Consumption risk sharing under private information when earnings are persistent

VERY PRELIMINARY AND INCOMPLETE

Marek Kapička
UC Santa Barbara
mkapicka@econ.ucsb.edu

Paul Klein
IIES, Stockholm University
paul.klein@uwo.ca

July 21, 2010

Abstract

We quantitatively investigate the implications of a model of consumption risk sharing where infinitely-lived households are subject to persistent idiosyncratic shocks to their earnings, and where the realization of these shocks are private information. We overcome the curse of dimensionality typically associated with persistent private information problems by developing a discrete-shock analogue to the first-order approach. Our approach has two benefits: First, the state space now has only two continuous dimensions. Second, under certain conditions the state space is a convex cone in $\mathbb{R}^2$, and we provide a complete characterization of the state space. These findings enable us to solve numerically for the optimal contract, and allow us to confront the implications of the model with micro data.

We analyze the risk sharing properties of the private information economy and find that the degree of risk sharing decreases when the persistence of shocks increases. We compare the risk sharing properties of the private information economy with risk sharing properties of a Bewley economy, and with the data. We find that i) while the degree of risk sharing in the private information economy overstates the degree of risk sharing in the data, the degree of risk sharing in the Bewley economy understates it, and that ii) in both data and the private information economy consumption is more skewed to the right than earnings, while the opposite is true in the Bewley economy. Those findings indicate that the private information model is a promising avenue in analyzing consumption insurance.
1 Introduction

In this paper we quantitatively investigate the implications of a model of consumption risk sharing where infinitely-lived households are subject to exogenous idiosyncratic shocks to their earnings, and where the realization of these shocks are private information. While most of the literature concludes that neither full insurance nor exogenously incomplete markets explain the extent of risk sharing well, the role of imperfect consumption insurance due to e.g. asymmetric information “has not been subject to a systematic empirical investigation” (Blundell, Pistaferri, and Preston (2008), page 1914). Our goal is to carry out precisely such a systematic investigation.

Our theoretical contribution relative to the existing literature is to allow for persistence in earnings. This creates some formidable obstacles to recursive computation of the optimal contract. If the earnings process follows a Markov chain whose state space has \( N \) elements, the state space for the optimal dynamic contracting problem has \( N \) continuous dimensions and is a nontrivial subset \( \mathcal{V} \subset \mathbb{R}^N \). Such a problem is prohibitively complex, first because of the dimension of the dimensionality of the state space and, second, because it is difficult to solve for \( \mathcal{V} \) itself. Previous research (for instance Thomas and Worrall (1990) and Phelan (1995)) has typically, for the sake of tractability, made the clearly counterfactual assumption that earnings are i.i.d.

However, as shown by Kapička (2008) for an environment with continuum of shocks, using a first-order approach makes the model tractable by reducing the dimension of the state space. Our approach in this paper to develop a discrete shock analogue to the first-order approach, interpreting a “local” deviation in the discrete context as a one-step deviation. This approach is based on the conjecture that if one-step deviations from truth-telling are not optimal then no deviations are optimal and truth-telling is optimal. That conjecture can be verified numerically ex post. As it turns out, our approach is able to resolve both of the obstacles in the recursive formulation. First, the state space now has only two continuous dimensions. Second, we show that if the period utility is bounded above and the Markov chain exhibits the monotone likelihood ratio property, then the state space is a convex cone in \( \mathbb{R}^2 \), and provide a complete characterization of the state space. These findings enable us to
solve numerically for the optimal contract, and allow us to confront the implications of the model with micro data.

To confront the quantitative predictions of the model with data, we need a panel dataset of individual consumption and earnings. Such panel data was not available until recently: Consumer Expenditure Survey (CEX) is traditionally used as a source of data on individual level consumption but contains imprecise measures of earnings, while the Panel Study of Income Dynamics (PSID) data are used as a source of information about individual level earnings, but contain a very limited measure of consumption, namely food consumption. Recently, however, progress has been made in constructing panel datasets that contain information about both earnings and consumption. Blundell, Pistaferri, and Preston (2005) suggest a procedure that imputes consumption from CEX dataset into PSID dataset. The principal benefit of having such datasets is that we are able to obtain covariances between consumption and income, and ultimately estimate parameters like marginal propensity to consume out of permanent and transitory shocks. We can then test the theories by comparing the estimates from the data with the estimates generated by the model. Such tests are not available if one does not have panel data with both consumption and income (as in Kaplan (2006), Heathcote, Storesletten, and Violante (2008), or Heathcote, Storesletten, and Violante (2009)).

We define a risk sharing coefficient to be one minus the expected percentage change in consumption when earnings change by one percent. That is, the risk sharing coefficient is zero in autarchy, and one under full insurance. We find that the risk sharing coefficient is 0.85 in the data. The private information model delivers a risk sharing coefficient of 0.94, while the Bewley economy delivers between 0.66 and 0.81, depending on the borrowing constraint. Thus, the Bewley economy delivers too little risk sharing, consistently with Kaplan and Violante (2009), while the private information economy delivers too much risk sharing.

We also investigate other properties of the equilibrium distribution of consumption,

\[\text{An alternative strategy would be to use datasets for foreign countries that contain both consumption and earnings, e.g. the Italian Survey of Household Income and Wealth, see Krueger and Perri (2009). See also Gervais and Klein (2008) who propose a new method of measuring earnings in CEX in a way that is consistent with the measurement of consumption.}\]
and find that both data and the private economy feature the excess skewness of consumption: The distribution of consumption is more right-skewed than earnings. The excess skewness of consumption is 0.70 in the data, and 0.65 in the private information economy. In contrast, in the Bewley economy the distribution of consumption is significantly less skewed than the distribution of earnings: the excess skewness of consumption is -1.10. We thus see the excess skewness of consumption as an important property that allows us to differentiate between different models of risk sharing.

1.1 Related literature

Traditionally, the economics profession has focused on two extreme theories of consumption insurance. A theory of complete markets assumes that consumers are fully insured against all shocks to earnings. A theory of exogenously incomplete markets typically assumes that people can only self-insure against earnings shocks by saving through risk free assets. Both of those extreme theories are usually rejected in the data (see Attanasio and Davis (1996), Hall and Mishkin (1982), Hayashi, Altonji, and Kotlikoff (1996) or Attanasio and Pavoni (2007)). Blundell, Pistaferri, and Preston (2008) find that households are able to insure 36% of the permanent earnings shocks, suggesting that people have access to more insurance than just self-insurance. The benchmark results of Kaplan and Violante (2009) confirm this finding.

The existing literature that examines the extent to which different theories of endogenously incomplete markets explain individual consumption fluctuations has recently significantly expanded. Typically, the literature tests one (or several selected) particular implications of a given theory. Ligon (1998) and Kocherlakota and Pistaferri (2009) study the ability of private information models and models with exogenously incomplete markets to explain the distortions in intertemporal Euler equation. Lustig (2007) and Krueger, Lustig, and Perri (2007) study the same problem in a model with limited enforcement. In an interesting paper, Kinnan (2009) distinguishes between limited enforcement and moral hazard models where past marginal utility of consumption is a sufficient statistics for current consumption (see Kocherlakota (1996)), and hidden income models where it is not. She uses Thai data to test those predictions.
Unlike our model, her hidden income model assumes i.i.d. shocks.

Other researchers are closer to our approach in that they compute structural models with explicit private information, moral hazard or limited enforcement frictions. Paulson and Townsend (2004) and Karaivanov, Paulson, and Townsend (2007) compute moral hazard models and limited enforcement models and compare them in their ability to explain patterns in Thai data. However, in order to do so, they assume a very simple structure of their model, which is essentially static. They are therefore unable to study dynamic implications of moral hazard and limited liability models, like frictions in the Euler equation. Krueger and Perri (2005) use a model with limited enforcement to explain why an increase in income inequality in the last 30 years has not been matched by the same increase in consumption inequality. Kaplan (2006) has a similar goal, but considers both models with private information and limited enforcement. Both papers also differ in the specification of the underlying stochastic process for shocks. While Krueger and Perri (2005) specify the stochastic process for earnings in essentially the same way as we do, Kaplan (2006) assumes, for the sake of tractability, only fixed-effect and transitory shocks. In addition, in case of the private information model, only the fixed effect is a private information. Neither of those two papers also tests the predictions on a joint panel data of consumption and earnings. Finally, Attanasio and Pavoni (2007) studies the ability of models with moral hazard and hidden savings to explain excess smoothness of consumption.

2 Setup

Time is infinite, \( t = 0, 1, \ldots, \infty \). There is a measure 1 of agents. The agents maximize the expected lifetime utility given by

\[
E_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \quad 0 \leq \beta < 1.
\]

where \( c_t \) is consumption in period \( t \). The period utility function is \( U(c) \), strictly increasing, convex, and bounded above by 0.
The agents face earnings shocks $y^i, i = 1 \ldots N$. The shocks are increasing in $i$, and equally spaced, with their difference $\delta = y^i - y^{i-1}$. Their probability distribution is given by a Markov transition matrix $\pi(i|j), i, j = 1 \ldots N$. The cumulative distribution function will be denoted by $\Pi(i|i-1) = \sum_{j=1}^{i} \pi(j|i-1)$.

The principal can borrow and lend at a gross interest rate $R \geq 1$.

3 Efficient Allocations

The social planner chooses a transfer scheme $\tau = \tau_t(i^t)$ where $i^t = (i_0, i_1, \ldots, i_t)$ is a history of reported earnings. The lifetime utility of an agent with period $-1$ shock $i_{-1}$ from a transfer scheme $\tau$ is

$$W(\tau, i_{-1}) = \sum_{t=0}^{\infty} \sum_{i^t} \beta^t U \left( y^{i^t} + \tau_t(i^t) \right) \mu(i^t|i_{-1})$$

where $\mu(i^t|i_{-1})$ is the probability of a sequence $i^t$ given the initial shock $i_{-1}$. The contract must deliver the agent a lifetime utility $v$:

$$W(\tau, i_{-1}) = v. \quad (1)$$

The earnings shocks are assumed to be the private information of the agent. The only exception is the initial shock $i_{-1}$, which is publicly known. Without loss of generality, a revelation principle is applied and the allocation is required to be incentive compatible. We impose the following temporary incentive compatibility constraint:

$$U \left( y^{i^t} + \tau_t(i^t) \right) + \beta W \left( \tau_t(i^t), i_t \right) \geq U \left( y^{i^t} + \tau_t(i^{t-1}, j) \right) + \beta W \left( \tau_t(i^{t-1}, j), i_t \right) \quad \forall j = 1, \ldots N \quad (2)$$

for all $i^t$, where $\tau_t(i^t)$ is the continuation of the transfer scheme after history $i^t$.

2That is, $\tau_{t+s}(i^t) = \tau_t(s)(i^t, i^t)$ for all $s \geq 0$. 

6
more primitive constraint that rules out all possible sequences of deviations, not just one period deviations.

It is also assumed that the agent cannot commit to the insurance contract in the sense that, at the beginning of each period, the agent can choose autarchy whenever he prefers to do so. The value of an autarchic allocation for an agent whose last period shock was \( i \) is denoted by \( V^{\text{AUT}}(i) \), and is given by

\[
V^{\text{AUT}}(i) = \sum_{j=1}^{N} [U(y_i) + \beta V^{\text{AUT}}(j)] \pi(j|i).
\]

The threat of autarchy effectively imposes a lower bound on the agent’s continuation utility. It is required that for any history of shocks, autarchy is not optimal:

\[
W(\tau_{i^t}, j) \geq V^{\text{AUT}}(j) \quad \forall j = 1, \ldots, N.
\] (3)

The social planner chooses a transfer scheme to minimize the costs:

\[
C^*(v, i_{-1}) = \min_{\tau} \sum_{t=0}^{\infty} \sum_{i^t} R^{-t} \tau_t(i^t) \mu(i^t|i_{-1})
\]

subject to the promise keeping constraint (1), incentive compatibility constraint (2) and the lower bound constraint (3).

To solve the model quantitatively, a recursive formulation is needed. One possible recursive formulation of the problem has been analyzed by Fernandes and Phelan (2000). However, for a number of shocks larger than two, this recursive formulation is prohibitively complex due to the size of its state space. In particular, the state space has \( N \) continuous dimensions, corresponding to the continuation utility for each possible shock value. This is so because, as one can see from the incentive compatibility constraint (2), one needs to know the continuation utility \( W(\tau^\infty_t(i^{t-1}, j), i_t) \) for each possible type \( j = 1, \ldots, N \). Appendix B shows the details of such a recursive formulation.
3.1 A Relaxed Problem

We therefore examine an alternative route by analyzing a relaxed problem, where only local one-step-down incentive compatibility constraints are assumed to bind: for all $i_{t-1}$ and for $i_t = 2, \ldots, N$, we only impose the following constraints:

$$U(y^{i_t} + \tau_t(i^t)) + \beta W(\tau_{i^t}, i_t) \geq U(y^{i_t} + \tau_t(i^{t-1}, i_t - 1)) + \beta W(\tau_{(i^{t-1}, i_t-1)}, i_t),$$

for all $i_t$, and that all the other constraints are slack. A relaxed problem also assumes that lifetime utility of the truthteller and one-step deviator cannot exceed the autarchic value:

$$W(\tau_{i^t}, i_t) \geq V_{\text{AUT}}(i_t) \tag{5}$$

$$W(\tau_{i^t}, i_t + 1) \geq V_{\text{AUT}}(i_t + 1). \tag{6}$$

As we shall show, the relaxed problem has an advantage that its recursive formulation involves only two continuous dimensions, one for the truthteller, and one for an agent whose shock is one step higher (since only this agent could have reported the same shock as the truthteller). The recursive formulation is studied in the next section in two steps. First, it is shown that the set of lifetime utility of the truthteller and the one-step-down deviator that are obtainable for some $i = 1, \ldots, N - 1$

$$\mathcal{V}_i^* = \{W(\tau, i), W(\tau, i + 1) \mid \text{there exists } \tau \text{ s.t. (4) -- (6) holds}\} \tag{7}$$

has a recursive structure in the style of Abreu, Pearce, and Stacchetti (1990). In the second step, the principal’s cost function of the relaxed problem is shown to satisfy a dynamic program. The state space of the dynamic program is be represented by the sets $\mathcal{V}_i^*$.
3.2 Recursive Formulation of the Relaxed Problem

An allocation in the recursive formulation is given by \( \{\tau_i, w_i, \hat{w}_i\}_{i=1}^N \), where \( \tau_j \geq 0 \) denotes a transfer to an agent who currently reports shock \( j \), \( w_i \) corresponds to the truth-teller’s continuation utility, and \( \hat{w}_{i-1} \) corresponds to a continuation utility of type \( i \) that reports \( i - 1 \). Incentive compatibility requires that the agents prefer to tell the truth about their shock to reporting one step lower shock:

\[
U(y^i + \tau_i) + \beta w_i^i \geq U(y^{i-1} + \tau_{i-1}) + \beta \hat{w}_{i-1}^i \quad \forall i = 2 \ldots N.
\]

Before proceeding further, we will change the notation as follows: Let \( u_i = U(y^i + \tau_i) \) for all \( i = 1, \ldots, N \). The current utility of an agent who receives income \( y^i \), but reports \( y^{i-1} \) is then given be \( \psi(u_{i-1}) \), where \( \psi(u) = U(\delta + U^{-1}(u)) \). The function \( \psi \) is increasing in \( u \), and satisfies \( \psi(0) = 0 \), \( \psi(-\infty) = U(\delta) \) and \( \psi(u) > u \) for any \( u < 0 \).

To get additional properties, we assume:

**Assumption 1.** Either \( -\frac{U''(c)}{U'(c)} \) is decreasing in \( c \), or \( -\frac{U''(c)}{U'(c)} \) is decreasing in \( c \).

We then have the following result:

**Lemma 2.** If Assumption 1 holds then \( \psi(u) \) is convex in \( u \).

Redefine the allocation by replacing the transfers by the current period utility \( u \). The allocation in the relaxed problem is thus given by \( (u, w, \hat{w}) = \{u_i, w_i, \hat{w}_i\}_{i=1}^N \). The incentive compatibility constraint of the relaxed problem can then be written as

\[
u_i + \beta w_i \geq \psi(u_{i-1}) + \beta \hat{w}_{i-1} \quad \forall i = 2 \ldots N. \tag{8}\]

Note that there is no incentive constraint for \( i = 1 \), because no agent has a lower income.

It is also required that reporting truthfully and choosing autarchy at the beginning of the next period cannot be optimal:

\[
w_i \geq V^{\text{AUT}}(i), \quad i = 1, \ldots, N. \tag{9}\]
In addition, misreporting the current shock and then choosing autarchy cannot be optimal as well:

\[ \hat{w}_{i-1} \geq V^{\text{AUT}}(i), \quad i = 2, \ldots, N. \]  

(10)

Let \( \mathcal{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_{N-1}) \) be an arbitrary collection of sets where \( \mathcal{V}_i \subseteq \mathbb{R}^2 \). An allocation is said to be \textbf{admissible} with respect to \( \mathcal{V} \) if it satisfies the incentive compatibility constraint (8), the lower bound constraints (9) and (10), and

\[ (w_i, \hat{w}_i) \in \mathcal{V}_i \quad \forall i = 1 \ldots N - 1. \]

That is, the set \( \mathcal{V}_i \) is the set of pairs of the continuation utility a type \( i \) agent gents, and the continuation utility that a type \( i + 1 \) gets, if he reports \( i \). Note that there is no set \( \mathcal{V}_N \), because no agent can lie downwards, and report \( N \) at the same time.

Denote the lifetime utility of the agent who truthfully reported to have a shock \( i_- \) last period by \( v \), and the lifetime utility of a liar who reported a shock \( i_- \) last period but in fact had a shock \( i_- + 1 \) by \( \hat{v} \). They are given by

\[ v = \sum_{i=1}^{N} (u_i + \beta w_i) \pi(i|i_-) \]  

(11)

\[ \hat{v} = \sum_{i=1}^{N} (u_i + \beta \hat{w}_i) \pi(i|i_- + 1). \]  

(12)

If an allocation \((u, w, \hat{w})\) is admissible with respect to \( \mathcal{V} \) and delivers such a pair of lifetime utilities, it is said to \textbf{support} \((v, \hat{v})\) given \( \mathcal{V} \) and \( i_- \). This defines an operator \( T \) that for each \( i_- \) generates a new set \( TV_{i_-} \) as follows:

\[ TV_{i_-} = \{(v, \hat{v}) | \text{There exists an allocation that supports } (v, \hat{v}) \text{ given } \mathcal{V} \text{ and } i_- \}. \]

A key result of this section shows that the collection of sets \( \mathcal{V}^* = (\mathcal{V}_1^*, \ldots, \mathcal{V}_{N-1}^*) \) defined in (7) is a fixed point of the operator \( T \):

\[ TV^* = V^*. \]
The set of lifetime utility pairs therefore has a recursive representation. This is necessary for the recursive formulation of the social planner’s problem, where the sets $V^*_i$ will form its state space.

### 3.2.1 Characterizing $V^*$

A convenient property of the collection of sets $V^*$ is that they are related in a simple one directional way. To solve for $V^*_1$, no other set is needed. To solve for $V^*_2$, only $V^*_1$ is needed, and so on. $V^*_i$ is also clearly nonempty, since $(0,0) \in V^*_i$ for all $i$. The following Proposition shows that $V^*_i$ is convex whenever the utility exhibits either decreasing absolute risk aversion, or decreasing relative risk aversion:

**Proposition 4.** If Assumption 1 holds then $V^*_i$ is convex for all $i$.

From now on, we will assume that Assumption 1 holds. Note that the convexity result depends crucially on the first-order approach. If there is any incentive compatibility constraint that binds in the opposite direction (e.g. agent $i$ lying to be $i+1$), then the result will not hold.

To further characterize the set $V^*_i$, we need to put more structure on the transition matrix. We assume that the transition matrix satisfies the *monotone likelihood ratio property*:

**Assumption 5 (MLRP).** $\frac{\pi(j|i+1)}{\pi(j|i)}$ is increasing in $j$.

MLRP in our context means that the difference between the deviator’s and the truth-teller’s probability will be the largest for the largest shock, regardless of what the last period shock was. As we shall see, this property will be critical for determining the upper contour of $V$. MLRP implies the following:

**Lemma 6.** If MLRP holds, then $\Pi(i|i_-) \geq \Pi(i|i_- + 1)$ $\forall i, i_-$. 

MLPR thus implies a very natural definition of persistence: For any shock $i$, the probability of getting shocks lower then $i$ is decreasing in $i_-$. In fact, MLRP is
stronger than persistence. For the results that follow it would not be enough to assume that $\Pi(i|i_-) \geq \Pi(i|i_- + 1) \quad \forall i, i_-.$

A very convenient way of characterizing the set of admissible utilities $V^*_i$ is to characterize its the lower and upper contour. Define them by

$$V^*(v,i) = \min\{\hat{v}|(v,\hat{v}) \in V^*_i\},$$
$$V'^*(v,i) = \max\{\hat{v}|(v,\hat{v}) \in V^*_i\}.$$ 

Next Proposition provides a sharp characterization of the sets $V^*$ if the lower bound on utility is minus infinity.

**Proposition 7.** Suppose that $V^{AUT}(i) = -\infty$ for all $i = 1, \ldots N$. Let $i \in 1, \ldots N - 1$. The boundaries of the set $V^*_i$ are given by

$$V^*(v,i) = q_i v$$
$$V'^*(v,i) = v,$$

where $q_i = \frac{\pi(1|i+1)}{\pi(1|i)}$.

The Proposition shows that the sets $V^*_i$ for $i = 1, \ldots, N - 1$ are cones, and provides a complete characterization of the cones. The blue cone in Figure (1) is a typical set of admissible utilities $V_i$.

The intuition behind the shape of the lower contour of $V^*_i$ is the following: Any feasible allocation that is independent of the report delivers $\hat{v} = v$. Hence it must be true that the lower bound is below $v$. On the other hand, the deviator has always the option of pretending he is of the lower type. If he does so, he consumes all the transfers of the lower type. However, since his past endowment is higher and MLRP holds, Lemma 11 implies that he can secure himself at least $\hat{v} \geq v$. Hence the lower contour is above $v$. Taken together, the lower contour of $V^*_i$ equals $v$.

To prove that the upper contour has the property given in Proposition 7, ignore first the incentive compatibility constraint and solve for the resulting upper bound.
The solution is to assign lifetime utility $u_i + \beta w_i$ as low as possible to a state where the deviator is the least likely to be relative to the truthteller, and zero otherwise. Given MLRP, this state is the lowest state $i = 1$. But such an allocation satisfies the incentive compatibility constraint (8) because agents with shock $i = 3, \ldots, N$ are indifferent between reporting their shock and a shock $i - 1$, while an agent with shock $i = 2$ strictly prefers to tell the truth to reporting $i = 1$. Hence the upper bound on the upper contour coincides with the upper contour. One can easily verify that the upper contour then takes the form given in Proposition 7.

Note that Proposition 7 depends critically on MLRP, but does not depend on the convexity assumption. On the other hand, Proposition 4 does not depend on MLRP, but depends on the convexity assumption. Depending on which assumption holds, it is possible to have a situation such that $V_i^*$ is convex but does not have linear boundaries, or such that $V_i^*$ has linear boundaries, but is not convex. Finally, note that if the shocks are i.i.d., $\pi i | j + 1 \pi i | j = 1$ and the cone shrinks to a line with a slope of
-1. That is, the deviator’s utility is always the same as the truthteller’s utility.

Allocations that support the upper and lower contour are given by the following corollary. The proof follows from the arguments in the proof of Proposition (7).

**Corollary 8.** Let $i \in 1, \ldots N - 1$.

1. Suppose that the promised utility pair $(v, \hat{v})$ is on the upper contour. Then any allocation such that $u_i$ and $w'_i$ solve $u_i + \beta w'_i = \frac{v}{\gamma}$, $\hat{w}'_1 = \frac{\pi_i(i + 1)}{\pi_i(1)} w'_1$, $u_i = w'_i = 0$ for all $i = 2, \ldots N$ supports $(v, \hat{v})$.

2. Suppose that the promised utility pair $(v, \hat{v})$ is on the lower contour. Then any allocation such that $u_1 = 0$, $w'_1 = \frac{v}{\beta}$, $\hat{w}'_1 = w'_1$, and $u_i$ and $w'_i$ solve $u_i + \beta w'_i = v$ for all $i = 2, \ldots N$ supports $(v, \hat{v})$.

Note also there is an asymmetry in the supporting allocations. For the upper boundary, one can choose the continuation utilities to be no smaller than $v$. That is, imposing a lower bound on promised utilities greater than minus infinity will not affect the upper contour above that lower bound. The same is not true of the lower bound. Imposing a lower bound on promised utility greater than minus infinity will therefore affect the whole lower contour. In particular, the boundary will shift inwards. It is going to be nonlinear. We can, however, partially characterize it as follows:

**Proposition 9.** Let $i \in 1, \ldots N - 1$. The upper and lower boundaries of the set $\mathcal{V}^*_i$ satisfy

$$\mathcal{V}^*(v, i) = q_i v,$$

$$\mathcal{V}^*(\mathcal{V}^{AUT}(i), i) = \mathcal{V}^{AUT}(i + 1).$$

In addition, the constraint (10) never binds.

The green set in Figure (1) is a typical set of admissible utilities $\mathcal{V}_i$ when the value of autarchy is finite. Note that $\mathcal{V}^*_i$ is inside of the set of admissible utilities when the lower bound is minus infinity. That is intuitive: decreasing the autarchic value relaxes the constraints on the insurance provider, and allows him to achieve more.
While there is no closed form solution for the lower boundary, we can still characterize its value at the autarchic promised utility \( V^{\text{AUT}}(i) \). The Proposition shows that the value of the lower boundary is the autarchic value of the deviator \( V^{\text{AUT}}(i + 1) \). Remarkably, the result holds even if the constraint (10) is not explicitly imposed. Since the lower boundary is increasing, it follows that once the constraint (9) is imposed, it is never optimal to deviate jointly by misreporting in the current period and then choosing autarchy. Constraint (10) can thus be ignored.

We also have the following corollary regarding monotonicity of lifetime utilities:

**Corollary 10.** \( u_i + \beta w_i \) is increasing in \( i \).

### 3.3 Principal’s problem after the initial period

After the agent has signed up with the principal, the efficient contract after the initial period solves the following dynamic program. For each \( i = 1, \ldots, N \) the value function satisfies

\[
C(v, \hat{v}, i_{\_}) = \min_{(u, w, \hat{w})} \sum_{i=1}^{N} [U^{-1}(u_i) - y^i + \frac{1}{R} C(w_i, \hat{w}, i)]\pi(i|i_{\_})
\]

subject to a constraint that

\[
(u, w, \hat{w}) \text{ supports } (v, \hat{v}) \text{ given } \mathcal{V}^* \text{ and } i_{\_}.
\]

The constraint set can be equivalently be written as a requirement that the constraints (8) - (12) all holds. Note that \( C(v, \hat{v}, N) \) is independent of \( \hat{v} \) since there is no threat keeping constraints if the last period shock is \( N \).

Principle of Optimality for the principal problem’s after the first period implies that \( C \) solves its corresponding sequence problem where the principal minimizes

\[
\sum_{t=0}^{\infty} \sum_{i_t} R^{t-t_0} \tau_t(i_t) \mu(i_t|i_{\_\_}) \text{ by choosing a transfer scheme subject to constraint (4) - (6), and a requirement that } W(\tau, i) = v \text{ and } W(\tau, i + 1) = \hat{v}.
\]
3.4 Principal’s problem in the initial period

In the initial period problem the principal is not bound by any past promises and so chooses a threat utility that minimizes the costs among all feasible threat utilities.

\[
C^*(v, i) = \min_{\hat{v}} \{ C(v, \hat{v}, i) \mid \hat{v} \in V^*_i \}.
\]

3.5 Validity of the Relaxed Problem

Once the relaxed problem has been computed, one can check numerically whether the relaxed problem is valid. Compute first the lifetime utility of an agent who has reported \( j \) in the previous period, but had in fact earnings \( i \):

\[
\hat{V}(v, \hat{v}, j|i) = \sum_{i=1}^{N} [u_i(v, \hat{v}, j) + \beta w_i(v, \hat{v}, j)]\pi(i|i) \quad \forall j, i = 1 \ldots N. \tag{13}
\]

Note that, by construction, \( \hat{V}(v, \hat{v}, j|j) = v \) and \( \hat{V}(v, \hat{v}, j|j+1) = \hat{v} \). But \( \hat{V} \) is important because it computes the continuation utility for all possible types.

Let also \( \tau_i = U^{-1}(u_i) - y^i \) be the optimal transfers. Then the incentive compatibility constraint can be checked by computing for all \( i, j = 1, \ldots, N \)

\[
D^i_j = U(y^i + \tau_i) + \beta w_i - U(y^j + \tau_j) - \beta \hat{V}(w_j, \hat{w}_j, j|i). \tag{14}
\]

The first-order approach is valid if for all state variable values

\[
D^i_j \geq 0 \quad \forall i, j = 1, \ldots, N.
\]

Note that a lie affects the agent in two ways. First, it changes the current transfer. Second, it changes the truth-teller’s continuation utility, and the continuation utility of an agent who lied one step down. That in turn affects the continuation utility of all possible types. While it is relatively easy to guess that a lie will increase the current transfer, it is much harder to predict a precise way in which the agents’ continuation utilities will be affected. Numerical verification is therefore critical.
4 General Equilibrium

We close the model by endogenizing the interest rate $R$. Output is produced by a standard Cobb-Douglas aggregate production function, and is divided between consumption and investment:

$$E[c_t] + K_{t+1} = AK_t^\alpha E[y_t]^{1-\alpha} + (1 - \delta)K_t,$$

where $\delta$ is the depreciation rate of capital, and $E[y_t] = 1$ are the aggregate units of labor. We focus on stationary equilibria, and without loss of generality choose $A$ to be such that the wage rate equals one.

5 Quantitative Evaluation

We redefine the state variable as follows. Define $h$ to be the relative position of $\hat{v}$ in the cone:

$$\hat{v} = v(1 - h + q_1 h)$$

That is, if $\hat{v} = v$ then $h = 0$ and if $\hat{v} = q_1 v$ then $h = 1$. We will call $h$ a relative threat utility. The advantage of working with a relative threat utility is that the magnitude of $h$ is independent of the magnitude of $v$, and one thus can easily compare the relative threat utility for different promised utilities.

We replicate Blundell, Pistaferri, and Preston (2008) imputation to create a panel data on consumption and earnings. The income process for each household $i$ is given by

$$\log Y_{it} = Z_{it}' \varphi_t + y_{it}$$

where $t$ indexes time and $Z$ is a set of income characteristics observable and known by consumers at time $t$. [details later] The residual $y_{it}$ follows an autoregressive process

$$y_{i,t} = \rho y_{i,t-1} + \varepsilon_{it}.$$
We estimate the persistence parameter \( \rho \) and the variance of the innovation \( \sigma_\varepsilon \), and obtain \( \rho = 0.777 \) and \( \sigma_\varepsilon = 0.438 \).

In the benchmark calibration we discretize the shock process set by two shock values \( y^L \) and \( y^H \) and, given the estimated process for earnings, determine the shock values, and the probability of remaining in the same state \( \gamma \).

The utility is given by \( U(c) = \frac{c^{1-\sigma}}{1-\sigma} \) with \( \sigma = 2 \), and the capital share is selected to be \( \alpha = 0.32 \). We choose the discount rate \( \beta \) and the depreciation rate \( \delta \) to be such that the equilibrium interest rate is 4\%, and the capital-output ratio is 2.6.

The promised utility is allowed to vary between the autarchic value \( V_{iAUT} \). We also impose an upper bound \( \bar{v} = -22 \) for computational reasons. The interval includes the lifetime utility that delivers zero costs in the first best allocation, which is \(-25\).

### 5.1 Private Information vs. data vs. Bewley

We estimate the marginal propensity to consume with respect to total earnings (as in Krueger and Perri (2004)):

\[
\phi^{\text{diff}} = 1 - \frac{\text{cov}(\Delta c_t, \Delta y_t)}{\text{var}(\Delta y_t)}.
\]

where \( \Delta c_t \) is the difference in log consumption, and \( \Delta y_t \) is the difference in log earnings. The measure is zero in autarky and one under perfect risk sharing. It therefore forms a convenient metric through which model predictions can be summarized and compared with the data.

<table>
<thead>
<tr>
<th></th>
<th>( \phi^{\text{diff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>0.85</td>
</tr>
<tr>
<td>Private Information</td>
<td>0.94</td>
</tr>
<tr>
<td>Bewley, zero borrowing constraint</td>
<td>0.66</td>
</tr>
<tr>
<td>Bewley, natural borrowing constraint</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Table 1: Risk Sharing Coefficients \( \phi^{\text{diff}} \)
Table (1) shows the risk sharing coefficients, and compares them with the degree of risk sharing found in the data. We compare the data with our private information economy and with models with exogenously incomplete markets (“Bewley models”), see Aiyagari (1994) or Huggett (1993). We analyze two versions of the Bewley models, one with a zero borrowing constraint and one with a natural borrowing constraint.

The private information economy exhibits more risk sharing than the data. On the other hand, the Bewley economy exhibits less risk sharing, and in case of zero borrowing constraint, significantly less so.

In addition to the risk sharing coefficients we investigate how skewed the resulting distribution of consumption is. In particular, we look at excess skewness of consumption $\mu$, defined as

$$\mu = \text{skew}(\ln c) - \text{skew}(\ln y).$$

Thus, if $\mu > 0$ then consumption is more right skewed than earnings, while is $\mu < 0$, it is less right skewed than earnings. Table 2 shows our findings:

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>0.65</td>
</tr>
<tr>
<td>Private Information</td>
<td>0.70</td>
</tr>
<tr>
<td>Bewley, zero borrowing constraint</td>
<td>-0.51</td>
</tr>
<tr>
<td>Bewley, natural borrowing constraint</td>
<td>-1.10</td>
</tr>
</tbody>
</table>

Table 2: Excess Skewness Coefficients $\mu$

Both the data and the private information economy exhibit a significantly positive excess skewness of consumption. In both cases, the excess skewness is quantitatively similar. Thus, the mean of the distribution of consumption is relatively more to the right of its mode. On the other hand, both Bewley economies exhibit a significantly negative excess skewness of consumption. The excess skewness of consumption allows us to sharply differentiate between both types of models, and shows that the distribution of the private information model is closer to the data.
6 Conclusions

[TBD]
References


Attanasio, O. and N. Pavoni (2007). Risk sharing in private information models with asset accumulation: Explaining the excess smoothness of consumption. working paper, University College London. 4, 5


7 Appendix

Proof of Lemma 2. Differentiating $\psi(u)$ twice one gets that

$$
\psi''(u) = \frac{U'(U^{-1}(u) + \delta)}{U'(C(u))^2} \left[ \frac{-U''(U^{-1}(u)) - U''(U^{-1}(u) + \delta)}{U''(U^{-1}(u))} \right].
$$

The assumption that $\frac{-U''(c)}{U'(c)}$ is decreasing in $c$ implies that $\psi''(u) > 0$. Rearranging terms, one gets

$$
\psi''(u) = \frac{U'(U^{-1}(u) + \delta)}{U^{-1}(u)U''(U^{-1}(u))^2} \times \left[ \frac{-U^{-1}(u)U''(U^{-1}(u))}{U'(U^{-1}(u))} - \frac{U^{-1}(u)}{U^{-1}(u) + \delta} \frac{U''(U^{-1}(u) + \delta)}{U'(U^{-1}(u) + \delta)} \right].
$$

The assumption that $\frac{-cU''(c)}{U'(c)}$ is decreasing in $c$ and the fact that $\frac{c}{c+\delta} < 1$ imply again that $\psi''(u) > 0$. \hfill \Box

Proof of Proposition 4. The result follows from the fact that the promise keeping constraint (11), the threat keeping constraint (12) and the left hand side of the incentive
compatibility constraint (8) are all linear in $u$ and $w$, and that the right-hand side of (8) is convex in $u$ and $w$ by Lemma 2 under the assumptions of the Proposition.

Proof of Proposition (3). Let $(v, \hat{v}) \in V^*_i$. Then there exists some transfer policy $\tau$ such that it satisfies (4) - (6), $W(\tau, i) = v$, and $W(\tau, i + 1) = \hat{v}$. Define an allocation $(u, w, \hat{w})$ by $u_i = U(y^i + \tau_i)$, $w_i = W(\tau^i, i)$ and $\hat{w}_i = W(\tau^i, i + 1)$. Since $\tau^i$ satisfies (4) - (6), one has $(w_i, \hat{w}_i) \in V^*_i \quad \forall i = 1, \ldots, N - 1.$ Since $\tau$ satisfies (4), the constraint (8) holds. Similarly, (9) and (10) holds as well. Hence $(u, w, \hat{w})$ is admissible w.r.t. $V^*$. One can easily show that it supports $(v, \hat{v})$ at $i$. Hence $(v, \hat{v}) \in TV^*_i$ and, consequently, $V^*_i \subseteq (TV^*)_{i + 1}$.

To show the reverse implication, suppose that $(v, \hat{v}) \in TV^*_i$. Then there exists some allocation $(u, w, \hat{w})$ such that the constraint (8) holds and $(w_i, \hat{w}_i) \in V^*_i$ for all $i = 1, \ldots, N - 1$. Define a transfer policy $\tau$ as follows. Let $\tau_1(i) = U^{-1}(u_i) - y^i$. Since $(w_i, \hat{w}_i) \in V^*_i$ for all $i = 1, \ldots, N - 1$, there is some transfer policy $\tilde{\tau}^i$ such that $W(\tilde{\tau}^i, i) = w_i$ and $W(\tilde{\tau}^i, i + 1) = \hat{w}_i$. Define $\tau^i = \tilde{\tau}^i$. It is easy to see that $\tau$ defined in this way satisfies (4) - (6), that $W(\tau, i) = v$, and that $W(\tau, i + 1) = \hat{v}$. Thus, $(v, \hat{v}) \in V^*_i$ and so $(TV^*)_{i + 1} \subseteq V^*_i$.

Proof of Lemma 6. Suppose that $j_1 \geq j_0$. Then

$$\pi(j_1 | i + 1)\pi(j_0 | i) \geq \pi(j_1 | i)\pi(j_0 | i + 1)$$

Summing those inequalities over $j_0 = 1 \ldots j_1$, one gets

$$\pi(j_1 | i + 1)\Pi(j_1 | i) \geq \pi(j_1 | i)\Pi(j_1 | i + 1).$$

Similarly, summing over $j_1 = j_0 + 1 \ldots N$,

$$(1 - \Pi(j_0 | i + 1))\pi(j_0 | i) \geq (1 - \Pi(j_0 | i))\pi(j_0 | i + 1).$$
Since both equations hold for any \( j_0 \) and \( j_1 \), we can write, for any \( j \),
\[
\frac{1 - \Pi(j|i + 1)}{1 - \Pi(j|i)} \geq \frac{\pi(j|i + 1)}{\pi(j|i)} \geq \frac{\Pi(j|i + 1)}{\Pi(j|i)},
\]
and so \( \Pi(j|i) \geq \Pi(j|i + 1) \) for any \( j \). \( \Box \)

To prove Proposition 7, the following lemma will be needed.

**Lemma 11.** If MLRP holds, then for any increasing vector \((f_1, \ldots, f_N)\) and any \( i_- = 1, \ldots, N - 1 \),
\[
\sum_{i=1}^{N} f_i \pi(i| i_- + 1) \geq \sum_{i=1}^{N} f_i \pi(i| i_-).
\]

**Proof.** Suppose by contradiction that
\[
\sum_{i=1}^{N} f_i \pi(i| i_- + 1) < \sum_{i=1}^{N} f_i \pi(i| i_-). \tag{15}
\]
Fix \( i_- \) and define \( g_i = \frac{\pi(i| i_- + 1)}{\pi(i| i_-)} \). Since MLRP holds, the vector \( g \) is increasing. Now rewrite (15) using \( g \):
\[
\sum_{i=1}^{N} f_i g_i \pi(i| i_-) < \sum_{i=1}^{N} f_i \pi(i| i_-) = \sum_{i=1}^{N} f_i \pi(i| i_-) \sum_{i=1}^{N} g_i \pi(i| i_-).
\]
That is, \( E(fg) < E(f)E(g) \), and so \( f \) and \( g \) are strictly negatively correlated. This is a contradiction, since both \( f \) and \( g \) are increasing. \( \Box \)

**Proof of Proposition 7.** The lower bounds on utility constraint can be dropped since the lower bound is minus infinity.

i) We will first prove that \( V(v, i_-) \leq v \). Consider an allocation that assigns \( u_i = 0 \) for all \( i = 1 \ldots N \) (by setting \( \tau_i = \infty \)) and \( w_i = \frac{v}{\beta} \). This allocation is trivially
incentive compatible since it is independent of the report. It also delivers \( \hat{v} = v \).

Hence \( V(v, i_-) \leq v \).

We will next show that \( V(v, i_-) \geq v \). The proof uses induction, and a limit argument.

Consider a truncated problem where the agents live only for \( T \) periods \( t = 1, \ldots, T \).

Let \( (u_{i,t}^{(T)}, w_{i,t+1}^{(T)}) \) be an allocation of the truncated problem in period \( t = 1, \ldots, T \), with \( w_{i,T+1}^{(T)} = 0 \) by truncation. Let also \( V^{(T)}_{t}(v, i_-) \) be the lower contour of the truncated problem in period \( t = 1, \ldots, T \). The incentive compatibility constraint (8) in the last period \( T \) together with properties of \( \psi \) imply that

\[
\frac{\alpha}{
\begin{array}{c}
u_{i,T} \geq \psi(u_{i-1,T}) \geq u_{i-1,T}, \\
\text{and so} (u_{1,T}, \ldots, u_{N,T}) \text{ is an increasing vector. By Lemma 11, for any } i_- = 1, \ldots, N - 1, \\
V^{(T)}_{t}(v, i_-) = \sum_{i=1}^{N} u_{i,T} \pi(i|i_- + 1) \geq \sum_{i=1}^{N} u_{i,T} \pi(i|i_-) = v.
\end{array}
\]

Now assume that \( V^{(T)}_{t+1}(v, i_-) \geq v \) for some \( t \in (1, \ldots, T) \), for all \( i_- = 1, \ldots, N - 1 \). The incentive compatibility constraint in period \( t \) implies

\[
u_{i,t} + \beta w_{i,t+1} \geq \psi(u_{i-1,t}) + \beta V^{(T)}_{t+1}(w_{i-1,t+1}, i-1) \geq u_{i-1,t} + \beta w_{i-1,t+1}.
\]

Hence \( u_{i,t} + \beta w_{i,t+1} \) increases in \( i \). By Lemma 11, for any \( i_- = 1, \ldots, N - 1, \\
V^{(T)}_{t}(v, i_-) = \sum_{i=1}^{N} (u_{i,t} + \beta w_{i,t+1}) \pi(i|i_- + 1) \geq \sum_{i=1}^{N} (u_{i,t} + \beta w_{i,t+1}) \pi(i|i_-) = v.
\]

Hence, by induction, \( V^{(T)}_{t}(v, i_-) \geq v \) for all \( t = 1, \ldots, T \), all \( i_- = 1, \ldots, N - 1 \). Since \( T \) was arbitrary we have

\[
V(v, i_-) = \lim_{T \to \infty} V^{(T)}_{1}(v, i_-) \geq v.
\]

ii) Consider an upper bound on the upper contour that ignores the incentive compat-
ibility constraint. The solution is such that the principal assigns the lowest possible utility to a state $i$ such that

$$i = \arg \min_{k=1 \ldots N} \frac{\pi(k|i_{-} + 1)}{\pi(k|i_{-})}.$$  

MLRP implies that $i = 1$. Thus

$$u_1 + \beta w_1 = \frac{v}{\pi(1|i_{-})}$$

$$u_i + \beta w_i = 0, \quad i \neq 1.$$  

It follows that $u_i = w_i = 0$ for all $i > 1$. Set $w_1 = 0$. Then $u_1 < 0$. Since $\psi(0) = 0$ and $\psi$ is increasing, the incentive compatibility constraint (8) is satisfied for all $i = 2, \ldots, N$. The upper contour is then given by

$$\overline{V}(v,i_{-}) = (u_1 + \beta w_1)\pi(1|i_{-} + 1) = \frac{\pi(1|i_{-} + 1)}{\pi(1|i_{-})}v,$$  

where the last equality follows from substituting in the expression for $u_1 + \beta w_1$.  

\textit{Proof of Proposition (9).} Ignore the constraint (10). Properties of the upper boundary follow from Corollary (8). For the lower boundary assume that $v = V^{\text{AUT}}(i_{-})$. Note that it follows from Corollary (8) that the lower bound on the continuation utility will bind, i.e.

$$w_i = V^{\text{AUT}}(i) \quad \forall i = 1, \ldots, N.$$  

It then follows from (11) and the definition of autarchy that the period utility has to satisfy

$$\sum_{i=1}^{N} u_i \pi(i|i_{-}) = \sum_{i=1}^{N} U(y^i)\pi(i|i_{-}).$$  

Since the continuation utility is independent of the current report, the allocation is
incentive compatible only if $u_i = U(y^i)$ for all $i = 1, \ldots, N$. It then follows that

$$V^* \left(V^{\text{AUT}}(i_-, i_-) = \sum_{i=1}^{N} (u_i + \beta w_i) \pi(i|i_- + 1)$$

$$= \sum_{i=1}^{N} (U(y^i) + \beta V^{\text{AUT}}(i)) \pi(i|i_- + 1)$$

$$= V^{\text{AUT}}(i_- + 1).$$

In addition, it follows that the constraint (10) is satisfied automatically, and hence does not bind. \qed

**Proof of Corollary 10.** We have for all $i = 2, \ldots, N$,

$$u_i + \beta w_i \geq \psi(u_{i-1}) + \beta \hat{w}_{i-1}$$

$$\geq \psi(u_{i-1}) + \beta \hat{V}(w_{i-1}, i - 1)$$

$$\geq u_{i-1} + \beta w_{i-1}$$

where the first inequality follows from (8), the second one from definition of the lower contour, and the third one from Proposition (7) and the properties of $\psi$. \qed

### 7.1 Appendix B: Fernandes-Phelan Recursive Formulation

Except for the initial period, the principal is constrained by a vector of promised utilities $(v_1, \ldots, v_N)$. An allocation is given by $\{\tau_j, w^i_j\}_{i,j=1}^{N}$, where $\tau_j \geq 0$ denotes a transfer to an agent who currently reports shock $j$, and $w^i_j \leq 0$ is his continuation utility, where $j$ is the report of the agent and $i$ is the shock. The incentive compatibility requires that the agents prefer to tell the truth about their shock to any other report:

$$U(y^i + \tau_i) + \beta w^i_i \geq U(y^j + \tau_j) + \beta w^i_j \quad \forall i, j = 1 \ldots N. \quad (16)$$

Let $\mathcal{V} \subseteq R^N_\infty$. An allocation is said to be admissible with respect to $\mathcal{V}$ if it satisfies the incentive compatibility constraint (16), and the continuation utilities are drawn
from the set $\mathcal{V}$:

$$(w_i^1, \ldots, w_i^N) \in \mathcal{V} \quad \forall i = 1 \ldots N.$$ 

An allocation generates a lifetime utility $v_{i-}$ to an agent who has received a shock $i-$ last period, given by

$$v_{i-} = \sum_{i=1}^{N} \left[ U(y^i + \tau_i) + \beta w_i^1 \pi(i|i-) \right] \quad \forall n = 1 \ldots N. \quad (17)$$

If an allocation admissible with respect to $\mathcal{V}$ satisfies (17), it is said to support $(v_1, \ldots, v_N)$ given $\mathcal{V}$. The set of all lifetime utility vectors that are supported by some allocation that is admissible with respect to $\mathcal{V}$ defines an operator $\mathcal{T}$:

$$\mathcal{T} \mathcal{V} = \{(v_1, \ldots, v_N) \mid \text{There exists an allocation that supports $(v_1, \ldots, v_N)$ given $\mathcal{V}$}\}.$$ 

Let $\mathcal{V}^*$ be the largest fixed point of the operator $\mathcal{T}$. The efficient contract can be found as follows. The social planner’s cost function $C : \mathcal{V}^* \times N \to R$ satisfies the following Bellman equation:

$$C(v_1, \ldots, v_N, i-) = \min_{(\tau_j, w_j^i)_{i,j=1}^N} \sum_{i=1}^{N} \left[ \tau_i + \frac{1}{R} C(w_i^1, \ldots, w_i^N, i) \pi(i|i-) \right]$$

subject to a constraint that

$$(\tau_j, w_j^i)_{i,j=1}^N \text{ supports $(v_1, \ldots, v_N)$ given $\mathcal{V}^*$}.$$ 

The value function therefore has $I$ continuous dimensions. That is in general intractable except for a case when $I$ is very small.