
\begin{align*}
    n^i_{ji}(\hat{n}_{ji}, \hat{N}) &= 0, \quad \text{if } \hat{N}^i_i = 0. \\
    n^i_{ji}(\hat{n}_{ji}, \hat{N}) &= \hat{n}^i_{ji}, \quad \text{if } \hat{N}^i_i > 0. \\
    n^k_{ji}(\hat{n}_{ji}, \hat{N}) &= \hat{n}^k_{ji} \min[1, \hat{N}^i_i/\hat{N}^k_i].
\end{align*}
\]

In words, if the aggregate action exhibits zero within-\(i\) links, then worker \(j\) forms zero such links regardless of his own action. If \(\hat{N}^i_i > 0\) then he can form as many links with his own type as he desires. Links between workers of different types are realized in a similar way, with the addition of rationing if the aggregate actions of the two types do not match: if more type-\(i\) workers desire to link with type-\(k\)-s than vice versa, then some of the desired links of type-\(i\)-s will not be formed. Note that the aggregate ratios only affect how high the announcement

\(^5\)Of course, off-the-equilibrium actions can be asymmetric.
needs to be in order to end up with a certain number of realized links: an individual type-\(i\) worker can have as many links with type \(k\)s as he desires if \(k\)-types desire to link with the \(i\)s in the aggregate.

This structure captures two important features. First, mutual consent is necessary to form a link between two workers. Second, an individual worker is small.

Let \(N_i^k\) denote the measure of realized links between workers of type \(i\) and type \(k\) and denote the aggregate realized network by \(\mathbf{N} = (N_i^H, N_i^L, N_i^H, N_i^L)\). The discussion above implies:

\[
\begin{align*}
N_i^i &= \hat{N}_i^i \\
N_i^k &= N_i^k = \min[\hat{N}_i^k, \hat{N}_i^i]
\end{align*}
\]  

Payoffs: The worker’s objective is to maximize the steady state utility of the labor market net of the cost of network formation. The choice of objective function is motivated by the idea that forming one’s network is a long-run choice and is therefore not affected by the short-run changes in one’s employment status or that of his friends. It would arise if workers did not discount the future (which they do).

The steady state utility of worker \(j\) of type \(i\) in the labor market depends on the aggregate network \(\mathbf{N}\) and on his own measure of links \(\mathbf{n}_{ji}(\hat{\mathbf{n}}_{ji}, \hat{\mathbf{N}})\) and it is denoted by \(\mathcal{L}_i(\mathbf{n}_{ji}(\hat{\mathbf{n}}_{ji}, \hat{\mathbf{N}}), \mathbf{N})\). The cost of his announcement is \(c(\hat{n}_{ji}^H + \hat{n}_{ji}^L)\) which is assumed to be strictly convex with \(c(0) = 0\), \(c'(0) = 0\) and \(\lim_{\hat{n} \to 1} c(\hat{n}) = \infty\).

Worker \(j\) of type \(i\) takes \(\hat{\mathbf{N}}\) as given and solves

\[
\max_{\hat{\mathbf{n}}_{ji}} \mathcal{L}_i(\mathbf{n}_{ji}(\hat{\mathbf{n}}_{ji}, \hat{\mathbf{N}}), \mathbf{N}) - c(\hat{n}_{ji}^H + \hat{n}_{ji}^L)
\]
The focus is on equilibria where

\[ \hat{n}_{ji}^k = \hat{N}_i^k, \forall i, j, k. \] (7)

2.2 The Labor Market

Time runs continuously, the horizon is infinite and the future is discounted at rate \( r \). The labor market is in steady state. Each firm hires one worker and it can be filled and producing or vacant and searching. A worker is either employed or unemployed. Vacancy creation occurs in two ways, both of which cost \( K \): a new firm enters the market or an existing firm expands (an expansion is followed by the spin-off of the new position to keep with one-worker firms).

A firm and a worker meet either through the market or through a referral. The market is described by a matching function. A referral occurs when a firm expands and the current worker refers an unemployed link.

When a worker and a firm meet, the match-specific productivity is drawn from a distribution that depends on the worker’s type and all payoff-relevant information (productivity, worker type, worker’s network) becomes common knowledge. The pair then decides whether to consummate their match and, if they do, production starts and the surplus is split through Nash bargaining. Matches are exogenously destroyed at rate \( \delta \) and there is no on the job search. The worker’s flow value when unemployed is \( b \) and the firm’s flow value when vacant is \( 0 \).

Vacancy creation, meeting through referrals, meeting in the market, and matching are now described in detail.

Vacancy Creation: A new firm pays \( K \) and starts searching for workers in the market. A producing firm expands at exogenous rate \( \rho \) where \( \rho < \delta \). In an expansion, the incumbent worker refers one of his unemployed links at random to the firm. If the referral results in \( \text{entry of new firms in steady state since the stock of producing firms is declining without entry}.\)
a hire, production begins; if it does not, then search in the market begins. Creating a new position through expansion differs from entry of a new firm in two crucial ways: first, a worker is immediately contacted; second, the worker is potentially drawn from a different pool than the market. The new position is sold off and the incumbent firm receives share \( \gamma \in [0, 1] \) of the referral’s value which is denoted by \( R_i \) when the referring worker is of type \( i \).

The value of a referral is given by

\[
R_i = q_i^H \bar{J}_H + q_i^L \bar{J}_L
\]

where \( \bar{J}_k \) is the expected capital gain to a firm of meeting a type-\( k \) worker (to be explicitly derived below) and \( q_k^i \) is the probability that a type \( i \) worker refers a type \( k \) worker:

\[
q_k^i = \frac{N_k^i u_k}{N_i^H u_H + N_i^L u_L}
\]

Meeting through Referrals: Worker \( j \) of type \( i \) is referred to a firm when the employer of one of his links expands and worker \( j \) is chosen among the referrer’s unemployed links. Worker \( j \) has \( n_{ji}^k \) links of type \( k \), each of whom is employed with probability \( 1 - u_k \) and is in the position to refer at rate \( \rho \). The referrer has \( N_k^H u_H + N_k^L u_L \) unemployed links and each of them is equally likely to receive the referral. Symmetry implies that \( n_{ji}^k = N_i^k \) but for now the two are distinguished to facilitate exposition. Worker \( j \) is referred to a job at rate

\[
\alpha_{Rji}(n_{ji}, N) = \frac{\rho n_{ji}^H (1 - u_H)}{N_i^H u_H + N_i^L u_L} + \frac{\rho n_{ji}^L (1 - u_L)}{N_i^H u_H + N_i^L u_L}
\]

A special case that will be of interest later is when the network is separated by type, i.e. \( N_k^i = 0 \) for \( k \neq i \). In that case

\[
\alpha_{Rji}(n_{ji}, N) = \frac{\rho n_{ji}^i (1 - u_i)}{N_i^i u_i}
\]
Meeting in the Market: The labor market is characterized by search and information frictions. Search frictions mean that it takes time for workers and firms to meet. Informational frictions mean that all workers search in the same market regardless of their productivity type. Even though all relevant information becomes common knowledge when a worker and a firm meet, firms cannot pre-screen workers by type. The different types capture heterogeneity that is present after controlling for workers’ observable characteristics (e.g. education, age, experience).

Three types of agents search in the market: measure $v$ of vacancies, measure $u_H$ high-type unemployed workers and measure $u_L$ low-type unemployed workers. Consequently, two kinds of meetings may occur in the market: meetings between a vacancy and a high type worker or between a vacancy and a low type worker. A meeting structure with ranking is considered, as in Blanchard and Diamond (1994). In our context, ranking means that a high type worker is congested by other high types in his search for vacancies but the low types do not affect his search outcomes. A low type worker, however, is congested by both high and other low types.

The flow of meetings between firms and high-type workers is described by a Cobb-Douglas function which only depends on $v$ and $u_H$:

$$M_H(v, u_H, u_L) = Av^n u_H^{1-\eta}$$  \hspace{1cm} (12)

where $A > 0$ and $\eta \in (0, 1)$.

The flow of meetings between firms and low-type workers has the same Cobb-Douglas form with the addition of the congestion generated by the high types.

$$M_L(v, u_H, u_L) = g(v, u_H)Av^n u_L^{1-\eta}$$  \hspace{1cm} (13)

The congestion is captured by $g(u_H, v)$ which has the following properties: $g(0, v) = 1$,
\( g(1, v) > 0, g_{wH} < 0, g_v > 0, g(\xi v, \xi u_H) = g(v, u_H). \)

Denote the rate at which a type-\( i \) worker meets with a firm through the market by \( \alpha_{Mi} \) and the rate at which a firm meets with a type-\( i \) worker by \( \alpha_{Fi} \).

\[
\alpha_{Mi} = \frac{M_i(v, u_H, u_L)}{u_i} \\
\alpha_{Fi} = \frac{M_i(v, u_H, u_L)}{v}
\]

It is additionally assumed that

\[
\frac{d(M_H + M_L)}{du_H} > 0 \\
\frac{d(\alpha_{FH} + \alpha_{FL})}{dv} > 0
\]

**Matching:** When a firm and a worker of type \( i \) meet, the match-specific productivity \( p \) of the match is drawn from distribution \( F_i(p) \) and it remains constant for the duration of the match. It is assumed that \( F_i(p) = 1 - e^{-p/p_i} \) and \( \pi_H > \pi_L \).\(^7\) The flow value of a match is given by the match-specific productivity plus the value of the referrals that is kept by the incumbent firm: \( p + \rho \gamma R_i \).

Every payoff relevant characteristic is common knowledge during the meeting of the worker and the firm. In particular, they both observe \( p \) and the firm knows the worker’s type and his network. They decide whether to form the match. If they do, the surplus is split through Nash bargaining where \( \beta \in (0, 1) \) denotes the worker’s bargaining power. Given the above assumptions, it is easy to see that there is a reservation productivity \( p_\beta \) such that a match between a type-\( i \) worker and a firm is consummated if and only if \( p \geq p_\beta \).

**Value Functions:** To summarize the labor market, the agents’ value functions are now de-

\(^7\)I.e. the distribution is exponential. The parameter of the exponential distribution equals its mean.
scribed. Consider worker $j$ of type $i$. To distinguish how a worker’s off-equilibrium network affects his labor market steady state, the $j$ subscript is retained in the referral rate. Denote his value of being unemployed by $U_{ji}$ and his value of employment at productivity $p$ by $W_{ji}(p)$. We have:

\[
  rU_{ji} = b + (\alpha_{Mi} + \alpha_{Rji}) \int_{\bar{p}_{ji}}^{\infty} [W_{ji}(p) - U_{ji}]dF_i(p) \tag{14}
\]

\[
  rW_{ji}(p) = w_{ji}(p) + \delta(U_{ji} - W_{ji}(p)) \tag{15}
\]

The unemployed worker’s flow utility is $b$. Job opportunities appear at rate $\alpha_{Mi} + \alpha_{Rji}$ and a match is formed if the match-specific productivity is above the reservation value. When employed, the worker’s flow utility is equal to the wage $w_{ji}(p)$ which is the outcome of Nash bargaining. The match is destroyed at rate $\delta$.

The worker’s steady state value in the labor market is given by

\[
  \mathcal{L}_{ji} = u_{ji}U_{ji} + (1 - u_{ji}) \int_{\bar{p}_{ji}}^{\infty} W_{ji}(p)dF_i(p)/(1 - F_i(p_{ji})) \tag{16}
\]

Now consider a firm. When vacant, it searches for workers and it meets with type-$i$ workers at rate $\alpha_{Fi}$.\(^8\) If the match-specific productivity is high enough the match is formed. When producing, the firm’s flow payoffs are $p + \rho\gamma R_i - w_i(p)$. The match is destroyed at rate $\delta$. The firm’s value of a vacancy and production are given by:

\[
  rV = \alpha_{FH}\bar{J}_H + \alpha_{FL}\bar{J}_L \tag{17}
\]

\[
  rJ_i(p) = p + \rho\gamma R_i - w_i(p) + \delta(V - J_i(p)) \tag{18}
\]

\(^8\)The probability that a firm meets an off-equilibrium worker is zero so all workers are symmetric and there is no $j$ subscript.
\[ J_k = \int_{p_k}^{\infty} (J_k(p) - V) dF_k(p) \]  \hspace{1cm} (19)

The wage is determined by Nash bargaining:

\[ w_i(p) = \arg\max_w (W_i(p) - U_i)^\beta (J_i(p) - V)^{1-\beta} \] \hspace{1cm} (20)

The flows in and out of unemployment are equal in steady state:

\[ u_i (\alpha_{Mi} + \alpha_{Ri})(1 - F(p_i)) = (1 - u_i)\delta \] \hspace{1cm} (21)

### 2.3 Equilibrium Definition

Given symmetric network $N$ the Labor Market Equilibrium is defined as follows.

**Definition 2.1** A Labor Market Equilibrium is the reservation productivity levels \{\(p_H, p_L\}\), the steady state unemployment levels \{\(u_H, u_L\}\} and the number of vacancies $v$ such that:

- All bilaterally efficient matches are created and the surplus is split according to (20).
- The labor market is in steady state as described in (21).
- There is free entry of firms: $V = K$.

The equilibrium of the network formation stage is defined as follows.

**Definition 2.2** An equilibrium is $\hat{n}$ which solves (6) subject to the symmetry restriction (7) where the payoffs are given by the steady state utility (16) as derived in the labor market equilibrium.
3 Equilibrium Analysis

The equilibrium is analyzed essentially by backwards induction: first, the existence of an equilibrium in the labor market is proven for an arbitrary symmetric network; then, network formation is considered.

3.1 Labor market equilibrium

Denote the surplus of a match between a firm and a type-\(i\) worker with productivity \(p\) by \(S_i(p)\):

\[
S_i(p) = W_i(p) - U_i + J_i(p) - V
\] (22)

Furthermore, let \(\bar{S}_i\) denote the expected surplus that is created when a type-\(i\) worker and a firm meet:

\[
\bar{S}_i = \int_{\mathbb{R}_+} S_i(p)dF_i(p)
\] (23)

Setting the first order conditions of the Nash bargaining equation (20) to zero yields:

\[
W_i(p) - U_i = \beta S_i(p)
\] (24)

\[
J_i(p) - V = (1 - \beta)S_i(p)
\] (25)
The value functions can be rewritten as

\[ rU_i = b + (\alpha_{Mi} + \alpha_{Ri})\beta\bar{S}_i \] (26)

\[ (r + \delta)W_i(p) = w_i(p) + \delta U_i \] (27)

\[ rV = \alpha_{FH}(1 - \beta)\bar{S}_H + \alpha_{FL}(1 - \beta)\bar{S}_L \] (28)

\[ (r + \delta)J_i(p) = p - w_i(p) + \rho(1 - \beta)\bar{q}_i^H \bar{S}_H + \bar{q}_i^L \bar{S}_L \] (29)

\[ (r + \delta)S_i(p) = p - b - rK + \rho(1 - \beta)(\bar{q}_i^H \bar{S}_H + \bar{q}_i^L \bar{S}_L) - (\alpha_{Mi} + \alpha_{Ri})\beta\bar{S}_i \] (30)

Free entry implies that \( V = K \).

**Proposition 3.1** A labor market equilibrium exists.

**Proof.** Note that equation (28) is strictly decreasing in \( v \). Equating (28) with \( rK \) yields the equation \( \hat{v}(p_H, p_L, u_H, u_L) \). Define \( \hat{p}_i \) by \( S_i(\hat{p}_i) = 0 \) and note that \( \hat{p}_i \) is unique for given \( \{v; u_H; u_L; p_k\} \). Let \( \hat{p}_i(p_k, v, u_H, u_L) = \max[0, \hat{p}_i] \). Equation (21) defines the equation \( \hat{u}_i(u_k, p_H, p_L, v) \).

The equations described above are continuous over a compact set. Using Brower’s fixed point theorem, a labor market equilibrium exists. \( \blacksquare \)

Additionally, the characterization is sharper if the network is separated by type. This case will turn out to be more relevant below.

**Proposition 3.2** The labor market equilibrium is unique when the network is separated by type.

**Proof.** See the Appendix. \( \blacksquare \)

### 3.2 Network formation equilibrium

The network formation stage is now examined. Agent \( j \) of type \( i \) takes as given the aggregate action and the equilibrium that will be played in the labor market. The first order conditions
which hold with equality if \( \hat{n}_{ji}^i > 0 \) or \( \hat{n}_{ji}^k > 0 \) for the first and second condition, respectively.

The first term describes how the steady state utility changes with the referral rate while the second term describes how the referral rate changes with the worker’s announcement. Both are analyzed in turn.

The next proposition proves that a worker’s steady state utility in the labor market is strictly increasing and strictly concave in the referral rate. This result is very intuitive: the worker benefits from a faster rate of meeting with potential employers; and the marginal benefit declines because he spends less time unemployed.

**Proposition 3.3** A worker’s labor market steady state utility is strictly increasing and strictly concave in his referral rate.

**Proof.** See the Appendix.

The way the referral rate is affected by the worker’s first stage action is now examined. Consider worker \( j \) of type \( i \) at the network formation stage who contemplates his action. Depending on the aggregate action, his referral rate is the following:

\[
\frac{d\alpha_{Rji}}{d\hat{n}_{ji}^i} = \frac{(1 - u_i) \rho}{N_i^H u_H + N_i^L u_L} \quad \text{if } N_i^i > 0
\]

\[
= 0 \quad \text{if } N_i^i = 0
\]

\[
\frac{d\alpha_{Rji}}{d\hat{n}_{ji}^k} = \frac{(1 - u_k) \min[1, N_k^i / \hat{N}_i^k] \rho}{N_k^H u_H + N_k^L u_L}
\]
Clearly, if \( \hat{N}_t = 0 \) for \( t \in \{H, L\} \) then the optimal action is \( \hat{n}_{ji}^t = 0 \).

The agent can be in one of four possible states with respect to the aggregate action.

First, no agent wants to link with a type-\( i \) agent: \( \hat{N}_H^i = \hat{N}_L^i = 0 \). In that event his optimal action is to set \( \hat{n}_{ji} = (0, 0) \). This case is not very interesting.

Second, the agents of his own type want to link with each other while the agents of the other type do not: \( \hat{N}_i^i > 0 \) and \( \hat{N}_k^i = 0 \). In that case, his referral rate will be

\[
\alpha_{Rji} = \frac{\hat{n}_{ji}^i (1 - u_i) \rho}{\hat{N}_i^i u_i} 
\]

(33)

The agent’s optimal action is to set \( \hat{n}_{ji}^k = 0 \) and \( \hat{n}_{ji}^i = \hat{n}_{ji}^{i*} \) where

\[
\frac{dL_{ji}}{d\alpha_{Rji}} \frac{(1 - u_i) \rho}{\hat{N}_i^i u_i} = c' (\hat{n}_{ji}^{i*}) 
\]

(34)

Third, the agents from the other type want to link with type-\( i \) agents and the type-\( i \) agents do not want to link with each other: \( \hat{N}_i^i = 0 \) and \( \hat{N}_k^i > 0 \). In that case his referral rate will be

\[
\alpha_{Rji} = \frac{\hat{n}_{ji}^k \min[1, \hat{N}_k^i / \hat{N}_k^k](1 - u_k) \rho}{\min[\hat{N}_k^i, \hat{N}_k^k] u_i + \hat{N}_k^k u_k} 
\]

(35)

The agent’s optimal action is to set \( \hat{n}_{ji}^i = 0 \) and \( \hat{n}_{ji}^k = \hat{n}_{ji}^{k*} \) where

\[
\frac{dL_{ji}}{d\alpha_{Rji}} \frac{(1 - u_k) \rho \min[\hat{N}_k^i, \hat{N}_k^k]}{\min[\hat{N}_k^i, \hat{N}_k^k] u_i + \hat{N}_k^k u_k} = c' (\hat{n}_{ji}^{k*}) 
\]

(36)

Fourth, both types want to link with type-\( i \) agents: \( \hat{N}_i^i > 0 \) and \( \hat{N}_k^i > 0 \). In this case the referral rate is

\[
\alpha_{Rji} = \frac{\rho \hat{n}_{ji}^i (1 - u_i)}{\hat{N}_i^H u_H + \hat{N}_i^L u_L} + \frac{\rho \hat{n}_{ji}^k \min[1, \hat{N}_i^i / \hat{N}_k^k](1 - u_k)}{\hat{N}_k^H u_H + \hat{N}_k^L u_L} 
\]

(37)
There are two possibilities to consider when increasing the desired links. If it is strictly preferable to link with one of the types, then case 2 or 3 provide the solution:

\[
\frac{\rho (1 - u_i)}{N_i^H u_H + N_i^L u_L} > \frac{\rho \min[1, \hat{N}_k^i / \hat{N}_i^k] (1 - u_k)}{N_k^H u_H + N_k^L u_L} \Rightarrow \text{Case 2 (38)}
\]

\[
\frac{\rho (1 - u_i)}{N_i^H u_H + N_i^L u_L} < \frac{\rho \min[1, \hat{N}_k^i / \hat{N}_i^k] (1 - u_k)}{N_k^H u_H + N_k^L u_L} \Rightarrow \text{Case 3 (39)}
\]

If the two types are equally desirable, then any combination of \( \hat{n}_{ji}^i + \hat{n}_{ji}^k = \hat{n}_{ji}^* \) is optimal, where:

\[
\frac{dL_{ji}}{d\alpha_{Rji}} \frac{d\alpha_{Rji}}{d\hat{n}} = c'(\hat{n}_{ji}^*)
\]  

(40)

Note, however, that this last possibility is a knife-edge case: if any of the terms changes ever so slightly (say \( u_k \) or \( \hat{N}_k \)) then the agent would strictly prefer linking with of the two types and his optimal action would look very different. For this reason, this type of equilibria will not be further analyzed.

From now on the focus is on the second type of equilibrium where the two types are separated from each other. The third case, where an \( H \)-worker only has \( L \) friends and vice versa, runs counter to the long-established idea that networks feature homophily.

The concavity of \( L_{ji} \) and strict convexity of \( c(\cdot) \) imply that the agent’s best response to the aggregate action is unique: \( \hat{n}_{ji}^*(\hat{N}) \). The last step is to prove:

**Proposition 3.4** There exists a unique symmetric equilibrium where types as separated in the network formation stage.

**Proof.** Rearranging the agent’s first order conditions (34) yields:

\[
\hat{N}_i^i c'(\hat{n}_{ji}^*) = \frac{dL_{ji}}{d\alpha_{Rji}} \frac{1 - u_i \rho}{u_i}
\]  

(41)
Evaluating the above expression at $\hat{n}_{ji}^* = \hat{N}_i$, the network size does not enter the right-hand side of the above equation. Furthermore, the left-hand size starts at zero and increases monotonically to infinity. Therefore, there exists a unique network size that satisfies that equation with equality.

4 Implications and Extensions

In this section the focus is on the equilibrium where the network is separated by type. Three predictions of the model are described.

**Prediction 1:** Matches created through referrals are more productive than the ones created through the market.

In equilibrium, high-type workers have lower unemployment rates. This is driven by the fact that they sample from a more favorable productivity distribution than the low-type workers. In addition, of course, they meet firms more frequently both in the market due to ranking and through referrals but they would have lower unemployment even if the meeting rates were the same across types. The higher employment rate means that high-type workers are in a position to refer a member of their social network more frequently than low-types do. A high-type worker’s network consists exclusively of other high-type workers and therefore a high-type is also referred to a job more often than an $L$. As a result, the first prediction of this model is that a referred worker is more likely to by $H$ and, therefore, to have higher productivity and receive a higher wage. There is plenty of evidence to support this prediction, as cited in the introduction.

**Prediction 2:** Matches created through referrals are more productive than the ones created through the market *only when certification is difficult*.

Assume that there is no heterogeneity in productivity or matching across workers: $\pi_H = \pi_L = \pi$ and $M_H(v, u_1, u_2) = M_L(v, u_2, u_1)$. Referrals are still used in equilibrium as they
alleviate search frictions but they do not lead to differences in productivity. This prediction may help interpret the evidence presented in Pistaferri (1999) and Pellizzari (2010) where, after controlling for worker characteristics, a referral has insignificant or even negative effect on wages.

One might be tempted to conclude that in this case the referrals play no role. However, this conclusion would be mistaken because referrals increase the meeting rate for workers. As a result, the reservation productivity is increasing in the referral rate and a higher referral rate leads to higher aggregate productivity.

**Prediction 3:** Job finding depends on stock of employed *conditional on labor market tightness*.

An unemployed worker’s job finding rate depends both on labor market tightness, which affects the rate he contacts firms in the market, and on the stock of employed workers, which affects the rate he contacts firms through referrals. Importantly, the job-finding rate depends on employment *conditional on tightness*, which distinguishes this model from the usual search and matching models of the labor market. As a result, this model’s aggregate matching function (including both the market and referrals) is pro-cyclical in that a given labor market tightness yields more matches when the unemployment rate is low than when it is high. Cheremukhin and Restrepo (2009) present evidence that the efficiency of the matching function is pro-cyclical which is consistent with the above prediction.

**5 Conclusions**

This paper introduces endogenous networks in a search model of the labor market. The model is very tractable and it is close to the most widely used models of the labor market. As a result it can be easily extended and compared with the predictions of models that do not include social networks.
6 Appendix

Proposition 3.2: The labor market equilibrium is unique when the network is separated by type.

Proof. It will prove convenient to have an interior solution (i.e. strictly positive) for the cutoff productivity. This is trivially the case in the typical models where the benefit of employing a worker arises from his output; here, however, it also arises from his ability to refer additional workers which might generate profits for the firm. Therefore, it is conceivable that, for instance, a high-type worker is always hired regardless of the match-specific productivity he draws, which leads to some technical inconveniences. The following lemma provides a condition that guarantees an interior cutoff.

Lemma 6.1 If \((r + \delta)(b + rK) > \rho \pi_H \gamma (1 - \beta)\) then \(p_i > 0\) for \(i \in \{H, L\}\).

Proof.

If \(S_i(0) < 0\), then \(p_i > 0\).

\[
(r + \delta)S_i(p) = p + \rho \gamma (1 - \beta)\bar{S}_i - rU_i - rV
\]

\[
= p + \rho \gamma (1 - \beta)\bar{S}_i - b - (\alpha_{Mi} + \alpha_{Ri})\beta \bar{S}_i - rK
\]

\[
= p - b - rK + (\rho \gamma (1 - \beta) - (\alpha_{Mi} - \alpha_{Ri})\beta)\bar{S}_i
\]

Integrate the above expression over \(p \in [\underline{p}_i, \infty)\).

\[
(r + \delta)\bar{S}_i = \int_{\underline{p}_i}^{\infty} pdF_i(p) + [-b - rK + (\rho \gamma (1 - \beta) - (\alpha_{Mi} - \alpha_{Ri})\beta)\bar{S}_i](1 - F_i(\underline{p}_i))
\]

\[
= [\pi_i - b - rK + (\rho \gamma (1 - \beta) - (\alpha_{Mi} - \alpha_{Ri})\beta)\bar{S}_i]e^{-\underline{p}_i/\pi_i}
\]

\[
\Rightarrow \bar{S}_i = \frac{\pi_i + \underline{p}_i - b - rK}{(r + \delta)e^{-\underline{p}_i/\pi_i} - \rho \gamma (1 - \beta) + (\alpha_{Mi} + \alpha_{Ri})\beta}
\]
Use the above result inside $S_i(p)$ to calculate $S_i(0)$ when $p = 0$:

$$(r + \delta)S_i(0) = -b - rK + \frac{(\rho(1 - \beta) - (\alpha_{Mi} - \alpha_{Ri})\beta))(\pi_i + -b - rK)}{(r + \delta) - \rho(1 - \beta) + (\alpha_{Mi} + \alpha_{Ri})\beta}$$

(42)

Manipulating the above expression yields the desired result. ■

From now we assume that the condition of the above lemma holds. As a result we have $S_i(p) = 0$ which leads to

$$p = rU_i + rK - \rho\gamma R_i$$

(43)

Furthermore, in a separated equilibrium we have

$$R_i = (1 - \beta)S_i$$

(44)

Introducing (43) into the definition of the match surplus yields

$$S_i(p) = \frac{p - p_i}{r + \delta}$$

(45)

and therefore

$$\bar{S}_i = \frac{\pi_i}{r + \delta} e^{-p_i/\pi_i}$$

(46)

Introducing (43) into the expression for the worker’s unemployment value yields:

$$p_i = b + rK + ((\alpha_{Mi} + \alpha_{Ri})\beta - \rho\gamma(1 - \beta)) \frac{\pi_i}{r + \delta} e^{-p_i/\pi_i}$$

(47)

which given an implicit definition of $p_i$ as a function of $u_i$, $v$ and parameters.

To calculate the flows note that network segregation and symmetry imply that $\alpha_{Ri} = (1 - u_i)\rho/u_i$. Therefore, the unemployment rate of each type of worker is implicitly defined by
the following equation as a function of the number of vacancies and reservation productivity of that type:

\[
u_i[M_i(v, u_H, u_L)/u_i + \rho(1 - u_i)/u_i]e^{-\pi_i/\pi_i} - \delta(1 - u_i) = 0\] (48)

Equations (47) and (48) imply that \(dp_i/dv > 0\). Therefore, if \(v\) is unique, then the equilibrium is unique.

We now turn to the determination of \(v\). The firm’s value of a vacancy can be rewritten as:

\[
rK = \frac{\alpha_{FH}(1 - \beta)}{r + \delta} \int_{Z_H}^{\infty} [1 - F_H(p)]dp + \frac{\alpha_{FL}(1 - \beta)}{r + \delta} \int_{Z_L}^{\infty} [1 - F_L(p)]dp\] (49)

which implies that \(dv/dp_i < 0\).

The rest of the proof is by contradiction. Suppose that \(v_A\) and \(v_B\) are two candidates for equilibrium with \(v_A > v_B\). For firms to optimize, we would need \(p_{Ai} < p_{Bi}\). However, worker optimization implies \(p_{Ai} > p_{Bi}\) leading to the desired contradiction.

The labor market equilibrium is unique when the network is separated by type. 

**Proposition 3.3:** A worker’s labor market steady state utility is strictly increasing and strictly concave in his referral rate.
Proof. First write the steady state utility of worker \( j \) of type \( i \) in a more convenient form:

\[
\begin{align*}
 r \mathcal{L}_{ji} &= u_{ji} r U_{ji} + (1 - u_{ji}) \int_{p_{ji}}^{\infty} r W_{ji}(p) dF_{i}(p)/(1 - F_{i}(p_{ji})) \\
 &= r U_{ji} + r (1 - u_{ji}) \int_{p_{ji}}^{\infty} (W_{ji}(p) - U_{ji}) dF_{i}(p)/(1 - F_{i}(p_{ji})) \\
 &= r U_{ji} + \frac{r (1 - u_{ji})}{1 - F(p_{ji})} \beta S_{ji} \\
 &= b + (\alpha_{Mi} + \alpha_{Rji}) \beta S_{ji} + \frac{r (\alpha_{Mi} + \alpha_{Rji}) \beta S_{ji}}{\delta + (\alpha_{Mi} + \alpha_{Rji})(1 - F(p_{ji}))}
\end{align*}
\]

There are two cases to consider: when \( p_{ji} = 0 \), then an increase in \( \alpha_{Ri} \) has no effect on the cutoff productivity which remains at zero; when \( p_{ji} > 0 \), an increase in \( \alpha_{Ri} \) makes the worker more picky and hence increases \( p_{ji} \).

Consider \( p_{ji} = 0 \). To calculate \( \bar{S}_{ji} \), integrate equation (30) over \([0, \infty)\) which yields, after rearranging:

\[
\bar{V}_{ji} = \frac{\pi_{i} + \rho R_{i} - b - r K}{r + \delta + (\alpha_{Mi} + \alpha_{Rji}) \beta} \tag{50}
\]

Note that the value of a referral \( (R_{i}) \) in the numerator includes some \( \bar{V}_{k} \) but for the other workers which is not affected by the change in \( \alpha_{Rji} \).

Introducing this expression to (50) yields:

\[
\begin{align*}
 r \mathcal{L}_{ji} &= b + \beta(\pi_{i} + \rho R_{i} - b - r K) \frac{r + \delta + \alpha_{Mi} + \alpha_{Rji}}{r + \delta + (\alpha_{Mi} + \alpha_{Rji}) \beta} \frac{\alpha_{Mi} + \alpha_{Rji}}{\delta + \alpha_{Mi} + \alpha_{Rji}} \\
 &= b + \beta(\pi_{i} + \rho R_{i} - b - r K) \frac{r + \delta + \alpha_{Mi} + \alpha_{Rji}}{r + \delta + (\alpha_{Mi} + \alpha_{Rji}) \beta} \frac{\alpha_{Mi} + \alpha_{Rji}}{\delta + \alpha_{Mi} + \alpha_{Rji}} \tag{51}
\end{align*}
\]

Differentiating with respect to \( \alpha_{Rji} \) leads to

\[
\begin{align*}
 \frac{dr \mathcal{L}_{ji}}{d\alpha_{Rji}} &= \beta(\pi_{i} + \rho R_{i} - b - r K) \frac{((r + \delta)(\delta + \alpha_{Mi} + \alpha_{Rji})^{2} + r \delta(r + \delta) - r \beta(\alpha_{Mi} + \alpha_{Rji})^{2})}{[r + \delta + (\alpha_{Mi} + \alpha_{Rji}) \beta]^{2}(\delta + \alpha_{Mi} + \alpha_{Rji})^{2}}
\end{align*}
\]

which is strictly positive.

For convenience, define by \( T_{1} \) and \( T_{2} \) the two terms in the numerator and by \( T_{3} \) and \( T_{4} \)
the two term in the denominator. Note that $T_1$ does not depend on $\alpha_{R ji}$ and the three terms are increasing in $\alpha_{R ji}$.

The second derivative is given by

$$\frac{d^2 r L_{ji}}{d\alpha_{R ji}^2} = \frac{T_1(T_2 T_3 T_4 - T_2 T_3 T_4 - T_2 T_3 T_4')}{T_3 T_4^2}$$ (52)

Since the middle term in the numerator is strictly negative, it suffices to show that $T = T_2 T_4 - T_2 T_4' < 0$ to arrive at strict concavity.

$$T = 2(r + \delta)\left(\delta + \alpha_{M i} + \alpha_{R ji}\right)^3 - 2r\beta(\alpha_{M i} + \alpha_{R ji})(\delta + \alpha_{M i} + \alpha_{R ji})^2 - 2(r + \delta)(\delta + \alpha_{M i} + \alpha_{R ji})^3$$

$$- 2r\delta(r + \delta)(\delta + \alpha_{M i} + \alpha_{R ji}) + 2r\beta(\alpha_{M i} + \alpha_{R ji})^2(\delta + \alpha_{M i} + \alpha_{R ji})$$

$$= -2r\beta(\alpha_{M i} + \alpha_{R ji})(\delta + \alpha_{M i} + \alpha_{R ji}) - 2r\delta(r + \delta)(\delta + \alpha_{M i} + \alpha_{R ji}) < 0$$

Consider the case where $p_{ji} > 0$. One implication is that $S_{ji}(p_{ji}) = 0$ which means that $p_{ji} + \rho R_i - RU_{ji} - rK = 0$. Combining this with equation (30) yields $S_{ji}(p) = (p - p_{ji})(r + \delta)$ which can be integrated over $[p_{ji}, \infty)$ to reach $\bar{S}_{ji} = \pi_i e^{-p_{ji}/\pi_i}/(r + \delta)$. Therefore, the cutoff productivity is defined by the following equation:

$$H(p_{ji}, \alpha_{R ji}) = b + rK - \rho R_i + (\alpha_{M i} + \alpha_{R ji}) \frac{\beta \pi_i}{r + \delta} e^{-p_{ji}/\pi_i} - p_{ji} = 0$$ (53)

Using the implicit function theorem it is straightforward to calculate:

$$\frac{dp_{ji}}{d\alpha_{R ji}} = \frac{\pi_i}{\alpha_{M i} + \alpha_{R ji} + e^{p_{ji}/\pi_i}(r + \delta)/\beta} > 0$$ (54)

$$\frac{d^2 p_{ji}}{d\alpha_{R ji}^2} = -\frac{\pi_i(1 + (dp_{ji}/d\alpha_{R ji})e^{p_{ji}/\pi_i}(r + \delta)/\beta)}{(\alpha_{M i} + \alpha_{R ji} + e^{p_{ji}/\pi_i}(r + \delta)/\beta)^2} < 0$$ (55)
Using the expression for \( \bar{S}_{ji} \), the steady state utility can be rewritten as

\[
 rL_{ji} = \rho R_i - rK + p_{ji} + \frac{r(p_{ji} + \rho R_i - rK - b)}{\delta + (p_{ji} + \rho R_i - rK - b)(r + \delta)/(\beta \pi)} \tag{56}
\]

which can be differentiated with respect to \( \alpha_{Rji} \) to yield:

\[
\frac{d(rL_{ji})}{d\alpha_{Rji}} = \frac{dp_{ji}}{d\alpha_{Rji}} (1 + \frac{\delta r}{(\delta + (p_{ji} + \rho R_i - rK - b)(r + \delta)/(\beta \pi))^2}) > 0
\]

\[
\frac{d^2(rL_{ji})}{d\alpha_{Rji}^2} = \frac{d^2p_{ji}}{d\alpha_{Rji}^2} (1 + \frac{\delta r}{(\delta + (p_{ji} + \rho R_i - rK - b)(r + \delta)/(\beta \pi))^2}) - \frac{dp_{ji}}{d\alpha_{Rji}} \frac{2\delta r(dp_{ji}/d\alpha_{Rji})(r + \delta)/(\beta \pi_i)}{\alpha_{Rji} (\delta + (p_{ji} + \rho R_i - rK - b)(r + \delta)/(\beta \pi))^2} < 0
\]
References


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