RANDOM LIMIT-ORDERS

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Abstract. We study a pure limit-order market in which strategic risk-neutral traders can post bids and offers at various prices, although they run the risk of being picked off by informed traders. We show that, in equilibrium, limit-order traders post offers at random prices and for random quantities. In equilibrium, a trader can increase his/her chance of making the market by submitting price-aggressive offers. The payoff associated with an aggressive offer is skewed to the left (i.e., there are small chances of large losses to informed traders) while the payoff associated with a non-aggressive offer is skewed to the right (i.e., there are small chances of large gains from uninformed traders). However, in equilibrium, expected payoffs are the same, so it is optimal to choose the offering price randomly. We also show that a trader who exposes his/her offer expects to lose, even though rational traders don't "penny jump" exposed offers.

Limit-orders are the source of liquidity in pure limit-order markets.[1] To understand liquidity in these markets, we have to understand the risks that limit-order traders bear. In the market microstructure literature, three risks have been studied extensively in the context of stylized models: 1) the execution risk, 2) the fundamental risk, and 3) the adverse selection risk. In this paper, we study the risk of being "pennied" and its implications.

A limit-order trader who desires immediacy faces an execution risk: it may take a long time for a matching order to hit the book. A trader who needs immediacy weighs the benefits of "making the market" against the execution risk, and accordingly chooses how aggressively to price the limit-order. Foucault (1999), Parlour (1998), Foucault, Kadan, and Kandel (2005), Goettler, Parlour, and Rajan (2005), and Rosu (2004) model the execution risk in dynamic environments.

A limit-order trader is also subject to a fundamental risk: the risk that market prices will move away from or through the limit price. In the former, the order will not be executed and in the latter it will be picked off by news traders. Copeland and Galai (1983) show that, because of the fundamental risk, limit-orders are equivalent to free options: buy limit-orders are put options while sell limit-orders are call options. To mitigate the fundamental risk, traders should monitor news and price their orders in a regret-free fashion.[2]

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[1]In this paper, a limit-order is an outstanding limit-order; i.e., a non-marketable limit-order.

[2]See also Brown and Holden (2005) who discuss pegged limit-orders.

The counter party's motive for trade is the adverse selection risk. Most of the literature in theoretical market microstructure focuses on the risk of being matched with insiders, although news traders also impose adverse selection risk on limit-order traders.[3] In the context of limit-order markets, Rock (1990) and Glosten (1994) are important papers. To mitigate the adverse selection risk, limit-order traders should post regret-free limit-orders.

Limit-orders are outstanding orders that, when matched, are executed at their limit-price. Based on precedence rules, limit-orders are placed in a queue that determines the order of execution. All else being equal, low precedence naturally increases the execution risk. To the extent that insiders and news trades are more likely to submit large orders, low precedence also increases the adverse selection risks.

Because (i) price priority is the primary precedence rule, and (ii) the minimum price variation (tick size) in most markets is now negligible, it is virtually costless for traders to undercut outstanding offers. This means that a limit-order that was placed at a price reflecting risks and benefits, once being pennied, most likely has to be repriced or cancelled.

We study the risk of being pennied using a stylized static model with n strategic value traders who can post multiple offers (limit-orders). Value traders are traders who trade only when there is a discrepancy between market prices and the perceived intrinsic value. Value traders are subject to adverse selection risk but, because they don't need immediacy, they only offer at prices that reflect the adverse selection risk (i.e., value traders "earn the spread"). In our model, the limit-orders are matched with either uninformed orders (submitted by liquidity traders) or informed orders (submitted by news traders). Because uninformed orders have finite sizes while news traders pick off all stale limit-orders, high precedence is a virtue. Thus each value trader likes to outguess the offering prices of the others.

Consider a simplified version of the value traders' problem in which only two value traders can post a single offer. Since the order size is correlated with information, there is a "capacity constraint" on the total number of shares that can be offered at aggressive prices without incurring losses. If one value trader posts an aggressive offer, the best response of the second value trader is to post an uncompetitive offer (i.e., to become a residual supplier of liquidity). On the other hand, if the second value trader posts an uncompetitive offer, the best response of the first value trader is then to offer a marginal price improvement (rather than to post an aggressive offer). This intuition suggests a non-existence of equilibrium when offers are forecastable (see Section 3). We then show that there is an equilibrium in which value traders use random limit-orders; i.e., no value trader can outguess the prices and quantities other traders offer (Sections 4 and 5).

To check that our results are not artifacts of the assumption that uninformed demand is inelastic, we extend the basic model and assume that each uninformed trader limits the price he/she is willing to pay for immediacy. We show (in Section 6) an equilibrium

[3] In this paper we use interchangeably the terms informed traders and news traders. See also Footnote (6).

in which (i) value traders use random offers and (ii) conditional on execution (that is, the uninformed trader's order is marketable), the order transacts at the same price it would in the model with inelastic liquidity demand.

In Section 7, we quantify the exposure risk discussed in Harris (1996). We introduce into the model a trader who posts a forcastable offer. We can interpret this offer as if it were an uninformed non-marketable order that is now simply exposed. Three things emerge in equilibrium: (i) the exposed order is not pennied. This is natural because to "penny jump" an exposed offer is a forecastable strategy, (ii) there is a hole behind the exposed offer in which no one posts offers, and (iii) the exposed offer has a negative expected payoff.

How relevant is our equilibrium to actual markets? If the risk of being pennied is a real concern to limit-order traders, what shall we then expect to see in markets? The static model predicts that traders mitigate the risk by using random limit-orders. Assuming that the necessity of unforecastable limit-orders is carried over to dynamic environments, then in an open, pure limit-order book, one can keep other traders guessing by using short-lived orders, and replacing cancelled orders with new ones that are (from the market point of view) drawn randomly.

Interestingly, Hasbrouck and Saar (2004) examine the INET ECN and find that one third of the limit-orders are canceled within 2 seconds. However, they do not report whether those "fleeting orders" are replaced by new unforecastable ones. Mitigating the exposure risk should be simpler if the trading platform allows hidden orders. Bessembinder, Panayides, and Venkataraman (2006) find that on Euronext, partially hidden orders represent about 44% of the volume (on Euronext one must reveal a portion of the quantity offered). Boehmer, Saar, and Yu (2005) find that after the NYSE introduced OpenBook, the cancellation rate of limit-orders increased. Those empirical findings are consistent with strategies aimed at reducing the risk of being undercut, although those empirical papers did not really test whether limit-orders are unforecastable.

Finally, consider a dealers market. This type of market is different from a pure limit-order market: for example dealers can quote better prices for larger orders, which is impossible in a pure limit-order market. Nevertheless, both markets still share many similarities. Dealers markets are sometimes opaque, as was traditionally the case for corporate bonds (see Bessembinder, Maxwell, and Venkataraman (2006)). In opaque dealers markets there is no continuous public dissemination of quotes, rather, dealers provide quotes to specific customers on request. But why don't they advertise their quotes? If exposure risk is relevant to dealers, then public quote dissemination would invite underbidding.

1. Motivation

Consider a trading platform organized as a pure limit-order book without a minimum price variation, so that bids and offers can be posted arbitrarily close to each other.

At any point in time, traders can post new bids/offers and cancel existing ones.[4] A marketable order is any order that can be executed immediately. The limit-order market is discriminatory because a large marketable order is executed at multiple prices; i.e., the marketable order "walks" the book until it is filled.

There is no minimum price variation, so it is meaningful to ask whether the expected payoff associated with a single offer is a continuous function of its offering price. The answer is negative because of the price priority rule: orders are first matched with the best offers, hence the expected payoff depends on whether the offer is marginally below an existing offer or marginally above it. The standard proof of existence of equilibrium requires the continuity of payoffs in the set of feasible actions, and indeed in many economic models with discontinuous payoffs an equilibrium fails to exist.[5]

Note that the lack of proof of existence does not imply that equilibrium fails to exist (though in Section 3 we show a non-existence result relevant to our model). Still, if equilibrium does fail to exist, we may successfully circumvent the problem if we proceed in the following way. We extend the set of feasible strategies so that when traders use the "new" strategies, the expected payoff function of each value trader is continuous in the set of all feasible strategies. We then try to find an equilibrium in which each value trader uses a strategy from the new set of feasible strategies. In the market microstructure literature, two such extensions were successfully utilized, both in models of dealers markets.

Biais, Martimort, and Rochet (2000) notice that they can define a strategy in terms of cumulative offers and bids. A feasible strategy is then any step function with the interpretation that an offer is a point of discontinuity of the cumulative function. Next, they extend the set of feasible strategies by allowing differentiable cumulative functions, effectively permitting each dealer to post a continuum of bids and offers, each good for only an infinitesimal number of shares. When traders only offer infinitesimal quantities, undercutting those offers does not introduce a discontinuity into the payoff function. In the information environment they considered, Biais, Martimort, and Rochet (2000) construct an equilibrium in the class of smooth strategies.

Dennert (1993) and this paper follow the literature on games with discontinuous payoffs by allowing traders to chose a strategy at random (see Dasgupta and Maskin (1986)). This approach is useful because if the randomization scheme is atomless (i.e., at any price, the probability that an offer or bid is posted is zero), then undercutting occurs with probability zero, so the expected payoff functions are continuous at any price.

In Dennert (1993), the liquidity providers are dealers, and each dealer makes his/her own market. The equilibrium was found in two special cases. In one case, a regulator does not allow dealers to post more than one quote, and in the other case, dealers

[4]An offer is a pair of quantity and price with the interpretation that the trader offers to buy the stock at the offered price and up to the offered quantity. In a pure limit-order market, an offer is any non-marketable sell limit-order. A bid is defined analogously.

[5]Two important examples of models with discontinuous payoffs are the Bertrand-Edgeworth duopoly model and the Hotelling model of price competition.

can post multiple quotes for different sizes and in equilibrium larger orders get better prices. These results are not relevant to pure limit-order markets because (i) traders can post multiple bids and offers, and (ii) regardless of their size, marketable orders are first matched with the best offers (thanks to the price priority rule), so value traders cannot implement a strategy where larger marketable orders get better prices.

2. The Model

We consider a pure-limit order market for a single risky asset in which n risk-neutral traders, hereafter value traders, post bids and offers at various prices. Liquidity traders arrive at the market at random times and submit buy or sell orders. Events relevant to the value of the asset occur at random times and the information generated by those events is conveyed to the market instantaneously by an unspecified number of news traders who pick off all stale offers and bids.[6]

As in Copeland and Galai (1983) and Glosten (1994), we postulate that we can study the problem of the value traders by looking at a single point in time. We further assume that with equal probability liquidity traders submit market orders for one or two shares (or round lots) and $\widetilde{v}$, the value of the asset if an information event has taken place, is either 1 or -1 (with equal probability). Under those simplifying assumptions, the only statistic relevant to the value traders is the probability that the next order is informed. We denote this probability by μ.[7] In the following, we look only at the offer side of the book, as the analysis of the bid is analogous.

Let $Q_i(x)$ be the cumulative number of shares offered by the i-th value trader up to the price x. A feasible strategy is any step function $Q(x)$ with finitely many jumps, with the interpretation that a jump at x, $\Delta Q(x)$, is the number of shares offered at the price x ($\Delta Q(x)$ can be any non-negative real number). Let $y(x) = \sum_{i=1}^{n} Q_i(x)$ be the aggregate number of shares offered up to the price x. We call $y(x)$ the book. An allocation rule is used when more than one offer is posted at the same price and the incoming market order is not large enough to fill up all offers at that price. All we assume is that the allocation rule is such that with a positive probability each offer gets a partial filling (e.g. a pro rata allocation rule). A competitive equilibrium is a book $y(x)$ such that the expected payoff of $y(x)$ equals zero for all x. This definition implies that the payoff of every single offer is in fact zero. It is immediate that the following book is a competitive equilibrium.

$$y(x) = \begin{cases} 0 & x < \mu \\ 1 & \mu \leq x < \frac{2\mu}{\mu+1} \\ 2 & \frac{2\mu}{\mu+1} \leq x \end{cases}$$

In the competitive equilibrium, one share is offered for sale at μ and an additional share is offered at $\frac{2\mu}{\mu+1}$.

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[6] Alternatively, we could think of multiple insiders who compete aggressively, as in Holden and Subrahmanyam (1992), so that effectively they trade as if their information is short-lived.

[7] If the time of the next event is exponentially distributed with parameter r, and the liquidity traders arrive at the market according to a Poisson process with intensity β, then $\mu = r/(r+\beta)$.

3. No Equilibrium in Pure Strategies

We present a sketch of a proof that here is no equilibrium in pure (i.e., forecastable) strategies.[8] Suppose, to the contrary, that there is an equilibrium in pure strategies, and let $x_1 < x_2 < \cdots < x_m$ be the prices at which offers are posted in the equilibrium. Also, let $y(x)$ denote the book. The proof has two steps. We first show that all offers must have zero expected payoffs and the allocation rule is applied with zero probability. We then show a profitable deviation for the trader who posts the best offer.

Say the i-th trader offers at x_m (we don't rule out the possibility that several value traders offer at x_m). The payoff associated with that offer cannot be negative because it is at the end of the queue (i.e., cancellation of this offer has no impact on the probability of execution and hence the payoff associated with the more aggressive offers). In fact, the expected payoff must be zero, since otherwise a profitable deviation for the j-th value trader would be to add a new offer for $\Delta y(x_m)$ shares at a price marginally below x_m. If the j-th trader also has an outstanding offer at x_m, that offer should be cancelled. This deviation is profitable because it does not affect the payoff associated with all other offers at $x_1, x_2, \ldots, x_{m-1}$. Thus, we conclude the payoff of the offer at x_m is zero.

We can say even more about the offer at x_m. Not only is the expected payoff of each offer at x_m zero, also the allocation rule should never be applied at x_m. Indeed, the allocation rule is applied only if the limit-orders are matched with uninformed orders (because informed traders pick off all stale offers). This means that if traders i and j post offers at x_m, and the allocation rule is applied with a positive probability, then the deviation suggested above is profitable even though the expected gain of the i-th trader's offer at x_m is zero. This is because the higher precedence of the deviation strategy increases the chances of trading with uninformed traders.

Next, we argue that the offer at x_{m-1} has a zero expected payoff. Say, to the contrary, that the expected payoff is positive and that the i-th trader posts an offer at x_{m-1}. The j-th trader could profitably deviate by placing an offer of size $\Delta y(x_{m-1})$, while canceling all his/her existing offers at prices x_{m-1} and above (if there are any). Having shown that offers at x_{m-1} have zero expected profits, we repeat the argument that shows that the allocation rule is never applied also at x_{m-1}. In a similar way, we show that all offers have zero expected profits and that the allocation rule is never applied in the conjectured equilibrium.

So far we have shown that if there is an equilibrium in pure strategies, then the expected payoff of every offer must be zero and the allocation rule is never applied. We now show a profitable deviation, and we focus on the beginning of the queue. The trader who posts the offer at x_1 (say for q_1 shares) must find it profitable to replace his/her strategy ("cancel" all his/her offers) with a single offer of size q_1 at a price marginally below x_2. This will not change the probability of execution, but

[8] We call it a sketch because we did not yet define an equilibrium.

will increase the payoff because of the higher price. We conclude that there is no equilibrium in pure strategies, and we thus turn our attention to mixed strategies.

A mixed strategy is a distribution over a set of feasible strategies, with the interpretation that the strategy employed is a draw from the distribution. An equilibrium in mixed strategies is an n-tuple of mixed strategies, where we associate the i-th strategy with the i-th value trader, such that if any $n-1$ value traders follow their mixed strategy, then the following two conditions regarding the mixed strategy of the n-th trader are satisfied: (i) any pure strategy in the support of the mixed strategy has the same expected payoff, and (ii) any feasible pure strategy in the complement of the support of the mixed strategy has a payoff that is not greater than the payoff of the strategies in the support. A symmetric equilibrium in mixed strategies is an equilibrium in which all traders use the same mixed strategy.

4. Two Value Traders

In this section we construct an equilibrium in mixed strategies when the number of value traders is 2. In this equilibrium, each value trader posts offers at random prices (up to the price of 1) and integer-valued random quantities. When constructing the equilibrium, it is convenient to treat each share offered as a separate offer. Let τ_i be the order statistic of the random prices at which the value trader posts offers; i.e. $\widetilde{\tau}_1 \leq \widetilde{\tau}_2$. If a value trader offers two shares at the same price, the choice of which one is the realization of τ_1 is arbitrary. In the equilibrium we are looking for, $\widetilde{\tau}_2 \leq 1$, the distribution of $\widetilde{\tau}_1$ is continuous, and that of $\widetilde{\tau}_2$ has a single atom at 1. In particular, with a strictly positive probability, both traders post offers at 1. However, because liquidity traders submit at most two units, the expected payoff of an offer posted at 1 is zero, and thus in this equilibrium the allocation rule is irrelevant. Also, we note that in this equilibrium, offers at prices greater than 1 will never be executed.

We define the survival functions

$$
\begin{aligned}
H_1(x) &= P\left(\widetilde{\tau}_1 \geq x\right) \\
H_2(x) &= P\left(\widetilde{\tau}_2 \geq x\right) \\
G(x) &= P\left(\widetilde{\tau}_1 \leq x, \widetilde{\tau}_2 > x\right) \\
&= 1 - P\left(\widetilde{\tau}_1 \leq x, \widetilde{\tau}_2 \leq x\right) - P\left(\widetilde{\tau}_1 \geq x, \widetilde{\tau}_2 \geq x\right) - P\left(\widetilde{\tau}_1 \geq x, \widetilde{\tau}_2 \leq x\right) \\
&= 1 - P(\widetilde{\tau}_2 \leq x) - P(\widetilde{\tau}_1 \geq x) - 0 = H_2(x) - H_1(x)
\end{aligned}
$$

We associate a feasible strategy $Q(x)$ with a curve in the xy-plane. This curve, which we denote by C_Q, has a direction corresponding to increasing values of x (See figure (1)). When one trader places offers consistent with the distributions of τ_1 and τ_2, the expected payoff to the other value trader who uses an arbitrary feasible strategy

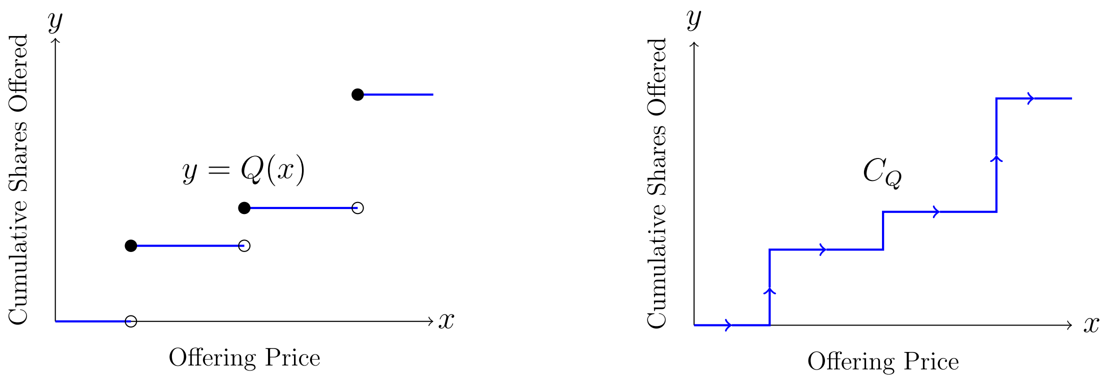

FIGURE 1. [A Feasible Strategy and its Corresponding Plane Curve.] This figure shows a typical feasible strategy (in the left panel) and the oriented plane curve associated with it (in the right panel).

$Q(x)$ is the line integral[9]

$$\int_{C_Q} u(x,y)dy$$

of the function

$$u(x,y) = \begin{cases} \mu(x-1) + (1-\mu)x\left[H_1(x) + G(x)\frac{1}{2}\right] & y < 1 \\ \mu(x-1) + (1-\mu)x\left[H_1(x)\frac{1}{2}\right] & 1 \le y < 2 \\ \mu(x-1) & 2 \le y \end{cases} \tag{1}$$

Theorem 1. *Let* $x^* = 2\mu/(1+\mu)$, *and for* $x \in [x^*, 1]$ *define*

$$H_1(x) = \begin{cases} 1 & x < x^* \\ \frac{2\mu(1-x)}{(1-\mu)x} & x \in [x^*, 1] \\ 0 & 1 < x \end{cases}$$

$$H_2(x) = H_1(x)\left(1 - \frac{2\mu}{1+\mu}\right) + \frac{2\mu}{1+\mu}$$

Then, a mixed strategy consistent with (H_1, H_2) *forms a symmetric equilibrium in which* $[x^*, 1]$ *is the support of* τ_1 *and* τ_2.

Before we prove the theorem, notice that $H_1(x)$ and $H_2(x)$ are decreasing (so they are feasible survival functions), $H_1(x) \le H_2(x)$, and $H_1(x^*) = H_2(x^*) = 1$ (so offers are never posted below x^*). Also, $H_1(1) = 0$ while $H_2(1) = \frac{2\mu}{1+\mu} > 0$. The latter implies that, with a strictly positive probability, shares are offered at the price one.

Consider the problem of one value trader, when the other trader follows the conjectured equilibrium strategy. We plug $H_1(x)$ and $G(x) = H_2(x) - H_1(x)$ into (1), and

[9]This line integral, sometimes referred to as a line integral of the second kind, is calculated in the following way. We divide the curve C_Q into segments of vertical and horizontals parts. The line integral along C_Q is the sum of line integrals along its segments. On each vertical segment, the integral is zero, and on each horizontal segment at x, the line integral is $\int_{Q(x-)}^{Q(x)} u(x,y)dy$ where $Q(x-)$ denotes the left limit of the $Q(\cdot)$ at x.

find that $u(x, y)$ is piecewise linear. Figure (2) is a graphical representation of $u(x, y)$. The following properties are important: (i) $u(x, u)$ attains its maximum value in the rectangle region $\{(x, y)|x^* \leq x < 1, 0 \leq y < 1\}$, (ii) $u(x, y)$ vanishes on the rectangular region $\{(x, y)|x^* \leq x < 1, 1 \leq y < 2\}$, and (iii) for $y > 2$, $u(x, y) = \mu(x - 1) \leq 0$. The conclusion is that any strategy in which $Q(x^*) = 0$ and $1 \leq Q(1) \leq 2$ is optimal. In particular, a randomization scheme that is consistent with the survival functions $H_1(x)$ and $H_2(x)$ is optimal. Thus, we have proved the theorem.

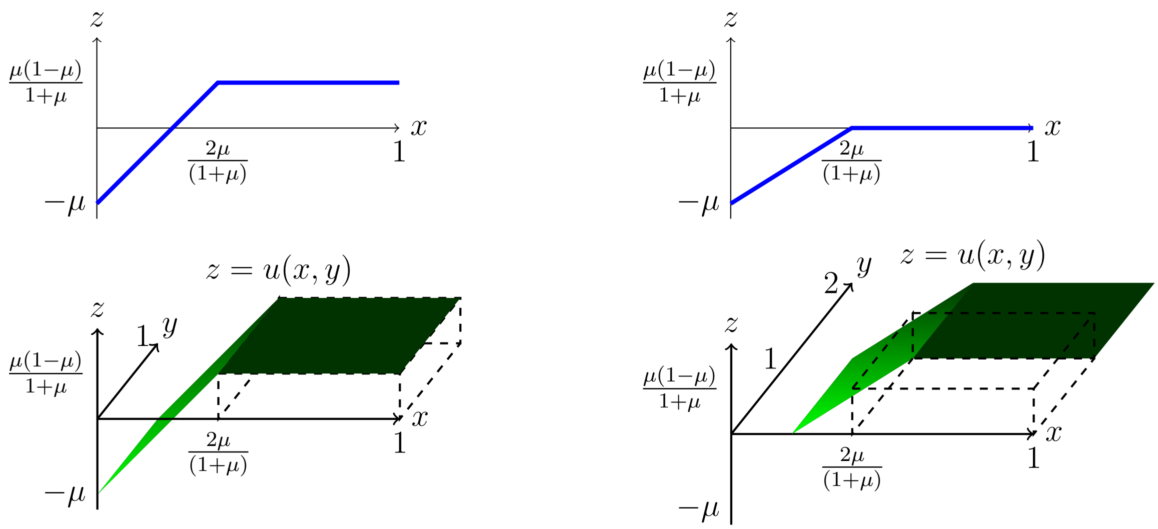

FIGURE 2. [The Function $u(x, y)$.] The equilibrium expected gain associated with a single offer of an infinitesimal size at a price x is proportional to $u(x, y)$, where y is the cumulative number of shares the trader offers at prices not greater than x. The lower left panel shows $u(x, y)$ for $y \in [0, 1]$ and the upper left panel shows the projections onto the y-plane. Similarly, the lower right panel shows $u(x, y)$ for $y \in [1, 2]$ and the upper right panel shows the projection onto the y-plane. To clarify the differences between the figures in the lower panels, an identical dashed box is drawn in both panels.

Corollary 1. *Let $H_1(x)$ and $H_2(x)$ be as in Theorem 1. Consider the following mixed strategy. Its support is the set of feasible strategies such that (i) $Q(1) = 2$ (i.e. a total quantity of 2 is offered up to the price 1); (ii) one and only one offer is posted at $\tilde{x} \in (x^*, 1)$ and the quantity offered is either 1 or 2. The randomization scheme is as follows: The distribution function of $\tilde{x}$ is $1 - H_1(x)$, and the probability that only one share is offered at x is $(2\mu)/(1 + \mu)$ (in particular, the decision of how many shares to offer is independent of the offering price). This mixed strategy defines a symmetric equilibrium.*

Treating an offer of size two as two separate offers of size one, the two prices at which offers are posted are random. Let τ_1 and τ_2 be the order statistics of those two prices. To prove the corollary, we only need to show that the survival functions of τ's are as

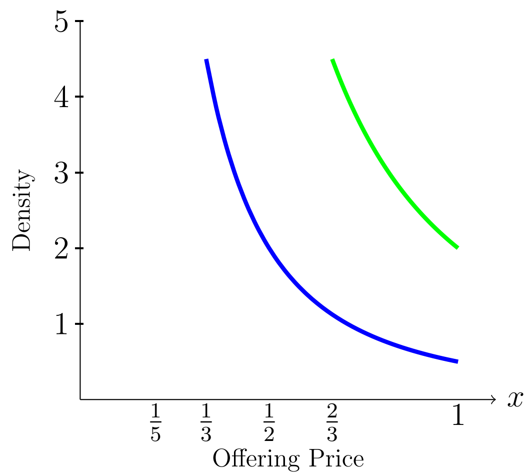

FIGURE 3. [Density of Random Offering Prices.] In equilibrium with two value traders, each trader offers one or two shares (the choice is random) at a random offering price. The density of the offering price is shown for two different values of μ, the probability that the next buy order is an informed order. In the lower curve $\mu = 1/5$ and in the upper it is $1/2$.

stated in Theorem 1. By construction, the survival function of τ_1 is H_1, and

$$\begin{aligned} Prob(\tau_2 > x) &= Prob(\tau_1 > x) + \frac{2\mu}{1+\mu} Prob(\tau_1 \leq x) \\ &= H_1(x) + \frac{2\mu}{1+\mu}(1 - H_1(x)) = H_1(x)\left(1 - \frac{2\mu}{1+\mu}\right) + \frac{2\mu}{1+\mu} \\ &= H_2(x) \end{aligned}$$

where the last equality is from the definition of $H_2(x)$ given in Theorem 1. This proves the corollary.

Figure (3) shows the density of τ_1 for two different values of μ. The figure shows that the larger the probability of trading with news traders, the less likely value traders are to post aggressive offers.

5. A Large Number of Value Traders

We now assume that n, the number of value traders, is strictly greater than two. We will show that there is a symmetric mixed strategy equilibrium in which each value trader offers one share at a random price. Again, in equilibrium, undercutting must not be profitable, so the randomization scheme traders use in equilibrium is atomless.[10]

[10]Note that unlike the case with two value traders, here the quantities value traders offer are fixed and the equilibrium expected gains of the value traders are zero. If we assume liquidity traders submit orders for a random number of shares up to m, then we conjecture that whenever $1 < n \leq m$, value traders will offer random quantities and will expect to make positive gains. In this paper, we verified the conjecture only for $m = 2$.

Let $H(x)$, with $H(1)=0$, be the equilibrium survival function we are looking for; i.e., it is the probability that the price (at which a value trader offers a single share) is greater than x. In this equilibrium, offers at prices greater than one can be ignored because they are never executed. The probability that $n-1$ traders do not post a single offer below x is $H^{n-1}(x)$, and the probability that $n-1$ traders post one and only one offer below x is $(n-1)H^{n-2}(x)(1-H(x))$. Thus, the expected payoff of a feasible strategy $Q(x)$ is the line integral $\int_{C_Q} u(x,y)dy$, where

$$u(x,y)=\begin{cases}\mu(x-1)+(1-\mu)x\left[H^{n-1}(x)+\frac{1}{2}(n-1)H^{n-2}(x)(1-H(x))\right] & y<1\\ \mu(x-1)+(1-\mu)x\left[\frac{1}{2}H^{n-1}(x)\right] & 1\le y<2\\ \mu(x-1) & 2\le y\end{cases}$$

To construct the equilibrium, we define the survival function in the following way. First, in the unit interval the function

$$\begin{aligned} x(h) &= \frac{1}{2}h^{n-1}(3-n)+\frac{1}{2}h^{n-2}(n-1)\\ &= h^{n-1}+\frac{1}{2}(n-1)h^{n-2}(1-h) \end{aligned} \tag{2}$$

is strictly increasing, and therefore has an inverse $h(x)$ that is strictly increasing in the unit interval. The inverse function also satisfies $h(0)=0$ and $h(1)=1$. We then define the survival function

$$H(x)=\begin{cases}1 & x<\mu\\ h\left(\frac{(1-x)\mu}{x(1-\mu)}\right) & \mu\le x<1\\ 0 & 1\le x\end{cases} \tag{3}$$

Because $H(\mu)=1$, $H(1)=0$ and $H(x)$ is strictly decreasing on $[\mu,1]$, $H(x)$ defines an atomless survival function.

Theorem 2. *Assume $n\ge 3$, and consider the family of feasible strategies comprised of a single offer of size one, parametrized by the offering price x. A mixing distribution over this family, consistent with the survival function $H(x)$, forms a symmetric equilibrium in mixed strategies.*

From (2), it follows that for all $x\in[\mu,1]$ we have $H^{n-1}(x)+\frac{1}{2}(n-1)H^{n-2}(x)(1-H(x))=\frac{(1-x)\mu}{x(1-\mu)}$, which implies that $u(x,y)=0$ on the rectangular region $\{(x,y)|\mu\le x<1, 0\le y<1\}$. Also, elsewhere in the positive quadrant $u(x,y)$ is strictly negative ($x<\mu$ or $y>1$) or zero ($x\ge 1$). It follows that when $n-1$ value traders follow the conjectured equilibrium strategy, it is optimal for the n-th trader to use any monotone increasing $Q(x)$ such that (i) $Q(\mu)=0$ and (ii) $Q(1)\le 1$. In particular, it is optimal to offer 1 share at any price $[\mu,1]$. This ends the proof.

Figure (4) shows the density of the random offering price for different values of μ, the probability of trading with informed traders. The figure is consistent with the intuition that offering prices are likely to be higher when the adverse selection is severe.

In a symmetric equilibrium, the value traders' offering prices are independently and identically distributed. Therefore, it is straightforward to find the density of the ordered prices; i.e., the best offer, the next best offer, etc. The density of the k-th best offer is

$$\frac{-n!}{(k-1)!(n-1)!}(1-H(x))^{k-1}H^{n-k}(x)H'(x).$$

Figure (5) depicts an example.

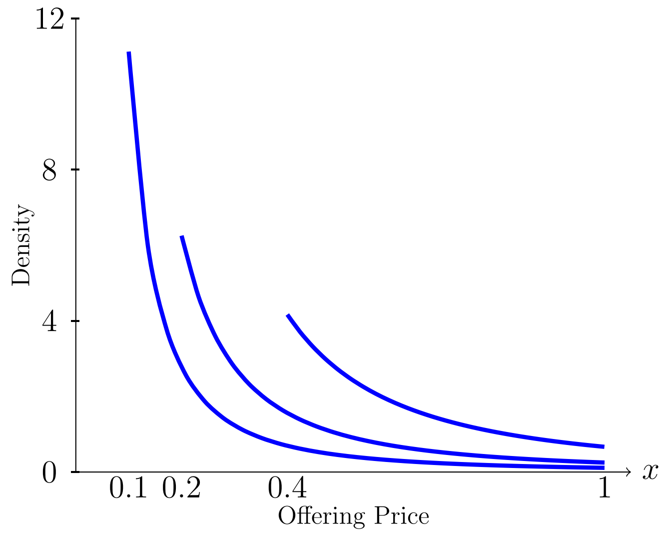

FIGURE 4. [Density of Random Offering Prices.] The figure shows the density of the random price when $n = 3$ and μ is either 0.1(the low curve), 0.2 (the middle curve), or 0.4 (the upper curve).

6. Price Elastic Liquidity Demand

The analysis thus far has relied on the assumption that uninformed traders' demand was irrespective of market liquidity. We would like now to relax this assumption.

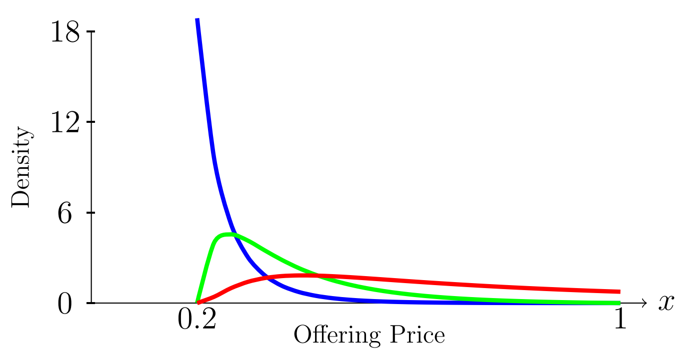

FIGURE 5. [Order Statistics.] The figure shows the densities of the best, second best, and third best offers in the book. The curves are ordered by their values at low prices: the density of the best offer gets the highest values at low prices. The parameters in this example are $n = 3$ and $\mu = 0.2$.

More specifically, we assume that liquidity traders attach to their order a limit price x^* (for simplicity, all liquidity traders use the same limit-price). The order is executed only if it is marketable; i.e., if there are offers at prices below x^*. We maintain the assumption that the size of a liquidity trader's order is equally likely to be 1 or 2. Clearly, if the limit-price is too low, the order will not execute. Also, we don't expect to see offers at prices above x^*, because those can be matched only with news traders.

We assume $n \geq 3$ and we define the survival function

$$H_e(x) = \begin{cases} H(x) & x < x^* \\ H(x^*) & x^* \leq x < 1 \\ 1 & 1 \leq x \end{cases} \tag{4}$$

where $H(x)$ (defined in (3)) is the equilibrium survival function when the liquidity demand is inelastic. If $x^* > 1$, then the two survival functions are identical. We state without proof the following simple corollary.

Corollary 2. *Assume $n \geq 3$, and consider the family of pure strategies (each consists of a single offer of size 1) parameterized by the offering price x. A mixing distribution over this family, consistent with the survival function $H_e(x)$, forms a symmetric equilibrium in mixed strategies.*

Interestingly, the equilibrium is very similar to the one with inelastic demand. Each value trader draws a single offering price from an identical distribution, and if the price falls below x^*, the trader posts the offer, otherwise the value trader posts the offer at 1.

7. Exposure Risk

In the previous section we saw that if the liquidity traders have a reservation price, then with a positive probability an incoming buy order is non-marketable. In real markets, if an aggressive limit-order is not executed it "makes the markets," i.e., it becomes the best quote in the book. It motivates us to consider the following question. How will the value traders react when an outstanding offer is exposed at x^*? We have the following result:

Corollary 3. *Assume there is an outstanding offer at x^*, and the number of value traders is at least 3. If the size of the exposed offer at x^* is two, then the mixed strategy stated in Corollary 2 forms a symmetric equilibrium in mixed strategies. If the size of the exposed offer is one, then the symmetric equilibrium is defined using the survival function*

$$H_o(x) = \begin{cases} H(x) & x < x^* \\ \min\left\{ H(x^*), \left(\frac{2\mu(1-x)}{(1-\mu)x}\right)^{\frac{1}{n-1}} \right\} & x^* \leq x < 1 \\ 1 & 1 \leq x \end{cases} \tag{5}$$

When the size of the exposed limit order is two, the mixed strategy played in this equilibrium is identical to the model with elastic demand. In particular, value traders don't place offers above a large exposed offer. The situation is slightly different when the size of the exposed offer is small: value traders may offer at prices above the offering price of the exposed limit-order; however, there is a hole in the book. Figure (6) shows the survival function when the exposed order is small and $n = 3$. The interval of prices where H_o is flat is not in the support of the mixed strategy; i.e., offers are posted there with probability zero. It is also interesting to note that nobody penny jumps the exposed offer; i.e. the function H_0 is continuous at x^*. This makes sense because a strategy that penny jumps an exposed offer is a forecastable strategy and thus cannot be present in equilibrium.

We only prove the corollary in the case that the exposed offer is small. Assume the first $n-1$ value traders play the equilibrium mixed strategy. We need to show that it is optimal for the n-th trader to post an offer of size 1 at all those prices where H_o is strictly decreasing. The n-th trader's expected payoff when he/she uses an arbitrary feasible strategy $Q(x)$ is the line integral $\int_{C_Q} u(x,y)dy$, where for $y \in [0,1)$ and $x \in [0,1)$ the function $u(x,y)$ is given by

(6)
$$u(x,y) = \begin{cases} \mu(x-1) + (1-\mu)x\left[H^{n-1}(x) + \frac{1}{2}(n-1)H^{n-2}(x)(1-H(x))\right] & x < x^* \\ \mu(x-1) + (1-\mu)x\left[\frac{1}{2}H_o^{n-1}(x)\right] & x^* \leq x \end{cases}$$

Let x_o be such that $H(x^*) = \left(\frac{2\mu(1-x_o)}{(1-\mu)x_o}\right)^{\frac{1}{n-1}}$, then

$$u(x,y) = \begin{cases} x-\mu & x < \mu \\ 0 & \mu \leq x < x^* \\ \mu(x-1) + (1-\mu)x\frac{1}{2}H^{n-1}(x^*) & x^* \leq x < x_o \\ 0 & x_0 \leq x \end{cases} \tag{7}$$

So, $u(x,y)$ is negative whenever $x < \mu$ or $x^* < x < x_0$ and zero elsewhere. Next, when $y > 1$, we have

$$u(x,y) = \begin{cases} \mu(x-1) + (1-\mu)x\left[\frac{1}{2}H^{n-1}(x)\right] & x < x^* \\ \mu(x-1) & x^* \leq x \end{cases} \tag{8}$$

and therefore $u(x,y)$ is negative when $y > 1$. We conclude that any strategy in which the trader offers at most a total of one share is optimal if all the offers are in $[\mu, x^*) \cup [x_o, 1]$. In particular, it is optimal to post a single offer of size 1 at any of those prices. This ends the proof.

Figure (7) shows the expected losses associated with an exposed offer. The figure illustrates that the losses of the exposed offer increase with the size of the exposed offer.

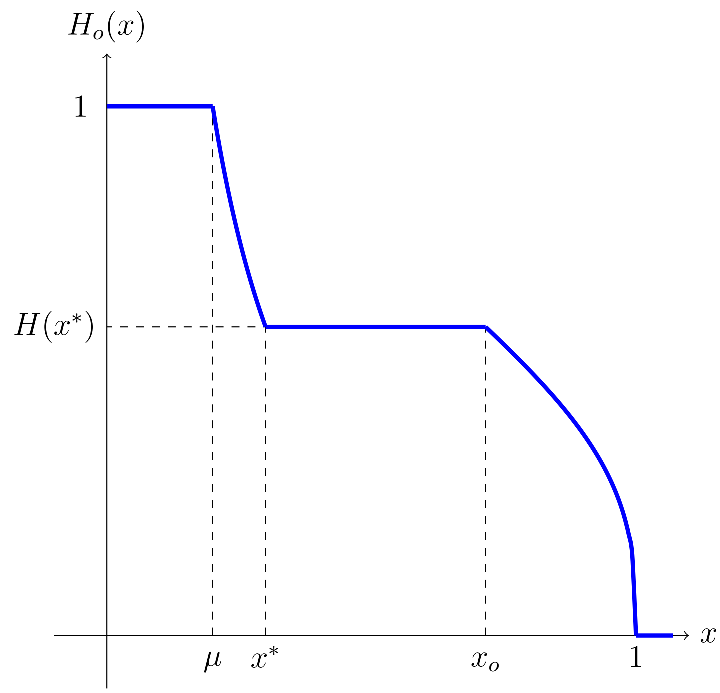

FIGURE 6. [A Limit-Order Book with a Hole.] Consider an equilibrium with an exposed offer of size 1 at x^*. This figure shows the survival function of the offering price of a value trader. In intervals where the survival function is flat, there is a zero probability that an offer will be posted. In particular, just behind the exposed offer there is a hole in the book. In this figure, $n = 3$ and the probability of informed trading is $\mu = 0.2$.

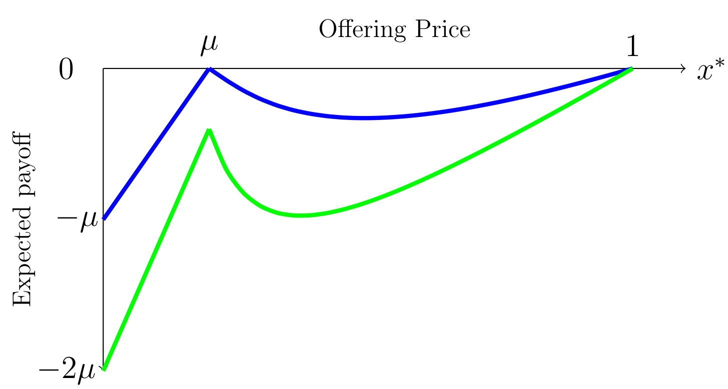

FIGURE 7. [Negative Expected Payoff of an Exposed Offer] The figure shows the expected payoff of an exposed offer as a function of its limit price. In the lower curve, the size of the exposed offer is 2 and in the upper curve only one share is offered. The number of value traders in this example is 3 and $\mu = 0.2$.

8. Conclusion

We have found an equilibrium in which n strategic value traders use random limit-orders. In this equilibrium, limit-orders that are "at the market" (i.e., the best offer and the best bid) are profitable (because the orders have a better chance of being matched with uninformed orders), while orders that are behind the market are not (because they are more likely to be matched with informed orders).

In equilibrium, a trader is never sure where his/her offer is in the queue. Posting an aggressive offer (i) increases the chance that the offer will have high precedence and hence increases the chance that the offer will be matched with an uninformed order, (ii) decreases the potential gains from trading with uninformed traders, and (iii) increases the potential losses to informed orders. Thus, the aggressive offer has a random payoff, skewed to the left (i.e., there are small chances of large losses to informed traders). On the other hand, an uncompetitive offer has a random payoff that is skewed to the right (i.e., there are small chances of high gains from uninformed traders). In equilibrium with random limit-orders, the skewed distributions have the same expected value.

We have also introduced into the model a trader who exposes his/her offer. In equilibrium, this trader expects to lose. Thus, we have demonstrated the importance of using unforecastable limit-orders.

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