1 Introduction

In this paper, we develop a theory of sales based on product market search by heterogeneous consumers. By a theory of sales we mean a theory that explains how it can be optimal for sellers to follow a repeated pattern of posting a high price for several periods and then posting a low price for one period. More formally, we demonstrate the existence of periodic nonstationary equilibria with self-generating cycles in a simple model of sequential search. While existence of equilibrium cross-sectional price dispersion is a well-understood feature of this class of model, the possibility of dispersion across dates - in the sense of recurring price cycles - has hardly been explored.

We consider a market in which consumers search for a good price for a single unit of a semi-durable good – we use shoes as an example. As is standard in equilibrium search theory, consumers know the distribution of prices across sellers but not the price charged by any particular seller, so it may take time for a consumer to find a new pair of shoes at an acceptable price. The framework we use is based on the Albrecht and Axell (1984) job search model. In that model, there are two worker types, who differ with respect to their values of leisure, which in turn leads to reservation wage heterogeneity. Similarly, in our model, we assume two consumer types – we call them fashionistas and sensible shoppers. Fashionistas differ from sensible shoppers in two ways. First, the fashionistas receive a higher utility from consuming the good (wearing their shoes) than do the sensible shoppers. This is the assumption made in Diamond (1987). Second, we assume that the fashionistas’ shoes depreciate faster than do the shoes of the sensible shoppers. Our interpretation of this assumption is that when styles change, fashionistas no longer receive any utility from wearing their old shoes. There is nothing essential about assuming that the two consumer types differ along two dimensions, as opposed to only one – this just allows us to make our point with a minimum of algebra.\(^1\)

Our model has both stationary and nonstationary equilibria. The stationary equilibria are as follows. For some parameter values – essentially, when the sensible shoppers aren’t worth bothering with – only a high price, which is equal to the fashionistas’ reservation price, is posted. Alternatively,

\(^{1}\)In the spirit of our example, however, we need to impose a restriction on the parameters of the model that ensure that fashionistas are more eager than sensible shoppers are to buy shoes. This is done in inequality (7) below.
when the sensible shoppers are relatively many and/or they are not so different from the fashionistas, only a low price, which is equal to the sensible shoppers’ reservation price, is posted. Finally, there is an in-between set of parameters that are consistent with stationary equilibrium price dispersion; that is, some sellers post a high price that is acceptable only to the fashionistas while the others post a low price that is acceptable to both consumer types. These stationary equilibrium possibilities are well understood in the equilibrium search literature. See, for example, Diamond (1987). However, the idea that there can be nonstationary equilibria in this type of model is new. We consider in particular periodic nonstationary equilibria in which all sellers post high prices (acceptable only to fashionistas) for \( T_h \) periods followed by \( T_c \) periods in which all sellers post low prices (acceptable to both consumer types). Our results are as follows. First, periodic equilibria exist for all parameter combinations that lead to stationary equilibrium price dispersion. Second, these equilibria exist for \( T_h = 1, 2, 3, \ldots \) but only in conjunction with \( T_c = 1 \). That is, the equilibrium pattern is one in which sellers post a high price for \( T_h \) periods followed by a single period in which all sellers post the low price. Finally, the value of \( T_h \) in a periodic equilibrium is uniquely determined by the parameters of the model. In particular, for those parameter combinations that are consistent with stationary equilibrium price dispersion in which relatively few sellers post the fashionistas’ reservation price, the periodic equilibrium is one in which all sellers post a high price for one period, followed by the low price, followed by the high price, etc. If the stationary dispersion equilibrium is such that sufficiently more sellers post the high price, then the periodic equilibrium is one with \( T_h = 2 \) (and \( T_c = 1 \)). Indeed, there are parameter combinations consistent with a periodic equilibrium in which sellers post high prices for an arbitrary number of periods, \( T_h \), followed by a single period in which the low price is posted. The higher is \( T_h \), the higher is the fraction of sellers who offer the high price in the corresponding stationary dispersion equilibrium.

At first glance, our results may seem to be only of technical interest. However, our model offers a simple and compelling theory of sales.\(^2\) Sellers post high prices for several periods and then “hold a sale” by posting a low price for a single period. Depending on parameters, these sales can occur at arbitrary intervals – weekly, monthly, quarterly, etc. Sales are, of course, worth studying in their own right, but they also play an important role in empirical macroeconomics. First, sales are important for the correct computation of price indices. Just as price indices need to account for the fact that consumers can substitute across different goods, so too do price indices need to account for the fact that consumers can substitute across the same good at different points in time. See, for example, Feenstra and Shapiro (2003). Second, there is an extensive empirical literature on price flexibility – how often are prices adjusted and by how much? To get accurate answers to these questions, researchers need to filter out transitory price changes due to sales. A theory of sales can help empirical researchers construct this filter in a sensible way. See, for example, Nakamura and Steinsson (2008) and Klenow and Malin (2010).

Turning back to the theory of sales, the mechanism that lies behind our model is straightforward. Demand from sensible shoppers accumulates during the high-price periods. At some point, which depends on the strength of the sensible shoppers’ demand and on how quickly they accumulate in the pool of shoppers, it makes sense to hold a sale to exploit the pent-up demand. Once that demand is satisfied, sellers revert to the high-price regime. This compositional mechanism is similar to the one present in Conlisk et al. (1984) and Sobel (1984). In Conlisk et al. (1984), a new cohort of consumers enters the market in each period, some with a low reservation price and some with a high reservation price. In that setting, a monopolist finds it optimal to charge a high price for

\(^2\)Some early papers in the equilibrium search literature, e.g., Varian (1980) and Salop and Stiglitz (1982), interpret stationary equilibrium price dispersion as a theory of sales. However, these models do not explain sales as they are usually understood, that is, a deterministic cycle in which several periods of “usual, regular” prices are followed by a single low-price period.
several periods before setting a low price for one period to take advantage of the build up of the low-reservation-price consumers. Sobel (1984) extends this result to an oligopolistic framework, showing that there is an equilibrium in which all sellers periodically cut their prices to the same level for one period. In common with our model, these papers are based on the possibility of deferred demand – the low-demand consumers can “do without” until the price falls. An alternative, but similar, mechanism is presented in Pesendorfer (2002). In his model, some consumers buy more than one unit when prices are low and then run down their inventories until the good is on sale again. For this pattern to be consistent with optimal price-setting behavior by sellers, the other, relatively high-demand, consumers need to exhibit “store loyalty” so that the sellers have a captive demand stock during high-price periods. Finally, Nakamura and Steinsson (2009) propose a model in which periodic sales are explained by switching costs. When a firm has a sale, it is offering a low price to attract a stock of consumers. These consumers then “get in the habit” of buying this firm’s good, so the firm can set a high price for several periods without losing too much of its customer base. Eventually, when the base gets low enough, the firm cuts its price and the cycle starts again.

We also contribute to the equilibrium search literature, where almost no work has been done on nonstationary equilibria. There are, of course, exceptions to this generalization. Fershtman and Fishman (1992) analyze a dynamic version of Burdett and Judd (1983). At the beginning of each period, consumers decide how many price quotes to sample, and sellers decide what prices to post. There is a degenerate stationary equilibrium in this model in which no consumer pays for a price quote and every seller posts a price greater than or equal to the common consumer reservation value. Another stationary equilibrium exhibits price dispersion, as in Burdett and Judd (1983). The dynamics in Fershtman and Fishman (1992) are driven by self-fulfilling expectations. When consumers and sellers believe that the period equilibrium will be degenerate, then it is rational for all agents to behave in a way that confirms those beliefs. When the common belief is that the period equilibrium will be the one with price dispersion, then again beliefs are self fulfilling. The nonstationary equilibrium is generated by the self-fulfilling belief that the economy will switch between the degenerate and nondegenerate phases. A similar mechanism – self-fulfilling expectations generate multiple equilibria and a common belief that the economy will move back and forth between the equilibria leads to dynamics – is also present in Diamond and Fudenberg (1989) and Burdett and Coles (1998). Our nonstationary equilibria are different. At any parameter configuration that is consistent with a periodic equilibrium in our model, there is one and only one corresponding stationary equilibrium. The periodic equilibria in our model are self generating. The high- and low-price phases lead to compositional changes in the pool of shoppers that naturally drive the market back and forth between the two regimes.

The outline of the rest of the paper is as follows. In the next section, we give the basics of our model. In Section 3, we present the stationary equilibria, and in Section 4 we analyze the nonstationary, periodic equilibria. Section 5 concludes.

2 The Basic Model

Environment. Consider the market for a consumption good that is semi-durable in the sense that it does not necessarily fully depreciate each period. We use shoes as an example although our theory would apply equally well (perhaps up to some minor formal changes) to cars, houses, domestic appliances, etc. Our model could also apply to the labor market, although this requires a more substantial reinterpretation; indeed the Albrecht and Axell (1984) model on which our theory is based is a model of the labor market.

We assume that time is discrete, indexed by $t$, and that all agents are infinitely lived and
discount the future at rate $\beta \leq 1$ per period. Agents either buy and wear shoes or run a shoe store. By wearing a pair of shoes, a consumer enjoys a constant utility of $v_k > 0$ each period, where $k = 0, 1$ is the consumer’s type (to be defined momentarily), until the shoes go out of fashion (or wear out), which occurs with constant probability $\delta_k$ per period. Once her shoes go out of fashion (or wear out), the consumer no longer enjoys wearing them (her utility per period falls to 0) and goes shopping for a new pair.

Shopping is modeled as a search process – the market for shoes is affected by search frictions in that every shopper finds one and only one pair of shoes that she likes in each shopping period. However, the consumer may or may not buy the shoes – that depends on the price. Shoe prices are posted by the shoe stores and cannot be bargained over. The consumer decides whether to buy the shoes or to continue shopping next period.

There are two consumer types. Type-1 consumers (fashionistas) enjoy a high utility, $v_1$, which we normalize to one, in each period that they wear shoes, whereas type-0 consumers (sensible shoppers) only enjoy $v_0 \equiv v$, where $0 < v < 1$. The two types also differ in terms of how quickly their shoes depreciate, i.e., go out of style. In particular, fashionistas’ shoes go out of style each period with probability $\delta_1 = 1$ while sensible shoppers’ shoes wear out (or go out of style) each period with probability $\delta_0 = \delta < 1$. The normalization $\delta_1 \equiv 1$ is not without loss of generality, but one can think of a period as the time it takes for the fashionistas to view their shoes as out of style. The total population of consumers is normalized to one, and the share of fashionistas in the population is $\lambda$.

Values and prices. Stores post prices and shoppers arbitrage between buying shoes and holding a numéraire good of which they receive a fixed endowment each period and from which they derive a utility that is normalized to zero. Because there are only two consumer types, each with a different reservation price, shoe stores either choose to post a high price, which is equal to the reservation price of fashionistas and which sensible shoppers would turn down, or a low price equal to the reservation price of sensible shoppers, which both consumer types would accept. Let $\gamma_t$ denote the fraction of stores posting the high price in period $t$.

For a type-$k$ consumer we denote the lifetime value of shopping by $S_k^t$ and that of wearing shoes by $W_k^t$. The net value of buying shoes at price $p_t$ is $W_k^t - p_t$, so that the reservation price $r_k^t$ of a type-$k$ consumer is defined by $W_k^t - r_k^t = \beta S_k^{t+1}$.

We now turn to formal definitions of the value functions. Starting with the value of wearing shoes for sensible shoppers,

$$W_t^0 = v + \beta \left( (1 - \delta) W_{t+1}^0 + \delta S_{t+1}^0 \right). \tag{1}$$

The first term is the current period utility. At the end of the period, the shoes remain wearable with probability $(1 - \delta)$, in which case the consumer continues without shopping, while with probability $\delta$ the shoes wear out and the individual becomes a shopper. The corresponding value of being a (sensible) shopper is

$$S_t^0 = \gamma_t \beta S_{t+1}^0 + (1 - \gamma_t) (W_t^0 - r_t^0) \tag{2}$$

In the current period, the shopper receives no utility. With probability $\gamma_t$, the shopper encounters a store with a high price and continues as a shopper into the next period. With probability $1 - \gamma_t$, she finds shoes at an acceptable price and purchases shoes, which generate value $W_t^0 - r_t^0$, the lifetime value of wearing the shoes less the price. The value functions (1) and (2) imply for all $t$
that
\[ S_t^0 = 0 \quad \text{and} \quad r_t^0 = W_t^0 = \frac{v}{1 - \beta (1 - \delta)} \equiv r^0. \quad (3) \]

Next consider the fashionistas. Their lifetime value of wearing shoes is
\[ W_t^1 = 1 + \beta S_t^1, \quad (4) \]
reflecting our assumption that a fashionista’s shoes go out of style with probability 1 at the end of the current period. This implies that the fashionistas’ reservation price is the same for all \( t \), namely,
\[ r_t^1 = 1 \equiv r^1. \quad (5) \]
The lifetime value of shopping for these consumers is
\[ S_t^1 = \gamma_t \left( W_t^1 - r^1 \right) + (1 - \gamma_t) \left( W_t^1 - r^0 \right) = \beta S_{t+1}^1 + (1 - \gamma_t) \left( r^1 - r^0 \right). \quad (6) \]
As was the case for the sensible shoppers, fashionistas receive no utility while searching for a new pair of shoes. With probability \( \gamma_t \), they encounter a store posting the high price, purchase the shoes, and receive the value of wearing new shoes less the high price. With probability \( 1 - \gamma_t \), they find the shoes at a store posting the low price and receive the value of wearing new shoes less the low price. The condition that ensures that fashionistas are willing to spend more than sensible shoppers are to buy shoes is simply \( r^1 = 1 > r^0 \); that is,
\[ v < 1 - \beta (1 - \delta). \quad (7) \]
If this restriction does not hold, then the roles of the fashionistas and sensible shoppers are reversed.

**Consumer stocks.** Let \( s_t \) denote the share of shoppers in the total consumer population and \( \varphi_t \) the share of fashionistas among the shoppers at the beginning of period \( t \). These shares depend on the history of stores’ pricing strategies. However, fashionistas never resist buying shoes and therefore always leave the population of shoppers with probability 1 at the end of each period no matter what fraction of stores post the high price. The stock of fashionistas is thus constant so
\[ s_t \varphi_t \equiv \lambda, \quad (8) \]
which is obtained from a simple flow-balance equation.

The total number of shoppers then evolves according to
\[ s_{t+1} = s_t - s_t \varphi_t - (1 - \gamma_t) s_t (1 - \varphi_t) + \lambda + \delta (1 - \lambda - s_t (1 - \varphi_t) \gamma_t) \]
\[ = \lambda + \gamma_t s_t (1 - \varphi_t) + \delta (1 - \lambda - s_t (1 - \varphi_t) \gamma_t) \]
\[ = \lambda + (1 - \lambda) \delta + \gamma_t s_t (1 - \varphi_t) (1 - \delta) \]
\[ = \lambda + (1 - \lambda) \delta + \gamma_t s_t (1 - \delta) - \lambda \gamma_t (1 - \delta), \quad (9) \]
where the second equality uses (8). The number of shoppers at the beginning of period \( t + 1 \) equals the number at the start of period \( t \) minus the \( s_t \varphi_t \) fashionistas who all buy shoes minus the \( (1 - \gamma_t) s_t (1 - \varphi_t) \) sensible shoppers who encountered stores posting the low price plus all the \( \lambda \) fashionistas whose shoes went out of fashion at the end of the period plus the \( \delta (1 - \lambda - s_t (1 - \varphi_t) \gamma_t) \) sensible shoppers who enjoyed their shoes during the period but whose shoes wore out at the end of the period.
Stores’ decisions. Stores seek to maximize expected sales revenue each period. This is proportional to the probability that the posted price will be accepted by a randomly met shopper times the sale price – the wholesale price at which shoe stores buy their stock is assumed constant and is normalized to zero. Hence a store chooses to post the high price \( r^1 \), at which only fashionistas are prepared to buy, rather than to post the low price \( r^0 \), which is acceptable to all consumers, if and only if \( r^1 \varphi_t > r^0 \). Of course, they make the converse choice (are indifferent between the two options) if and only if this latter inequality holds in reverse (becomes an equality).

Finally note that by using (3) and (8), one can rewrite the condition under which stores post the high price, both prices or the low price as

\[
v < \left( \frac{\lambda}{s_t} \right) (1 - \beta (1 - \delta)) \quad v = \left( \frac{\lambda}{s_t} \right) (1 - \beta (1 - \delta)) \quad v > \left( \frac{\lambda}{s_t} \right) (1 - \beta (1 - \delta)),
\]

respectively.

3 Stationary Equilibria

In this section we investigate the existence and properties of stationary long-run equilibria in which all endogenous variables remain constant over time.

Stationary equilibrium with \( \gamma = 1 \) (all stores post the high price) Equation (9) implies that if such a stationary equilibrium exists, it features a constant number of shoppers \( s^*_h \) given by

\[
s^*_h = 1. \tag{11}
\]

Condition (10) further implies that it is indeed optimal for stores to post the high price in this situation if and only if

\[
v < \lambda (1 - \beta (1 - \delta)). \tag{12}
\]

Stationary equilibrium with \( \gamma = 0 \) (all stores post the low price) Following similar steps, if such a stationary equilibrium exists, the number of shoppers \( s^*_c \) is given by

\[
s^*_c = \lambda + (1 - \lambda) \delta, \tag{13}
\]

and the consistency condition (10) for this to be an equilibrium is

\[
v > \frac{\lambda (1 - \beta (1 - \delta))}{\lambda + (1 - \lambda) \delta}. \tag{14}
\]

Stationary equilibrium with \( \gamma \in (0, 1) \) (price dispersion) If a stationary equilibrium with price dispersion exists, the number of shoppers \( s^*_y \) is given by

\[
s^*_y = \frac{\lambda + (1 - \lambda) \delta - \lambda \gamma (1 - \delta)}{1 - \gamma (1 - \delta)}. \tag{15}
\]

3 This assumes that finding a good deal at a particular store does not lead a consumer to return to the same store again after her shoes have depreciated. In the equilibrium that we analyze, this assumption is sensible – there is no reason for a consumer to patronize the same store more than once. It is also worth noting that the assumption that sellers maximize expected current period revenue distinguishes our model, which is set in a product market, from labor market models like Albrecht and Axell (1984) and Gaumont, et al. (2006). In a labor market, the relationship between the buyer (the firm) and the seller (the worker) extends beyond the current period.
For this to be an equilibrium, (10) must hold as an equality. Substituting the above expression for $s^*_\gamma$, this condition becomes

$$v = \frac{\lambda (1 - \beta (1 - \delta)) (1 - \gamma (1 - \delta))}{(\lambda + (1 - \lambda) \delta - \lambda \gamma (1 - \delta))}.$$

(16)

This defines a unique share of firms $\gamma^* \in (0, 1)$ if and only if

$$\frac{\lambda (1 - \beta (1 - \delta))}{\lambda + (1 - \lambda) \delta} > v > \lambda (1 - \beta (1 - \delta)).$$

(17)

To see that $\gamma^*$ is indeed unique, note that the function $v(\gamma) = \frac{\lambda (1 - \beta (1 - \delta)) (1 - \gamma (1 - \delta))}{(\lambda + (1 - \lambda) \delta - \lambda \gamma (1 - \delta))}$ decreases monotonically from $\frac{\lambda (1 - \beta (1 - \delta))}{\lambda + (1 - \lambda) \delta}$ to $\lambda (1 - \beta (1 - \delta))$ as $\gamma$ increases from 0 to 1. Condition (17) defines a nonempty set of parameter values for which stationary equilibria with price dispersion exist. Inspection of (16) further reveals that $\gamma^*$, the equilibrium share of stores posting the high price, is a decreasing function of $v$ and an increasing function of $\lambda$. Posting the high price is more likely to be profitable if there are more fashionistas around or if sensible shoppers are relatively less eager to buy shoes.

**Example**
Let $\beta = \frac{1}{1 + 0.012} = 0.988$, so we think of a period as one quarter, and let $\lambda = \delta = 0.5$. Then, from (17), $0.253 < v < 0.337$ is consistent with equilibrium price dispersion. If, for example, $v = 0.3$, then (from equation (16)) a fraction $\gamma = 0.544$ of the sellers post the fashionista reservation price ($r^1 = 1$). The other sellers post the sensible shopper reservation price, $r^0 = 0.593$ (from equation (3)). If $v < 0.253$, then only the high price is posted, while if $v > 0.337$, only the reservation price of the sensible shoppers, which is increasing in $v$ up to the point that inequality (7) binds (when $v = 0.506$ for this example), is posted.

**Stationary equilibria: Summary.** Conditions (12), (14), and (17) define the pattern of stationary equilibria. These conditions partition the parameter space, so a unique stationary equilibrium always exists. This equilibrium only features price dispersion (in the form of a two-point distribution) when condition (17) holds. Other stationary equilibria are single-price. This is essentially the result presented in inequality (5) of Diamond (1987) and in Proposition 1 of Gaumont, et al. (2006). The latter paper is set in a labor market context, as is Albrecht and Axell (1984), which gives the original result of this type.

### 4 Nonstationary Equilibria

The existence of stationary equilibria with price dispersion in this type of model is well known. What has not been understood thus far is that these equilibria coexist with nonstationary equilibria. We now investigate the existence of such equilibria.

#### 4.1 A Simple ‘Bang-Bang’ Example

The simplest example of a nonstationary equilibrium is a periodic one in which all stores post the high price in even periods and the low price in odd periods (or vice versa), i.e. $\gamma_{2k} = 1$ and $\gamma_{2k+1} = 0$. 


Let $s_h$ ($s_\ell$) denote the number of shoppers at the beginning of a period in which $\gamma = 1$ ($\gamma = 0$). Equation (9) implies

\[
\begin{cases}
  s_h = \lambda + (1 - \lambda)\delta \\
  s_\ell = (1 - \delta) s_h + \delta = (1 - \delta) (\lambda + (1 - \lambda)\delta) + \delta.
\end{cases}
\quad (18)
\]

Our candidate nonstationary equilibrium has the economy alternating between one period in which the number of shoppers takes on the value $s_h$ and one period featuring $s_\ell$. For such a situation to be an equilibrium, the stores’ optimality condition (10) must be satisfied in both types of period; i.e.,

\[
v > \frac{\lambda}{s_\ell} (1 - \beta (1 - \delta)) \quad \text{and} \quad v < \frac{\lambda}{s_h} (1 - \beta (1 - \delta))
\]

must hold simultaneously. Substituting for $s_h$ and $s_\ell$, these conditions imply the following restriction on the parameters:

\[
\frac{\lambda (1 - \beta (1 - \delta))}{\lambda + (1 - \lambda)\delta} > v > \frac{\lambda (1 - \beta (1 - \delta))}{(1 - \delta) (\lambda + (1 - \lambda)\delta) + \delta}.
\quad (19)
\]

This restriction defines a nonempty set of parameter values. This type of nonstationary equilibrium only coexists with stationary equilibria of the $\gamma \in (0, 1)$ type, i.e. stationary equilibria featuring price dispersion. More precisely, the maximum value of $v$ that is consistent with stationary price dispersion is the same as the maximum $v$ that is consistent with a bang-bang equilibrium, while the minimum $v$ that is consistent with stationary price dispersion is less than the minimum $v$ that is consistent with a bang-bang equilibrium. That is, the upper bound of (19) equals the upper bound of (17), while the lower bound of (19) is above the lower bound of (17).

The intuition underlying the bang-bang equilibrium is straightforward. If only the high price is posted in period $t$, then the sensible shoppers will delay replacing their shoes. This changes the composition of the pool of shoppers moving into period $t + 1$; in particular, the fraction of fashionistas in the shopper pool falls between $t$ and $t + 1 < (\varphi_{t+1} < \varphi_t)$. In period $t + 1$, firms react to the increased presence of sensible shoppers by posting the lower price, etc.

### 4.2 A Theory of Sales

The simple bang-bang example shows that an equilibrium with self-generating cycles exists, but this example is limited as a theory of sales. A theory of sales should allow for richer nonstationary periodic equilibria in which the market spends, e.g., $T_h$ periods in a high-price regime ($\gamma = 1$) and then spends one period in a low-price regime ($\gamma = 0$). Equilibria of this type explain sales more generally, e.g., weekly, monthly or quarterly sales. We start with a more general setup and consider the possibility of equilibria with $T_h$ periods in a low-price regime ($\gamma = 0$) followed by $T_h$ periods ($\gamma = 1$) in a high-price regime, with $T_h$ and $T_\ell$ left unrestricted for now. We show that equilibria exist in which $T_h$ takes on values greater than one but that no equilibria exist with $T_\ell > 1$. Thus, we indeed have a theory of periodic price reductions rather than one of cycles with several low price periods followed by high price periods. We also show that $(T_h, 1)$ equilibria exist for $T_h = 1, 2, 3, \ldots$. Furthermore, there is a $(T_h, 1)$ equilibrium for each $v$ satisfying (17). These equilibria are ordered in the sense that the lower is $v$ (equivalently, the higher is the fraction of firms, $\gamma$, that would post the high price in the stationary dispersion equilibrium), the higher is $T_h$. The length of the interval of $v$’s consistent with a $(T_h, 1)$ equilibrium decreases with $T_h$, and as $T_h \to \infty$, the value of $v$ consistent with periodic equilibria collapses to $\lambda (1 - \beta (1 - \delta))$, the lower limit of the interval given by (17).
Now we turn to the analysis. Because we are interested in periodic equilibria, it is convenient to reset the time subscript $t$ to zero every time the economy switches regimes. Moreover, in what follows we add an $\{h, \ell\}$ superscript to the endogenous variable $s$ to indicate the economy’s current regime (high- or low-price, $\gamma = 1$ or $0$). With these notational conventions, characterization of the economy’s dynamic behavior follows from equation (9).

**The high- and low-price regimes.** In the high-price regime, for $t \in \{1, \cdots, T_h - 1\}$, the market behaves following

$$s_{t+1}^h = (1 - \delta) s_t^h + \delta. \quad (D_h)$$

Equation $(D_h)$ implies that the number of shoppers keeps increasing so long as the economy stays in the high-price regime. In the limit, if the high price prevailed forever, the number of shoppers would approach one – all sensible shoppers would eventually go barefoot, unable (or unwilling) to afford shoes, while fashionistas would continue shopping every period.

In the low-price regime, for $t \in \{1, \cdots, T_c - 1\}$, the economy is simply characterized by

$$s_{t+1}^\ell \equiv \lambda + (1 - \lambda) \delta. \quad (D_\ell)$$

The number of shoppers stays constant (after one period) in the low-price regime. As always, all fashionistas shop in every period, while sensible shoppers are now willing to buy the first pair of shoes they sample – and they sample one with certainty in their first period of shopping. Therefore, in this regime, the number of shoppers each period is made up of the $\lambda$ fashionistas, who shop no matter what, and the fraction $\delta$ of the $1 - \lambda$ sensible shoppers whose shoes just wore out.

The steady-state values of $s^h$ and $s^\ell$ were already given in equations (11) and (13).

**Switching points.** The dynamic systems $(D_h)$ and $(D_\ell)$ only apply in the “interior” of each regime, i.e., when the economy is not about to switch from one regime to the other. At switching points – i.e. at $t = T_h$ in the high-price regime and $t = T_\ell$ in the low-price regime – equation (9) implies the following. At a switch from the high-price into the low-price regime, one has

$$s_1^\ell = (1 - \delta) s_{T_h}^h + \delta, \quad (S_{h \rightarrow \ell})$$

and at a switch from the low- into the high-price regime

$$s_1^h = \lambda + (1 - \lambda) \delta. \quad (S_{\ell \rightarrow h})$$

**Candidate nonstationary equilibrium for given $T_h$ and $T_\ell$.** In order to construct a candidate nonstationary equilibrium, we solve systems $(D_h)$, $(D_\ell)$, $(S_{h \rightarrow \ell})$, and $(S_{\ell \rightarrow h})$ recursively for the sequence of populations of shoppers $s_t$. This gives

$$s_1^\ell = 1 - (1 - \delta)^{T_h+1} (1 - \lambda)$$

$$s_t^\ell \equiv \lambda + (1 - \lambda) \delta \quad \text{for } t \in \{2, \cdots, T_\ell\}$$

$$s_1^h = \lambda + (1 - \lambda) \delta$$

$$s_t^h = 1 - (1 - \delta)^t (1 - \lambda) \quad \text{for } t \in \{2, \cdots, T_h\}.$$ 

Note that any candidate nonstationary equilibrium characterized by the set of equations above is independent of the number of periods spent in the low-price regime, $T_\ell$. For given values of $T_h$, we have therefore constructed a family of candidate nonstationary equilibria, each member of which is characterized by a duration of the high-price regime, $T_h$. We now turn to the consistency requirements for these candidate equilibria to indeed constitute valid equilibria of our model.
Consistency conditions. For the candidate equilibrium described above to be valid, it is necessary that the following inequalities hold:

$$(\frac{\lambda}{s^h_t})(1 - \beta (1 - \delta)) > v > (\frac{\lambda}{s^l_{T_t}})(1 - \beta (1 - \delta))$$  \hspace{1cm} (20)

Note that we only need to check these conditions at dates $T_h$ and $T_l$, respectively. Thanks to the monotonicity properties of $s^h_t$ and $s^l_t$ in the candidate periodic equilibrium, inequality (20) ensures that $(\frac{\lambda}{s^h_t})(1 - \beta (1 - \delta)) > v$ holds for all $t < T_h$ when the market is in the high-price regime and that $v > (\frac{\lambda}{s^l_{T_t}})(1 - \beta (1 - \delta))$ holds for all $t < T_l$ when the market is in the low-price regime.

Substitution of the various expressions that characterize our candidate equilibrium into (20) leads to conditions on the parameters that are slightly different depending on whether one considers $T_l = 1$ or $T_l \geq 2$. The difference is due to the fact that it takes two periods in the low-price regime for the number of shoppers to reach its constant value of $\lambda + (1 - \lambda) \delta$. For $T_l = 1$, the consistency conditions can be rewritten as:

$$\frac{\lambda(1 - \beta (1 - \delta))}{1 - (1 - \delta)(1 - \lambda)^{T_h}} > v > \frac{\lambda(1 - \beta (1 - \delta))}{(1 - (1 - \delta)(1 - \lambda)^{T_h+1}}$$ \hspace{1cm} (21)

This latter condition coincides with (19) when $T_h = 1$. In addition, condition (21) defines a nonempty set of parameter values for $T_h = 2, 3, \ldots$. Note that the maximum value of $v$ consistent with the periodic equilibrium that spends two periods in the high-price regime followed by one period in the low-price regime, i.e., the $(2,1)$ equilibrium, equals the minimum value of $v$ consistent with the bang-bang equilibrium; the maximum value of $v$ consistent with the $(3,1)$ equilibrium equals the minimum value of $v$ consistent with the $(2,1)$ equilibrium, etc. Further, as $T_h \rightarrow \infty$, the minimum and maximum values of $v$ consistent with a $(T_h,1)$ equilibrium both converge to $\lambda(1 - \beta (1 - \delta))$, i.e., the lowest $v$ consistent with stationary price dispersion. That is, the parameter ranges for the various $(T_h,1)$ cycles are non-overlapping, ordered, and cover the entire interval of $v$’s for which stationary price dispersion equilibria exist.

Finally, for a cyclical equilibrium with $T_l > 1$ to exist, the following inequalities must hold:

$$\frac{\lambda(1 - \beta (1 - \delta))}{1 - (1 - \delta)(1 - \lambda)^{T_h}} > v > \frac{\lambda(1 - \beta (1 - \delta))}{\lambda + (1 - \lambda) \delta}$$ \hspace{1cm} (22)

This, however, is clearly not possible. When $T_h = 1$, the left- and right-hand sides of inequality (22) are equal; and when $T_h > 1$, the left-hand side of inequality (22) is less than the right-hand side.

To summarize, we find that nonstationary equilibria with periodic cycles in which stores charge the high price for some number of periods followed by the low price for one period exist. These equilibria characterize periodic sales. We thus find that periodic sales result from search behavior in a market in which consumers differ in terms of their willingness to pay for a single unit of the good and in terms of their desire to switch to the latest style. After several periods of high prices, the pool of searching consumers accumulates a large enough fraction of the sensible shoppers that a sale is optimal for the stores.

Example Consider again the case of $\beta = 0.988$ with $\lambda = \delta = 0.5$, and recall that the stationary equilibrium exhibits price dispersion for $0.253 < v < 0.337$. The bang-bang equilibrium ($T_h = 1, T_l = 1$) exists for $0.289 < v < 0.337$, the $(2,1)$ equilibrium exists for $0.270 < v < 0.289$, the
(3, 1) equilibrium exists for $0.261 < v < 0.270$, etc. Skipping ahead, the (10, 1) equilibrium exists for $0.25306 < v < 0.25313$, and the range of $v$ for which the (100, 1) equilibrium exists essentially collapses to the point $v = 0.253$. Periodic equilibria exist, however, for arbitrarily large $T_h$.

5 Concluding Remarks

In this paper, we propose a new theory of sales. We do this by proving the existence of nonstationary periodic equilibria in a model of product market search. We show that these periodic equilibria coexist with the stationary equilibria of the model that exhibit price dispersion. More precisely, consider any parameter configuration that leads to a stationary equilibrium in which a fraction $\gamma \in (0, 1)$ of the sellers post a price that only the high-demand consumers accept while the other sellers post a lower price that both consumer types accept. Then there is a corresponding nonstationary equilibrium in which the following pattern repeats – all sellers post a high price for $T_h$ periods and then post a low price for one period. Equilibria of this $(T_h, 1)$ type exist for $T_h = 1, 2, 3, ...$ and these equilibria are ordered in the sense that the higher is $T_h$, the higher is the value of $\gamma$ in the corresponding stationary equilibrium. An equilibrium in which all sellers post a high price for several periods and then post a low price for one period before returning to the high-price regime is exactly what one means by “sales,” and, depending on parameters, our theory is rich enough to generate sales at arbitrary frequencies.
References


