# Pivots Versus Signals in Elections*

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**Abstract**

We consider a two-period model of elections in which voters have private information about their policy preferences. A first-period vote can have two types of consequences: it may be pivotal in the first election and it provides a signal that affects candidates' positions in the second election. Pivot events are exceedingly unlikely, but when they occur the effect of a single vote is enormous. In contrast, vote totals always have some signaling effect, but the effect of a single vote is small. We investigate which effect – pivot or signaling – drives equilibrium voting behavior in large electorates.

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# 1 Introduction

In nearly all models of voting the payoff from casting a particular ballot hinges exclusively on pivot events. These are events in which the election is tied or nearly tied, so that a single vote can determine the outcome. In decision-theoretic models a voter decides whether and how to vote based on exogenous probabilities of ties between candidates. Game-theoretic models endogenize the equilibrium probability that a vote is pivotal. Several recent influential papers focus on information that a voter can infer from the fact that he is pivotal, and analyze electoral equilibria when voters condition on being pivotal.[1]

The pivot-based literature on elections is vast, but the models have two features in common: (i) when a voter is pivotal, the action she takes has a large impact on her payoff, but (ii) pivot events are very unlikely. The large impact is due to the fact that in a pivot event, a single vote can determine the outcome of the election. The low probability arises from the fact that in a large election it is exceedingly unlikely that two candidates will receive the same number of votes or differ by exactly one vote.

Although pivot based models dominate the game-theoretic literature on elections, the infrequency of pivot events in all but the smallest elections raises a natural question: is electoral behavior driven by more than just concerns about being pivotal? If pivot events do not actually drive the calculus of voters then a large and growing literature on voting theory may be focused on second-order concerns.

Why would voters care about anything other than a pivot event? Consider, for example, the buildup to the 2006 midterm election in the United State. Pundits speculated that voters' dissatisfaction with President Bush's handing of the war in Iraq would cost the Republican party its majority in Congress. While Republicans' electoral losses may in fact have been a direct result of voters' desires to change the composition of the legislature, another explanation is that voters cast ballots for Democrats in order to send Bush a message, and encourage him to change policy.

An emerging literature, based on the intuition that vote totals matter in elections that don't end in a tie, offers an alternative perspective to the dominant pivot-based theories of elections. In this paper,

[1]Decision theoretic models include Downs [10], Tullock [25], Riker and Ordeshook [22], and Myerson and Weber [16]. Examples of game theoretic models include Palfrey and Rosenthal [19], Myerson [17] [18], Campbell [6], and Borgers [4]. Models involving information aggregation include Feddersen and Pesendorfer [11] [12] [13], Austen-Smith and Banks [1], Dekel and Piccione [9], and Battaglini [3].

we analyze a model of elections with both pivot and signaling motivations and show that the latter dominates with a large electorate.

Theorists of elections have explored several different ways that vote totals could affect downstream electoral or policy outcomes. One possible modelling approach is a common values setup, in which voters use their votes to convey information about the state of the world. Piketty [20] develops a two period model of referendum voting in which voters communicate policy information to each other as they vote. Razin [21] analyzes a model of mandates in which vote totals convey information to the winner of an election and thus affect the policies she enacts. In this model, the functional form of the election winner's response to signals is a primitive of the game. Razin investigates limiting behavior for large electorates and characterizes two types of equilibria. In one type of equilibrium voters' behavior is "conventional" in the sense that a voter whose private signal indicates that liberal policies are good tends to vote for a liberal candidate. In any limit of these conventional equilibria, the behavior of voters converges to coin flipping [21, Proposition 4(i)]. The other equilibria are "unconventional" since voters respond perversely to their private signals: upon observing information that favors liberal policies, a voter becomes more likely to vote for the *conservative* candidate.

The equilibria of the Piketty and Razin models suggest that a desire to influence the decisiveness of victory, and not just the identity of the winner, can remain in large elections. However, in both models all voters have identical preferences and the effects that their votes can have on future policy outcomes are determined directly by modelling assumptions, rather than being determined by equilibrium choices made by competing political elites who observe election outcomes.

Another modelling approach is a private values setup, in which voters can use their votes to affect future candidates' positions, and thus policy outcomes. Each election thus serves two purposes: to select a winner and to act as a poll about voters' preferences. In this vein, Meirowitz and Tucker [15] analyze a model of alternating parliamentary and presidential elections, in which voters use their votes to signal dissatisfaction with an incumbent and thereby induce him to exert costly effort to make himself more appealing in a subsequent election.

Castanheira [7] adapts Piketty's model to a private values setting, and uses it to analyze voting for losers in an election with four candidates. In his model, there are four possible distributions of voters in the electorate, and voters may choose to vote for a candidate who is almost certain to lose in the first election since their vote may determine which of four positions candidates will adopt in a second period election. In Castanheira's model, as in Piketty's model, the signaling effect of first period electoral behavior is based on the very low probability event that a vote is informationally pivotal, i.e., although there is signaling in these models, the signaling is fundamentally based on pivot events, which are extremely unlikely to occur.

In contrast, Shotts [24] develops a private values model of repeated elections in which each voter's actions, by conveying information about his preferences, always have some small effect on future policy outcomes. Shotts's main result is that there exists an equilibrium in which moderates abstain to signal that they are moderate even though voting is costless. In the model, small magnitude, high probability signaling effects work quite differently from large magnitude low probability pivot effects. However, in large elections, a single vote has a vanishingly small effect on politicians' beliefs about voter preferences, and Shotts does not address the question of whether signaling motivations are actually relevant in large elections.

Moreover, a voter may find herself cross-pressured when a vote for the candidate she prefers sends the wrong message to politicians, i.e., she may wish to vote for one candidate for signaling reasons and a different one for pivot reasons. Because both the likelihood that she is pivotal and the effect of a single vote on candidates' vote shares are small it is not clear how tradeoffs between pivot and signaling effects balance out in equilibria for large elections. Thus, although recent papers have moved beyond pivot-based theories of voting, they do not shed light on the question of whether equilibrium behavior in large repeated elections with private values is driven by pivot or signaling considerations.

In this paper, we analyze a private values model of repeated elections with both pivot and signaling motivations and show that the latter dominates with a large electorate. While our paper is related to that of Meirowitz [14], which focuses on signaling motivations in public opinion polls – showing that respondents are typically dishonest in equilibrium – it is, however, closest to that of Shotts [24]. Shotts analyzes elections with a fixed population and focuses on equilibria in which moderates abstain to signal that they are moderate. He does not focus on the tradeoff between pivot and signaling motivations and does not provide any asymptotic analysis.

Here, we analyze limiting behavior for large electorates, under the assumption that abstention is not an option. For two reasons, it turns out that the assumption that voters cannot abstain is not crucial for our results. First, in the Shotts model there also exist equilibria in which no types of voters abstain.[2] Second, an unpublished paper by Patrick Hummel builds on the present paper and Shotts [24] to study the limiting behavior of equilibria in which voters abstain. Hummel shows that although the type of equilibrium with abstention characterized by Shotts exists for any finite population, in the limit abstention vanishes and voter behavior converges to the behavior we characterize here.[3]

Having discussed the relevant literature, we now summarize our contribution. The key elements of our model are as follows. Each voter in our model has private information about his own policy preferences, and in each of two elections he casts a ballot for one of two available alternatives. Following the first election, two office-motivated candidates compete for a second office, by staking policy positions, and the second election is held. Candidates in the second period base their policy positions on beliefs about the distribution of preferences in the electorate. In equilibrium these beliefs are informed by the vote totals in the first election, and each voter's vote in the first period thus has a small signaling effect on second period policy.

Our main result is that as the electorate gets large the equilibrium converges to the equilibrium of a slightly different game in which the first period outcome is payoff irrelevant, as in the case of a first period poll. In other words, in large elections behavior is driven by the signaling motivation and not the pivot effect.

This result has potentially important implications for the literature on pivot-based models of elections, because most of the interesting equilibria in such models rely heavily on the fact that a voter only cares about events in which his vote is pivotal. In our model, in contrast, the effect of pivot events on equilibrium voter behavior is relatively unimportant compared to the effect of signaling concerns. At the very least, future research needs to take seriously the possibility that pivot events are not of first-order importance when rational voters take into account the future effects of their votes.

The paper proceeds as follows. Section 2 introduces the model and in Section 3 we present two

[2] This sort of equilibrium is supported by off-the-path beliefs that a voter who abstained is just like a voter who cast a ballot for one of the candidates.

[3] Hummel also derives a useful lower bound for the magnitude of signaling effects in large elections.

concrete examples of how signaling and pivot effects work. Section 4 proves equilibrium existence. Section 5 describes the intuition behind our main result, which is proved in Section 6. Section 7 discusses the result.

## 2 The Model

Consider an electorate with an odd number of voters $n \geq 3$. It will be convenient to use the fact that $n = 2m+1$ for some integer $m$. Let the set of voters be $N$, and let each voter $i \in N$ have an ideal point, $v_i \in [0,1]$. We assume that the ideal points are i.i.d. draws from a distribution function, $F(\cdot)$, which is strictly increasing and continuously differentiable, and has a continuous density, $f(\cdot)$, on the support $[0,1]$. Each voter's utility over policy, $x$, in a given period is $u_i(x) = -\gamma(|x - v_i|)$ where $\gamma : [0,1] \rightarrow \mathbb{R}_+$ is strictly increasing, convex and differentiable. Expected utility is defined only up to positive affine transformations, so the convenient assumption that $\gamma'(1) = 1$ is innocuous. The voter's total utility is simply the sum of his policy utility in the two periods.

In the first period election, two fixed alternatives are available. We denote the locations of the alternatives by $L, R \in [0,1]$ (with $L \leq R$). If voters care only about the first period, or are myopic, elimination of weakly dominated strategies yields a unique equilibrium, in which all voters to the left of $x_p = \frac{L+R}{2}$ vote $L$ and all voters to the right of $x_p$ vote $R$. We call this the *pivot cutpoint.* We, however, are interested in the dependencies across elections, and thus consider a model with two periods, building on Shotts [24].

In the second period, two office motivated candidates select policy platforms and then the electorate votes. The candidates are assumed to know only the distribution, $F(\cdot)$, from which the $n$ ideal points are drawn, the size of the electorate, $n$, and the voters' first period actions. From Calvert [5], we know that for a game in which two office motivated candidates believe that $F_{median}(\cdot)$ is the distribution of the median voter's ideal point, in any Nash equilibrium with weakly undominated voting the candidates will both locate at $F^{-1}_{median}(\frac{1}{2})$.[4] In the two-period signaling game that we study, in any Perfect Bayesian equilibrium, the distribution of the median depends on the first-period votes via Bayes' Rule. At any

[4] One such equilibrium has each voter flipping a fair coin when indifferent. Given strategies for the first period, the second period behavior is standard, and well understood (Calvert [5], Shotts [24]).

history in which $F_{median}(\cdot)$ is a distribution consistent with Bayes' Rule following the observed first period voting, the second period candidates both locate at the point $F^{-1}_{median}(\frac{1}{2})$. While Shotts [24] focuses on equilibria with abstention in the first period, we restrict the set of actions available to voters so that they must vote either $L$ or $R$; this enables us to focus on a particularly simple class of equilibria, involving only a single cutpoint. In particular, we focus on a class of equilibria in which all voters use the same type specific monotone voting strategy.[5]

In such an equilibrium, first-period voting strategies are characterized by a cutpoint $x_c$ with voters to the left ( $v_i < x_c$) voting $L$ and voters to the right ($v_i > x_c$) voting $R$. If voting satisfies this cutpoint, i.e. it is monotone, then the number of votes for $R$, denoted $\#R$, captures all of the publicly available information about voter ideal points, and is a sufficient statistic for the second-period candidates' problem of inferring the distribution of the median voter's ideal point from first-period behavior. We denote such a posterior distribution as $F_{median}(\cdot \mid \#R; x_c)$.

Before proceeding we provide a few comments about this function. Given that $\#R$ of $n$ voters have ideal points to the right of (greater than) $x_c$, the median is less than $x_c$ if and only if $\#R \leq m = \frac{n-1}{2}$. Similarly the median is greater than $x_c$ if and only if $\#R \geq m+1$. In the former case, the median is the $(m+1)$'th lowest ideal point of the $n - \#R$ voters with ideal points less than $x_c$, i.e., the median ideal point is the $(m+1)$'th order statistic from $n - \#R$ draws from the distribution $H^{-}(x; x_c) = \frac{F(x)}{F(x_c)}$ with support $[0, x_c]$. Similarly in the latter case, the median is the $(m+1-(n-\#R))$'th order statistic from $\#R$ draws from the distribution $H^{+}(x; x_c) = \frac{F(x)-F(x_c)}{1-F(x_c)}$ with support $[x_c, 1]$.

As previously mentioned, Calvert's result shows that, given $x_c$, $\#R$, and a belief mapping $F_{median}\left(\cdot|\#R; x_c\right)$, sequential rationality of the candidates and weakly undominated voting by the voters implies that the second period policy is $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$. In characterizing a cutpoint perfect Bayesian equilibrium with weakly undominated second period voting strategies it is sufficient to characterize a first period cutpoint $x_c \in [0,1]$ such that if every voter other than $i$ is using the strategy with cutpoint $x_c$, it is optimal for voter $i$ to do so as well. Checking this condition is facilitated by the fact that in an equilibrium of this form, second period candidates both locate at the point $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$.

[5] As mentioned above in footnote 2, when abstention is allowed there are equilibria without abstention that look exactly like the equilibria characterized here. This extension hinges only on specifying off the path beliefs and offers no additional insights.

The equilibrium cutpoint balances two effects that influence first period voting. The pivot effect captures the incentive to vote for $L$ if $|L - v_i| < |R - v_i|$ and $R$ if the opposite is true. The signaling motivation captures the incentive to vote for $R$ if, given $i$'s expectations about the actions of the other voters, increasing $\#R$ is likely to move the second period policy $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$ towards $v_i$, and to vote for $L$ if increasing $\#R$ is likely to move the second period policy away from $v_i$. The pivot effect is the product of the probability that $i$ is pivotal and the payoff difference between the policy $L$ and $R$. In contrast to the pivot effect, which captures a low probability event with a non trivial payoff in that event, the signaling motivation takes into account the fact that $i$'s vote will *always* have an effect on the second period policy. However, the signaling effect is small for each of the possible realizations of the votes cast by $N\backslash\{i\}$. For some realizations of the votes by $N\backslash\{i\}$, increasing $\#R$ will be attractive to $i$, whereas for other realizations of the these votes, increasing $\#R$ will be unattractive to $i$.

Before analyzing the model in more depth, we note one feature that some readers may find unnatural: the fact that the first period election is between two divergent alternatives whereas candidate positions in the second period converge. We focus on a model with exogenous first period divergence, to facilitate a comparison between the pivot and signaling motivations. Although it is possible to add features to the model, e.g., candidate valence, that result in equilibrium divergence in the second period, we find the current framework defensible for three reasons. First, the current model is already used in the literature on signaling and thus the results have a natural connection with extant work. Second, divergence in the second period would substantially complicate the signaling motivation by adding an additional source of uncertainty about second period payoffs, which, for each of $N + 1$ possible first period election outcomes, would depend on the endogenously-determined location of two second period candidates as well as the endogenously-determined probability that each wins the election. Thus, we find the current framework much more elegant.

Third, there are, in fact, important empirical referents for our model. State referenda in the shadow of elections are a natural application. Referenda involve divergent policy alternatives, whereas the subsequent election may involve convergence. Another application would be a contest between an incumbent and a challenger followed by a contest between two new candidates (e.g., a presidential election with an incumbent in office and a subsequent senatorial election with no incumbent). In any case, there is no reason to believe that a model with second-period divergence would produce results substantially different from the ones we obtain here.

The goal of this paper is to compare high impact, low probability pivot events with low impact, high probability signaling effects and determine which type of effect dominates in large elections. In particular we investigate the limiting behavior of the cutpoint $x_c$ as $n$ tends to infinity. We find that the limiting cutpoint corresponds to the equilibrium cutpoint in a game in which the first period is irrelevant (or, equivalently, $L = R$) so that, in the limit, the cutpoint for voter behavior is identical to what it would be if voters were motivated purely by signaling concerns. Thus, we find that while equilibrium voting involves a balancing of these two motivations, in a very strong sense, equilibrium voting in large elections is driven by voters' desire to influence the inferences of observers and not by their desire to influence the election at hand.

# 3 Two Examples

Before analyzing the model, we illustrate its structure and incentives with two simple examples.

**Example 1.** We start with what is essentially a decision-theoretic version of the model, in which there is just one voter, with ideal point $v_i$ and a linear loss function $\gamma(|x - v_i|) = |x - v_i|$. Suppose the exogenously-fixed first period candidate locations are $L = \frac{1}{2}$ and $R = 1$. The second period candidates believe that the single voter's ideal point is drawn from a uniform distribution on $[0, 1]$. Thus, if the voter's strategy is monotone, with cutpoint $x_c$, the second period policy will be $\frac{x_c}{2}$ if $i$ votes for $L$ and $\frac{1+x_c}{2}$ if $i$ votes for $R$. For $i$ to be indifferent between voting $L$ and $R$ when her ideal point is $v_i = x_c$, the following equality must hold:

$$-|x_c - L| - \left| x_c - \frac{x_c}{2} \right| = -|R - x_c| - \left| x_c - \frac{1 + x_c}{2} \right|.$$

For $L = \frac{1}{2}$ and $R = 1$ this equality is solved at $x_c = \frac{2}{3}$.

**Example 2.** To illustrate how pivot and signaling effects work in the model, we now consider the simplest variant where a vote has a probabilistic effect on both first and second period outcomes. While this example cannot resolve the horse race between the signaling and pivot effects as the number of voters gets large, all of the relevant incentives and quantities of interest are present.

Consider $n = 3$ and assume that voters $i \in \{1, 2, 3\}$ have ideal points that are i.i.d. draws from a uniform distribution on $[0, 1]$. Assume that each voter has a linear loss function $\gamma(|x - v_i|) = |x - v_i|$. The first-period election is between two candidates, with exogenously-fixed policy positions $L = \frac{1}{2}$ and $R = 1$.

We first consider two benchmark cases: a pure pivot model and a pure signaling model. In a pure pivot model there is a unique voting equilibrium in weakly undominated strategies: a voter votes for the closer candidate, i.e., she votes for $L$ if her ideal point is to the left of $\frac{L+R}{2} = 0.75$ and votes for $R$ if her ideal point is to the right of 0.75. So $x_p = 0.75$ is the *pivot cutpoint.*

For a pure signaling model, all that matters is how a vote affects $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$ through $\#R$. If voters only care about the outcome of the second period election then in the three voter example there is a unique equilibrium, specified by the *signaling cutpoint,* $x_s = 1/2$.

To check that this cutpoint is an equilibrium in the game in which only the second period outcome affects voter payoffs, we confirm that a voter with $v_i = 1/2$ is indifferent between voting $L$ and $R$, given the other actors' strategies. Focusing on voter $i = 1$, assume that the other voters are using this cutpoint strategy. The signaling effect of $i's$ first-period vote thus depends on the other voters' first-period actions:

- With probability $F(x_s) \cdot F(x_s) = \frac{1}{2} \cdot \frac{1}{2} = 0.25$ the other two voters vote $L$. In this case, the second-period policy outcome will be 0.25, if $i$ votes $L$. This is true because the second-period candidates' posterior belief given $\#R = 0$ and $x_s = 1/2$ is that all three voters' ideal points are uniform draws from $[0, 0.5]$. If $i$ votes $R$ the second-period policy outcome will be $F^{-1}_{median}\left(\frac{1}{2}|1; 1/2\right) = 0.35$, since there is a 50% chance that both of the $L$ voters, and hence the median, will be to the left of 0.35. Thus, the signaling effect of voting $R$ if both other voters vote $L$ is to move the second-period policy outcome from 0.25 to 0.35.

- With probability $F(x_s) \cdot (1 - F(x_s)) + (1 - F(x_s)) \cdot F(x_s) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = 0.5$ the other two voters split their votes. In this case, if $i$ votes $L$ the second-period policy outcome will be 0.35, and if she votes $R$, the policy outcome will be 0.65.

- With probability $(1 - F(x_s)) \cdot (1 - F(x_s)) = \frac{1}{2} \cdot \frac{1}{2} = 0.25$ the other two voters vote $R$. In this case, if $i$ votes $L$ the second-period policy outcome will be 0.65, and if $i$ votes $R$, the policy outcome will be 0.75.

Thus, if a voter with ideal point $v_i$ votes $L$, her expected second-period utility is $-0.25 \cdot |v_i - 0.25| - 0.5 \cdot |v_i - 0.35| - 0.25 \cdot |v_i - 0.65|$. If she votes $R$, her expected utility is $-0.25 \cdot |v_i - 0.35| - 0.5 \cdot |v_i - 0.65| - 0.25 \cdot |v_i - 0.75|$. A voter at $v_i = x_s = 1/2$ is indifferent between voting $L$ and voting $R$. It is straightforward to confirm that any voter left of $1/2$ strictly prefers to vote $L$, and a voter to the right prefers to vote $R$.

It is worth noting three features of signaling effects that will show up in our later analysis of large elections. First, which action, $L$ or $R$, better promotes the voter's policy interests in the second period depends on the other voters' actions, as well as the cutpoint $x_s$. For a voter with $v_i = 1/2$, if the other two voters vote $L$, then voting $R$ is optimal, whereas if the others vote $R$, then voting $L$ is optimal, and if the others split their votes, then the voter is indifferent. Second, the different signaling effects are not equally likely to occur, but rather occur with different probabilities. Third, since the other voters' actions are simply draws from a binomial distribution, in a large election, the most likely realized vote totals are those where $L$ receives a share close to $F(x_s)$ of the votes and $R$ receives a share close to $1 - F(x_s)$ of the votes. All three of these properties of signaling effects hold regardless of the cutpoint for voter behavior in the first period.

In a model with both pivot and signaling effects, equilibrium behavior hinges on the *combined cutpoint* $x_c$, which, not surprisingly, lies between the pivot cutpoint and the signaling cutpoint. If $x_c$ were equal to the signaling cutpoint, $\frac{1}{2}$, then a voter at $x_c$ would have a strict incentive to vote $L$, in the hopes of helping the candidate at $L = \frac{1}{2}$ win the first-period election. If $x_c$ were equal to the pivot cutpoint, 0.75, then a voter at $x_c$ would have a strict incentive to vote $R$ in the hopes of shifting second period policies to the right.

As shown in Figure 1, in the three-voter example, the equilibrium combined cutpoint is $x_c \approx 0.65$.[6] For this $x_c$ the *pivot probability* is $2 \cdot 0.65 \cdot (1 - 0.65) = 0.455$. For a voter with $v_i = x_c$ the utility difference between the two possible first-period policy outcomes, $L$ and $R$, is $-|v_i - L| + |v_i - R| = -\left|0.65 - \frac{1}{2}\right| + |0.65 - 1| = 0.2$. So $i$ receives, in expectation, $0.2 \cdot 0.455 \approx 0.09$ more first-period utility

[6]It is just coincidence that this value for $x_c$ is the same, to two decimal places, as the second period policy outcome under one of the action profiles in the pure signaling model. For the general model, they need not be the same.

by voting $L$ than by voting $R$.

[Insert Figure 1 about here]

The second-period signaling effect is a bit more complicated to compute:

- With probability $F(x_c) \cdot F(x_c) = 0.65 \cdot 0.65 = 0.4225$ the other two voters vote $L$. In this case, the second-period policy outcome will be 0.325 if $i$ votes $L$. If $i$ votes $R$, the second-period policy outcome will be 0.463.
- With probability $F(x_c) \cdot (1 - F(x_c)) + (1 - F(x_c)) \cdot F(x_c) = 2 \cdot 0.65 \cdot (1 - 0.65) = 0.455$ the other two voters split their votes. In this case, if $i$ votes $L$, the second-period policy outcome will be 0.463, and if $i$ votes $R$, the policy outcome will be 0.756.
- With probability $(1 - F(x_c)) \cdot (1 - F(x_c)) = (1 - 0.65) \cdot (1 - 0.65) = 0.1225$ the other two voters vote $R$. In this case, if $i$ votes $L$, the second-period policy outcome will be 0.756, and if $i$ votes $R$, the policy outcome will be 0.825.

Thus, if a voter with ideal point $v_i$ votes $L$, his expected second-period utility is $-0.4225 \cdot |v_i - 0.325| - 0.455 \cdot |v_i - 0.463| - 0.1225 \cdot |v_i - 0.756|$, which equals $-0.24$ for $v_i = 0.65$. And if $i$ votes $R$, his expected utility is $-0.4225 \cdot |v_i - 0.463| - 0.455 \cdot |v_i - 0.756| - 0.1225 \cdot |v_i - 0.825|$, which equals $-0.15$ for $v_i = 0.65$. The difference is equal to 0.09, and for $v_i = x_c$ it exactly counteracts the first-period utility gain that the voter receives by voting $L$ rather than $R$. Thus, at $x_c$ the pivot and signaling effects cancel each other out and the voter is indifferent.

This example illustrates the basic tension between pivot and signaling effects in our model. In this three voter example, the equilibrium cutpoint is $x_c \approx 0.65$, which lies between the signaling cutpoint, $x_p = 0.5$, and the pivot cutpoint, $x_s = 0.75$. The question is how a sequence of equilibrium cutpoints $\{x_m\}$ will behave in the limit as the population size gets large.

The difficulty in answering this question is that in large elections both the pivot effect and the signaling motivation become small – the probability of a pivot event goes to zero and the distance that second-period candidates move in response to a single vote also goes to zero. The question is which converges faster.

## 4 Preliminary Results

In this section we establish two lemmas that are useful in establishing existence of a particular type of equilibrium for any $n$ as well as in proving the main result about the limiting behavior of this type of equilibrium. We then present the existence result. Our analysis focuses on a particular class of equilibria.

**Definition 1 (Symmetric Cutpoint Strategy)** *Voters use a symmetric cutpoint strategy if there exists a point $x_c \in [0, 1]$ such that for all $i \in N$*

*(1) if $x_c = 0$, then $i$ votes $L$ if $v_i = 0$, and $i$ votes $R$ if $v_i > 0$*

*(2) if $x_c \in (0, 1]$, then $i$ votes $L$ if $v_i < x_C$, and $i$ votes $R$ if $v_i \geq x_c$.*

Given that all other voters use a symmetric cutpoint strategy with cutpoint $x_c$, optimal behavior for a voter with ideal point $v_i$ depends on the difference in her expected utility between voting $R$ and voting $L$ in the first period. Using $a_i^1 \in \{L, R\}$ to denote voter $i's$ first period action, we can express this difference as

$$u_{dif}(v_i) \equiv u(a_i^1 = R|v_i) - u(a_i^1 = L|v_i) = u_{dif1}(v_i) + u_{dif2}(v_i) \tag{1}$$

where

$$u_{dif1}(v_i) \equiv \binom{2m}{m} F(x_c)^m (1 - F(x_c))^m (\gamma(|L - v_i|) - \gamma(|R - v_i|)) \tag{2}$$

and

$$u_{dif2}(v_i) \equiv \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \begin{pmatrix} \gamma(\left|F_{median}^{-1}(\frac{1}{2}|k, x_c) - v_i\right|) \\ -\gamma(\left|F_{median}^{-1}(\frac{1}{2}|k+1, x_c) - v_i\right|) \end{pmatrix}. \tag{3}$$

Thus $u_{dif1}(v_i)$ captures the first period effect of voting: the pivot probability is $\binom{2m}{m} (F(x_c))^m (1 - F(x_c))^m$ and the utility difference between the two candidates is $\gamma(|L - v_i|) - \gamma(|R - v_i|)$ for a voter with ideal point $v_i$. Likewise, $u_{dif2}(v_i)$ captures the second period effect: the probability that $k$ other voters vote $R$ is $\binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k$ and the utility difference between voting $R$ versus $L$ in this event is $\gamma(\left|F_{median}^{-1}(\frac{1}{2} \mid k; x_c) - v_i\right|) - \gamma(\left|F_{median}^{-1}(\frac{1}{2} \mid k+1; x_c) - v_i\right|)$.

**Remark 1:** Deriving $F_{median}(y|\#R; x_c)$ for $x_c \in (0, 1)$

To understand $u_{dif2}(v_i)$ it is important to see how $F_{median}(y|\#R;x_c)$ depends on $\#R$ and $x_c$. This function can be characterized in terms of order statistics. We note that for fixed $x_c \in (0,1)$, the distribution of the median is constructed as follows. Given that there are $n - \#R$ draws with values strictly less than $x_c$ and $\#R$ draws with values greater than or equal to $x_c$, we know that the median is less than $x_c$ if $\#R$ is strictly less than $m+1$, and it is greater than $x_c$ otherwise. If $\#R < m+1$, the median is the $m+1$ smallest of $n - \#R$ draws from the conditional (on $y < x_c$) distribution $H^-(y;x_c) = \frac{F(y)}{F(x_c)}$. Accordingly if $\#R < m+1$, $F_{median}(y|\#R;x_c) = H^-_{m+1,n-\#R}(y;x_c)$, which is the distribution of the $(m+1)$'th order statistic from $n - \#R$ draws from the distribution function $H^-(\cdot;x_c)$. Similarly, if $\#R \geq m+1$, the median is the $(m+1-(n-\#R))$'th order statistic from $\#R$ draws from the conditional (on $y > x_c$) distribution $H^+(y;x_c) = \frac{F(y)-F(x_c)}{1-F(x_c)}$. So if $\#R \geq m+1$ then $F_{median}(y|\#R;x_c) = H^+_{m-(n-\#R),\#R}(y;x_c)$, which is the distribution of the $(m+1-(n-\#R))$'th order statistic from $\#R$ draws from the distribution function $H^+(\cdot;x_c)$.■

**Remark 2:** Deriving $F_{median}(y|\#R;x_c)$ for $x_c \in \{0,1\}$

Now consider extremal cutpoints $x_c \in \{0,1\}$. If $x_c = 0$, then according to Definition 1, all voters with ideal points in $(0,1]$ vote $R$ and voters with ideal point $v_i = 0$ vote $L$. Accordingly, if $\#R < m+1$, then the median voter's ideal point is 0 with probability 1 and $F_{median}(y|\#R;0)$ is constant at 1 for all $y \in [0,1]$. In this case, we define $F^{-1}_{median}(\frac{1}{2} \mid \#R;0) = 0$, and it is clear that equilibrium second period candidate locations are at 0. If $\#R \geq m+1$, then the median is the $(m+1-(n-\#R))$'th order statistic from $\#R$ draws from $F(\cdot)$. This distribution corresponds to $H^+(y;x_c) = \frac{F(y)-F(0)}{1-F(0)}$ with support $[0,1]$. If $x_c = 1$, then, according to Definition 1, all voters with ideal point 1 vote $R$ and all voters with ideal points $v_i \in [0,1)$ vote $L$. Accordingly, if $\#R \geq m+1$, then the median voter's ideal point is at 1 with probability 1 and $F_{median}(y|\#R;1)$ is constant at 0 for all $y \in [0,1)$ and equal to 1 at $y = 1$. In this case we define $F^{-1}_{median}(\frac{1}{2} \mid \#R;1) = 1$. If $\#R < m+1$ then the median is the $(m+1)$'th order statistic from $n - \#R$ draws from $F(\cdot)$. This distribution corresponds to $H^-(y;x_c) = \frac{F(y)}{F(1)}$ on $[0,x_c]$.■

One way to see how the distribution function $F_{median}(\cdot|\cdot;\cdot)$ behaves as the arguments $\#R$ and $x_c$ change is to consider the case of the uniform, $F(y) = y$ on $[0,1]$. Figure 2 plots the function $H^{-}(y;x_c)$ for $x_c \in \{0, 0.65, 1\}$. Figure 3 plots $H^{-}_{m+1,n-\#R}(y;x_c)$ for $n = 11, m = 5$ and $\#R \in \{2,3\}$ when $x_c = 0.65$. This figure shows how increasing $\#R$ shifts the second period candidates' beliefs about the location of the median to the right, thereby causing $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$ to increase.

[Insert Figures 2 and 3 about here]

Combining Remarks 1 and 2, we can express the second period policy location as a function of $x_c$ and $\#R$ when voters use a symmetric cutpoint strategy.

$$\chi(x_c,\#R) = \begin{cases} 0 \text{ if } x_c = 0 \text{ and } \#R < m+1 \\ 1 \text{ if } x_c = 1 \text{ and } \#R \geq m+1 \\ \{y : H^{-}_{m+1,n-\#R}(y;x_c) = \frac{1}{2}\} \text{ if } x_c \in (0,1] \text{ and } \#R < m+1 \\ \{y : H^{+}_{m+1-(n-\#R),\#R}(y;x_c) = \frac{1}{2}\} \text{ if } x_c \in [0,1) \text{ and } \#R \geq m+1. \end{cases}$$

The first lemma builds on this derivation to establish properties of the distribution of the median and the above mapping.

**Lemma 1 (Properties of Second Period Policy Outcomes)** *If voters use a symmetric cutpoint strategy with cutpoint $x_c$, then*

*(1) For each $\#R < m+1$ and $y \in [0,1]$, $F_{median}(y|\#R;x_c)$ is weakly decreasing in $x_c$ for $x_c \in [0,y)$ and strictly decreasing for $x_c \in [y,1]$. For each $\#R \geq m+1$ and $y \in (0,1)$, $F_{median}(y|\#R;x_c)$ is weakly decreasing in $x_c$ for $x_c \in (y,1]$ and strictly decreasing in $x_c \in [0,y]$ .*

*(2) If $\#R_1 < \#R_2$ (both in $0,1,2,...,n$) then for each $x_c \in (0,1)$, for some set $A_{x_c} \subset [0,1]$ with positive lebesgue measure, $F_{median}(y|\#R_1;x_c) > F_{median}(y|\#R_2;x_c)$ if $y \in A_{x_c}$ and $F_{median}(y|\#R_1;x_c)$ $\geq F_{median}(y|\#R_2;x_c)$ for all $y \in [0,1]$. For $x_c \in \{0,1\}$ , $F_{median}(y|\#R_1;x_c) \geq F_{median}(y|\#R_2;x_c)$ for all $y \in [0,1]$.*

*(3) For any $\#R \in \{0,1,2,...,n-1\}$ and $x_c \in [0,1]$, $F^{-1}_{median}(\frac{1}{2} \mid \#R;x_c) \leq F^{-1}_{median}(\frac{1}{2} \mid \#R+1;x_c)$.*

*(4) $F_{median}(y|\#R;x_c)$ is continuous in $x_c$ on $(0,1)$, for each $\#R \in \{0,...,n\}$ and $y \in [0,1]$ as well as continuous in $y$ on $[0,1]$ for each $\#R \in \{0,...,n\}$ and $x_c \in (0,1)$.*

*(5) The mapping $\chi(x_c,\#R)$ is a function from $[0,1] \times \{1,2....,n\}$ into $[0,1]$ and it is continuous in $x_c$.*

*Proof:*

(1) Assume $\#R < m+1$. From our derivation of $F_{median}(y|\#R; x_c)$ in Remark 1 this distribution takes on the value 1 if $y \geq x_c$ and $H^-_{m+1,n-\#R}(y; x_c)$ otherwise. Thus the conclusion that it is weakly decreasing for $x_c \in [y, 1]$ is immediate. Consider $x_c < x'_c$. Since

$$\frac{F(y)}{F(x'_c)} < \frac{F(y)}{F(x_c)}$$

$H^-(y; x'_c) < H^-(y; x_c)$ for all $x < x_c$ and thus the former first order stochastically dominates the latter on $[0, x_c]$. This ordering of $H^-(y; x'_c)$ and $H^-(y; x_c)$ implies that the distributions of order statistics, $H^-_{m+1,n-\#R}(y; x'_c)$ and $H^-_{m+1,n-\#R}(y; x_c)$ are also ordered by first order stochastic dominance (see for example David and Nagaraja [8, Theorem 4.4.1]). An analogous argument holds in the case of $\#R \geq m+1$, establishing that $H^+_{m+1-(n-\#R),\#R}(y; x'_c)$ and $H^+_{m+1-(n-\#R),\#R}(y; x_c)$ are ordered by first order dominance.

(2) To establish strict monotonicity in $\#R$, consider two integers, $\#R_1$ and $\#R_2$, with $0 \leq \#R_1 < \#R_2 \leq n$. If $\#R_1 < m+1 \leq \#R_2$ then the support of $F_{median}(\cdot|\#R_1; x_c)$ is $[0, x_c]$ and the support of $F_{median}(\cdot|\#R_2; x_c)$ is $[x_c, 1]$. Since the distribution $F(\cdot)$ is strictly increasing on $[0, 1]$, $F_{median}(y|\#R_1; x_c) > 0$ for all $y \in (0, x_c)$ while $F_{median}(y|\#R_1; x_c) = 1$ for all $y \geq x_c$. Similarly, $F_{median}(y|\#R_2; x_c) = 0$ for all $y \leq x_c$ and $F_{median}(y|\#R_2; x_c) < 1$ for $y \in (x_c, 1)$. Thus, $F_{median}(y|\#R_2; x_c) \leq F_{median}(y|\#R_1; x_c)$, with a strict inequality for any $y \notin \{0, 1\}$.

Suppose instead that $\#R_1 < \#R_2 < m+1$. The relevant comparison is now between $H^-_{m+1,n-\#R_1}(y; x_c)$ and $H^-_{m+1,n-\#R_2}(y; x_c)$. To see that these two distributions are ordered by first order stochastic dominance, we can partition $n - \#R_1$ draws from $F(\cdot)$ into two sets: first $n - \#R_2$ draws are taken and then another $\#R_2 - \#R_1$ are taken. Because $F(\cdot)$ is strictly increasing on $[0, 1]$, the probability that one of the $\#R_2 - \#R_1$ draws is less than the $m+1$ highest draw of the first $\#R_2 - \#R_1$ draws is strictly positive, and thus $H^-_{m+1,n-\#R_2}(y; x_c) < H^-_{m+1,n-\#R_1}(y; x_c)$ for $y$ on $[0, x_c]$. This implies that $F_{median}(y|\#R_2; x_c) \leq F_{median}(y|\#R_1; x_c)$ with a strict inequality if $y \in A_{x_c} = [0, x_c)$ if $\#R_1 < \#R_2 < m+1$. A similar argument holds for $A_{x_c} = (x_c, 1]$ and $m+1 \leq \#R_1 < \#R_2$.

The result for $x_c \in \{0, 1\}$ follows from Remark 2.

(3) Follows immediately from (2).

(4) Continuity of $F_{median}(y|\#R;x_c)$ in $x_c$ on $(0,1)$ for each $\#R \in \{0,...,n\}$ and $y \in [0,1]$ as well as continuity in $y$ on $[0,1]$ for each $\#R \in \{0,...,n\}$ and $x_c \in (0,1)$ follows from the assumption that $F(\cdot)$ is strictly increasing and continuously differentiable and the fact that the distribution of an order statistic from a differentiable distribution function has a density. In particular, for $\#R < m+1$ the distribution $F_{median}(y|\#R;x_c)$ has density $h^-_{m+1,n-\#R}(y;x_c) = k\left[\frac{\partial}{\partial y}\left(\frac{F(y)}{F(x_c)}\right)\right]\frac{F(y)}{F(x_c)}^a(1-\frac{F(y)}{F(x_c)})^b$ for integers $k,a,b$. For $\#R \geq m+1$, the distribution $F_{median}(y|\#R;x_c)$ has density $h^+_{m+1-(n-\#R),\#R}(y;x_c) = k'\left[\frac{\partial}{\partial y}\left(\frac{F(y)-F(x_c)}{1-F(x_c)}\right)\right]\left(\frac{F(y)-F(x_c)}{1-F(x_c)}\right)^{a'}(1-\left(\frac{F(y)-F(x_c)}{1-F(x_c)}\right))^{b'}$ for some $k', a', b'$. Since we have assumed that $F(\cdot)$ has a continuous density, for fixed $\#R$, as long as $x_c \in (0,1)$ the above densities are well defined and thus the distribution functions are continuous.

(5) To show that $\chi(x_c,\#R)$ is defined on its domain we must show that $\{y : H^-_{m+1,n-\#R}(y;x_c) = \frac{1}{2}\}$ is non-empty if $x_c \in (0,1]$ and $\#R < m+1$ and that $\{y : H^+_{m+1-(n-\#R),\#R}(y;x_c) = \frac{1}{2}\}$ is non-empty if $x_c \in [0,1)$ and $\#R \geq m+1$. In the first case, consider $x_c \in (0,1]$ and $\#R < m+1$. From the proof of part 4 of this lemma we see that $H^-_{m+1,n-\#R}(y;x_c)$ has a continuous density function that is strictly positive as long as $y < x_c$. So the function $H^-_{m+1,n-\#R}(y;x_c)$ is continuous and strictly increasing in $y$ on $[0,x_c]$ with $0 = H^-_{m+1,n-\#R}(0;x_c) < \frac{1}{2} < H^-_{m+1,n-\#R}(x_c;x_c) = 1$. This means that the set $S^-(x_c,\#R) = \{y \in [0,1] : H^-_{m+1,n-\#R}(y;x_c) \in (0,1)\}$ is non-empty for $x_c \in (0,1]$ and $\#R < m+1$. Moreover, by the intermediate value theorem this means that the set $\{y : H^-_{m+1,n-\#R}(y;x_c) = \frac{1}{2}\}$ is non-empty if $x_c \in (0,1]$ and $\#R < m+1$. An analogous argument establishes that $\{y : H^+_{m+1-(n-\#R),\#R}(y;x_c) = \frac{1}{2}\}$ is non-empty if $x_c \in [0,1)$ and $\#R \geq m+1$. To show that $\chi(x_c,\#R)$ is a function it is sufficient to note that $F_{median}(\cdot|\#R;x_c)$ has the property that for each value of $x_c$ and $\#R$ there are 2 numbers, $a_1, a_2 \in [0,1]$ such that $F_{median}(\cdot|\#R;x_c)$ is constant at 0 on $[0,a_1]$, $F_{median}(\cdot|\#R;x_c)$ is strictly increasing on $[a_1,a_2]$ and $F_{median}(\cdot|\#R;x_c)$ is constant at 1 on $[a_2,1]$. This means that the equation $F_{median}(\cdot|\#R;x_c) = \frac{1}{2}$ has at most one solution (and it is in $[a_1,a_2]$).

To establish continuity we consider two cases. First assume that $\#R < m+1$. By part 4 of this lemma, for a fixed $y$, $H^-_{m+1,n-\#R}(y;x_c)$ is continuous in $x_c$ on $(0,1)$ and thus this and the fact that it is strictly increasing (and has a density) in $y$ on a neighborhood of the point $\{y : H^-_{m+1,n-\#R}(y;x_c) = \frac{1}{2}\}$ implies by way of the implicit function theorem that the solution $\chi(x_c,\#R)$ is continuous in $x_c$ if $x_c \in (0,1)$ and $\#R < m+1$. Continuity at $x_c = 0$ follows from the fact $\chi(x_c,\#R) \leq x_c$ if $\#R < m+1$

and thus $\lim_{x_c \to 0} \chi(x_c, \#R) = 0$ and $\chi(0, \#R) = 0$. Continuity at $x_c = 1$ follows from the fact that $H^{-}_{m+1,n-\#R}(y; 1)$ is defined and for each $y$, $H^{-}_{m+1,n-\#R}(y; x_c)$ is continuous in $x_c$ at 1. An analogous argument about $H^{+}_{m+1-(n-\#R),\#R}(y; x_c)$ establishes continuity in the case of $\#R \geq m+1$.■

The next result establishes some properties of the utility difference function in Eq. (1). Since this result simply uses conclusions from Lemma 1 in standard ways the proof is in the appendix.

**Lemma 2 (Properties of Utility Difference Function)** *If voters use a symmetric cutpoint strategy with cutpoint $x_c$, then*

*(1) $u_{dif}(v_i)$ is continuous and weakly increasing in $v_i$.*

*(2) (Lipschitz property) $\forall \tilde{v}_i, \hat{v}_i \in [0,1]$, $|u_{dif}(\tilde{v}_i) - u_{dif}(\hat{v}_i)| \leq 4 |\tilde{v}_i - \hat{v}_i|$.*

*(3) $u_{dif}(0) \leq 0$ and $u_{dif}(1) \geq 0$.*

We can now state our first main result. The proof, which applies a standard fixed point argument to the function $u_{dif}(\cdot)$, is in the appendix.

**Proposition 1** *There exists an equilibrium in which voters use a symmetric cutpoint strategy in the first period.*

# 5 Intuition for the Convergence Result

Having established existence, we now turn to the question of equilibrium behavior in large electorates, i.e., as $m \to \infty$. We suppress the $c$ subscript and let $x_m$ denote the cutpoint in a symmetric cutpoint strategy equilibrium with $n = 2m+1$ voters. Our interest is then in $\lim_{m\to\infty} x_m$ (if it exists). We show that this limit is equal to the point $F^{-1}(\frac{1}{2})$. Since this limit does not depend on the first period candidate locations $L$ and $R$, it is also the limit of cutpoints for equilibria in the pure signaling game.

The proof proceeds by contradiction. We show that if a sequence of cutpoints does not converge to $F^{-1}(\frac{1}{2})$, then these cutpoints cannot be equilibrium cutpoints for infinitely many values of $m$ because voters at the cutpoint $x_m$, who must be indifferent in equilibrium, will strictly prefer to vote for one candidate over the other. To be more precise, we show that if any subsequence of equilibrium cutpoints converges to a point other than $F^{-1}(\frac{1}{2})$, then for $m$ sufficiently large a voter with ideal point $x_m$ will strictly prefer to vote for one candidate over the other. Once it is established that no subsequence converges to a point other than $F^{-1}(\frac{1}{2})$ it follows that every subsequence, and thus the actual sequence, converges to $F^{-1}(\frac{1}{2})$. This brief section serves as a roadmap for the proof, presenting an informal version of the argument. The next section contains a proof of the main result.

Suppose that in a large electorate voters behave according to a cutpoint $x_m$ which is converging to a number $Z > F^{-1}(\frac{1}{2})$. We show that for large values of $m$ a voter at $x_m$ will strictly prefer to vote $R$. There are three types of effects that the voter must consider.

The first consideration is a pivot effect, which we label $PV$. Since the election is not expected to be a tie, i.e., $x_m \neq F^{-1}(\frac{1}{2})$, and the population size is large, the probability of this pivot event is exceedingly small in a large electorate.

The second consideration involves bad signaling effects from voting $R$. Whenever more than half of the other voters vote $R$, the second period policy will be to the right of $x_m$, so if a voter with $v_i = x_c$ votes $R$, this will move second period policy to the right, i.e., away from his ideal point, as established in part 3 of Lemma 1. However, because $x_m > F^{-1}(\frac{1}{2})$, more than half of the votes are expected to go to $L$, and thus in a large electorate bad signaling effects are extremely unlikely to occur. We find an upper bound on the probability-weighted sum of these bad signaling effects, and label it $UBBS$ (upper bound for bad signaling).

The third consideration involves good signaling effects from voting $R$. Whenever more than half of the other voters vote $L$, the second period policy will be to the left of $x_m$, so if a voter with $v_i = x_m$ votes $R$ this will move second period policy to the right, i.e., towards his ideal point. Since $x_m > F^{-1}(\frac{1}{2})$, more than half of the votes are expected to go to $L$, and thus in a large electorate it is extremely likely that the signaling effect of voting $R$ will be good. We find a lower bound on the probability-weighted sum of these good signaling effects, and label it $LBGS$ (lower bound for good signaling).

We consider the ratio of bad signaling plus pivot effects to good signaling effects, and show that this ratio

$$\frac{PV + UBBS}{LBGS}$$

can be expressed as a limit of the form

$$\lim_{m\to\infty} \frac{P_{tie} + (m+1)\, P_{tie}}{cP}. \tag{4}$$

In this expression, $P_{tie} = \binom{2m}{m} F(x_m)^m \,(1 - F(x_m))^m$ is the probability of an exact tie among the other

$2m$ voters given the cutpoint $x_m$. In the denominator, $P$ is the probability of a certain type of good signaling effect, and $P$ goes to zero much more slowly than $P_{tie}$. The $c$ in the denominator is a constant that does not depend on $m$. At the end of the proof we show that Eq. (4) is bounded by an expression of the form $(m+2)\,q^{\left(\frac{1}{2}-c_1\right)m}$ with constants $q \in (0,1)$, and $c_1 \in (0, 1/2)$. Thus, the limit of Eq. (4) is 0, which means that for a voter with ideal point $x_m$ (and, by continuity, voters with ideal points slightly to the left of $x_m$) it will be optimal to deviate and vote $R$.

Although a full account of why $P$ vanishes much more slowly than $P_{tie}$ requires working through the proof, an informal assessment can be provided. When $x_m > F^{-1}(\frac{1}{2})$, a tie is extremely unlikely, and the most likely first period election outcomes are ones in which $L$ receives more than half of the votes. In such events, voting $R$ has a good signaling effect, from the perspective of a voter at $x_m$.

# 6 The Convergence Result

Our main result is:

**Proposition 2** *For any sequence of equilibrium cutpoints,* $\lim_{m\to\infty} x_m = F^{-1}(\frac{1}{2})$.

*Proof:* For all $m$, $x_m \in [0,1]$, a bounded set, so the Bolzano-Weierstrass Theorem implies that there exists some number $Z \in [0,1]$ such that a subsequence $\{x_{m'}\}$ of $\{x_m\}$ converges to $Z$. Because the proposition is true if and only if every subsequence converges to $M := F^{-1}(\frac{1}{2})$, it is thus sufficient to show that **(i)** no subsequence converges to a number $Z \neq M$ and **(ii)** $\{x_m\}$ cannot possess both convergent and non convergent subsequences with all of the convergent subsequences converging to the same limit. We dispense with point **(ii)** via the following lemma, which is proved in the appendix.

**Lemma 3** *If* $\{x_m\}$ *is a sequence with* $x_m \in [0,1]$ *for every* $m$ *then it cannot be the case that* $\{x_m\}$ *contains both convergent and non convergent subsequences and that every convergent subsequence has the same limit.*

The lemma and the Bolzano-Weierstrass Theorem imply that if $\{x_m\}$ has no subsequence converging to a point other than $M$ then $\{x_m\}$ converges to $M$. For the remainder of the proof of the proposition, we deal with point **(i)**. By way of a contradiction, assume that there is a subsequence converging to

$Z \neq M$. The remainder of the proof focuses on such a subsequence, ignoring the residual portion of the original sequence, and establishes a contradiction. Either $Z < M$ or $Z > M$, and in the remainder of the proof we focus on the latter case. The argument for the former case is virtually identical and is thus omitted.

Our goal is to show that there exists a $\bar{m}$ such that if $m > \bar{m}$ then a voter with ideal point $x_m$ has a strict preference to vote for $R$. Once this claim is established, the continuity of the utility functions established in Lemma 2 implies that for $m > \bar{m}$ there exists a $\delta_m > 0$ such that if $v_i \in (x_m - \delta_m, x_m + \delta_m)$ a voter with ideal point $v_i$ prefers to vote for $R$ when everyone else uses the cutpoint $x_m$. Thus, for some voters to the left of $x_m$ voting $L$ is not a best response, contradicting the hypothesis that $x_m$ is an equilibrium cutpoint when the population size is $2m + 1$. This contradiction means that we cannot have a subsequence of cutpoints converging to any $Z \neq M$ and thus the sequence of cutpoints converges to $M$.

For each $m$, consider a voter, $i$, with ideal point $x_m$. Given that voters to the left of $x_m$ vote $L$ and voters to the right of $x_m$ vote $R$, the probability of any individual voting $R$ is

$$p_m \equiv 1 - F(x_m).$$

Since $x_m > F^{-1}(1/2)$ we know that $p_m < \frac{1}{2}$.

We start by analyzing the utility function of a voter with ideal point $x_m$. Following Eq.'s (1), (2), and (3), and given the conjectured equilibrium for population size $n = 2m+1$, the utility difference between voting $R$ versus voting $L$ for a voter with ideal point $v_i = x_m$ in the equilibrium with population size $2m + 1$ is

$$u_{dif}^{m}(x_m) \equiv u_{dif1}^{m}(x_m) + u_{dif2}^{m}(x_m)$$

Our ultimate goal is to show that there exists an $\bar{m}$ such that for $m > \bar{m}$, $u^m_{dif}(x_m) > 0$. We can re-write $u^m_{dif}(x_m)$ as

$$\begin{aligned} u^m_{dif}(x_m) &= \binom{2m}{m}(1-p_m)^m {p_m}^m (\gamma(|L-x_m|) - \gamma(|R-x_m|)) \\ &+ \sum_{k=0}^{m-1}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k;x_m\right)-x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1;x_m\right)-x_m\right|)\right) \\ &+ \binom{2m}{m}(1-p_m)^m {p_m}^m \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|m;x_m\right)-x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|m+1;x_m\right)-x_m\right|)\right) \\ &+ \sum_{k=m+1}^{2m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k;x_m\right)-x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1;x_m\right)-x_m\right|)\right). \end{aligned} \tag{5}$$

We will proceed by focusing on the distinct lines of this expression separately. If exactly $m$ voters other than $i$ vote $R$, then the election is tied, and $i$'s vote is pivotal in determining the first period policy. However, in terms of first period motivations, which depend on the candidate locations $L$ and $R$, it is not clear whether $i$ prefers to vote $L$ or vote $R$ in the event that he is pivotal. The first line of Eq. (5) is the **pivot effect**.

In terms of second period motivations, which depend on candidate locations given $\#R$, it is also unclear whether $i$ prefers to vote $L$ or vote $R$ in the event that he is pivotal. In contrast, the voter's preferences are clear for events in which he is not pivotal. If $m-1$ or fewer of the other voters vote for $R$ the second period policy will be to the left of $x_m$ regardless of $i$'s vote and if at least $m+1$ of the other voters vote for $R$ then the second period policy will be to the right of $x_m$ regardless of $i$'s vote (as established in Remark 1). These facts and the monotonicity of the second period policy in $\#R$ (part 3 of Lemma 1) imply that if $m-1$ or fewer of the other voters vote for $R$ then a vote for $R$ moves the second period policy closer to $i$'s ideal point. On the other hand, if at least $m+1$ of the other voters vote for $R$, then a vote for $R$ moves the second period away from $i$'s ideal point. Note that in either of these cases, $i$'s vote cannot move policy far enough to leapfrog her ideal point, $x_m$ (see Remark 1). Translating this intuition to Eq. (5) is straightforward. The second line represents **good signaling effects** of voting $R$ when $m-1$ or fewer other voters vote $R$. The third line represents the **indeterminate signaling effect** when the $2m$ other voters split their votes equally between $L$ and $R$. The fourth line represents **bad signaling effects**, when $m+1$ or more other voters vote $R$.

In order to establish that $u^m_{dif}(x_m)$ is eventually strictly positive it is helpful to establish three lemmas. The first (Lemma 4), states an upper bound on how much the pivot effect can favor voting for $L$. Lemma 5 provides an upper bound for how much the bad and indeterminate signaling effects can favor voting for $L$ and Lemma 6 provides a lower bound on how much the good signaling effect can favor voting for $R$.

Before stating the lemmas, we introduce some additional notation. Fix any points $A$ and $B$ in the unit interval such that $M < A < B < Z$. For any $m$, let $B_m$ represent the largest number less than $B$ such that for some integer $b_m < 2m+1$ it is the case that $B_m = F^{-1}_{median}\left(\frac{1}{2}|b_m; x_m\right)$. Let $\underline{\gamma} := \gamma'(\frac{Z-B}{2})$, and note that because $\gamma(\cdot)$ is strictly increasing $\underline{\gamma} > 0$.

**Lemma 4** *The pivot effect has magnitude of at most* $\binom{2m}{m}(1-p_m)^m {p_m}^m$.

**Lemma 5** *The bad and indeterminate signaling effects have magnitude of at most* $(m+1)\binom{2m}{m}(1-p_m)^m {p_m}^m$.

**Lemma 6** *The good signaling effects have a magnitude of at least* $\underline{\gamma}(B-A)\binom{2m}{b_m}(1-p_m)^{2m-b_m} p_m^{b_m}$.

We now substitute these bounds into the utility difference expression in Eq. (5) to get

$$u^m_{dif}(x_m) > -\binom{2m}{m}(1-p_m)^m {p_m}^m - (m+1)\binom{2m}{m}(1-p_m)^m {p_m}^m + \underline{\gamma}(B-A)\binom{2m}{b_m}(1-p_m)^{2m-b_m} p_m^{b_m}.$$

To show that there exists an $\bar{m}$, such that for $m > \bar{m}$, $u^m_{dif}(x_m) > 0$, it is sufficient to show that

$$\lim_{m\to\infty} \frac{\binom{2m}{m}(1-p_m)^m {p_m}^m + (m+1)\binom{2m}{m}(1-p_m)^m {p_m}^m}{\underline{\gamma}(B-A)\binom{2m}{b_m}(1-p_m)^{2m-b_m} p_m^{b_m}} = 0.$$

Combining terms in the numerator, and noting that $\underline{\gamma}(B-A)$ is strictly greater than zero and unaffected by $m$, it is sufficient to show that $\lim_{m\to\infty}(m+2)\frac{\binom{2m}{m}(1-p_m)^m {p_m}^m}{\binom{2m}{b_m}(1-p_m)^{2m-b_m} p_m^{b_m}} = 0$. For convenience, define $\Psi_m = (m+2)\frac{\binom{2m}{m}(1-p_m)^m {p_m}^m}{\binom{2m}{b_m}(1-p_m)^{2m-b_m} p_m^{b_m}}$. Rearranging yields

$$\begin{aligned}
\Psi_m &= (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{\binom{2m}{m}}{\binom{2m}{b_m}} \\
&= (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{\frac{2m!}{m!m!}}{\frac{2m!}{b_m!(2m-b_m)!}} \\
&= (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{b_m!(2m-b_m)!}{m!m!} \\
&= (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{\prod_{j=1}^{m-b_m}(2m-b_m-j+1)}{\prod_{j=1}^{m-b_m}(m-j+1)}.
\end{aligned}$$

Taking the largest of the $m-b_m$ terms on the top of the product and the smallest of the $m-b_m$ terms on the bottom we see that

$$\Psi_m < (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{\prod_{j=1}^{m-b_m}(2m-b_m)}{\prod_{j=1}^{m-b_m}(b_m+1)} = (m+2)\left[\left(\frac{p_m}{1-p_m}\right)\left(\frac{2m-b_m}{b_m+1}\right)\right]^{m-b_m}. \quad (6)$$

Although we do not establish convergence of the right hand side of Eq. (6), it is sufficient to show that for a subsequence $\{m'\}$ it converges to 0. This implies that a voter whose type corresponds to the conjectured equilibrium cutpoint will have a strict incentive to vote for $R$, infinitely often. From this we can conclude that the sequence $\{x_m\}$ converging to $Z$ cannot be a sequence of equilibrium cutpoints. In particular for a subsequence $\{m'\}$ we characterize the limit of $\left(\frac{p_{m'}}{1-p_{m'}}\right)\left(\frac{2m'-b_{m'}}{b_{m'}+1}\right)$ and an eventual lower bound for the exponent, $m' - b_{m'}$, and then prove that for this subsequence $\lim_{m'\to\infty} \Psi_{m'} = 0$, a contradiction.

**Lemma 7** *$\lim_{m\to\infty} B_m = B \in (M, Z)$ and for some subsequence $\{m'\}$, $\lim_{m'\to\infty} \frac{b_{m'}}{2m'} = c_1 \in (1 - F(Z), \frac{1}{2}]$.*

For convenience we henceforth label this subsequence with $m$ (dropping the prime) and ignore the residual terms in the sequence.

**Lemma 8** $\lim_{m\to\infty} \left(\frac{p_m}{1-p_m}\right)\left(\frac{2m-b_m}{b_m+1}\right) = \frac{1-F(Z)}{F(Z)} \cdot \frac{1-c_1}{c_1} \in (0, 1)$.

**Lemma 9** *There is some $\bar{m}$ s.t. if $m \geq \bar{m}$ then $m - b_m \geq \left(\frac{1}{2} - c_1\right) m$, where $c_1 \in \left(0, \frac{1}{2}\right)$.*

To conclude the argument we can combine results from Lemmas 8 and 9 with Eq. (6) to get our result:

$$\begin{aligned}\lim_{m\to\infty} \Psi_m &\leq \lim_{m\to\infty} (m+2)\left[\frac{1-F(Z)}{F(Z)} \cdot \frac{1-c_1}{c_1}\right]^{\left(\frac{1}{2}-c_1\right)m} \\ &= 0.\end{aligned}$$

This last step follows from the observation that the relevant limit is of the form $\lim_{m\to\infty} (m+2)\left(\zeta(m)\right)^{\eta(m)}$ where $\lim_{m\to\infty} \zeta(m) \in (0,1)$ and $\lim_{m\to\infty} \eta(m) = \infty$.■

## 7 Discussion

One can imagine other signaling motivations in elections. In fact, Razin [21] discusses a model in which the signaling motivation is exogenous. The goal of our paper, in contrast, is to consider a game where the signaling motivation is endogenous and see which effect – pivot or signaling – dominates. A natural extension of our analysis would be a model in which the second period candidates do not converge in equilibrium, e.g., because they face uncertainty about voter preferences and have policy motivations. We conjecture that the proof technique employed in establishing Proposition 2 could be extended to address this case, but such an analysis faces additional complications and is beyond the scope of this paper.

A second natural extension would be to consider a model in which there is uncertainty about the exact distribution from which voter ideal points are drawn, as in Meirowitz [14].[7] In such a model both the signaling and pivot effects converge more slowly than in the present paper. While this extension is interesting, it is somewhat striking that in the current model without aggregate uncertainty, and thus less for candidates to learn, the signaling motivation still dominates the pivot effect. We conjecture that in at least some reasonable models with uncertainty about the distribution the signaling motivation dominates, but this analysis is left for future work. Another area for future work is the study of turnout and costly voting. As previously noted, Patrick Hummel has extended our Proposition 2 to a model with abstention – subsequent extension of his model to deal with costly voting seems possible.

Finally, as is the case in many, if not most, models of elections, we have abstracted away from complications like the fact that candidate platforms rarely fit perfectly on a one dimensional policy space. While we do not believe that the priority of the signaling motivation hinges on this feature, it is fair to say that future work on electoral competition in higher dimensional spaces will also need to rethink this question. Multidimensional models are particularly likely to be interesting when studying multicandidate elections, in which citizens may vote for minor party candidates to signal that they care about issues unaddressed by the major parties.

At a broader level, our result has important implications for theories of elections. Put bluntly, it may be the case that existing electoral models' focus on pivot events is misplaced. Of course, the purely pivot-based variant of our model is substantially more simplified than the sophisticated one-shot pivot-based models that other authors have used to analyze issues such as turnout, multiple candidates, sequential versus simultaneous voting, and voters' correlated private information. However, all of these analyses are fundamentally based on low-probability pivot events, so our main result suggests that it might be fruitful to rethink some of the more sophisticated pivot-based models, as well as the insights

[7] Note that almost all pivot based models of elections with uncertainty about voter preferences assume that the distribution from which voters are drawn is known with certainty. Thus, our setup here parallels standard assumptions in previous research.

about representation and efficiency that they yield, when there is a large electorate and a signaling motivation is present.

# 8 Appendix

The following lemma is used in the proof of Proposition 1.

**Lemma 2**

*(1)* $u_{dif}(v_i)$ *is continuous and weakly increasing in* $v_i$.

*(2) [Lipschitz property]* $\forall \tilde{v}_i, \hat{v}_i \in [0,1]$, $|u_{dif}(\tilde{v}_i) - u_{dif}(\hat{v}_i)| \leq 4\ |\tilde{v}_i - \hat{v}_i|$.

*(3)* $u_{dif}(0) \leq 0$ *and* $u_{dif}(1) \geq 0$.

*Proof:* We first prove separate versions of this result for $u_{dif1}(v_i)$ and $u_{dif2}(v_i)$, then combine them to get the desired result for $u_{dif}(v_i) = u_{dif1}(v_i) + u_{dif2}(v_i)$.

For $u_{dif1}(v_i)$, note that $\gamma(|L - v_i|) - \gamma(|R - v_i|)$ is continuous and weakly increasing in $v_i$ since $\gamma$ is continuous and strictly increasing and $L \leq R$. Thus, because $\binom{2m}{m}(F(x_c))^m (1 - F(x_c))^m \in [0,1]$, $u_{dif1}(v_i)$ is continuous and weakly increasing in $v_i$. For the Lipschitz property, note that

$$\begin{aligned} |u_{dif1}(\tilde{v}_i) - u_{dif1}(\hat{v}_i)| &= |\gamma(|L - \tilde{v}_i|) - \gamma(|R - \tilde{v}_i|) - \gamma(|L - \hat{v}_i|) + \gamma(|R - \hat{v}_i|)| \\ &= |\gamma(|L - \tilde{v}_i|) - \gamma(|L - \hat{v}_i|) - \gamma(|R - \tilde{v}_i|) + \gamma(|R - \hat{v}_i|)| \end{aligned}$$

which is less than or equal to 2 $|\tilde{v}_i - \hat{v}_i|$ because $\gamma'(1) = 1$ and $\gamma(\cdot)$ is convex. Finally, since $L \leq R$, $u_{dif1}(0) \leq 0$ and $u_{dif1}(1) \geq 0$.

For $u_{dif2}(v_i)$, note that by part 3 of Lemma 1, $\forall k \in \{1, ..., 2m\}$, $F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) \leq F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right)$, so $\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - v_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - v_i\right|)$ is continuous and weakly increasing in $v_i$. Thus the probability-weighted sum,

$$\sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - v_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - v_i\right|)\right)$$

is continuous and weakly increasing in $v_i$. For the Lipschitz property, note that $|u_{dif2}(\tilde{v}_i) - u_{dif2}(\hat{v}_i)|$ equals

$$\left| \begin{array}{c} \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - \tilde{v}_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - \tilde{v}_i\right|)\right) - \\ \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - \hat{v}_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - \hat{v}_i\right|)\right) \end{array} \right|$$

which simplifies to

$$\left| \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left( \begin{array}{c} \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - \tilde{v}_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - \hat{v}_i\right|) \\ -\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - \tilde{v}_i\right|) + \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - \hat{v}_i\right|) \end{array} \right) \right|.$$

Recall that $\gamma'(1) = 1$ and $\gamma(\cdot)$ is convex, so the above less than or equal to

$$\sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \cdot 2 |\tilde{v}_i - \hat{v}_i| \leq 2 \cdot |\tilde{v}_i - \hat{v}_i|$$

For $u_{dif2}(0) \leq 0$, recall from part 3 of Lemma 1 that $\forall k$, $F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) \leq F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right)$ so

$$\begin{aligned} u_{dif2}(0) &= \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right)\right) \\ &\leq 0 \end{aligned}$$

By a similar argument $u_{dif2}(1) \geq 0$.

Because $u_{dif1}(v_i)$ and $u_{dif2}(v_i)$ are continuous and weakly increasing in $v_i$, so is $u_{dif}(v_i) = u_{dif1}(v_i) + u_{dif2}(v_i)$. And because $|u_{dif1}(\tilde{v}_i) - u_{dif1}(\hat{v}_i)| \leq 2|\tilde{v}_i - \hat{v}_i|$ and $|u_{dif2}(\tilde{v}_i) - u_{dif2}(\hat{v}_i)| \leq 2|\tilde{v}_i - \hat{v}_i|$, we have $|u_{dif}(\tilde{v}_i) - u_{dif}(\hat{v}_i)| \leq 4|\tilde{v}_i - \hat{v}_i|$. Finally, $u_{dif1}(0) \leq 0$ and $u_{dif2}(0) \leq 0$ imply that $u_{dif}(0) \leq 0$, and likewise $u_{dif1}(1) \geq 0$ and $u_{dif2}(1) \geq 0$ imply $u_{dif}(1) \geq 0$.■

**Proposition 1:** *There exists an equilibrium in which voters use a symmetric cutpoint strategy in the first period.*

*Proof:* Consider the correspondence

$$\phi(x_c) = \{v_i \in [0,1] : u_{dif}(v_i) = 0 \text{ when } N \backslash i \text{ use the symmetric cutpoint strategy specified by } x_c\}$$

Note that $\phi(x_c) : [0,1] \to [0,1]$ is nonempty for all $x_c \in [0,1]$, by parts 1 and 3 of Lemma 2 and the Intermediate Value Theorem. Also, since $u_{dif}(x)$ is continuous and weakly increasing, $\phi(x_c)$ is convex-valued. So, to apply Kakutani's fixed point theorem, and conclude that there exists an equilibrium, i.e., an $x_c^* \in \phi(x_c^*)$, all we need to do is to establish that $\phi(x_c)$ is upper hemi-continuous.

Consider a sequence of points $\{x_c^t\} \to \tilde{x}_c$ and a sequence $\{y^t\} \to \tilde{y}$ where $y^t \in \phi(x_c^t), \forall t$. We need to show that $\tilde{y} \in \phi(\tilde{x}_c)$.

For each $t$, following the definition of $u_{dif}(v_i)$ in Eq. (1), let $u^t_{dif}(v_i)$ be the utility difference function given cutpoint $x_c^t$ and let $\tilde{u}_{dif}(v_i)$ be the utility difference function given cutpoint $\tilde{x}_c$.

We first note that $\left\{u^t_{dif}(v_i)\right\}$ converges pointwise to $\tilde{u}_{dif}(v_i)$. The first part of the utility difference function is

$$u^t_{dif1}(v_i) = \binom{2m}{m} \left(F(x_c^t)\right)^m \left(1 - F(x_c^t)\right)^m (\gamma(|L - v_i|) - \gamma(|R - v_i|))$$

which converges pointwise to

$$\tilde{u}_{dif1}(v_i) = \binom{2m}{m} (F(\tilde{x}_c))^m (1 - F(\tilde{x}_c))^m (\gamma(|L - v_i|) - \gamma(|R - v_i|))$$

since $\{x_c^t\} \to \tilde{x}_c$. The second part is

$$\begin{aligned} u_{dif2}^t(v_i) \quad &\equiv \quad \sum_{k=0}^{2m} \binom{2m}{k} \left(F(x_c^t)\right)^{2m-k} \left(1 - F(x_c^t)\right)^k \\ &\cdot \left(\gamma(\left|F_{median}^{-1}\left(\frac{1}{2}|k; x_c^t\right) - v_i\right|) - \gamma(\left|F_{median}^{-1}\left(\frac{1}{2}|k+1; x_c^t\right) - v_i\right|)\right) \end{aligned}$$

which (by part 5 of Lemma 1, continuity of $\gamma(\cdot)$, and the fact that $\{x_c^t\} \to \tilde{x}_c$) converges pointwise to

$$\begin{aligned} \tilde{u}_{dif2}(v_i) \quad &\equiv \quad \sum_{k=0}^{2m} \binom{2m}{k} (F(\tilde{x}_c))^{2m-k} (1 - F(\tilde{x}_c))^k \\ &\cdot \left(\gamma(\left|F_{median}^{-1}\left(\frac{1}{2}|k; \tilde{x}_c\right) - v_i\right|) - \gamma(\left|F_{median}^{-1}\left(\frac{1}{2}|k+1; \tilde{x}_c\right) - v_i\right|)\right). \end{aligned}$$

Now we suppose that $\tilde{y} \notin \phi(\tilde{x}_c)$, and derive a contradiction. If $\tilde{y} \notin \phi(\tilde{x}_c)$ then either $\tilde{u}_{dif}(\tilde{y}) > 0$ or $\tilde{u}_{dif}(\tilde{y}) < 0$. Without loss of generality suppose the former. Then since $u_{dif}^t(v_i)$ converges pointwise to $\tilde{u}_{dif}(v_i)$ there exists $T$ such that for all $t > T$, $u_{dif}^t(\tilde{y}) > \frac{\tilde{u}_{dif}(\tilde{y})}{2}$. By the Lipschitz property in part 2 of Lemma 2, for all $t > T$, $u_{dif}^t(\tilde{y}) - u_{dif}^t(\tilde{y} - \delta) \leq 4\delta$ for any $\delta > 0$. Setting $\delta = \frac{\tilde{u}_{dif}(\tilde{y})}{8}$ we have that for $t > T$, $u_{dif}^t(\tilde{y}) - u_{dif}^t(\tilde{y} - \delta) < \frac{\tilde{u}_{dif}(\tilde{y})}{2}$, so $u_{dif}^t(\tilde{y} - \delta) > u_{dif}^t(\tilde{y}) - \frac{\tilde{u}_{dif}(\tilde{y})}{2} > 0$. Thus, since $y^t \in \phi(x_c^t)$ or, equivalently, $u_{dif}^t(y^t) = 0$, and $u_{dif}^t(v_i)$ is weakly increasing in $v_i$, we conclude that $y^t < \tilde{y} - \delta$ for all $t > T$, which means that $\{y^t\}$ cannot converge to $\tilde{y}$, a contradiction.■

**Lemma 3:** *If $\{x_m\}$ is a sequence with $x_m \in [0,1]$ for every $m$ then it cannot be the case that $\{x_m\}$ contains both convergent and non convergent subsequences and that every convergent subsequence has the same limit.*

*Proof:* The proof is by contradiction. Assume $\{x_m\}$ is a sequence with both convergent and non convergent subsequences and that all of its convergent subsequences converge to $Y$. Since $\{x_m\}$ has subsequences that are not convergent (and thus not convergent to $Y$), for a small $\delta > 0$, it is the case that infinitely many elements of $\{x_m\}$ fall outside of the ball $B(Y, \delta)$. Let $\{x_k\}$ denote the subsequence of $\{x_m\}$ that contains all of the elements that are not in $B(Y, \delta)$. Since this subsequence does not converge to $Y$ it must not be convergent. Since $\{x_k\}$ is bounded the Bolzano-Weierstrass Theorem implies that it has a convergent subsequence. But since no element of $\{x_k\}$ is in $B(Y, \delta)$ for any $\varepsilon < \delta$ it must be the case that the limit of this convergent subsequence is not in $B(Y,\varepsilon)$. We have thus established that there is a convergent subsequence that does not converge to $Y$, contradicting the assumption.■

**Lemma 4:** *The pivot effect has magnitude of at most* $\binom{2m}{m}(1-p_m)^m {p_m}^m$.

*Proof:* The pivot effect can be either positive or negative, depending on the positions of the two first-period candidates. A bound based on the fact that $L$, $R$, and $x_m$ are all in the interval $[0,1]$ will be sufficient. Recall that we have normalized utility so that $\gamma'(1)=1$ and we have assumed that this function is convex, allowing us to treat the maximal possible utility difference between $L$ and $R$ as 1. Thus substitution yields:

$$\left|\binom{2m}{m}(1-p_m)^m {p_m}^m \left(\gamma(|L-x_m|)-\gamma(|R-x_m|)\right)\right| < \binom{2m}{m}(1-p_m)^m {p_m}^m\gamma'(1) = \binom{2m}{m}(1-p_m)^m {p_m}^m.\blacksquare \tag{7}$$

**Lemma 5:** *The bad and indeterminate signaling effects have magnitude of at most* $(m+1)\binom{2m}{m}(1-p_m)^m {p_m}^m$.

*Proof:* For any $k\in\{1,..,n-1\}$, $F^{-1}_{median}\left(\frac{1}{2}|k;x_m\right)\in(0,1)$ and $F^{-1}_{median}\left(\frac{1}{2}|k+1;x_m\right)\in(0,1)$, so

$$\begin{aligned}&\left|\sum_{k=m+1}^{2m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k\left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k;x_m\right)-x_m\right|)-\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1;x_m\right)-x_m\right|)\right)\right.\\&\left.+\binom{2m}{m}(1-p_m)^m{p_m}^m\left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|m;x_m\right)-x_m\right|)-\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|m+1;x_m\right)-x_m\right|)\right)\right|\\&\qquad<\sum_{k=m}^{2m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k\gamma'(1)=\sum_{k=m}^{2m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k.\end{aligned}$$

In this and subsequent proofs we will use repeatedly the fact that the *binomial expansion is monotonic*. In particular, since $p_m<1/2$, for any $k\in\{m+1,...,2m\}$, $\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k<\binom{2m}{m}(1-p_m)^m{p_m}^m$. As a consequence of this fact, the total magnitude of the bad and indeterminate signaling effects must be less than

$$(m+1)\binom{2m}{m}(1-p_m)^m{p_m}^m.\blacksquare \tag{8}$$

**Lemma 6:** *The good signaling effects have a magnitude of at least* $\underline{\gamma}(B-A)\binom{2m}{b_m}(1-p_m)^{2m-b_m}p_m^{b_m}$.

*Proof:* We now develop a lower bound on good signaling effects. Fix any points $A$ and $B$ such that $M<A<B<Z$. For any $m$, let $A_m$ represent the largest number less than $A$ such that for some integer $a_m<2m+1$ it is the case that $A_m=F^{-1}_{median}\left(\frac{1}{2}|a_m;x_m\right)$. Likewise, let $B_m$ represent the largest number less than $B$ such that for some integer $b_m<2m+1$ it is the case that $B_m=F^{-1}_{median}\left(\frac{1}{2}|b_m;x_m\right)$. For the $b_m$ identified in the definition of $B_m$, let $C_m=F^{-1}_{median}\left(\frac{1}{2}|b_m+1;x_m\right)$.

For $m$ sufficiently large it turns out that $M < A_m < B_m < B \leq C_m < Z$, and since we are interested in the limit as $m \to \infty$ we henceforth focus on values of $m$ that are large enough for this inequality to hold. Because, by construction, $A_m \leq B_m < B < Z$ and $B_m < B \leq C_m$, we only need to show that, for $m$ large enough, (i) $M < A_m$, (ii) $C_m < Z$, and (iii) $A_m < B_m$.

Each of these inequalities follows from the fact that for any $q \in (0, 1)$ the sequence

$$\left| F^{-1}_{median}\left(\frac{1}{2} | \lfloor (2m+1)q \rfloor + 1; x_m \right) - F^{-1}_{median}\left(\frac{1}{2} | \lfloor (2m+1)q \rfloor ; x_m \right) \right| \tag{9}$$

converges to 0.[8] For inequality (i), note that because $M < A$ there exists a $\delta > 0$ such that $A - M > \delta$. The fact that the sequence in Eq. (9) converges to zero implies that for every $\varepsilon > 0$, $A_m - A < \varepsilon$ eventually. Thus that for any $\epsilon > 0$, $A - A_m < \epsilon$ eventually and thus $A_m > M + \delta$ eventually. Thus, $A_m > M$ eventually. For inequality (ii), because $B_m < B < Z$, there exists a $\delta$ such that $Z - B_m < \delta$. The fact that the sequence in Eq. (9) converges to zero implies that for any $\delta$, $C_m - B_m < \delta$ eventually, and thus $C_m < Z$ eventually. For inequality (iii), the fact that $A < B$ implies that if $A_m \geq B_m$, then $A_m > A$ or $F^{-1}_{median}\left(\frac{1}{2}|b_m + 1; x_m\right) < B$ for infinitely many $m$ (contradicting the definition of $A_m$ or $B_m$).

For fixed $m$ the set of profiles for other voters that, given $i's$ vote, can result in a policy between $A_m$ and $C_m$ consists of profiles for which the number of other voters that vote $R$ is in the set $\{a_m, a_m + 1, ...., b_m - 1, b_m\}$. Although we cannot analytically solve for the policy distance between $F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right)$ and $F^{-1}_{median}\left(\frac{1}{2}|k + 1; x_m\right)$ for particular values of $k$, we do know that

$$\sum_{k=a_m}^{b_m} \left( F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) \right) = C_m - A_m > B - A.$$

Since we have $A_m < B_m < B \leq C_m$, the last inequality above is due to the fact that $A_m < A < B$. We re-write the good signaling effects term from Eq. (5) as

$$\sum_{k=0}^{m-1} \binom{2m}{k} (1 - p_m)^{2m-k} {p_m}^k \left( \gamma(\left| F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) - x_m \right|) - \gamma(\left| F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - x_m \right|) \right)$$

[8] By $\lfloor y \rfloor$ we denote the greatest integer less than or equal to $y$.

$$
\begin{aligned}
= \quad & \sum_{k=0}^{a_m-1} \binom{2m}{k} (1-p_m)^{2m-k} {p_m}^k \left( \gamma(\left| F_{median}^{-1}\left(\frac{1}{2}|k;x_m\right) - x_m \right|) - \gamma(\left| F_{median}^{-1}\left(\frac{1}{2}|k+1;x_m\right) - x_m \right|) \right) \\
& + \sum_{k=a_m}^{b_m} \binom{2m}{k} (1-p_m)^{2m-k} {p_m}^k \left( \gamma(\left| F_{median}^{-1}\left(\frac{1}{2}|k;x_m\right) - x_m \right|) - \gamma(\left| F_{median}^{-1}\left(\frac{1}{2}|k+1;x_m\right) - x_m \right|) \right) \\
& + \sum_{k=b_m+1}^{m-1} \binom{2m}{k} (1-p_m)^{2m-k} {p_m}^k \left( \gamma(\left| F_{median}^{-1}\left(\frac{1}{2}|k;x_m\right) - x_m \right|) - \gamma(\left| F_{median}^{-1}\left(\frac{1}{2}|k+1;x_m\right) - x_m \right|) \right).
\end{aligned}
$$

Because $B < Z$ and $\gamma(\cdot)$ is convex, the slope of the utility difference to $i$ with $v_i > \frac{Z+B}{2}$ for points in $[0, B]$ is at least $\underline{\gamma} := \gamma'(\frac{Z-B}{2})$. Since $\gamma(\cdot)$ is strictly increasing $\gamma'(\frac{Z-B}{2}) > 0$. Because $\{x_m\}$ is assumed to converge to $Z$, for $m$ sufficiently large $x_m > \frac{Z+B}{2}$. Thus, the good signaling effects term is eventually greater than

$$
\sum_{k=a_m}^{b_m} \binom{2m}{k} (1-p_m)^{2m-k} {p_m}^k \left( \left| F_{median}^{-1}\left(\frac{1}{2}|k;x_m\right) - x_m \right| - \left| F_{median}^{-1}\left(\frac{1}{2}|k+1;x_m\right) - x_m \right| \right) \underline{\gamma}.
$$

Since $M < A_m < C_m$ and $p_m < 1/2$, we know from monotonicity of the binomial expansion that the event in which $b_m$ others vote for $R$ is the least likely of the set of profiles of the $2m$ other voters in which $i$'s vote can result in a policy in the interval $[A_m, C_m]$. Thus the above expression is greater than

$$
\binom{2m}{b_m} (1-p_m)^{2m-b_m} p_m^{b_m} \sum_{k=a_m}^{b_m} \left( \left| F_{median}^{-1}\left(\frac{1}{2}|k;x_m\right) - x_m \right| - \left| F_{median}^{-1}\left(\frac{1}{2}|k+1;x_m\right) - x_m \right| \right) \underline{\gamma}. \tag{10}
$$

Note that because $\forall k \in \{a_m, a_m+1, ..., b_m\}$, $F_{median}^{-1}\left(\frac{1}{2}|k;x_m\right) < F_{median}^{-1}\left(\frac{1}{2}|k+1;x_m\right) < x_m$,

$$
\begin{aligned}
& \sum_{k=a_m}^{b_m} \left( \left| F_{median}^{-1}\left(\frac{1}{2}|k;x_m\right) - x_m \right| - \left| F_{median}^{-1}\left(\frac{1}{2}|k+1;x_m\right) - x_m \right| \right) \\
= \quad & F_{median}^{-1}\left(\frac{1}{2}|b_m+1;x_m\right) - F_{median}^{-1}\left(\frac{1}{2}|a_m;x_m\right) \\
= \quad & C_m - A_m \\
> \quad & B - A.
\end{aligned}
$$

Thus, we can rewrite Eq. (10) to get the following lower bound for good signaling effects:

$$
\underline{\gamma}(B-A) \binom{2m}{b_m} (1-p_m)^{2m-b_m} p_m^{b_m}. \blacksquare \tag{11}
$$

**Lemma 7:** $\lim_{m\to\infty} B_m = B \in (M, Z)$ *and for some subsequence* $\{m'\}$, $\lim_{m'\to\infty} \frac{b_{m'}}{2m'} = c_1 \in (1 - F(Z), \frac{1}{2}]$.

*Proof:* Recall that $B_m$ is the largest number less than $B$ such that for some integer $b_m < 2m+1$, it is the case that $B_m = F^{-1}_{median}\left(\frac{1}{2}|b_m; x_m\right)$. We now claim that $B_m \to B$. To see this note that if for some $\varepsilon > 0$, $\left|F^{-1}_{median}\left(\frac{1}{2}|b_m; x_m\right) - B\right| > \varepsilon$ infinitely often we obtain the following contradiction. Since $\max_{x\in[0,1]} |F_{median}(x|b_m; x_m) - F_{median}(x|b_m+1; x_m)| \to 0$ there is some $\bar{m}$ such that for all $m > \bar{m}$, $\left|F^{-1}_{median}\left(\frac{1}{2}|b_m; x_m\right) - F^{-1}_{median}\left(\frac{1}{2}|b_m+1; x_m\right)\right| < \varepsilon$ and thus $F^{-1}_{median}\left(\frac{1}{2}|b_m+1; x_m\right) < B$, contradicting the definition of $B_m$. We have thus shown that $B_m = F^{-1}_{median}\left(\frac{1}{2}|b_m; x_m\right)$ is converging to $B$, which is in $(M, Z)$.

The sequence $\frac{b_m}{2m}$ is bounded in $[0,1]$ and thus by the Bolzano-Weierstrass theorem a convergent subsequence exists. For the remainder of the proof we focus on any such convergent subsequence (and for convenience we use $m$ as the index).

We now establish eventual bounds on the ratio $\frac{b_m}{2m}$. Suppose now that $\frac{b_m}{2m}$ converges to a number strictly greater than $\frac{1}{2}$. By Remark 1, $b_m > m+1$ implies that $F^{-1}_{median}\left(\frac{1}{2}|b_m; x_m\right) > x_m$ and since we have assumed that $x_m \to Z$, and we have $F^{-1}_{median}\left(\frac{1}{2}|b_m; x_m\right) \to B$ we must have $B \geq Z$ contradicting the definition of $B$ (that $B < Z$). Thus $c_1 \leq \frac{1}{2}$.

We now show that $c_1 > 1 - F(Z)$, by showing that if it were the case that $c_1 \leq 1 - F(Z)$, then $B \leq M$, which contradicts the definition of $B$ (that $B > M$).

From David and Nagaraja [8, Theorem 2.5] the distribution of the $(m+1)$'th ideal point (i.e. the median) from $2m+1$ (i.e. $n$) draws from $F(\cdot)$ conditional on the fact that $b_m < m+1$ realized ideal points are greater than $x_m$ is just the distribution of the $(m+1)$'th draw from $2m+1-b_m$ draws from the distribution $\frac{F(\cdot)}{F(x_m)}$ on $[0, x_m]$, which we denote as $X_{m+1,2m+1-b_m}$.

To disprove the possibility that $c_1 \leq 1 - F(Z)$ we assume that there is a $\delta \geq 0$ such that $c_1 + \delta = 1 - F(Z)$. We use a very weak consequence of Sen's result on the asymptotic normality of sample quantiles for non i.i.d. draws [23, Theorem 2.1] [9] to conclude that $X_{m+1,2m+1-b_m}$ is asymptotically

[9] Sen extends Bahadur's [2] result on the asymptotic normality of a quantile of a sample to the case of non i.i.d. random variables. Bahadur's result states that if $F(x)$ is a probability distribution that is twice differentiable and $F(\xi) = p$ (with density $f(\xi) > 0$) and $X_{r,n}$ is the $r$ th order statistic from $n$ draws and $Z_n$ is the number of observations greater than $\xi$ then for $r/n \to p \in (0,1)$ the random variable $X_{r,n}$ is equal to $\xi + \frac{Z_n - n(1-p)}{nf(\xi)} + R_n$, where $R_n$ vanishes, and this random variable is asymptotically normal. Sen's generalization goes well beyond, but includes the case of independent draws from distributions that vary with $n$, $F_n(\cdot)$. This is the only part of the extension we need. The challenge of these results is pinning down the

normal with mean $\xi$ where $\xi$ solves $\frac{F(\xi)}{F(Z)} = \frac{1}{2F(Z)+2\delta} < \frac{1}{2}$. This is true since we have just established that $X_{m+1,2m+1-b_m}$ is an order statistic from $\frac{F(\cdot)}{F(x_m)}$, which is converging to $\frac{F(\cdot)}{F(Z)}$ by assumption, and $\lim \frac{m+1}{2m+1-b_m} = \frac{1}{2(1-c_1)} = \frac{1}{2(F(Z)+\delta)}$. Thus $X_{m+1,2m+1-b_m}$ has the same limiting distribution as $X_{\frac{n}{2F(Z)+2\delta},n}$. This means that $X_{m+1,2m+1-b_m}$ is asymptotically normal with mean $\xi$ where $\xi$ solves $\frac{F(\xi)}{F(Z)} = \frac{1}{2F(Z)+2\delta} \leq \frac{1}{2}$, and thus, because $F(\cdot)$ is strictly increasing on its support, $F(\xi) \leq \frac{1}{2}$ implies that $\xi \leq M$. But this means that $B \leq M$, which is a contradiction.■

**Lemma 8:** $\lim_{m\to\infty} \left(\frac{p_m}{1-p_m}\right)\left(\frac{2m-b_m}{b_m+1}\right) = \frac{1-F(Z)}{F(Z)} \cdot \frac{1-c_1}{c_1} \in (0,1)$.

*Proof:* We now work on the limit of the terms in brackets from Eq. (6), using the fact that $\lim_{m\to\infty} p_m = 1 - F(Z)$ and $\lim_{m\to\infty} \frac{b_m}{2m} = c_1$,

$$\begin{aligned}\lim_{m\to\infty}\left(\frac{p_m}{1-p_m}\right)\left(\frac{2m-b_m}{b_m+1}\right) &= \lim_{m\to\infty}\left(\frac{p_m}{1-p_m}\right)\left(\frac{1-\frac{b_m}{2m}}{\frac{b_m+1}{2m}}\right) \\ &= \frac{1-F(Z)}{F(Z)} \cdot \frac{1-c_1}{c_1}. \end{aligned} \tag{12}$$

Because, from Lemma 7 and the fact that $Z > M$, $0 \leq 1 - F(Z) < c_1 \leq \frac{1}{2}$ we conclude that $\frac{1-F(Z)}{F(Z)} \cdot \frac{1-c_1}{c_1} \in (0,1)$.■

**Lemma 9:** *There is some $\bar{m}$ s.t. if $m \geq \bar{m}$ then $m - b_m \geq \left(\frac{1}{2} - c_1\right) m$,* where $c_1 \in \left(0, \frac{1}{2}\right)$.

*Proof:* We now find a lower bound, as a function of $m$, on the exponent $m - b_m$ in Eq. (6). We first tighten the bound on $c_1$ showing that in fact $c_1 < \frac{1}{2}$. By way of a contradiction, suppose now that $\frac{b_m}{2m} \to \frac{1}{2}$. We can divide this sequence into two subsequences (ignoring the trivial possibility that one of these is finite). The first subsequence contains $\{\frac{b_t}{2t}\}_{t=1}^{\infty}$ with $t$ satisfying $b_t \leq t$, and the second contains $\{\frac{b_j}{2j+1}\}_{j=1}^{\infty}$ with $j$ satisfying $b_j > j$. So the first subsequence consists of the cases where the median ideal point is an intermediate order statistic of the form $X_{b_t,t}$ from the parent distribution $\frac{F(\cdot)}{F(x_m)}$ on $[0, x_m]$ with $\frac{b_t}{t} \to 1$ and the second subsequence consists of the cases where the median ideal point is an intermediate order statistic of the form $X_{b_j-j,j}$ from the parent distribution $\frac{F(y)-F(x_m)}{1-F(x_m)}$ on $[x_m, 1]$ with $\frac{b_j-j}{j} \to 0$. For intermediate order statistics, the relevant analogue to the result by Sen, used above, is due to Watts [26, Theorem 1]. To apply Watts's result, we first consider a slight modification of the subsequences with the parent distributions $\frac{F(\cdot)}{F(Z-\varepsilon)}$ on $[0, Z-\varepsilon]$ and $\frac{F(y)-F(Z+\varepsilon)}{1-F(Z+\varepsilon)}$ on $[Z+\varepsilon, 1]$ respectively. Here $\varepsilon$ is chosen to satisfy $Z - B > 2\varepsilon$. Watts's result, restated in our notation, states that as long as the convergence of $\frac{b_t}{t} \to 1$ and $\frac{b_j-j}{j} \to 0$ is slow enough, the relevant order

rate of convergence of $R_n$. We care only about the fact that $X_{r,n}$ is asymptotically normal with mean $\xi$.

statistics converge to $Z-\varepsilon+\delta_t$ and $Z+\varepsilon+\delta_j$ in probability (where both $\delta_t$ and $\delta_j$ vanish at a known rate).[10] These two convergence results imply that even when $\frac{b_t}{t} \to 1$ and $\frac{b_j - j}{j} \to 0$ converge slowly $F_{median}(Z-2\varepsilon|b_t; Z-\varepsilon) \to 0$ and $F_{median}(Z+2\varepsilon|b_j; Z+\varepsilon) \to 1$. Accordingly it is not possible for $B_m$ to converge to $B$ with $B < Z-2\varepsilon$. Now we relax the assumption that the sequence of draws is from identical parent distributions and allow for the fact that the draws are from $\frac{F(\cdot)}{F(x_m)}$ on $[0, x_m]$ and $\frac{F(y)-F(x_m)}{1-F(x_m)}$ on $[x_m, 1]$. To accommodate this fact it is sufficient to observe that the actual order statistics in the subsequence $\{\frac{b_t}{2t}\}_{t=1}^{\infty}$ (as well as $\{\frac{b_j}{2j+1}\}_{j=1}^{\infty}$) first order stochastically dominate the subsequence $\{\frac{b_t}{2t}\}_{t=1}^{\infty}$ with parent distribution $\frac{F(\cdot)}{F(Z-\varepsilon)}$ on $[0, Z-\varepsilon]$.[11] Thus, we cannot have $B_m \to B$ if $\frac{b_m}{2m} \to \frac{1}{2}$, so it cannot be that $\frac{b_m}{2m} \to \frac{1}{2}$, so $c_1 \in (0, \frac{1}{2})$.

Thus, $\lim_{m\to\infty} \frac{m-b_m}{2m} = \frac{1}{2} - c_1$, and if we fix a $\delta = \frac{\frac{1}{2}-c_1}{2}$ there exists a $\bar{m}$ such that for all $m > \bar{m}$, $\frac{m-b_m}{2m} > \frac{1}{2} - c_1 - \delta$, i.e.,

$$m - b_m > \left(\frac{1}{2} - c_1\right) m. \blacksquare \tag{13}$$

[10] Bounds on the rate of $\frac{b_t}{t} \to 1$ and $\frac{b_j - j}{j} \to 0$ are critical to showing that a limiting distribution exists. For example when $\frac{b_j - j}{\log^3 j}$ has a finite limit the intermediate order statistic behaves like an extreme value and may not possess a limiting distribution. For our purposes the worst case scenarios are the ones when the convergence is slow. If $\frac{b_m}{2m} \to \frac{1}{2}$ more quickly, it is even harder to support $B_m \to B$ with $B \neq Z$ while $x_m \to Z$.

[11] This additional, rather trivial, step is neccesitated by our inability to find an extension of Watts's result to the case of non i.i.d. draws from a parent distribution.

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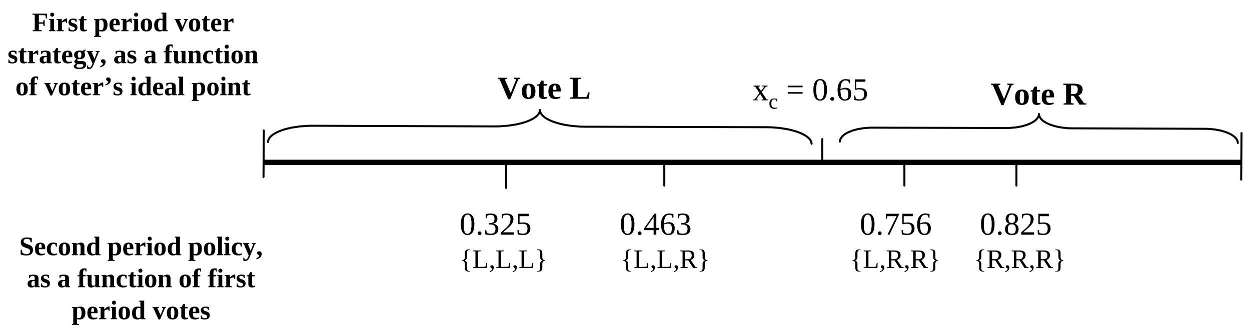
[[Top of Figure]]

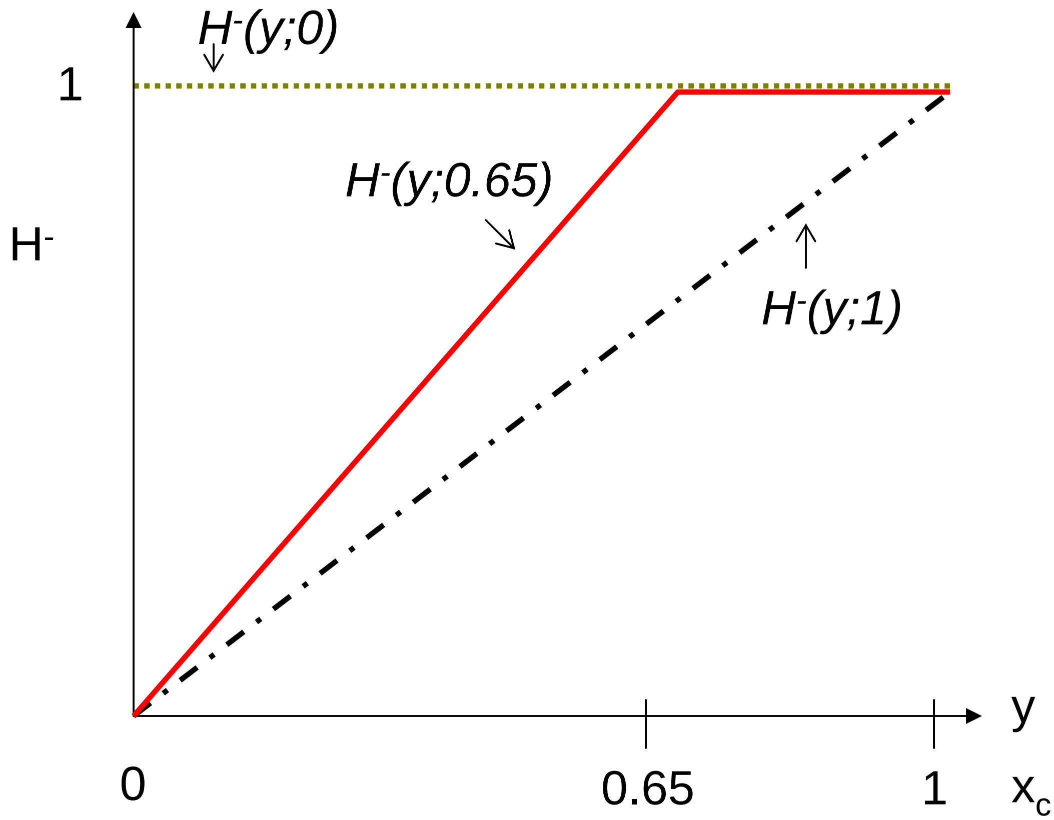
**Figure 1: Equilibrium in Example 2, Combined Model**

**First period candidates: L=1/2, R=1**



[[Top of Figure]]

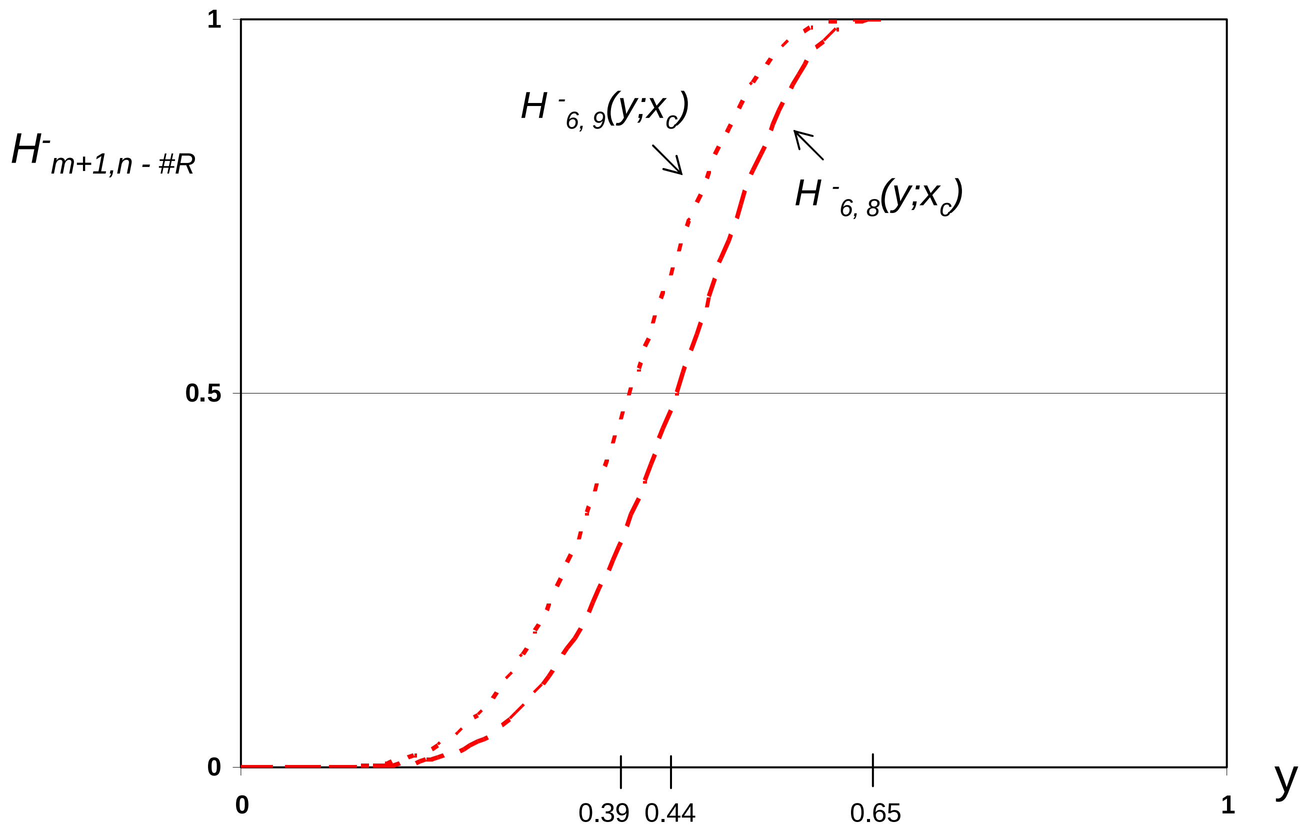
# Figure 2




Examples of $H^{-}(y;x_c)$ for values of $x_c$ in $\{0,0.65,1\}$

[[Top of Figure]]

# Figure 3




Examples of $H^{-}_{m+1,n\ -\ \#R}(y;x_c)$ for $x_c = 0.65$

N=11, m+1=6, and #R is either 2 (for the left dashed line) or 3 (for the right dashed line)

For $\#R = 3,\ F^{-1}_{median}(1/2|\#R;x_c) = 0.39$

For $\#R = 4,\ F^{-1}_{median}(1/2|\#R;x_c) = 0.44$