

Liquidity and Information in Order Driven Markets

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October 16, 2008

Abstract

This paper analyzes the interaction between liquidity traders and informed traders in a dynamic model of an order-driven market. Agents freely choose between limit and market orders by trading off execution price and waiting costs. In equilibrium, informed patient traders generally submit limit orders, except when their privately observed fundamental value of the asset is far away from the current market-inferred value, in which case they become impatient and submit a market order. As a result, a market buy order is interpreted as an unambiguously positive signal; by contrast, a limit buy order is typically a weaker positive, and in some cases even negative, signal. The model generates a rich set of relationships among prices, spreads, trading activity, and volatility. In particular, the order flow is autocorrelated if and only if there are informed traders in the market, and the order flow autocorrelation increases with the percentage of informed traders. Higher volatility and smaller trading activity generate larger spreads, while a higher percentage of informed traders, controlling for volatility and trading activity, surprisingly generates smaller spreads.

KEYWORDS: Bid-ask spread, price impact, volatility, trading volume, limit order book, waiting costs.

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1 Introduction

This article studies the role of information in order-driven markets, where trading is done via limit orders and market orders in a limit order book.¹ Today more than half of the world's stock exchanges are order-driven, with no designated market makers (e.g., Euronext, Helsinki, Hong Kong, Tokyo, Toronto), while in many hybrid markets designated market makers have to compete with a limit order book (NYSE, Nasdaq, London).

Given the importance of order-driven markets, there have been relatively few models which describe price formation in these markets. This is partly due to the difficulty of the problem. Since there is no centralized decision maker, prices arise from the interaction of a large number of traders, each of which can be fully strategic. The presence of traders who are informed about the asset's fundamental value complicates the problem even more.

This paper proposes a dynamic model of an order-driven market where agents can strategically choose between limit and market orders. The model builds on the framework proposed by Roşu (2008), but modifies it in several important ways. First, the model introduces a fundamental value $v(t)$ of an asset. This is assumed to follow an exogenous diffusion process. Upon arrival, the informed traders observe the fundamental value, and decide whether they want to trade, and, if so, whether to use a limit or a market order. Because of anonymity reasons, the market cannot distinguish the informed traders from those who only trade for liquidity reasons. But, based on the observed order flow, the market forms an expectation $v^e(t)$ of the fundamental value. We call v^e the “efficient” price.

Second, the liquidity traders use the efficient price v^e to make an initial choice whether or not to trade. This is done by comparing their expected profit with a private one-time cost, uniformly and independently distributed on an interval $[-C, C]$. This is done to avoid the no-trade theorem of Milgrom and Stokey (1982). For example, a liquidity trader with a positive cost must make a positive expected profit in order to enter the market, while a trader with negative cost tolerates an expected loss as long as it is not below a given threshold. This assumption also generates a downward-sloping demand function from liquidity buyers, and an upward-sloping supply function from liquidity sellers: for example, the probability that

¹A limit order is a price-contingent order to buy (sell) if the price falls below (rises above) a prespecified price. A sell limit order is also called an *offer*, while a buy limit order is also called a *bid*. The limit order book (or simply the book) is the collection of all outstanding limit orders. The lowest offer in the book is called the *ask price*, or simply *ask*, and the highest bid is called the *bid price*, or simply *bid*.

a liquidity seller enters the market increases with his expected utility (expected price minus waiting costs), i.e., a higher price attracts more order flow from liquidity sellers. This has two main implications. One, the limit order book naturally becomes resilient, i.e., bid and ask prices tend to gravitate around the center of the book, and the bid-ask spread tends to revert to small values. The other consequence is that the bounds of the limit order book become endogenously determined, thus removing the need for an exogenous fundamental band as in Foucault, Kadan and Kandel (2005), or Roşu (2008).

Most other results from Roşu (2008) hold true in the current model, although some of them may now have another interpretation due to asymmetric information. For example, in Roşu (2008) the “comovement effect” of Biais, Hillion and Spatt (1995)² is explained only using competition among strategic liquidity providers that have waiting costs: A decrease in the bid price lowers the reservation value for the sellers, so competition between them also drives the ask price down. In the present paper, the comovement effect can also be explained using asymmetric information: a sell market order contains the possibility that an informed trader has negative information about the fundamental value of the asset. Therefore, the whole limit order book, including the ask price, should go down by the same amount. In a direct comparison, the comovement effect due to waiting costs and competition is most likely of an order of magnitude smaller than the comovement effect based on asymmetric information.

Another consequence of having privately informed traders is that the bid-ask spread now has two components: one due to waiting costs, and one due to asymmetric information. Their relative importance depends on the level of asymmetric information, although it turns out that above a certain level of asymmetric information the waiting costs component becomes at least an order of magnitude smaller than the asymmetric information component.

Next, we discuss in more detail the effect of private information on the limit order book. First, the limit order book is always centered around the efficient price v^e , but otherwise depends only on the number of sellers m and number of buyers n in the book. The strategy of a patient informed trader depends on how far the fundamental price v is from v^e . If $v - v^e$

²The comovement effect is the fact that, e.g., a sell market order not only lowers the bid price—this can in part be due to the mechanical execution of limit orders on the buy side—but also lowers the ask price. Moreover, the decrease in the bid price is larger than the decrease in the ask price, which leads to a wider bid-ask spread.

is above or below two cutoffs (that depend on the state of the book), the patient informed trader optimally behaves in an impatient way and submits a market order. Otherwise, the patient trader uses a limit order.

As a result, the efficient price v^e always converges towards the fundamental value v . The speed of convergence depends on how the market reacts to market and limit orders. Since a fundamental value v away from v^e makes market orders more likely, market orders are correctly interpreted by the market to contain a lot of information about v . For example, a buy market order is a clear positive signal: with positive probability it comes from a patient informed trader, and so the fundamental value v is higher than the efficient price v^e plus a cutoff. By contrast, it can be shown that a limit buy order is a weaker positive signal, and in some cases it can even be a negative signal. Notice that a higher percentage of informed traders means a higher adjustment of the efficient price v^e to a market order, which means that prices converge faster when there are more informed traders. A testable implication of the model is that, as long as the percentage of informed traders is not too high, regardless of how many informed traders there are, the expected adjustment of the efficient price after a market order should be (in absolute value) about 4 times the expected adjustment after a limit order.

In general, the model generates a rich set of relationships between prices, spreads, trading activity, volatility, and information asymmetry (measured by the percentage of informed traders). In particular, consistent with previous literature, one can show that smaller trading activity and higher fundamental volatility generate larger spreads.³ Smaller trading activity implies that market orders are more rare, and so the volatility between two consecutive market orders is higher. Since this fundamental uncertainty is higher, both the adjustment of the efficient price v^e after a market order, and the size of the bid-ask spread are wider. Higher fundamental volatility leads to higher spreads for a similar reason. These predictions have been tested by Linnainmaa and Roşu (2008), who use instrumental variables to generate exogenous variation in trading activity and fundamental volatility.

A surprising new prediction is that, controlling for volatility and trading activity, a higher percentage of informed traders should generate *smaller* spreads. This is because a higher percentage of informed traders generates a quicker adjustment of prices to fundamentals, and

³See Foucault, Kadan and Kandel (2005), Foucault (1999), and Roşu (2008).

therefore to *less* asymmetric information. This prediction raises some interesting questions about why spreads are larger around earnings announcements. According to this paper, it is not the amount of asymmetric information, since this by itself would generate smaller spreads. Then it must be due to either lower trading activity, or to larger volatility, or perhaps a combination of both. While it is not clear why there would necessarily be lower trading activity in times of high uncertainty, one may argue that the high volatility surrounding these events causes higher spreads just by itself.

Also, this model contributes towards an explanation of the “diagonal effect” of Biais, Hillion and Spatt (1995), namely that the order flow is positively autocorrelated (e.g., a market buy order makes a future market buy order more likely). In this model one can show that the order flow is autocorrelated if and only if there exist informed traders, with a higher autocorrelation when there is a larger percentage of informed traders.

One potential use of this paper is to estimate the extent of information asymmetry in the market, e.g., by using the fact that the level of autocorrelation of order flow increases with the percentage of informed traders. The problem is that some traders may have private information not about the fundamental value v , but about the future order flow, i.e., they are informed in the sense of Evans and Lyons (2002). These agents would behave the same as the informed agents in our model, and become impatient if the information about the future order flow is extreme enough. Naturally, one would like to distinguish between the agents with information about the order flow from those informed about the fundamental value. Our suggestion is to look at price reversals. If the order-flow-informed traders do not really have information about v , then the efficient price v^e would then revert to the fundamental value v after the truly informed traders would have brought prices in line with fundamentals.⁴

Notice that this model generates two types of market resilience. One is a “micro” resilience, relative to the limit order book: bid and ask prices tend to stay close to the efficient price,

⁴Information about order flow is not the only alternative explanation of the diagonal effect. Another explanation comes from the possibility of large orders—if one relaxes the assumption that agents can only trade one unit of the asset. Then an agent who wants to trade a large quantity and is patient enough to work the order (divide it into smaller orders) and thus take advantage of the resilience of the limit order book would also generate a positively autocorrelated order flow. We could test if this is the right explanation if we had access to identity of the order flow: simply check if there is just one trader working the order. Theoretically, one could argue that the present model is able to incorporate working orders: by treating each order as coming from a separate trader. In that case, one finds again the problem of distinguishing information about fundamentals from information about order flow.

and the bid-ask spread tends to stay small. This type of resilience is due to the action of uninformed but strategic liquidity traders. At the same time, there is also a “macro” resilience, due to the action of informed traders, which makes the efficient price eventually converge to the fundamental value.

The choice between market orders and limit orders has been analyzed in various contexts, see, e.g., Chakravarty and Holden (1995), Cohen, Maier, Schwartz and Whitcomb (1981), Handa and Schwartz (1996), Kumar and Seppi (1993). Dynamic models of order-driven markets include Foucault (1999), Foucault, Kadan and Kandel (2003), Parlour (1998), and Roşu (2008). The price behavior in limit order books has been analyzed theoretically by Biais, Martimort and Rochet (2000), Glosten (1994), OHara and Oldfield (1986), Rock (1990), and Seppi (1997). Models that analyze liquidity traders, the dynamics of prices and trades and the convergence of prices to the fundamental value include Glosten and Milgrom (1985), Kyle (1985), Admati and Pfleiderer (1988), Easley and O’Hara (1987).

Empirical papers include Biais, Hillion, and Spatt (1995), who document the diagonal effect (positive autocorrelation of order flow) and the comovement effect (e.g., a downward move in the bid due to a large sell market order is followed by a smaller downward move in the ask – which increases the bid-ask spread); Sandas (2001), who uses data from the Stockholm exchange to reject the static conditions implied by the information model of Glosten (1994), and also finds that liquidity providers earn superior returns; Harris and Hasbrouck (1996) who obtain a similar result for the NYSE SuperDOT system; Hollifield, Miller and Sandas (2004) who test monotonicity conditions resulting from a dynamic model of the limit order book and provides some support for it; Hollifield, Miller, Sandas and Slive (2006) who use data from the Vancouver exchange to find that agents supply liquidity (by limit orders) when it is expensive and demand liquidity (by market orders) when it is cheap.

2 The Model

This section describes the assumptions of the model. Consider a market for an asset which pays no dividends, and whose fundamental value, or *full-information* price, v_t moves according to a diffusion process with no drift and constant volatility σ : $dv_t = \sigma dW_t$, where W_t is a standard Brownian motion. Based on all available public information until t , the market

forms an estimate, the *efficient* price: $v_t^e = \mathbf{E}\{v_t \mid \text{Public Information at } t\}$. For simplicity, we assume that v_t is believed each period to be normally distributed: $v_t \sim N(v_t^e, \sigma_t^e)$, with mean v_t^e and standard deviation σ_t^e .

The buy and sell prices for this asset are determined as the bid and ask prices resulting from trading based on the rules given below. Prices can take any value, i.e., the tick size is zero.

Trading

The time horizon is infinite, and trading in the asset takes place in continuous time. The only types of trades allowed are market orders and limit orders, which are executed with no delays. There is no cost of cancellation for limit orders.⁵

Trading is based on a publicly observable limit order book, which is the collection of all the limit orders that have not yet been executed. The limit orders are subject to the usual price priority rule; and, when prices are equal, the time priority rule is applied. If several market orders are submitted at the same time, only one of them is executed, at random, while the other orders are canceled.

Agents

The market is composed of two types of agents: liquidity traders and informed traders. Both types of traders can be patient and impatient, in a sense to be precisely described below. They trade at most one unit, after which they exit the model forever. The traders' types are fixed from the beginning and cannot change.

All agents in this model are risk-neutral, so their instantaneous utility function (felicity) is linear in price. By convention, felicity is equal to price for sellers, and minus the price for buyers. Traders discount the future in a way proportional to the expected waiting time. If τ is the random execution time and P_τ is the price obtained at τ , the expected utility of a seller is $f_t = \mathbf{E}_t\{P_\tau - r(\tau - t)\}$. (The expectation operator takes as given the strategies of all the players.) Similarly, the expected utility of a buyer is $-g_t = \mathbf{E}_t\{-P_\tau - r(\tau - t)\}$, where

⁵In most financial markets cancellation of a limit order is free, although one may argue that there are still *monitoring* costs. The present model ignores such costs, but one can take the opposite view that there are infinite cancellation / monitoring costs. See, e.g., Foucault, Kadan and Kandel (2005).

by notation $g_t = \mathbf{E}_t\{P_\tau + r(\tau - t)\}$. One calls f_t the value function, or utility, of the seller at t ; and similarly g_t is the value function, or utility, of the buyer, although in fact g_t equals minus the expected utility of a buyer.

The discount coefficient r is constant.⁶ It can take only two values: if it is low, the corresponding traders are called *patient*, otherwise they are *impatient*. As in Roşu (2008), we assume that the discount coefficient of the impatient trader is high enough, which implies the impatient agents always submit market orders. Then by r we denote only the time discount coefficient of the patient agents.

Liquidity Traders

These are of four types: patient buyers, patient sellers, impatient buyers, and impatient sellers. All types of liquidity traders arrive to the market according to independent Poisson processes with the same arrival intensity rate λ^U .⁷ They are liquidity traders, in the sense that they want to trade the asset for reasons exogenous to the model. But they do have discretion whether to enter the market, and once they enter, whether to use market or limit orders.

The decision to enter the market is based on a private one-time cost of trading c , uniformly distributed on the interval $[-C, C]$. This means that each trader who arrives at the market makes the entry decision based on the private cost c . For example, a seller who expects utility f from trading must satisfy $f - v^e \geq c$.⁸ This assumption generates an upward-sloping supply of sell orders: the probability that a liquidity seller enters the market increases with expected utility f . Similarly, this also generates a downward-sloping supply of buy orders: the probability that a liquidity buyer enters the market decreases with their expected utility.

⁶The nature of waiting costs is intentionally vague in this paper. One can interpret it as an opportunity cost of trading. Another interpretation is that waiting costs reflect traders' uncertainty aversion: if uncertainty increases with the time horizon, an uncertainty averse trader loses utility by waiting.

⁷By definition, a Poisson arrival with intensity λ implies that the number of arrivals in any interval of length T has a Poisson distribution with parameter λT . The inter-arrival times of a Poisson process are exponentially distributed with the same parameter λ . The average time until the next arrival is then $1/\lambda$.

⁸Alternatively, one can think of $v^e + c$ as the seller's private opinion about the fundamental value; the seller trades only if the expected utility f is above the opinion $v^e + c$. There are two problems with that interpretation. First, the efficient price can change and so one must assume that the opinion $v^e + c$ also moves, and by the same amount (maybe one can think of traders as constantly optimistic or pessimistic). Second, the seller's expected utility f can drop, in which case the decision to stay in the market may change if f is no longer above $v^e + c$. This may account for the large number of limit order cancellations typically observed in order-driven markets, but would make the model more complicated.

Informed Traders

For simplicity, we assume that there are only *patient* informed traders. Unlike uninformed traders, informed traders can be both buyers and sellers.⁹ This can be modeled by assuming that each trader either (1) is endowed with one unit of the asset, and can either sell it, or buy another unit, after which exits the model; or (2) is initially endowed with no unit, and can buy or short sell (borrow and sell) one unit of the asset. We choose option (2), although the two modeling choices lead to essentially identical results. Similar to the uninformed traders, the informed agents can trade at most one unit of the asset, after which they exit the model. The informed traders arrive to the market according to independent Poisson processes with the arrival intensity rate $2\lambda^I$.

When informed traders arrive at time t , they observe the fundamental value v_t , and decide whether they want to enter the market, and if so, whether to use market or limit orders. After the initial entry decision, it is further assumed that the informed agents do not use their information anymore. So we assume that, after their initial decision to enter the market, the informed traders either places a market order, or hires an uninformed patient broker to monitor the market and handle the limit order appropriately. This is a strong assumption, but at the end of Section 4 it is argued that removing this assumption does not significantly affect the solution.

Strategies

Since this is a model of continuous trading, it is desirable to set the game in continuous time. There are also technical reasons why that would be useful: in continuous time, with Poisson arrivals the probability that two agents arrive at the same time is zero, and this simplifies the analysis of the game. But setting the game in continuous time requires extra care, see Roşu (2008) for details.

⁹Alternatively, one may consider a version of the model where informed traders are *ex ante* either buyers or sellers, just as the uninformed traders. We describe this alternative in Proposition 12 in the Appendix.

3 Benchmark: No Informed Traders

Without informed traders, the efficient price v^e does not evolve in time, and so one can assume that $v^e = 0$. The intuition for the solution follows that of the model in Roşu (2008). In equilibrium, the limit order book is formed only of limit orders submitted by patient liquidity traders. If limit orders are pictured on the vertical axis, the limit order book is formed by two queues: the sell (or ask, or offer) side, with sell limit orders descending to a minimum price, called the *ask price*; and the buy (or bid) side, with buy limit orders ascending to a maximum price, called the *bid price*. The two sides are disjoint, i.e., the ask price is above the bid price.

3.1 Equilibrium

In the equilibrium limit order book, the patient limit sellers compete for the incoming market orders from impatient buyers, and the patient buyers compete for the order flow from impatient sellers. The sellers for example have their limit orders placed at different prices, but they get the same expected utility: otherwise, they would undercut by a penny those with higher utility. Thus, the sellers with a higher limit order obtain in expectation a higher price, but also have to wait longer. Similarly, all the buyers have the same expected utility. This makes the equilibrium Markov, and the numbers of buyers and sellers in the book becomes a state variable.

Denote by m the number of sellers, and by n the number of buyers in the limit order book at a given time. Denote by $a_{m,n}$ the ask price, $b_{m,n}$ the bid price, $f_{m,n}$ the expected utility of the sellers, and $g_{m,n}$ (minus) the expected utility of the buyers, as defined in Section 2. One can prove the following formulas:

$$a_{m,n} = f_{m-1,n} \quad \text{and} \quad b_{m,n} = g_{m,n-1}, \quad (1)$$

which follows from the fact that, e.g., all sellers have the same utility in state (m, n) : if an impatient buyer comes, then the bottom seller gets the ask price $a_{m,n}$, while all the other sellers get $f_{m-1,n}$. As in Roşu (2008), one defines the *state region* Ω as the collection of all pairs (m, n) where in equilibrium the m sellers and the n buyers wait in expectation for some

positive time. Also one defines the *boundary* γ of Ω as the set of (m, n) where at least some agent has a mixed strategy. This is the set of states where the limit order becomes full, and any extra incoming patient seller or buyer would immediately place a market order.

Recall that the each agent's decision to enter the market is based on a private one-time cost of trading c , uniformly distributed on the interval $[-C, C]$. Take for example an impatient seller with cost c who arrives at the market in state (m, n) . If the seller placed a sell market order, he would get the bid price $b_{m,n}$. According to the assumption, the seller then decides to enter the market and place a market order if and only if $b_{m,n} - v^e = b_{m,n} \geq c$. The ex-ante probability of this event is $P(c \leq b_{m,n}) = \frac{1+b_{m,n}/C}{2}$. Similarly, if a patient seller decided to place a limit order, the book would go to the state $(m+1, n)$ with one extra limit seller, which yields expected utility of $f_{m+1,n}$. This even happens with ex-ante probability of $P(c \leq f_{m+1,n}) = \frac{1+f_{m+1,n}/C}{2}$. Also, an impatient buyer would place a market order at the ask price $a_{m,n}$ if the gain $v^e - a_{m,n} = -a_{m,n} \geq c$. This event has the ex-ante probability $P(c \leq -a_{m,n}) = \frac{1-a_{m,n}/C}{2}$.

From now on normalize $C = 1$. From a typical state (m, n) the system can go to the following neighboring states:

- $(m-1, n)$, if an impatient buyer arrives and places a market order at the ask;
- $(m+1, n)$, if a patient seller arrives and submits a limit order;
- $(m, n-1)$, if an impatient seller arrives and places a market order at the bid;
- $(m, n+1)$, if a patient buyer arrives and submits a limit order.

If a trader arrives at the market, but decides to place no order, the book remains in state (m, n) .

Any of these four events arrives at a random time which is exponentially distributed with parameter λ^U , therefore the first of them arrives at a random time which is exponentially distributed with parameter $4\lambda^U$. This means that a patient seller loses an expected utility of $r \cdot \frac{1}{4\lambda^U}$ from waiting in state (m, n) . Now, conditional on the first event, the arrival, e.g., of an impatient buyer happens with probability $1/4$. Once the impatient buyer arrives, he places a market with probability $\frac{1-a_{m,n}}{2}$; otherwise, with probability $\frac{1+a_{m,n}}{2}$ he decides not to

do anything, in which case the market remains in the same state. Doing this analysis for all states, we get

$$\begin{aligned}
f_{m,n} = & \frac{1}{4} \left(\frac{1 - a_{m,n}}{2} f_{m-1,n} + \frac{1 + a_{m,n}}{2} f_{m,n} \right) + \frac{1}{4} \left(\frac{1 + f_{m+1,n}}{2} f_{m+1,n} + \frac{1 - f_{m+1,n}}{2} f_{m,n} \right) \\
& + \frac{1}{4} \left(\frac{1 + b_{m,n}}{2} f_{m,n-1} + \frac{1 - b_{m,n}}{2} f_{m,n} \right) + \frac{1}{4} \left(\frac{1 - f_{m,n+1}}{2} f_{m,n+1} + \frac{1 + f_{m,n+1}}{2} f_{m,n} \right) \\
& - r \cdot \frac{1}{4\lambda U}. \tag{2}
\end{aligned}$$

A similar recursive system of difference equations holds for g , the expected utility of a buyer in state (m, n) . It is not difficult to see that a solution to a system of recursive equations leads to a Markov equilibrium of this market. (See Theorem 3 in Roşu (2008).) The description of the solution is very difficult, because of the need to simultaneously determine the state space Ω and its boundary γ (where the book becomes full), and find a solution to the recursive system. To get a better understanding what the solution looks like, one can simplify the problem by looking only at the sell side of the book. This is a model where the liquidity traders are only patient sellers and impatient buyers.

3.2 One-Sided Limit Order Book

Assume now that the liquidity traders arriving at the market are only patient sellers and impatient buyers. Moreover, assume that there is a large supply of limit buy orders at zero, so that prices never go below zero. This is an artificial assumption (similar to the assumption of exogenous bounds for the limit order book, as in Foucault, Kadan and Kandel (2005) and Roşu (2008)) which ensures that the book always has a bid price of zero.

The model for the one-sided book is easier to solve (although it still has to be solved numerically), and it provides important intuition for the two-sided case. In equilibrium, each new patient seller arrives to the market and places a limit order inside the bid-ask spread, thus lowering the ask price.¹⁰ There is only one exception: there is a state where all the patient sellers have zero utility, in which case a new incoming patient seller has no incentive to wait and instead places a market order at the bid price and exits. In this case the limit order book is called “full,” and there is a maximum number M of sellers, while the bid-ask

¹⁰Biais, Hillion and Spatt (1995) empirically show in their study of the Paris Bourse (now Euronext) that the majority of limit orders are spread improving.

spread is at a minimum.

As in the two-sided case, if the book has m patient sellers, they all have the same expected utility f_m . Then m becomes a state variable and f_m satisfies a recursive system (a_m is the ask price in state m): $f_m = \frac{1}{2} \left(\frac{1-a_m}{2} f_{m-1} + \frac{1+a_m}{2} f_m \right) + \frac{1}{2} \left(\frac{1+f_{m+1}}{2} f_{m+1} + \frac{1-f_{m+1}}{2} f_m \right) - r \cdot \frac{1}{2\lambda U}$. Equation (1) in the one-sided case implies that the ask price a_m when there are m sellers in the book equals to the expected utility f_{m-1} in the state $m - 1$ with one less seller.¹¹

$$a_m = f_{m-1} \quad \text{for all } m = 2, \dots, M. \quad (3)$$

This is true for $m < M$, when the limit order book is not full. In the case when $m = M$ any new seller submits a market order at zero and exits, and so the state $M + 1$ never exists. Therefore the recursive equation becomes $f_M = 0 = \frac{1}{2} \left(\frac{1-f_{M-1}}{2} f_{M-1} \right) - \frac{r}{2\lambda U}$. Now define the *granularity* parameter

$$\varepsilon = \frac{r}{\lambda U}. \quad (4)$$

One gets the recursive equation

$$f_m = \frac{1}{2} \left(\frac{1-f_{m-1}}{2} f_{m-1} + \frac{1+f_{m-1}}{2} f_m \right) + \frac{1}{2} \left(\frac{1+f_{m+1}}{2} f_{m+1} + \frac{1-f_{m+1}}{2} f_m \right) - \frac{\varepsilon}{2}. \quad (5)$$

with $f_M = 0$ and $f_{M-1} = \frac{\varepsilon}{1+\sqrt{1-2\varepsilon}}$. This suggests that one could find f_m numerically by rewriting the recursive equation in terms of f_{m-1} as a function of f_m and f_{m+1} :

$$f_{m-1} = \frac{1}{2} \left(1 + f_m - \sqrt{(1+f_m)^2 - 4[f_m(2+f_{m+1}) - f_{m+1}(f_{m+1}+1) + 2\varepsilon]} \right). \quad (6)$$

What are the bounds of the limit order book? In Roşu (2008), as in Foucault, Kadan and Kandel (2005), the bounds are assumed to be exogenous. The present model gives a way to endogenize them. In the one-sided case, the lower bound is exogenous (zero), so one must show only how the upper bound is determined.

Proposition 1. *Let f_m be the utility of sellers in state $m = 1, 2, \dots, M$, given by the solution of the recursive equation (5). Then the level a_1 of the sole limit order in state $m = 1$ is set at*

¹¹Just like in Theorem 1 of Roşu (2008), the equation is true when $m < M$. When $m = M$ the equation is true only if one assumes that the bottom agent does not have a mixed strategy. Since we are interested here only in what happens to the average bid-ask spread, the single state $m = M$ does not affect such calculations.

the monopoly price

$$a_1 = \frac{1 + f_1}{2}.$$

Proof. See the Appendix. □

3.3 Resilience

The next numerical result shows that the average bid-ask spread and price impact are of the order of the granularity parameter $\varepsilon = \frac{r}{\lambda^U}$ to some power less than one. This shows that a higher trading activity λ^U and higher patience (lower r) indeed generate smaller spreads. This is because when agents do not have to wait much (high λ) or do not mind waiting (low r), they tolerate staying closer to each other, which generates smaller spreads.

Proposition 2. *Let granularity ε run over $10^{-1}, 10^{-2}, \dots, 10^{-16}$. For each ε compute the solution f_m to the recursive system that describes the utility of the m sellers in the book. Denote by x_m the Markov stationary probability that the limit order book is in state m (has m sellers). Since f_m is the bid-ask spread in state m , denote by $\bar{s} = \sum_{m=0}^M \pi_m f_m$ the average bid-ask spread. Then regressing $\log(\bar{s})$ on $\log(\varepsilon)$ gives the approximate formula $\log(\bar{s}) \approx -0.42 + 0.34 \log(\varepsilon)$, with $R^2 = 0.9996$. This means that with a very good approximation the average spread*

$$\bar{s} \approx 0.66 \varepsilon^{0.34}. \tag{7}$$

Moreover, denote by $I_m = a_{m-1} - a_m = f_{m-2} - f_{m-1}$ the price impact of a one-unit market order in state m . Denote by $\bar{I} = \sum_{m=0}^M \pi_m I_m$ the average price impact I_m . Then regressing $\log(\bar{I})$ on $\log(\varepsilon)$ gives the approximate formula $\log(\bar{I}) \approx 1.15 + 0.68 \log(\varepsilon)$, with $R^2 = 0.9998$. This means that with a very good approximation the average one-unit price impact

$$\bar{I} \approx 3.15 \varepsilon^{0.68}. \tag{8}$$

Proof. See the Appendix to see how the stationary probabilities π_m are computed. □

Proposition 2 shows that the average spread gets very close to zero when the granularity $\varepsilon = r/\lambda^U$ is small. Moreover, one can check that the system is more likely to remain in states with more sellers, i.e., π_m is increasing in m . This indicates that the one-sided limit order book is resilient: the bid-ask spreads tend to revert to the minimum value $a_M = f_{M-1} = \frac{\varepsilon}{1 + \sqrt{1 - 2\varepsilon}}$.

Now, coming back to the two-sided case, recall that the mid-point of the book is zero (the mid-point of the exogenous private cost interval $[-1, 1]$ for the liquidity traders). Now, suppose the limit order book has m sellers and n buyers, i.e., is in state (m, n) . Then the probability of entry for an impatient buyer (conditional on arrival to the market) is $(1 - a_{m,n})/2$, where $a_{m,n}$ is the ask price; and the probability of entry for a patient seller is $(1 + f_{m+1,n})/2$ which is the utility of patient sellers in state $(m + 1, n)$ with one more seller. When the granularity ε is small, $a_{m,n} = f_{m-1,n}$ and $f_{m+1,n}$ are very close to each other, with a difference of the order of $\varepsilon^{0.68}$.¹² If these two numbers are not close to zero, for example if $a_{m,n}$ is much significantly larger than zero, then $(1 - a_{m,n})/2$ is significantly smaller than $(1 + f_{m+1,n})/2$, which means that patient sellers arrive significantly faster than impatient buyers. This drives the ask price down to the point $a_{m,n}$ is not significantly larger than zero. A similar argument works for patient buyers and impatient sellers, which also brings the bid price $b_{m,n}$ close to zero. In conclusion, the two-sided limit order book is also resilient, and moreover tends to be centered around zero. According to Proposition 2, the average bid-ask spread is of the order of $\varepsilon^{0.34}$, and the distance between the mid-point of the bid-ask spread and zero is also of the order of $\varepsilon^{0.34}$. This result is to be compared with that of Farmer, Patelli and Zovko (2003), who in their cross-sectional empirical analysis of the London Stock Exchange find that the average bid-ask spread varies proportionally to $\varepsilon^{0.75}$.

In the theoretical model of Roşu (2008) there are two cases, depending on the competition parameter c , which is the ratio of arrival rate of patient traders to the impatient traders. When $c = 1$, the average bid-ask spread does not depend on the granularity parameter ε , and is in fact very large: $\bar{s} = (A - B)/2$, where A and B are the bounds of the limit order book. When $c > 1$, i.e., patient traders arrive faster than impatient traders, the book becomes resilient, and the average bid-ask spread becomes of the order of $\varepsilon \ln(1/\varepsilon)$. Notice that the present model is a mixture of the cases $c = 1$ and $c > 1$: when the bid and ask prices are close to zero, c is close to one, while when, e.g., the ask price is significantly larger than zero, c is significantly larger than one.

Notice that a consequence of this section is that the entry probabilities of various types of uninformed agents are close to $1/2$. For example, the arrival rate of an impatient buyer

¹²According to Proposition 2, $f_{m-1,n} - f_{m+1,n} = (f_{m-1,n} - f_{m,n}) + (f_{m,n} - f_{m+1,n})$, and each term is of the order of $\varepsilon^{0.68}$.

who decides to place a market order equals $\lambda^U(1 - a_{m,n})/2$. Since the ask price $a_{m,n}$ is close to zero, it follows that the arrival rate $\lambda^U(1 - a_{m,n})/2$ is close to $\lambda^U/2$.

4 General Case: Informed Traders

Now we discuss the general case, when besides the discretionary uninformed liquidity traders, there are also informed traders. To simplify the formulas, we are going to assume that there are only *patient* informed buyers and sellers. A newly arrived informed trader observes the fundamental value (or *full-information price*) v_t , which is assumed to follow a diffusion process with constant volatility σ . The market forms an estimate v_t^e (called the *efficient price*) of v based on all publicly available information. It is further assumed that in each state of the book the fundamental value v is believed by traders to be normally distributed with mean v^e and volatility σ^e .¹³ From now on, to simplify notation, we use subscript notation: $\sigma_e = \sigma^e$.

In Section 4.2 it is shown that the strategy of a patient informed trader depends on how far the fundamental value v is from the efficient price v^e every time there is a market order.¹⁴ In most cases, the informed trader compares $v - v^e$ to three cutoffs that depend on the state of the book. If $v - v^e$ is below the lower cutoff, the patient informed trader optimally behaves in an impatient way and submits a market sell order. If $v - v^e$ is above that lower cutoff but still below the middle cutoff, the patient seller submits a sell limit order and waits. Above the middle cutoff but below the upper cutoff, the trader places a limit buy order and waits, and finally above the upper cutoff the patient traders submits a market buy order.¹⁵

4.1 Uninformed (Liquidity) Traders

In Section 3 we described the equilibrium in which there are only uninformed traders. In this section, we analyze how the equilibrium changes when informed traders are also present. The answer is relatively simple: the limit order book shifts up and down along with the

¹³Proposition 9 shows that the approximation is good.

¹⁴The efficient price should also change when there is a limit order, but the change implied by a limit order is so small that it can be neglected.

¹⁵If we allowed impatient informed traders, their behavior is even simpler: an impatient trader submits a buy market order if v is larger than the ask; a sell market order if v is smaller than the bid; and does nothing if v is between the bid and the ask.

efficient price v^e . Moreover, a translation by $-v^e$ moves the limit order book to a canonical one centered at zero, which can be analyzed separately.

There is one more difference: the bid-ask spread becomes wider, as the uninformed traders with limit orders at the ask or bid must protect themselves from the information contained in market orders from potentially informed traders. To understand this more clearly, consider the case when there are m patient sellers and n patient buyers in the limit order book. Suppose the efficient price at that time is v^e . Then the canonical equilibrium yields numbers $f_{m,n}$, $g_{m,n}$ that do not depend on v^e , so that $v^e + f_{m,n}$ is the expected utility of the patient sellers, and $v^e + g_{m,n}$ is (minus) the expected utility of the patient buyers. Unlike the uninformed case where the ask price $a_{m,n}$ equals $f_{m-1,n}$, here $a_{m,n} = f_{m-1,n} + \Delta$, where Δ is the change in the efficient price due a market buy order (see Propositions 6 and 10). Similarly, $b_{m,n} = g_{m,n-1} - \Delta$. Define:

$$a_{m,n}^0 = f_{m-1,n}, \quad b_{m,n}^0 = g_{m,n-1}, \quad (9)$$

In the absence of private information, $a_{m,n}^0$ and $b_{m,n}^0$ would be the ask and bid prices, respectively. Then the bid-ask spread with private information is $s_{m,n} = a_{m,n}^0 - b_{m,n}^0 + 2\Delta$.

Proposition 3. *There is a canonical equilibrium in which the shape of the limit order book does not depend on the efficient price v^e , and up to a translation only depends on the number of sellers m and number of buyers n in the book. The limit order book is centered at v^e .*

Proof. The shape of the book depends on the arrival rates and strategies of the various market participants. The assumptions of Section 2 imply that the behavior of both the uninformed and informed traders only depends on the state of the book when they arrive: For the uninformed liquidity traders, entry is based on a one-time private cost that is compared to the expected profit from entering the book. For the informed traders, it was assumed that after the initial entry decision they do not use their information anymore. So as long as the strategy of the informed traders involves only the difference between v and v^e , the existence of the canonical equilibrium follows in a straightforward way. In Section 4.2 it is shown that indeed the strategy of the informed trader only depends on the difference $v - v^e$ and the state of the book (m, n) . \square

The exact formulas for the canonical equilibrium are given in equations (44) in the Appendix. They are similar to the equations for the equilibrium with no informed traders from

Section 3.1, except for the presence of Δ (the absolute value of the adjustment of the efficient price to a market order), and the market-inferred probabilities P_j that an informed trader submits an order of type $j \in \{BMO, BLO, SMO, SLO, NO\}$ (buy market order, buy limit order, sell market order, sell limit order, or no order). These probabilities can be easily derived from Proposition 5, since $v - v^e$ is assumed to be normally distributed with mean zero and standard deviation σ_e .

4.2 Informed Traders

The strategy of an informed trader depends on how far the fundamental price v is from v^e , and on the current state (m, n) of the book, with m sellers and n buyers. To understand better what this strategy is based on, consider an informed trader who contemplates the choice of an order $j \in \{BMO, BLO, SMO, SLO, NO\}$. As mentioned in Section 2, this can be modeled by assuming that the patient trader either has one unit already, and can sell it or buy an extra one; or has no unit of the asset, and can buy one unit or short sell one unit. The next result assumes the latter case. The formulas for the former case are the same, except that v has to be added to each equation.

Proposition 4. *Consider the decision of an informed trader to enter a limit order book in state (m, n) , with m sellers and n buyers. Denote by $\Delta = \Delta_{m,n}$ the absolute value of the adjustment in the efficient price after a market order. Define*

$$k = \frac{3\lambda^U + 2\lambda^I}{4\lambda^U + 4\lambda^I} \in \left(\frac{1}{2}, \frac{3}{4}\right), \quad (10)$$

$$a'_{m,n} = a_{m,n} - k\Delta, \quad b'_{m,n} = b_{m,n} + k\Delta. \quad (11)$$

Then the expected utility u_j from submitting an order of type $j \in \{BMO, BLO, SMO, SLO, NO\}$ (buy market order, buy limit order, sell market order, sell limit order, or no order) is:

- $u_{BMO} = v - (v^e + a_{m,n}) = (v - v^e) - a_{m,n}$;
- $u_{BLO} \approx v - (v^e + b'_{m,n} + k(v - v^e)) = (1 - k)(v - v^e) - b'_{m,n}$;
- $u_{SMO} = (v^e + b_{m,n}) - v = b_{m,n} - (v - v^e)$;
- $u_{SLO} \approx (v^e + a'_{m,n} + k(v - v^e)) - v = a'_{m,n} - (1 - k)(v - v^e)$;
- $u_{NO} = 0$.

Proof. See the Appendix. □

The intuition, e.g., for the expected utility u_{BMO} of the informed trader from a market buy order is: the buyer gets an asset that is worth v , but must pay the ask price $v^e + a_{m,n}$. If the same trader submits a limit buy order, the limit order book moves to the state with one more seller: $(m+1, n)$. He gets an asset worth v , but must pay—including waiting costs—the expected utility of a patient seller: $g_{m+1,n}$. According to Section 3.3, this is approximately equal to $g_{m-1,n} = b_{m,n}^0$. On top of this, the agent is informed, and so he knows the efficient price will change in the direction of the fundamental value. The overall price paid turns out to be approximately $v^e + b'_{m,n} + k(v - v^e)$, where k is some constant between $1/2$ and $3/4$.

If one follows the proof of this Proposition more closely, it turns out that the above formulas would be more accurate if one replaced k by k_v , the function of v given by equation (45) in the Appendix. At the same time, it is also shown in the proof that replacing k by k_v does not alter the strategies of the informed trader in the next Proposition, at least not the choice between market orders and limit orders.

Proposition 5. *For a limit order book with m sellers and n buyers, denote by $\Delta = \Delta_{m,n}$ the absolute value of the adjustment of the efficient price to a market order. Define also*

$$s'_{m,n} = s_{m,n} - k\Delta = a_{m,n} - b'_{m,n} = a'_{m,n} - b_{m,n}, \quad (12)$$

$$p_{m,n} = \frac{a_{m,n} + b_{m,n}}{2} = \frac{a'_{m,n} + b'_{m,n}}{2}, \quad (13)$$

where $k = \frac{3\lambda^U + 2\lambda^I}{4\lambda^U + 4\lambda^I} \in (\frac{1}{2}, \frac{3}{4})$. Then the optimal strategy of an informed trader is given by some cutoffs, described in the Appendix. For example, consider the case when $a_{m,n}(1-k) - b'_{m,n} > 0$, $a'_{m,n} - b_{m,n}(1-k) > 0$, and $-\frac{s'_{m,n}}{k} < \frac{p_{m,n}}{1-k} < \frac{s'_{m,n}}{k}$.¹⁶ Then the optimal strategy of an informed trader is:

- Submit a sell market order (SMO) if $v - v^e < -\frac{s'_{m,n}}{k}$;
- Submit a sell limit order (SLO) if $-\frac{s'_{m,n}}{k} < v - v^e < \frac{p_{m,n}}{1-k}$;
- Submit a buy limit order (BLO) if $\frac{p_{m,n}}{1-k} < v - v^e < \frac{s'_{m,n}}{k}$;

¹⁶According to Section 3, these conditions are satisfied with large probability. This follows from the fact that with large probability: $a_{m,n}^0$ and $a'_{m,n} = a_{m,n}^0 + k\Delta$ are positive, $b_{m,n}^0$ and $b'_{m,n} = b_{m,n}^0 - k\Delta$ are negative, and $p_{m,n} = (a_{m,n}^0 + b_{m,n}^0)/2$ is close to zero.

- Submit a buy market order (*BMO*) if $\frac{s'_{m,n}}{k} < v - v^e$.

Moreover, under no conditions is it optimal for an informed trader to submit no order (*NO*).

Proof. See the Appendix. □

The strategies described above are true in most states of the book, but in some rare situations the strategies of the informed traders are different. For example, in Proposition 7 an example is given where submitting a sell limit order (*SLO*) is not optimal, regardless of the fundamental value v . However, the previous Proposition shows that submitting no order (*NO*) is never optimal. To see why this is important, suppose in some case *NO* is optimal. Then the time between transactions becomes an informative signal. For example, if a long time elapses between transactions, it may be due to informed traders choosing the *NO* option. This means the informed traders are likely to observe a fundamental value that is not too extreme (otherwise they would place a market order). Hence the next market order is less likely to be informed, and so the adjustment to a market order should be smaller.

We now discuss what happened to informed traders' optimal strategies under several modifications of the model. First, if one allowed *impatient* informed traders, then their strategies would be similar, except that they would only submit market orders. For example, the optimal strategy of an informed impatient trader would be: submit a buy market order (*BMO*) if $v - v^e > a_{m,n}$ (v is larger than the ask price), a sell market order (*SMO*) if $v - v^e < b_{m,n}$ (v is smaller than the bid price), or no order (*NO*) if $b_{m,n} < v - v^e < a_{m,n}$. Introducing impatient informed traders makes the efficient price converge faster to the fundamental value.

The model assumes that informed traders can be both buyers and sellers. If one assumes that *ex ante* a trader must be either a buyer or seller, then the optimal strategies are different. Proposition 12 in the Appendix discusses this case. It turns out that, e.g., an informed patient buyer does not place any order if the fundamental value is below a certain threshold. This is because they would prefer to place a sell market order or a sell limit order, but since they are not allowed to do that, they place no order instead.

Finally, we discuss the assumption that, after the initial order submission decision, informed traders do not use their information anymore. Reassuringly, the equilibrium does not appear to change in a significant way if we removed this assumption. To see that, consider an informed trader who observes a high fundamental value v , and must decide what order to

place. According to Proposition 5, the trader should place a buy market order. The intuition is that the future informed traders are also likely to observe high fundamental values, and therefore will push prices up as they trade. Then the current informed trader is better off placing a buy market order and get the ask price before it goes further up. One may think though that, armed with the information about v , the informed trader can handle the limit order better than an uninformed patient trader, and thus make a higher profit than u_{BLO} defined in Proposition 4. We argue that this is not the case. To see why, consider what the informed trader could do differently. First, the buy limit order should not be placed lower than the theoretical bid price, because then the profit of the informed trader would be lower. Second, if the informed trader placed the order higher than the theoretical bid price, this is out-of-equilibrium behavior for uninformed traders, so all the traders in the book will deduce that the order must come from an informed trader. The other buyers in the book will then do well to overbid the informed trader, and the efficient price will also probably be adjusted upwards. In the end, it seems that being an informed trader does not bring any advantage while waiting in the limit order book, and thus the most important decision is the initial one: whether to place a market or a limit buy order.

4.3 Equilibrium

Given the arrival rate of uninformed sellers λ^U , the arrival rate of (patient) informed traders λ^I , the time discount coefficient of patient traders r , and the volatility σ of the fundamental value v_t , define the following variables:¹⁷

$$\lambda = 2\lambda^U + 2\lambda^I = \text{total trading activity}, \quad (14)$$

$$i = \frac{\lambda^I}{\lambda^U} = \text{information ratio}, \quad (15)$$

$$\varepsilon = \frac{r}{2\lambda^U + 2\lambda^I} = \text{granularity parameter}. \quad (16)$$

We assume that λ , i , and σ are exogenous, and study how the equilibrium shape of the limit order book, and the behavior of the efficient price v^e depends on them. In particular, we

¹⁷We could define total trading activity as the $\lambda = 4\lambda^U + 2\lambda^I$, as the sum of the arrival rates of all types of agents. But only half of the uninformed traders who arrive to the market decide to place an order (see Section 3.3), so it is more appropriate to define total trading activity as $\lambda = 2\lambda^U + 2\lambda^I$.

want to compute (i) the probabilities of various types of orders; (ii) the change in the efficient price due to a market order or a limit order; (iii) the efficient volatility σ_e , i.e., the standard deviation of the fundamental value v_t conditional on the information available until t ; (iv) the speed of convergence of the efficient price v_t^e to the fundamental value v_t ; (v) the bid-ask spread; and (vi) the order flow autocorrelation, i.e., the probability of an order of a certain type, e.g., a buy market order (*BMO*), conditional on the previous order being of the same type. Then we study their dependence on λ , i , σ .

The next result gives the exact formula for the adjustment of the efficient price to an order of type $j \in \{BMO, BLO, SMO, SLO\}$.

Proposition 6. *In the context of Proposition 5, suppose the limit order book is in state (m, n) with m sellers and n buyers, and the efficient price equals v^e . Assume further that the market satisfies the technical assumptions from Proposition 5. Denote by Δ the absolute value of the adjustment to a market order (to be computed in equation (17)), by $p_{m,n} = \frac{a_{m,n} + b_{m,n}}{2}$, $s'_{m,n} = s_{m,n} - k\Delta$, and $k = \frac{3\lambda^U + 2\lambda^I}{4\lambda^U + 4\lambda^I} \in (\frac{1}{2}, \frac{3}{4})$. Then, following a buy market order (*BMO*), the efficient price v^e changes to*

$$E(v|BMO) = v^e + \Delta = v^e + \sigma_e \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{s'_{m,n}}{k\sigma_e} \right)^2}}{\Phi \left(-\frac{s'_{m,n}}{k\sigma_e} \right) + \frac{\lambda^U}{4\lambda^I}}, \quad (17)$$

where $\Phi(x)$ is the cumulative density function for the standard normal distribution. Following a buy limit order (*BLO*), the efficient price v^e changes to

$$E(v|BLO) = v^e + \sigma_e \frac{\frac{1}{\sqrt{2\pi}} \left(e^{-\frac{1}{2} \left(\frac{p_{m,n}}{(1-k)\sigma_e} \right)^2} - e^{-\frac{1}{2} \left(\frac{s'_{m,n}}{k\sigma_e} \right)^2} \right)}{\Phi \left(\frac{s'_{m,n}}{k\sigma_e} \right) - \Phi \left(\frac{p_{m,n}}{(1-k)\sigma_e} \right) + \frac{\lambda^U}{4\lambda^I}}, \quad (18)$$

The formulas for sell market orders (SMO) and sell limit orders (SLO) are similar:

$$E(v|SMO) = v^e - \Delta = v^e - \sigma_e \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{s'_{m,n}}{k\sigma_e} \right)^2}}{\Phi \left(-\frac{s'_{m,n}}{k\sigma_e} \right) + \frac{\lambda^U}{4\lambda^I}}; \quad (19)$$

$$E(v|SLO) = v^e - \sigma_e \frac{\frac{1}{\sqrt{2\pi}} \left(e^{-\frac{1}{2} \left(\frac{p_{m,n}}{(1-k)\sigma_e} \right)^2} - e^{-\frac{1}{2} \left(\frac{s'_{m,n}}{k\sigma_e} \right)^2} \right)}{\Phi \left(\frac{s'_{m,n}}{k\sigma_e} \right) - \Phi \left(-\frac{p_{m,n}}{(1-k)\sigma_e} \right) + \frac{\lambda^U}{4\lambda^I}}. \quad (20)$$

Proof. See the Appendix. □

This result shows that a buy market order represents a positive signal, and that the impact of a sell market order has the same magnitude, but the opposite sign to that of a buy market order. One can see from equation (18) that the effect of a buy limit order (BLO) on the efficient price is also positive. But this was made under the technical assumptions of Proposition 5: $a_{m,n}(1-k) - b'_{m,n} > 0$, $a'_{m,n} - b_{m,n}(1-k) > 0$, and $-\frac{s'_{m,n}}{k} < \frac{p_{m,n}}{1-k} < \frac{s'_{m,n}}{k}$. These assumptions hold with large probability (see Footnote 16), but they are not always true. For example, suppose the last condition was replaced by $\frac{p_{m,n}}{1-k} < -\frac{s'_{m,n}}{k}$. The next result shows that under these conditions a buy limit order has a *negative* impact on the efficient price.

Proposition 7. *In the context of Proposition 5, suppose the limit order book is in state (m, n) with m sellers and n buyers, and the efficient price equals v^e . Assume that the following technical conditions hold: $a_{m,n}(1-k) - b'_{m,n} > 0$, $a'_{m,n} - b_{m,n}(1-k) > 0$, and $\frac{p_{m,n}}{1-k} < -\frac{s'_{m,n}}{k}$. Then this implies $\frac{p_{m,n}}{1-k} < \frac{b'_{m,n} - (k/2)\Delta}{1-k/2} < -\frac{s'_{m,n}}{k}$, and the optimal strategy of an informed trader is:*

- Submit a sell market order (SMO) if $v - v^e < \frac{b'_{m,n} - (k/2)\Delta}{1-k/2}$;
- Submit a buy limit order (BLO) if $\frac{b'_{m,n} - (k/2)\Delta}{1-k/2} < v - v^e < \frac{s'_{m,n}}{k}$;
- Submit a buy market order (BMO) if $\frac{s'_{m,n}}{k} < v - v^e$.

Moreover, a buy limit order in this case has a negative impact on the efficient price.

Proof. See the Appendix. □

For simplicity, we now assume that limit orders have no effect on the efficient price. We do this partly because of the sign ambiguity just described, and partly because, as it will be seen later in a calibration, the effect of a limit order is significantly smaller than the effect of a market order.¹⁸

Assumption. Limit orders have no effect on the efficient price.

Next, we discuss the properties of the efficient price process. In order to do that, we consider a filtering problem that closely resembles our problem.

Proposition 8. *Suppose that the market tries to estimate a fundamental value v_t which follows a diffusion process: $v_t = v_{t-1} + \eta_t$, with η_t IID normal, with mean zero and constant standard deviation σ_0 . Each period, the market gets a public signal $s_t = v_t + \delta_t$, with the noise δ_t IID normal, with mean zero and constant standard deviation σ_δ . Denote by $\mathcal{I}_t = \{s_1, \dots, s_t\}$ the public information set available at t , so that subscript t indicates conditioning on the information set \mathcal{I}_t . Denote by $v_t^e = \mathbf{E}_t(v_t)$ the efficient price at time t , i.e., the expected fundamental value given all public information. Define also*

$$a = \frac{\sigma_\delta^2}{\sigma_0^2}. \quad (21)$$

Then the following statements are true:

1. The efficient price v_t^e follows a quasi Brownian motion, i.e., it has independent increments, and $\mathbf{Var}_{t-1}(v_t^e) \approx \sigma_0^2$ (for t large);
2. The speed of convergence $\frac{\mathbf{E}_t(|v_{t+1}^e - v_t^e|)}{\mathbf{E}_t(|v_{t+1} - v_t^e|)} \approx \left(\frac{2}{1+\sqrt{1+4a}}\right)^{1/2}$ (for t large);
3. The pricing error variance $\mathbf{Var}_t(v_t - v_t^e) = \mathbf{Var}_t(v_t) \approx \sigma_0^2 \frac{\sqrt{1+4a}-1}{2}$ (for t large);
4. The pricing error autocorrelation $\mathbf{Corr}_t(v_{t+1} - v_{t+1}^e, v_t - v_t^e) \approx \frac{a}{a + \frac{1+\sqrt{1+4a}}{2}}$ (for t large).

Proof. See the Appendix. □

In our case, the process does not have normally distributed signals, but instead has discrete signals: buy market orders and sell market orders. Our strategy is to use the formulas from

¹⁸In the calibration, for typical values of the parameters, the effect is about 4 times higher for a market order than for a limit order.

Proposition 8 to get approximate results for our filtration problem. (Proposition 9 below shows that this approximation is good.) So consider our modified filtration problem, and assume that the only possible signals at t are

- $s_t = BMO_t$ (buy market order) if $v_t > v_t^e + \alpha$, for $\alpha = \frac{s'_{m,n}}{k}$;
- $s_t = SMO_t$ (sell market order) if $v_t < v_t^e - \alpha$.

The fundamental value v_t follows a continuous diffusion process with constant volatility σ . Define time t as the “transaction time,” i.e., $t - 1$ changes to t if a market order is submitted at time t . As before, define by $\mathcal{I}_t = \{s_1, \dots, s_t\}$ the public information set available at t , and using the subscript t indicates conditioning on the information set \mathcal{I}_t . Assume that, conditional on \mathcal{I}_t , v_t is normally distributed with mean $v_t^e = \mathbf{E}_t(v_t)$ and variance $\sigma_e^2 = \mathbf{Var}_t(v_t)$.

We next define σ_0 , the volatility of the change in fundamental value between $t - 1$ and t . In our case, this corresponds to the volatility of the change in v_t between two market orders. To determine the arrival rate of a market order, consider a buy market order. This arrives either when an uninformed impatient buyer arrives, or an informed patient trader arrives and observes a fundamental value v_t above the cutoff $v_t^e + \alpha$. A buy market order then arrives at a rate of $\frac{\lambda^U}{2} + 2\lambda^I \Phi\left(-\frac{\alpha}{\sigma_e}\right)$, and this is the same as the arrival rate of a sell market order. Therefore, a market order arrives at twice that rate: $\lambda^U + 4\lambda^I \Phi\left(-\frac{\alpha}{\sigma_e}\right)$. Denote by

$$\sigma_0 = \frac{\sigma}{\left(\lambda^U + 4\lambda^I \Phi\left(-\frac{s'_{m,n}}{k\sigma_e}\right)\right)^{1/2}}, \quad (22)$$

Now, statement 1 from Proposition 8, says that $\mathbf{Var}_{t-1}(v_t^e) = \sigma_0^2$. In our case, $\mathbf{Var}_{t-1}(v_t^e) = \mathbf{Var}_{t-1}(v_{t-1}^e \pm \Delta) = \mathbf{Var}(\pm\Delta) = \Delta^2$. This means that we can impose

$$\Delta = \sigma_0. \quad (23)$$

Since we already have a formula for Δ :

$$\Delta = \sigma_e \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{s'_{m,n}}{k\sigma_e}\right)^2}}{\Phi\left(-\frac{s'_{m,n}}{k\sigma_e}\right) + \frac{\lambda^U}{4\lambda^I}}. \quad (24)$$

the extra equation $\Delta = \sigma_0$ helps us determine the efficient volatility σ_e , and closes the model.

Given the equation $\Delta = \sigma_0$, it is a good strategy to measure our other variables in Δ units. Define the numbers h and τ by:

$$h = \frac{s'_{m,n}}{\Delta}. \quad (25)$$

$$\tau = \frac{\sigma_e}{\Delta}. \quad (26)$$

From equation (24), it follows that (recall $i = \frac{\lambda^I}{\lambda^U}$, $k = \frac{3+2i}{4+4i}$):

$$1 = \tau \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{h}{k\tau}\right)^2}}{\Phi\left(-\frac{h}{k\tau}\right) + \frac{1}{4i}}, \quad (27)$$

which gives τ as a function of i and h . (Later, when we analyze the bid-ask spread, we remove the dependence of τ on h .)

Before we start using the formulas from Proposition 8, we need to show that the approximation is good, i.e., a buy market order, the posterior distribution for v_{t+1} is approximately equal to a normal one: $N(v_{t+1}^e, \sigma_e^2)$.

Proposition 9. *Assume that the market is in the context of Proposition 5. Suppose a buy market order arrives at time $t + 1$ (BMO_{t+1}). Let $k = \frac{3+2i}{4+4i}$, where $i = \frac{\lambda^I}{\lambda^U}$ is the information ratio. Denote by Δ as in equation (24), by $h = \frac{s'_{m,n}}{\Delta}$, and by $\tau = \frac{\sigma_e}{\Delta}$. Then*

$$E(v_{t+1} | v_t^e, BMO_{t+1}) = v_t^e + \Delta. \quad (28)$$

$$\text{Var}(v_{t+1} | v_t^e, BMO_{t+1}) = \sigma_e^2 \left(1 + \frac{h}{k\tau^2}\right). \quad (29)$$

Moreover, the density function of $v_{t+1} - (v_t^e + \Delta)$ conditional on v_t^e and BMO_{t+1} is:

$$f(x) = \frac{1}{\sqrt{2\pi} \sqrt{\sigma_0^2 + \sigma_e^2}} e^{-\frac{1}{2} \frac{(x+\Delta)^2}{\sigma_0^2 + \sigma_e^2}} \frac{\Phi\left(-\frac{h\Delta/k - \lambda - \Delta}{\omega}\right) + \frac{1}{4i}}{\Phi\left(-\frac{h}{k\tau}\right) + \frac{1}{4i}}, \quad (30)$$

$$\text{where } \lambda = \frac{x\sigma_e^2 - \Delta\sigma_0^2}{\sigma_0^2 + \sigma_e^2} \quad \text{and} \quad \omega = \frac{\sigma_0\sigma_e}{\sqrt{\sigma_0^2 + \sigma_e^2}}. \quad (31)$$

Proof. See the Appendix. □

We now compare the posterior density $f(x)$ for $v_{t+1} - (v_t^e + \Delta)$ with the normal density

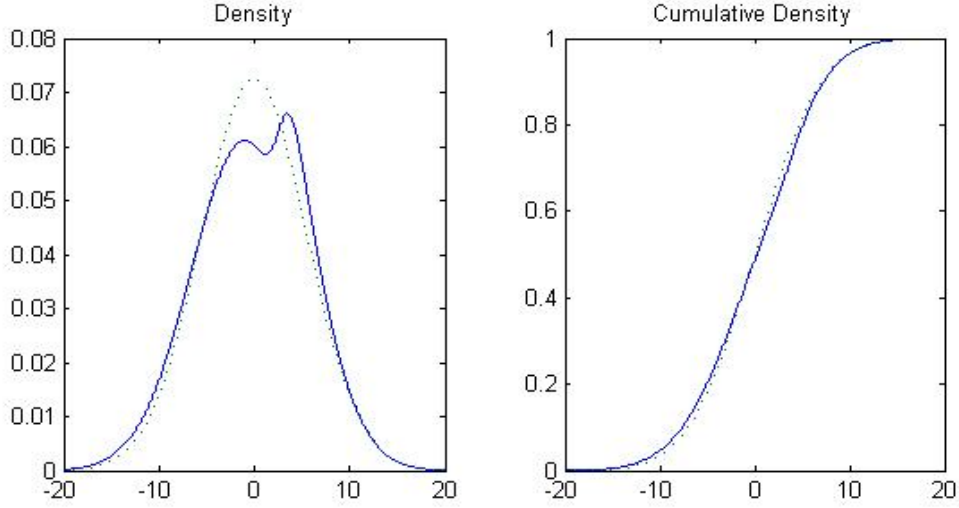


Figure 1: The posterior distribution of the pricing error immediately after a buy market order, compared with a benchmark normal distribution. The plot on the left represents the probability density function (pdf), while the plot on the right represents the cumulative density function (cdf). The normal densities are drawn with dotted lines. The pricing error at time $t + 1$ equals $v_{t+1} - v_{t+1}^e = v_{t+1} - (v_t^e + \Delta)$, where Δ is the change in the efficient price v^e . All variables are measured in Δ units, i.e., $\Delta = 1$. The variables for this particular case are: information ratio $i = \frac{\lambda^t}{\lambda^U} = \frac{1}{6}$, $k = \frac{3+2i}{4+4i} = 0.714$, adjusted bid-ask spread $s'_{m,n} = s_{m,n} - k\Delta = s\Delta = 2.654$ (where $s_{m,n} = 3.368$ is the actual bid-ask spread), conditional fundamental volatility $\sigma_e = \tau\Delta = 5.507$. The benchmark density is normal with mean zero and standard deviation $\sigma_e = 5.507$.

$N(0, \sigma_e^2)$, and we also compare the cumulative densities. As parameters, we choose $i = 1/6 = 0.167$ and $h = 2.654$ (one can use Proposition 11 to show that h indeed corresponds to $i = 1/6$). We compute $k = 0.71$, and from equation (27) we compute $\tau = 5.507$. We use Δ units, or, equivalently, we set $\Delta = 1$. The difference between true posterior cumulative density and the approximate normal cumulative density is anywhere between -0.037 and 0.022 . This leads to an error in estimating interval probabilities of at most $3.7\% + 2.2\% = 5.9\%$, which indicates that the approximation is reasonable.¹⁹ Graphically, this is done in Figure 1.

Another issue that must be clarified before we search for the equilibrium is the identification of σ_δ from Proposition 8, which is the standard deviation of the noise δ_t corresponding to the signal $s_t = v_t + \delta_t$. Equivalently, we need to find $a = \sigma_\delta^2 / \sigma_0^2$. For this, we use statement

¹⁹The error in estimating interval probabilities can be as low as 4% when the information ratio equals $i = 0.1$, and as high as 9% when $i = 0.378$ (the highest information ratio for which the current solution works: see Proposition 11 below). The approximation is less good for very low or very high values of i . For example, when $i < 0.03$, the error in estimating interval probabilities starts being larger than 10%, and when $i = 20$ the error goes above 30%.

3 from Proposition 8: $\mathbf{Var}_t(v_t) = \sigma_0^2 \frac{\sqrt{1+4a}-1}{2}$ (this is true in the limit when t approaches infinity). Then $\mathbf{Var}_t(v_{t+1}) = \mathbf{Var}_t(v_t) + \sigma_0^2 = \sigma_0^2 \frac{\sqrt{1+4a}+1}{2}$. But $\mathbf{Var}_t(v_{t+1}) = \sigma_e^2$, so $\sigma_e^2 = \sigma_0^2 \frac{\sqrt{1+4a}+1}{2}$. Since $\sigma_e = \tau\Delta = \tau\sigma_0$, one gets $\tau^2 = \frac{\sqrt{1+4a}+1}{2}$. So we can finally identify

$$a = \tau^2(\tau^2 - 1). \quad (32)$$

Finally, we discuss the bid-ask spread, and we discover that in the presence of information, it has two components: one due to asymmetric information, and another one due to waiting costs.

Proposition 10. *In a limit order book with m sellers and n buyers, denote by $a_{m,n}^0 = f_{m-1,n}$ and $b_{m,n}^0 = g_{m,n-1}$, where f and g are the expected utilities of the sellers and buyers, respectively, which satisfy equations (44) in the Appendix. Then the equilibrium bid-ask spread in the general case with informed traders is*

$$s_{m,n} = a_{m,n}^0 - b_{m,n}^0 + 2\Delta. \quad (33)$$

Proof. Consider the patient seller with the most aggressive limit order, at the ask price. Since market orders carry information, the seller must be protected from ex post regret. Since in equilibrium all sellers must have the same expected utility (otherwise they will immediately undercut a seller with higher utility), it follows that the ask price $a_{m,n}$ equals the sellers' utility in the state with one less seller, *plus* the increase in price corresponding to a buy market order: $a_{m,n} = f_{m-1,n} + \Delta$. Similarly, the bid price equals $b_{m,n} = g_{m,n-1} - \Delta$. Since $s_{m,n} = a_{m,n} - b_{m,n}$, we get the desired formula. \square

We are now ready to prove the main result of this section, which computes the dependence of several variables of interest on the exogenous variables of our model: the fundamental volatility σ , i.e., the volatility of the diffusion process followed by the fundamental value v_t ; the trading activity $\lambda = 2\lambda^I + 4\lambda^U$, which is the sum of the arrival rates of the informed and uninformed traders; and the information ratio $i = \frac{\lambda^I}{\lambda^U}$. Recall that we also defined the granularity parameter $\varepsilon = \frac{r}{\lambda}$, where r is the waiting cost of patient traders.

Proposition 11. *For a limit order book with m sellers and n buyers, denote by Δ the absolute value of the adjustment of the efficient price after a market order; by σ_0 the volatility during*

the expected period between two consecutive market orders; by $\sigma_e = \tau\Delta$ the volatility of the fundamental value v_t conditional on the information available until t ; by $s_{m,n} = a_{m,n} - b_{m,n}$ the bid-ask spread; and by $s'_{m,n} = h\Delta = s_{m,n} - k\Delta$ the adjusted bid-ask spread, where $k = \frac{3+2i}{4+4i}$. Assume that limit orders have no effect on the efficient price v_t^e , and that our filtration problem approximates well the one described in Proposition 8. Suppose the information ratio $i = \frac{\lambda^I}{\lambda^V} \in (0.05, 20)$, and Δ is sufficiently small (e.g., of the order of $\sqrt{\varepsilon}$, the square root of the granularity parameter).²⁰

Then for $i < 0.378$ we have the following approximate results:

1. All four types of orders BMO, BLO, SMO, SLO are equally likely, i.e., they all have probability $\frac{1}{4}$. For example, the probability of a buy market order

$$P_{BMO} = \Phi\left(-\frac{s'_{m,n}}{k\sigma_e}\right) = \Phi\left(-\frac{h}{k\tau}\right) = \frac{1}{4}. \quad (34)$$

2. The efficient price adjustment after a buy market order

$$\Delta = \Delta_{BMO} = \sigma_0 = \frac{\sigma}{\sqrt{\lambda/2}}. \quad (35)$$

The efficient price adjustment after a buy limit order is a constant multiple of Δ : $\Delta_{BLO} = 0.256 \Delta$, which implies that

$$\frac{\Delta_{BMO}}{\Delta_{BLO}} = 3.912. \quad (36)$$

3. The efficient volatility

$$\sigma_e = \tau\Delta = 0.787 \frac{i+1}{i} \Delta. \quad (37)$$

4. The speed of convergence

$$\frac{E_t(|v_{t+1}^e - v_t^e|)}{E_t(|v_{t+1} - v_t^e|)} = \frac{\sigma_0}{\sigma_e} = \frac{1}{\tau} = 1.271 \frac{i}{i+1}. \quad (38)$$

²⁰Recall that ε is a measure of the waiting costs of patient traders, who lose utility proportionally to the expected time until execution. Suppose this utility loss comes from uncertainty aversion. Then the loss is proportional to the variance of the fundamental value, since variance increases linearly with time. So $\varepsilon = \frac{\tau}{\lambda}$ is proportional to $\frac{\sigma^2}{\lambda}$, which according to equation (35) is proportional to Δ^2 .

5. *The bid-ask spread*

$$s_{m,n} = (0.675\tau + 1)k\Delta = \left(0.133\left(2 + \frac{3}{i}\right) + \frac{3 + 2i}{4 + 4i}\right) \Delta. \quad (39)$$

So 2Δ is the component of the bid-ask spread due to asymmetric information, and $\left(0.133\left(2 + \frac{3}{i}\right) + \frac{3+2i}{4+4i} - 2\right) \Delta$ is the component due to waiting costs. The second component is positive as long as $i < 0.378$.

6. Define the order flow autocorrelation $A_{OF} = P(BMO_{t+1}, BMO_t) - P(BMO_t)^2$. Then

$$A_{OF} = \int \frac{1 + 4i I(x > 0.675)}{4 + 4i} \frac{1 + 4i I(y > 0.675)}{4 + 4i} f_\rho(x, y) dx dy - \frac{1}{16}, \quad (40)$$

$$\text{where } \rho = 1 - 1.616 \left(\frac{i}{i+1}\right)^2, \quad (41)$$

and $f_\rho(x, y)$ is the standard bivariate normal density with correlation ρ . A_{OF} is zero when $i = 0$ and increases with the information ratio i .

We also briefly describe the case when the information ratio $i > 0.378$, while details are left to the Appendix. The number $\tau = \frac{\sigma_e}{\Delta}$ is a decreasing function $\tau(i)$ given by an implicit equation (48) in the Appendix. The probability of a buy market order $P_{BMO} = \Phi\left(-\frac{2-k}{k\tau(i)}\right)$ is less than $\frac{1}{4}$, and decreases with i . The efficient price adjustment is $\Delta = \frac{\sigma}{\sqrt{\lambda}} / \left(\frac{1}{2+2i} + \frac{4i}{2+2i} P_{BMO}\right)^{1/2}$, which decreases with i . The ratio of the price adjustments after a market order and limit order $\frac{\Delta_{BMO}}{\Delta_{BLO}}$ is less than 3.912, and decreases with i . The bid-ask spread is approximately equal to 2Δ , which is the asymmetric information component; the bid-ask spread component due to waiting costs is negligible.

Proof. See the Appendix □

In the paper, we mostly refer to the results corresponding to an information ratio $i = \frac{\lambda I}{\lambda U}$ less than 0.378. This assumes that the market has significantly more uninformed liquidity traders than informed traders. The reason, although outside our model, can be given by studying the optimal behavior of informed traders who want to trade a large quantity. As in Kyle (1985), they would not trade all at once, but instead use small quantities and disguise their trades.

Using the formulas from Proposition 11, we can also tackle the issue of entry of new informed traders. Suppose information acquisition is costly, e.g, has a positive fixed unit cost. Then the unit profit of an informed trader is proportional to the bid-ask spread, which according to equation (39) equals $s_{m,n} = \left(0.133\left(2 + \frac{3}{i}\right) + \frac{3+2i}{4+4i}\right) \Delta$, which is a decreasing function of i . As more informed traders enter the market to take advantage of this, i increases, so the spread decreases. Moreover, recall that $\Delta = \frac{\sigma}{\sqrt{\lambda/2}}$. If more *new* informed traders enter the market total trading activity λ also goes up, so Δ goes down. So the equilibrium information ratio i takes place at a point where the unit profit from participating in the market becomes equal to the unit cost of information.

Finally, the proposition above assumes that the stock is liquid enough so that the efficient price adjustment Δ is small compared to the width of the limit order book, which should be a number of the order of one (one can choose the units to make sure this is the case). Given a stock with daily fundamental volatility of about 2%, and a trading activity λ of about 100 order arrivals per day (which is not very liquid), we get an efficient price adjustment Δ of at most $\frac{2\sigma}{\sqrt{\lambda}} = 0.004$.²¹ So our results apply even to stocks that are not very liquid.

4.4 Empirical Discussion

In Section 3.3 we saw that, as a result of the uninformed traders, the limit order book is resilient, i.e., in a typical configuration with m sellers and n buyers, the bid-ask spread $s_{m,n} = a_{m,n} - b_{m,n}$ is close to zero, and the mid-point $p_{m,n} = \frac{a_{m,n} + b_{m,n}}{2}$ is close to the efficient price v^e (the difference is on average of the order of $\varepsilon^{0.34}$, where $\varepsilon = \frac{r}{4\lambda\sigma + 2\lambda\Gamma}$ is the granularity parameter). So even though the efficient price v^e is not observable, our results show that the bid-ask spread mid-point is a good proxy for the efficient price.²²

The previous section shows that the limit order book is also resilient in a different sense. As a result of the strategies of the informed traders, the efficient price v^e always converges towards the fundamental value v , up to an error of the magnitude of the bid-ask spread. To see

²¹This assumes a stock price of one. If instead the stock price is 10, the price adjustment Δ is at most $10 \times 0.004 = 0.04$.

²²In principle, the efficient price could be derived by the econometrician, as the center of the equilibrium limit order book. The notion of “center” is somewhat improper, since the outer bounds of the book given by the top seller and the bottom buyer need not be symmetric around v^e . So one may safely assume that the actual position of the efficient price is not known by the econometrician, since it is difficult even for the author to find the actual formula.

why this is true, suppose that, under the assumptions of Proposition 5, the fundamental value is higher than $v^e + \frac{s'_{m,n}}{k}$. Then, according to Proposition 5, whenever informed traders arrive to the market, they would submit buy market orders. But each buy market order pushes up the efficient price v^e , since the uninformed traders correctly infer that the fundamental value must be on average higher than the current efficient price v^e . A similar situation occurs if v is lower than $v^e - \frac{s'_{m,n}}{k}$, so eventually the efficient price v^e converges to a value inside the interval $(-\frac{s'_{m,n}}{k}, \frac{s'_{m,n}}{k})$.²³

We now use Proposition 11 to derive empirical implications of our model. To begin with, putting together Statements 5 and 2, we find that the average bid-ask spread approximately equals

$$\bar{s} = \left(0.133 \left(2 + \frac{3}{i} \right) + \frac{3 + 2i}{4 + 4i} \right) \frac{\sigma}{\sqrt{\lambda/2}}. \quad (42)$$

Consistent with Foucault, Kadan and Kandel (2005) and Roşu (2008), this formula implies that lower trading activity causes higher spreads: a slow market increases the waiting costs of traders, and so they must be compensated with higher spreads. Consistent with Foucault (1999), we also find that a higher fundamental volatility σ causes higher spreads.²⁴ This operates both via the asymmetric information channel and the waiting cost channel, since $\Delta = \frac{\sigma}{\sqrt{\lambda/2}}$ is a factor in both components (see Proposition 11, statement 5), and a higher fundamental volatility σ leads to a higher efficient price adjustment Δ .

To understand the role played in our model by the information ratio $i = \lambda^I/\lambda^U$, consider first how quickly the efficient price v_t^e adjusts to the fundamental value v_t . This is measured by the speed of convergence $\frac{1}{\tau}$, which by equation (38) equals $1.271 \frac{i}{i+1}$, so it increases with the information ratio i .

A new, surprising prediction of the model is that if one controls for fundamental volatility and trading activity, a higher information ratio $i = \frac{\lambda^I}{\lambda^U}$ (the informed-to-uniformed arrival rate ratio) generates *smaller* spreads. One might expect the opposite to be true: the presence of more informed traders should generate more asymmetric information, which should amount to higher spreads. But, in our model, having more informed traders generates *less* asymmetric

²³If there are also impatient informed traders, the convergence is even faster: every time the fundamental value is not in the bid-ask spread, an impatient informed trader who arrives to the market would place a market order. This brings the efficient price v^e in the confines of the bid-ask spread.

²⁴Both these predictions have been tested by Linnainmaa and Roşu (2008), by using instrumental variables to generate exogenous variation in trading activity and fundamental volatility.

information. To get some intuition for this finding, consider equation (39): $s_{m,n} = (0.675\tau + 1)k\Delta$. We show that $(0.675\tau + 1)k$ decreases with i . As before, having more informed traders make the efficient price converge faster, so $\frac{1}{\tau}$ increases with i , i.e., τ decreases with i . In other words, a higher information ratio i leads to *less* asymmetric information.²⁵

Another interesting implication of the model, that can in principle be tested, is that the adjustment of the efficient price to a market order is in absolute value about 4 times higher than the adjustment corresponding to a limit order. This ratio (the exact figure is 3.912) is independent of all our exogenous variables: fundamental volatility σ , trading activity λ , and information ratio i .

To understand why all types of orders are equally likely in equilibrium, which is true as long as the information ratio $i < 0.378$, consider, e.g., the dependence of the probability of a buy market order (*BMO*) on the information ratio i . As before, a higher information ratio makes the speed of convergence $\frac{1}{\tau}$ higher, which leads to less extreme fundamental values, and this makes the probability of buy market orders $P_{BMO} = \Phi(-\frac{h}{k\tau})$ (see equation (34)) smaller. The probability does not stay smaller for long, as it should revert to its long-term equilibrium value of $\frac{1}{4}$. Indeed, a smaller ratio of expected market orders per limit order, or equivalently a larger ratio of limit orders per market order means, in the language of Roşu (2008), a larger trading competition. This is shown in that paper to generate a smaller bid-ask spread. In fact, the (adjusted) spread $s'_{m,n} = h\Delta$ should decrease exactly to the point where P_{BMO} rises again to the equilibrium level of $\frac{1}{4}$.

One may wonder what causes liquidity to dry up (and spreads to increase) on days before the earnings announcements. This paper suggests that it is not the amount of asymmetric information that causes illiquidity, since that by itself would generate smaller spreads. This model provides two alternative explanations: spreads are higher either due to higher fundamental volatility or to lower trading activity, or perhaps a combination of both. The first explanation seems more reasonable: surely, there is more uncertainty about the fundamental value before earnings announcements. The second explanation begs the question why fewer traders would arrive before earnings announcements. In the model this is assumed exogenous, but one could argue that the arrival rate of uninformed traders should be proportional

²⁵The factor $k = \frac{3+2i}{4+4i}$ also depends on the information ratio, and is increasing in i , but the dependence on i is weaker, which generates a negative overall effect of i on the bid-ask spread.

to their expected profit – and that the uninformed profits would be smaller before earnings announcements, due to the fact there is more information asymmetry then. This is however not true in this model. The uninformed traders still have the discretion whether to enter the market or not: if their expected profit given their information is not above the private cost (which is an exogenous constant), they would simply not enter the market. So in the end the volatility story seems more plausible: higher fundamental volatility makes extreme events more possible, hence the market more illiquid. The presence of informed traders turns out in fact to alleviate this effect, and make the market *more* liquid.

Another contribution of this paper is to help understand better the “diagonal effect” of Biais, Hillion and Spatt (1995), i.e., that the order flow in order-driven markets is positively autocorrelated (a market buy order makes a future market buy order more likely, etc). In this model one can show that the order flow is autocorrelated if and only if there exist informed traders. Indeed, consider the case where there are only uninformed traders. Then we saw in Section 3.3 that the limit order book is resilient and tends to revert to the efficient price v^e , with a very small bid-ask spread. This means that the ex-ante entry probability is approximately equal for all types of traders, and is equal to $1/2$. Therefore, since market arrivals are independent and entry decisions are made with the same probability (based on an independently distributed private cost), it follows that the order flow is approximately uncorrelated. Now, when there are very few informed traders, market orders reflect a fundamental price v far away from the efficient price v^e , but the probability of the market order coming from an informed trader is small. This means that the adjustment of the efficient price v^e to market orders is relatively small. This leads to a staggered price adjustment, which means that more informed traders are likely to arrive with the same extreme information, hence more market orders in the same direction. This leads to a large order flow autocorrelation among informed traders. But since there are very few informed traders, the overall order flow autocorrelation is also small. As the percentage of informed traders increases, the order flow autocorrelation from informed trader decreases, but the contribution from the increase in the information ratio is stronger, which leads to a larger order flow autocorrelation. This can be verified numerically using equation (40).

This paper suggests that order flow autocorrelation could be a good proxy for the level of information asymmetry in the market. But the diagonal effect can be explained in at least

two other ways. One is by taking the view of Evans and Lyons (2002). They point out that in foreign exchange markets information about fundamentals does not seem to be much related to prices. Instead, knowledge about the order flow has a very high explanatory power for future returns in this market. So what would happen if one replaced agents with private information about the fundamental value with private information about future order flow? They would appear to behave very similar to the agents in the current model: they become impatient when the information about the future order flow is extreme enough. One way to separate the two ways of being informed is by looking at price reversals. If traders were truly informed about the fundamental value, then price moves would not revert on average. If instead the traders who are informed about the order flow do not have fundamental information, then the truly informed would eventually bring prices in line with fundamentals and we would observe price reversals.

Another explanation of the diagonal effect (probably the most common one) comes from the possibility of large orders – if one relaxes the assumption that agents only trade one unit. Then a patient trader with a large order would want to take advantage of the resilience of the book and work the order, i.e., divide the order into smaller ones. This clearly would create autocorrelation in the order flow. Is this the right explanation? If one had information about trader identity, then one could check if there was just the same trader working the order.²⁶ In the end, the three explanations for the diagonal effect (truly informed traders, traders informed about order flow, and agents with multi-unit demand) are very hard to distinguish.

5 Conclusions

This paper proposes a tractable models of an order-driven market with both liquidity traders and informed traders. The resulting equilibrium is intuitive. Informed patient traders generally submit limit orders, except when their privately observed fundamental value of the asset deviates a lot from the current market-inferred value, the efficient price. In that case, they become impatient and submit a market order. As a result, e.g., market buy orders are interpreted as unambiguously positive signals; by contrast, a limit buy order are weaker signals,

²⁶Even then, one could argue that the current model can be used by treating individual orders as coming from separate traders. But this model would not capture an informed trader who gets such an extreme signal that he would place a multi-unit market order.

and in some cases can even be negative signals. The market displays two types of resilience: (i) a “micro” resilience, related to the shape of the limit order book: the bid and ask prices tend to stay close to the efficient price, which also leads to small values of the bid-ask spread; and (ii) a “macro” resilience: the efficient price converges to the fundamental value. The more informed traders as a percentage of the total number, the quicker the speed of convergence.

The model derives various testable implications, relating prices, spreads, trading activity, and volatility. In particular, the order flow is autocorrelated only in the presence of informed traders, and the order flow autocorrelation is higher whenever there is larger percentage of informed traders, or a larger volatility. Higher volatility and smaller trading activity are showed to generate larger spreads. By contrast with some of the existing market microstructure literature, a higher percentage of informed traders actually generates smaller spreads. The effect of a market order on the efficient price is roughly 4 times larger in absolute value than for a limit order.

Some of the limitations of this model point towards future directions for research. In particular, the model assumes that everybody monitors the market at all times and makes the right inferences. It would be useful to see how the results of this paper would change in the presence of monitoring costs. This would also create the risk of limit orders being picked off, which does not occur in this model beyond information asymmetry.

Also, the model assumes that the private cost (or valuation) of the liquidity traders does not change. Similarly, informed traders do not use their information past the entry decision. While relaxing these assumptions arguably does not change the qualitative predictions of the model, it would probably lead to a richer model, which could also generate order cancellations.

A Proofs of Results

PROOF OF PROPOSITION 1: Recall the recursive equation (5) for f_m :

$$f_m = \frac{1}{2} \left(\frac{1 - f_{m-1}}{2} f_{m-1} + \frac{1 + f_{m-1}}{2} f_m \right) + \frac{1}{2} \left(\frac{1 + f_{m+1}}{2} f_{m+1} + \frac{1 - f_{m+1}}{2} f_m \right) - \frac{\varepsilon}{2}.$$

When $m = 1$ one gets: $f_1 = \frac{1}{2} \left(\frac{1-a_1}{2} a_1 + \frac{1+a_1}{2} f_1 \right) + \frac{1}{2} \left(\frac{1+f_2}{2} f_2 + \frac{1-f_2}{2} f_1 \right) - \frac{\varepsilon}{2}$, which implies

$$f_1 = \frac{f_2(1+f_2) + a_1(1-a_1)}{2+f_2-a_1}.$$

Since the seller is a monopolist, he can choose an ask price a_1 that maximizes f_1 above. The solution is

$$a_1^* = 2 + f_2 - \sqrt{2(1+f_2)}.$$

Substituting a_1^* in the formula for f_1 , one gets $f_1 = 2a_1^* - 1 = 3 + 2f_2 - 2\sqrt{2(1+f_2)}$, and so

$$a_1^* = \frac{1+f_1}{2}.$$

One can check that indeed $f_2 < f_1 < f_0 = a_1 < 1$. □

PROOF OF PROPOSITION 2: The one-sided market with patient sellers and impatient buyers is a Markov system with transition matrix P , where P_i^j is the probability of transition from state i to state j , with $i, j = 0, 1, \dots, M-1, M$. Then P can be written as

$$P = \begin{bmatrix} P_0^0 & P_1^0 & 0 & 0 & \cdots & 0 & 0 \\ P_0^1 & 0 & P_2^1 & 0 & \cdots & 0 & 0 \\ 0 & P_1^2 & 0 & P_3^2 & \cdots & 0 & 0 \\ 0 & 0 & P_2^3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & P_M^{M-1} \\ 0 & 0 & 0 & 0 & \cdots & P_{M-1}^M & P_M^M \end{bmatrix}. \quad (43)$$

To determine P_i^j , use the recursive equation (5) for f_m :

$$f_m + \frac{2\varepsilon}{2+f_{m+1}-f_m} = f_{m+1} \frac{1+f_{m+1}}{2+f_{m+1}-f_m} + f_{m-1} \frac{1-f_{m-1}}{2+f_{m+1}-f_m}.$$

This means that for $m = 1, 2, \dots, M-1$,

$$P_{m+1}^m = \frac{1+f_{m+1}}{2+f_{m+1}-f_{m-1}}, \quad P_{m-1}^m = \frac{1-f_{m-1}}{2+f_{m+1}-f_{m-1}}.$$

For $m = 1$, $P_1^0 = 1$ and $P_0^0 = 0$. For $m = M$, $P_M^M = \frac{1}{2-f_{M-1}}$ and $P_{M-1}^M = \frac{1-f_{M-1}}{2-f_{M-1}}$.

To calculate the distribution of the bid-ask spread or price impact, one needs to know the stationary probability that the system is in state m . Denote this by x_m . Consider the row vector X with entries x_m . From the theory of Markov matrices, one knows that $XP = X$. Solving for X , one gets

$$\begin{aligned} x_0 &= x_0 P_0^0 + x_1 P_0^1, \\ x_1 &= x_0 P_1^0 + x_2 P_1^2, \\ &\vdots \\ x_{M-1} &= x_{M-2} P_{M-1}^{M-2} + x_M P_{M-1}^M, \\ x_M &= x_{M-1} P_M^{M-1} + x_M P_M^M. \end{aligned}$$

Starting with an arbitrary value of $x_0 > 0$, this generates a recursive system of equations in x_m . The resulting vector is then normalized so that the components sum to one. These are the stationary probabilities x_m . One can numerically check that they increase with state m , which is to be expected since the limit order book is resilient, so it is more likely that the system is in a state m with higher m . \square

For reference, one can write down the recursive equations that the expected utility $f_{m,n}$ satisfies in the canonical equilibrium in state (m, n) with m limit sellers and n limit buyers. Recall that λ^U is the arrival rate of uniformed traders, λ^I is the arrival rate of informed traders, and P_j is the market-estimated probability that an informed trader makes the decision $j \in \{BMO, BLO, SMO, SLO, NO\}$ (buy market order, buy limit order, sell market order, sell

limit order, or no order).

$$\begin{aligned}
f_{m,n} &= \frac{\lambda^U}{4\lambda^U + 2\lambda^I} \left(\frac{1 - a_{m,n}}{2} f_{m-1,n} + \frac{1 + a_{m,n}}{2} f_{m,n} \right) \\
&+ \frac{\lambda^U}{4\lambda^U + 2\lambda^I} \left(\frac{1 + f_{m+1,n}}{2} f_{m+1,n} + \frac{1 - f_{m+1,n}}{2} f_{m,n} \right) \\
&+ \frac{\lambda^U}{4\lambda^U + 2\lambda^I} \left(\frac{1 + b_{m,n}}{2} f_{m,n-1} + \frac{1 - b_{m,n}}{2} f_{m,n} \right) \\
&+ \frac{\lambda^U}{4\lambda^U + 2\lambda^I} \left(\frac{1 - f_{m,n+1}}{2} f_{m,n+1} + \frac{1 + f_{m,n+1}}{2} f_{m,n} \right) \\
&+ \frac{2\lambda^I}{4\lambda^U + 2\lambda^I} \left(P_{BMO} f_{m-1,n} + P_{BLO} f_{m,n+1} \right. \\
&\quad \left. + P_{SMO} f_{m,n-1} + P_{SLO} f_{m+1,n} + P_{NO} f_{m,n} \right) \\
&\quad - \frac{r}{4\lambda^U + 2\lambda^I}, \\
g_{m,n} &= \frac{\lambda^U}{4\lambda^U + 2\lambda^I} \left(\frac{1 - a_{m,n}}{2} g_{m-1,n} + \frac{1 + a_{m,n}}{2} g_{m,n} \right) \\
&+ \dots, \\
a_{m,n} &= f_{m-1,n} + \Delta, \\
b_{m,n} &= g_{m,n-1} - \Delta.
\end{aligned} \tag{44}$$

To be more precise, the recursive formula for $f_{m,n}$ should be written including the Δ changes after market orders: $f_{m,n} = \frac{\lambda^U}{4\lambda^U + 2\lambda^I} \left(\frac{1 - a_{m,n}}{2} (f_{m-1,n} + \Delta) + \frac{1 + a_{m,n}}{2} f_{m,n} \right) + \dots$, but all these terms cancel out since the expected change in the efficient price must be at each point be zero.

PROOF OF PROPOSITION 4: Let us first show that the utility of an informed trader from submitting a sell market order (*SMO*) is $u_{SMO} = (v^e + b_{m,n}) - v$. It was assumed that this order is in fact a short sell: the trader does not actually have a unit of the asset, and has to borrow it from someone else. This means that the trader owes the fundamental value v , while the proceeds from the market sell order equal the bid price $v^e + b_{m,n}$.

The more difficult part is to show that the utility of an informed trader from submitting a sell limit order (*SLO*) is $u_{SLO} \approx (v^e + a'_{m,n} + k(v - v^e)) - v$, with $k = \frac{3\lambda^U + 2\lambda^I}{4\lambda^U + 4\lambda^I}$. Normally, the expected utility of a trader that arrives in state (m, n) and places a sell limit order is $f_{m+1,n}$.

But here, since we are giving approximate formulas, it is enough to compute $f_{m,n}$.²⁷ The expected utility of the informed trader in various states satisfies the recursive formula (44), except that this is seen from the perspective of an informed trader, who has better information about what other future informed traders are likely to do. Denote by $f_{i,j}^*$ the expected utility of the informed trader in a future state (i, j) . To compute $f_{m,n}^*$, we approximate all the probabilities such as $\frac{1-a_{m,n}}{2} \approx \frac{1}{2}$, and the granularity parameter $\frac{r}{4\lambda^U+2\lambda^I} \approx 0$ (see Footnote 27). Equation (44) for $f_{m,n}^*$ becomes $(4\lambda^U + 2\lambda^I)f_{m,n}^* \approx \frac{\lambda^U}{2}(f_{m-1,n}^* + f_{m+1,n}^* + f_{m,n-1}^* + f_{m,n+1}^* + 4f_{m,n}^*) + 2\lambda^I(P_{BMO}^* f_{m-1,n}^* + P_{BLO}^* f_{m,n+1}^* + P_{SMO}^* f_{m,n-1}^* + P_{SLO}^* f_{m+1,n}^* + P_{NO}^* f_{m,n}^*)$. The probabilities P_j^* for an order of type j are also different from the market-inferred probabilities P_j , since the informed trader knows better the distribution of the fundamental value v . The informed trader expects all f^* to eventually change on average by $v - v^e$, e.g., $f_{m+1,n}^* = f_{m+1,n} + (v - v^e)$. The only exception to this formula is when a market buy order arrives, and sends the limit order book in state $(m-1, n)$. Since in the canonical equilibrium the informed trader submits a limit sell order at the ask price, it follows that $f_{m-1,n}^* = a_{m,n}$. Assume further that $P_{NO}^* = 0$, which means that submitting no order is not optimal for an informed trader (Proposition 5 shows that this is indeed the case). We obtain $(2\lambda^U + 2\lambda^I)f_{m,n}^* \approx \frac{\lambda^U}{2}(a_{m,n} + f_{m+1,n} + f_{m,n-1} + f_{m,n+1}) + \frac{3\lambda^U}{2}(v - v^e) + 2\lambda^I(P_{BMO}^* a_{m,n} + P_{BLO}^* f_{m,n} + P_{SMO}^* f_{m,n} + P_{SLO}^* f_{m,n}) + 2\lambda^I(P_{BLO}^* + P_{SMO}^* + P_{SLO}^*)(v - v^e)$. We know that $a_{m,n} = f_{m-,n} + \Delta$, so by approximating all $f_{i,j}$ in the formula with $f_{m,n}$, we get $(2\lambda^U + 2\lambda^I)f_{m,n}^* \approx (2\lambda^U + 2\lambda^I)f_{m,n} + \frac{\lambda^U}{2} \Delta + \frac{3\lambda^U}{2}(v - v^e) + 2\lambda^I P_{BMO}^* \Delta + 2\lambda^I(1 - P_{BMO}^*)(v - v^e)$. If one defines

$$k_v = \frac{3\lambda^U + 4\lambda^I(1 - P_{BMO}^*)}{4\lambda^U + 4\lambda^I}, \quad (45)$$

we get $f_{m,n}^* \approx f_{m,n} + (1 - k_v)\Delta + k_v(v - v^e)$. Since $a_{m,n} = f_{m-1,n} + \Delta$, this implies

$$u_{SLO} \approx (v^e + f_{m,n}^*) - v = (v^e + a_{m,n} - k_v\Delta + k_v(v - v^e)) - v. \quad (46)$$

If we set $P_{BMO}^* \approx \frac{1}{2}$, we obtain $k_v \approx k = \frac{3\lambda^U + 2\lambda^I}{4\lambda^U + 4\lambda^I}$, so using $a'_{m,n} = a_{m,n} - k\Delta$, we finally obtain the desired equation: $u_{SLO} \approx (v^e + a'_{m,n} + k(v - v^e)) - v$.

We now show that replacing k_v by the constant k does not change the cutoffs that an informed trader uses to determine the choice between a limit order and a market order.

²⁷Recall that $f_{m,n}$ is typically of the order of $\varepsilon^{0.34}$, where $\varepsilon = r/(4\lambda^U + 2\lambda^I)$ is the granularity parameter, and $f_{m,n} - f_{m+1,n}$ is of the order of $\varepsilon^{0.68}$.

According to Proposition 5, the cutoff between, e.g., a buy market order and a buy limit order is $v = v^e + \frac{s'_{m,n}}{k}$. To show that $k_v = k$ at that cutoff, we must show that $P_{BMO}^* = \frac{1}{2}$ at that cutoff. Note that P_{BMO}^* is the probability that the *next* trader which arrives to the market, provided he is informed, submits a buy market order. This trader's decision depends not on the current fundamental value $v = v_t$, but on $v_{t+1} = v_t + \eta_{t+1}$, where the innovation η_{t+1} is normally distributed with mean zero and standard deviation σ_0 (see equation (22)). Then $P_{BMO}^* = P(v_{t+1} > v^e + \frac{s'_{m,n}}{k}) = P(v + \eta_{t+1} > v^e + \frac{s'_{m,n}}{k}) = \Phi((v - v^e - \frac{s'_{m,n}}{k})/\sigma_0)$. Now, if one sets $v = v^e + \frac{s'_{m,n}}{k}$, it follows that $P_{BMO}^* = \Phi(0) = \frac{1}{2}$. \square

PROOF OF PROPOSITION 5: From Proposition 4, one only needs to compare the utilities corresponding to the various types of orders:

$$\begin{aligned}
u_{BMO} > u_{BLO} &\iff (v - v^e) - a_{m,n} > (1 - k)(v - v^e) - b'_{m,n} \iff v > v^e + \frac{s'_{m,n}}{k}; \\
u_{BMO} > u_{NO} &\iff (v - v^e) - a_{m,n} > 0 \iff v > v^e + a_{m,n}; \\
u_{BMO} > u_{SLO} &\iff (v - v^e) - a_{m,n} > a'_{m,n} - (1 - k)(v - v^e) \iff v > v^e + \frac{a'_{m,n} - (k/2)\Delta}{1 - k/2}; \\
u_{BMO} > u_{SMO} &\iff (v - v^e) - a_{m,n} > b_{m,n} - (v - v^e) \iff v > v^e + p_{m,n}; \\
u_{BLO} > u_{NO} &\iff (1 - k)(v - v^e) - b'_{m,n} > 0 \iff v > v^e + \frac{b'_{m,n}}{1 - k}; \\
u_{BLO} > u_{SLO} &\iff (1 - k)(v - v^e) - b'_{m,n} > a'_{m,n} - (1 - k)(v - v^e) \iff v > v^e + \frac{p_{m,n}}{1 - k}; \\
u_{BLO} > u_{SMO} &\iff (1 - k)(v - v^e) - b'_{m,n} > b_{m,n} - (v - v^e) \iff v > v^e + \frac{b'_{m,n} - (k/2)\Delta}{1 - k/2}; \\
u_{NO} > u_{SLO} &\iff 0 > a'_{m,n} - (1 - k)(v - v^e) \iff v > v^e + \frac{a'_{m,n}}{1 - k}; \\
u_{NO} > u_{SMO} &\iff 0 > b_{m,n} - (v - v^e) \iff v > v^e + b_{m,n}; \\
u_{SLO} > u_{SMO} &\iff a'_{m,n} - (1 - k)(v - v^e) > b_{m,n} - (v - v^e) \iff v > v^e - \frac{s'_{m,n}}{k}.
\end{aligned}$$

This gives ten cutoffs that have to be compared in order to determine which strategy gives a higher expected utility. The first condition $a_{m,n}(1 - k) - b'_{m,n} > 0$ can be shown to imply the cutoff inequality $\frac{b'_{m,n}}{1 - k} < a_{m,n} < \frac{s'_{m,n}}{k}$ (the differences between the cutoffs are all constant multiples of each other, and the sign is determined by the first condition). This determines the choice between *BMO*, *BLO* and *NO*: *NO* is the best of the three when $v - v^e$ is below $\frac{b'_{m,n}}{1 - k}$; *BLO* between $\frac{b'_{m,n}}{1 - k}$ and $\frac{s'_{m,n}}{k}$; and *BMO* above $\frac{s'_{m,n}}{k}$. The second condition, $a'_{m,n} - b_{m,n}(1 - k) > 0$, implies the cutoff inequality $-\frac{s'_{m,n}}{k} < b_{m,n} < \frac{a'_{m,n}}{1 - k}$. This determines the choice between

NO, *SLO* and *SMO*: *SMO* is the best of the three below $-\frac{s'_{m,n}}{k}$; *SLO* between $-\frac{s'_{m,n}}{k}$ and $\frac{a'_{m,n}}{1-k}$; and *NO* above $\frac{a'_{m,n}}{1-k}$. The condition $-\frac{s'_{m,n}}{k} < \frac{p_{m,n}}{1-k} < \frac{s'_{m,n}}{k}$ and the cutoff inequalities above, together with the inequality $\frac{b'_{m,n}}{1-k} < \frac{p_{m,n}}{1-k} < \frac{a'_{m,n}}{1-k}$, imply the following cutoff inequality: $-\frac{s'_{m,n}}{k} < \frac{b'_{m,n}}{1-k} < \frac{p_{m,n}}{1-k} < \frac{a'_{m,n}}{1-k} < \frac{s'_{m,n}}{k}$.

Now one can see that *BMO* is optimal above $\frac{s'_{m,n}}{k}$: we saw it is better than *BLO* and *NO*. It is also better than *SLO*, since *BLO* is better than *SLO* above $\frac{p_{m,n}}{1-k}$, hence also above $\frac{s'_{m,n}}{k}$. By the same type of reasoning, *BMO* is better than *SMO*, since *SLO* is better than *SMO* above $-\frac{s'_{m,n}}{k}$. Next, we show that *BLO* is optimal between $\frac{p_{m,n}}{1-k}$ and $\frac{s'_{m,n}}{k}$: As shown above, *BLO* is better than both *BMO* and *NO*. It is better than *SLO*, since it is above $\frac{p_{m,n}}{1-k}$, and it is better than *SMO*, since *SLO* is better than *SMO* above $-\frac{s'_{m,n}}{k}$. The proof for the optimality of *SLO* and *SMO* on their corresponding intervals is symmetric. \square

The next Proposition discusses what happens if we required that informed traders can execute only either as buyers or sellers, and their type does not change. Then we have to discuss two types of traders

Proposition 12. *In the context of Proposition 5, consider a limit order book with m sellers and n buyers. Then the optimal strategy of an informed patient buyer (*IPB*) is given by the following (approximate) cutoffs:*

- *Case 1: $a_{m,n}(1-k) - b'_{m,n} > 0$. Then there are two cutoffs $\frac{b'_{m,n}}{1-k} < \frac{s'_{m,n}}{k}$, so that the informed patient buyer submits: a buy market order (*BMO*) if $v - v^e > \frac{s'_{m,n}}{k}$; a buy limit order (*BLO*) if $\frac{b'_{m,n}}{1-k} < v - v^e < \frac{s'_{m,n}}{k}$; and no order (*NO*) if $v - v^e < \frac{b'_{m,n}}{1-k}$.*
- *Case 2: $a_{m,n}(1-k) - b'_{m,n} < 0$. Then there is a cutoff $a_{m,n}$, so that the informed patient buyer submits: a buy market order (*BMO*) if $v - v^e > a_{m,n}$; and no order (*NO*) if $v - v^e < a_{m,n}$.*

*The optimal strategy of an informed patient seller (*IPS*) is similar.*

Proof. From Proposition 4, we only need to compare the utilities corresponding to the various types of orders. For example, consider an informed patient buyer. Then, as in the proof of Proposition 5, $u_{BMO}^{IPB} > u_{BLO}^{IPB} \iff v > v^e + \frac{s'_{m,n}}{k}$, $u_{BMO}^{IPB} > u_{NO}^{IPB} \iff v > v^e + a_{m,n}$, and $u_{BLO}^{IPB} > u_{NO}^{IPB} \iff v > v^e + \frac{b'_{m,n}}{1-k}$. Comparing the three cutoffs $\frac{s'_{m,n}}{k}$, $a_{m,n}$, and $\frac{b'_{m,n}}{1-k}$, there are two cases:

- Case 1: $a_{m,n}(1-k) - b'_{m,n} > 0$. Then the order of cutoffs is $\frac{b'_{m,n}}{1-k} < a_{m,n} < \frac{s'_{m,n}}{k}$, and the optimal strategy is as stated.
- Case 2: $a_{m,n}(1-k) - b'_{m,n} < 0$. Then the order of cutoffs is $\frac{s'_{m,n}}{k} < a_{m,n} < \frac{b'_{m,n}}{1-k}$, and the proof is complete.

Notice that here an informed patient buyer who sees a low fundamental value does not have the option to short sell, and chooses not to submit any order. \square

PROOF OF PROPOSITION 6: Recall that we only look at traders that arrive to the market, and for uninformed traders that happens only to approximately half of them: for example, the probability of a buy market order conditional on a trader being an uninformed impatient buyer is $\frac{1-a_{m,n}}{2} \approx \frac{1}{2}$ (use Proposition 2 which shows that $a_{m,n}$ is small). Now, a buy market order (*BMO*) can come either from an uninformed impatient buyer, or an informed patient trader if the fundamental value is above a certain cutoff. The probability of this latter event can be computed as $I(v \geq v^e + \frac{s'_{m,n}}{k})$, where I is the indicator function. Then $P(BMO | v_t = v) = \frac{\lambda^U/2 + 2\lambda^I I(v \geq v^e + s'_{m,n}/k)}{2\lambda^U + 2\lambda^I}$. Using Bayes' rule, $P(v_t = v | BMO) = \frac{P(BMO | v)P(v)}{\int P(BMO | v)P(v) dv} = \frac{\lambda^U/2 + 2\lambda^I I(v \geq v^e + s'_{m,n}/k)}{\lambda^U/2 + 2\lambda^I \Phi(-s'_{m,n}/(k\sigma_e))}$, where $\Phi(\cdot)$ is the cumulative density function for the standard normal distribution. One then computes $\mathbf{E}(v | BMO) = v^e + \sigma_e \frac{\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(s'_{m,n}/(k\sigma_e))^2)}{\Phi(-s'_{m,n}/(k\sigma_e)) + \lambda^U/(4\lambda^I)}$. The formulas for $E(v | BLO)$, $E(v | SMO)$, and $E(v | SLO)$ are derived in a similar way. \square

PROOF OF PROPOSITION 7: Just as in the proof of Proposition 5, from the two conditions $a_{m,n}(1-k) - b'_{m,n} > 0$ and $a'_{m,n} - b_{m,n}(1-k) > 0$, we derive the cutoff inequalities: $\frac{b'_{m,n}}{1-k} < \frac{s'_{m,n}}{k}$ and $-\frac{s'_{m,n}}{k} < \frac{a'_{m,n}}{1-k}$. The first inequality defines the intervals on which one can compare *BMO*, *BLO* and *NO*, and the second inequality defines the intervals on which to compare *SMO*, *SLO* and *NO*. The condition $\frac{p_{m,n}}{1-k} < -\frac{s'_{m,n}}{k}$ implies the cutoff inequality: $\frac{p_{m,n}}{1-k} < \frac{b'_{m,n} - (k/2)\Delta}{1-k/2} < -\frac{s'_{m,n}}{k}$ (their differences are constant multiples of each other). Using also the inequality $\frac{b'_{m,n}}{1-k} < \frac{p_{m,n}}{1-k} < \frac{a'_{m,n}}{1-k}$, we get the following cutoff inequality: $\frac{b'_{m,n}}{1-k} < \frac{p_{m,n}}{1-k} < \frac{b'_{m,n} - (k/2)\Delta}{1-k/2} < -\frac{s'_{m,n}}{k} < \frac{s'_{m,n}}{k}$.

Now if $v - v^e$ is above the cutoff $\frac{s'_{m,n}}{k}$, we show that *BMO* is optimal: From the first condition, *BMO* is better than both *BLO* and *NO*. Also, *BMO* is better than *SLO* and *SMO*: it is better than *BLO*, and *BLO* is better than *SLO* above $\frac{p_{m,n}}{1-k}$, and *SLO* is better than *SMO* above $-\frac{s'_{m,n}}{k}$. Next, we show that *BLO* is optimal for $v - v^e$ between $\frac{b'_{m,n} - (k/2)\Delta}{1-k/2}$ and $\frac{s'_{m,n}}{k}$: *BLO* is clearly better than *BMO* and *SMO*, by just looking at the cutoffs. It is

better than *SLO*, since $\frac{b'_{m,n}-(k/2)\Delta}{1-k/2}$ is above $\frac{p_{m,n}}{1-k}$; and better than *NO*, since $\frac{b'_{m,n}-(k/2)\Delta}{1-k/2}$ is above $\frac{b'_{m,n}}{1-k}$. Finally, we show that *SMO* is optimal for $v - v^e$ below $\frac{b'_{m,n}-(k/2)\Delta}{1-k/2}$: *SMO* is better than *SLO* and *NO*, since this is true below $-\frac{s'_{m,n}}{k}$. *SMO* is better than *BLO*, since this is true below $\frac{b'_{m,n}-(k/2)\Delta}{1-k/2}$. It is better than *BMO*, since *BLO* is better than *BMO* below $\frac{s'_{m,n}}{k}$.

To show that a buy limit order has a negative price impact, recall that *BLO* is optimal between $\frac{b'_{m,n}-(k/2)\Delta}{1-k/2}$ and $\frac{s'_{m,n}}{k}$. Then the inequality: $\frac{b'_{m,n}-(k/2)\Delta}{1-k/2} < -\frac{s'_{m,n}}{k} < \frac{s'_{m,n}}{k}$ shows that the center of the interval over which *BLO* is optimal is negative. Using the formula in equation (18) for the price impact corresponding to a buy limit order, it follows that the price impact corresponding to a *BLO* is negative.

We now determine more explicit conditions under which a buy limit order would have a negative price impact. Rewrite the above conditions: $a'_{m,n} + k(1-k)\Delta - b'_{m,n} > 0$, $a'_{m,n} - b'_{m,n}(1-k) + k(1-k)\Delta > 0$, $\frac{a'_{m,n}+b'_{m,n}}{2(1-k)} < -\frac{a'_{m,n}-b'_{m,n}+k\Delta}{k}$. The first two inequalities are satisfied if one assumes $a'_{m,n} > 0$ and $b'_{m,n} < 0$, which is true with a large probability. The third inequality is more restrictive: $(1-\frac{k}{2})a'_{m,n} + k(1-k)\Delta < (\frac{3k}{2}-1)(-b'_{m,n})$. Taking k close to $\frac{3}{4}$ (which happens when there are almost no informed traders, i.e., $\lambda^I \approx 0$), one gets $5a'_{m,n} + \frac{3}{2}\Delta < (-b'_{m,n})$. So if k is close to the upper bound, and $b'_{m,n}$ is large and negative, this inequality is true, and it leads to *BLO* having a negative price impact. \square

PROOF OF PROPOSITION 8: Write $v_t = v_t^e + \mu_t$, where μ_t is the pricing error. The efficient price is $v_t^e = \mathbf{E}_t(v_t \mid s_1, \dots, s_t) = \mathbf{E}_t(v_t \mid \mathcal{I}_t)$. All variables are normal with mean zero, so we can apply the standard regression techniques, by regressing v_t on the vector of signals $S = [s_1, \dots, s_t]$. We get a linear relation: $v_t = v_t^e + \mu_t = S\beta + \mu_t$. Here β is the column vector $[\beta_1, \dots, \beta_t]'$ given by the formula $\beta = \Sigma_S^{-1}\Sigma_{Sv_t}$, where $\Sigma_S = \mathbf{Var}(S)$, and $\Sigma_{Sv_t} = \mathbf{Cov}(S, v_t)$. So we finally get $\mathbf{Var}(v_t) = \mathbf{Var}(v_t^e) + \mathbf{Var}(\mu_t) = \Sigma'_{Sv_t}\Sigma_S^{-1}\Sigma_{Sv_t} + \sigma_{\mu_t}^2$. Similarly, one gets $\mathbf{Var}(v_{t+1}) = \Sigma'_{Tv_{t+1}}\Sigma_T^{-1}\Sigma_{Tv_{t+1}} + \sigma_{\mu_{t+1}}^2$, where $T = [s_1, \dots, s_t, s_{t+1}] = [S, s_{t+1}]$.

Define the following matrices: $A = \Sigma_S$ a positive definite symmetric $t \times t$ -matrix, $b = \Sigma_{Ss_{t+1}}$ a t -column vector, and $c = \sigma_{s_{t+1}}^2$ a positive scalar. Notice that $\Sigma_T = \begin{bmatrix} A & b \\ b' & c \end{bmatrix}$. By matrix theory, $\begin{bmatrix} A & b \\ b' & c \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{c-b'A^{-1}b} \begin{bmatrix} A^{-1}bb'A^{-1} & -A^{-1}b \\ -b'A^{-1} & 1 \end{bmatrix}$. Denote by

$d = \mathbf{Var}(v_{t+1}) = c - \sigma_\delta^2$. We need to compute $\mathbf{Var}(v_{t+1}^e) = \begin{bmatrix} b' & d \end{bmatrix} \begin{bmatrix} A & b \\ b' & c \end{bmatrix}^{-1} \begin{bmatrix} b \\ d \end{bmatrix} = b'A^{-1}b + \frac{(b'A^{-1}b-d)^2}{d-b'A^{-1}b+\sigma_\delta^2} = \mathbf{Var}(v_t^e) + \frac{(\mathbf{Var}(v_t^e)-d)^2}{d-\mathbf{Var}(v_t^e)+\sigma_\delta^2}$. We compute $d = (t+1)\sigma_0^2$. So if $x_t = \frac{\mathbf{Var}(v_t^e)}{\sigma_0^2}$, we obtain the following recursive relation: $x_{t+1} = x_t + \frac{(t+1-x_t)^2}{(t+1-x_t+a)}$, with $x_0 = 0$ (recall that $a = \sigma_\delta^2/\sigma_0^2$). Define $y_t = (t+1)-x_t$ for $t \geq 0$. We get the recursive relation $y_{t+1} = 1+y_t + \frac{y_t^2}{a+y_t}$, with $y_0 = 1$, $y_1 = 2 - x_1 = \frac{1+2a}{1+a} > 1$. If y_t converges to some finite limit L when $t \rightarrow \infty$, it must satisfy $L = 1 + L - \frac{L^2}{a+L}$. The unique solution $L > 1$ is $L = \frac{1+\sqrt{1+4a}}{2}$. One can show that y_t is an increasing sequence that converges to L . Therefore, $\mathbf{Var}(v_{t+1}^e) - \mathbf{Var}(v_t^e) = \sigma_0^2 \frac{y_t^2}{a+y_t}$, which converges to σ_0^2 .

Now subtract the equation $\mathbf{Var}(v_t) = \mathbf{Var}(v_t^e) + \mathbf{Var}_t(v_t)$ from $\mathbf{Var}(v_{t+1}) = \mathbf{Var}(v_{t+1}^e) + \mathbf{Var}_{t+1}(v_{t+1})$, and obtain $\sigma_0^2 = \mathbf{Var}(v_{t+1}^e) - \mathbf{Var}(v_t^e) + \mathbf{Var}_{t+1}(v_{t+1}) - \mathbf{Var}_t(v_t)$. From this and the results in the previous paragraph, $\mathbf{Var}_{t+1}(v_{t+1}) - \mathbf{Var}_t(v_t) = \sigma_0^2 - \sigma_0^2 \frac{y_t^2}{a+y_t} = \sigma_0^2 \left(1 - \frac{y_t^2}{a+y_t}\right) = \sigma_0^2(y_{t+1} - y_t)$. Adding these equations from 1 to t , we obtain $\mathbf{Var}_t(v_t) = \sigma_0^2(y_t - y_0) = \sigma_0^2(y_t - 1)$ (since $\mathbf{Var}(v_0) = \mathbf{Var}(0) = 0$). We know that $y_t \rightarrow \frac{\sqrt{1+4a}+1}{2}$, so $\mathbf{Var}_t(v_t) \rightarrow \sigma_0^2 \frac{\sqrt{1+4a}-1}{2}$. This proves statement 3.

To prove statement 1, we need to analyze the evolution of the efficient price in more detail. Consider the equation $v_t^e = S\beta = S\Sigma_S^{-1}\Sigma_{Sv_t} = S\Sigma_S^{-1}\Sigma_{Ss_{t+1}} = SA^{-1}b$. Similarly, $v_{t+1}^e = \begin{bmatrix} S & s_{t+1} \end{bmatrix} \begin{bmatrix} A & b \\ b' & c \end{bmatrix}^{-1} \begin{bmatrix} b \\ d \end{bmatrix} = v_t^e + \frac{(s_{t+1}-SA^{-1}b)(t+1-b'A^{-1}b)}{c-b'A^{-1}b} = v_t^e + (s_{t+1} - v_t^e) \frac{y_t}{a+y_t}$. So we have shown

$$v_{t+1}^e = v_t^e + (s_{t+1} - v_t^e) \frac{y_t}{a + y_t}. \quad (47)$$

Compute $\mathbf{Var}_t(v_{t+1}^e) = \mathbf{Var}_t(s_{t+1}) \frac{y_t^2}{(a+y_t)^2}$. But $\mathbf{Var}_t(s_{t+1}) = \sigma_\delta^2 + \sigma_0^2 + \mathbf{Var}_t(v_t) = \sigma_0^2(a + 1 + y_t - 1)$, so $\mathbf{Var}_t(v_{t+1}^e) = \sigma_0^2 \frac{y_t^2}{a+y_t} \rightarrow \sigma_0^2$. Since $s_{t+1} - v_t^e$ is uncorrelated to v_t^e and they are both normally distributed, they are also independent. So v_t^e is a quasi Brownian motion, i.e., a process with independent increments, and an innovation variance that converges to σ_0^2 .

For statement 2, use the fact that, if X is normally distributed with mean zero and variance σ^2 , $\mathbf{E}(|X|) = \sqrt{\frac{2}{\pi}} \sigma$. So the speed of convergence equals $\left(\frac{\mathbf{Var}_t(v_{t+1}^e)}{\mathbf{Var}_t(v_{t+1})}\right)^{1/2} = \left(\frac{\sigma_0^2}{\sigma_0^2 \frac{\sqrt{1+4a}+1}{2}}\right)^{1/2} = \left(\frac{2}{\sqrt{1+4a}+1}\right)^{1/2}$. For statement 4, use the recursive formula for v_{t+1}^e , and write $v_{t+2} - v_{t+1}^e = v_{t+2} - v_{t+1} + (v_{t+1} - p_t) \frac{a}{a+y_t} - \delta_{t+1} \frac{y_t}{a+y_t}$. From this, we get $\mathbf{Cov}_t(v_{t+2} - p_{t+1}, v_{t+1} - p_t) =$

$$\frac{a}{a+y_t} \text{Var}_t(v_{t+1} - p_t), \text{ so } \text{Cov}_t(v_{t+2} - p_{t+1}, v_{t+1} - p_t) = \frac{a}{a+y_t} \rightarrow \frac{a}{a + \frac{\sqrt{1+4a+1}}{2}}.$$

□

PROOF OF PROPOSITION 9: As in the proof of Proposition 6, we compute the density of the fundamental value v_t conditional on a buy market order at $t+1$ (BMO_{t+1}) and on the efficient price being v_t^e : $P(v_t = v \mid v_t^e, BMO_{t+1}) = \frac{\lambda^U/2 + 2\lambda^I I(v \geq v^e + s'_{m,n}/k)}{\lambda^U/2 + 2\lambda^I \Phi(-s'_{m,n}/(k\sigma_e))}$, where I is the indicator function, and $\Phi(\cdot)$ is the cumulative density function for the standard normal distribution. We then compute $\mathbf{E}(v_t \mid v_t^e, BMO_{t+1}) = v_t^e + \Delta$, and then use $\mathbf{E}(v_{t+1} \mid v_t^e, BMO_{t+1}) = \mathbf{E}(v_t \mid v_t^e, BMO_{t+1})$. To compute $\text{Var}(v_t \mid v_t^e, BMO_{t+1})$, use the same method as above, together with equation (27): $\frac{\Delta}{\sigma_e} = \frac{1}{\tau} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}}}{\Phi(-\alpha) + \frac{1}{4i}}$, where $\alpha = \frac{s'_{m,n}}{k\sigma_e} = \frac{h}{k\tau}$. We obtain $\text{Var}(v_t \mid v_t^e, BMO_{t+1}) = \sigma_e^2 \left(1 + \frac{h}{k\tau^2}\right) - \Delta^2$. So we have $\text{Var}(v_{t+1} \mid v_t^e, BMO_{t+1}) = \text{Var}(v_t \mid v_t^e, BMO_{t+1}) + \sigma_0^2 = \sigma_e^2 \left(1 + \frac{h}{k\tau^2}\right)$.

Finally, to determine the posterior distribution for v_{t+1} , given v_t^e and BMO_{t+1} , note that $v_{t+1} = v_t + \eta_{t+1}$. Therefore, the density of v_{t+1} is the convolution of v_t and η_{t+1} . We know the density for v_t : $\frac{I\left(\frac{v-v^e}{\tau} \geq \alpha\right) + \frac{1}{4i}}{\Phi(-\alpha) + \frac{1}{4i}} \frac{1}{\sigma_e \sqrt{2\pi}} e^{-\frac{(x-v_t^e)^2}{2\sigma_e^2}}$. The density for η_{t+1} is normal with mean zero and variance σ_0^2 . A straightforward calculation shows that their convolution yields the required density function $f(x)$. □

PROOF OF PROPOSITION 11: Equation (44) shows that the expected utility of a patient uninformed seller in the limit order book satisfies:

$$\begin{aligned} & \left(1 - \frac{1}{2+2i} \left(\frac{a_{m,n}}{2} - \frac{f_{m+1,n}}{2} - \frac{b_{m,n}}{2} + \frac{f_{m,n+1}}{2}\right)\right) f_{m,n} + \frac{r}{2\lambda^U + 2\lambda^I} \\ &= \frac{1}{2+2i} \left(\frac{1-a_{m,n}}{2} + 2iP_{BMO}\right) f_{m-1,n} + \frac{1}{2+2i} \left(\frac{1+f_{m+1,n}}{2} + 2iP_{SLO}\right) f_{m+1,n} \\ &+ \frac{1}{2+2i} \left(\frac{1+b_{m,n}}{2} + 2iP_{SMO}\right) f_{m,n-1} + \frac{1}{2+2i} \left(\frac{1-f_{m,n+1}}{2} + 2iP_{BLO}\right) f_{m,n+1} \end{aligned}$$

(submitting no order (NO) is never optimal for an informed trader). We search among resilient equilibria (see Section 3.3), and so $a_{m,n}$, $b_{m,n}$, $f_{m+1,n}$, $f_{m,n+1}$ are small. The equation becomes

$$\begin{aligned} f_{m,n} + \frac{r}{2\lambda^U + 2\lambda^I} &= \frac{1}{4+4i} \left(1 + 4iP_{BMO}\right) f_{m-1,n} + \frac{1}{4+4i} \left(1 + 4iP_{SLO}\right) f_{m+1,n} \\ &+ \frac{1}{4+4i} \left(1 + 4iP_{SMO}\right) f_{m,n-1} + \frac{1}{4+4i} \left(1 + 4iP_{BLO}\right) f_{m,n+1}. \end{aligned}$$

Recall that the ask price $a_{m,n} = f_{m-1,n} + \Delta$ and the bid price $b_{m,n} = g_{m,n-1} - \Delta$, so the bid-ask

spread $s_{m,n} = s_{m,n}^0 + 2\Delta$, where $s_{m,n}^0 = f_{m-1,n} - g_{m,n-1}$ is the component due to waiting costs.

If $P_{BMO} = P_{SMO} > \frac{1}{4}$, then market orders are more likely than limit orders, which implies a trading competition parameter less than one (this is the ratio of the arrival rates of limit orders to market orders). Roşu (2008, Appendix, proof of Proposition 2) shows that this leads to large average bid-ask spreads, so the equilibrium cannot be resilient.

So either $P_{BMO} = P_{SMO}$ are either both strictly smaller than $\frac{1}{4}$, or they are equal to $\frac{1}{4}$. If the P_{BMO} strictly smaller than $\frac{1}{4}$, limit orders arrive faster than market orders, i.e., the trading competition parameter is higher than one. Roşu (2008, Proposition 2) shows that the average bid-ask spread component $s_{m,n}^0$ due to waiting costs is then of the order of $\varepsilon \ln(\frac{1}{\varepsilon})$, where $\varepsilon = \frac{r}{2\lambda^I + 2\lambda^U}$ is the granularity parameter. But Δ was assumed to be of the order of $\sqrt{\varepsilon}$, so the component $s_{m,n}^0$ is negligible compared to the component 2Δ due to asymmetric information. Therefore, we can assume that in the case $P_{BMO} > \frac{1}{4}$ the bid-ask spread is approximately 2Δ . Then $s'_{m,n} = s_{m,n} - k\Delta = (2 - k)\Delta$, and the equation that determines $\tau = \frac{\sigma\varepsilon}{\Delta}$ becomes

$$1 = \tau \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{2-k}{k\tau}\right)^2}}{\Phi\left(-\frac{2-k}{k\tau}\right) + \frac{1}{4i}}. \quad (48)$$

This leads to $\tau = \tau(i)$ as function of i . It turns out that $P_{BMO} = \Phi\left(-\frac{2-k}{k\tau}\right)$ is larger than $\frac{1}{4}$ if and only if $i > 0.378$. One can check that $\tau(i)$ decreases with i , and therefore P_{BMO} also decreases with i . The efficient price adjustment Δ equals σ_0 , the volatility of the fundamental value over a time period between two consecutive market orders. Market orders arrive at twice the rate of BMO , i.e., at $\lambda^U + 4\lambda^I P_{BMO}$, so $\Delta = \frac{\sigma}{(\lambda^U + 4\lambda^I P_{BMO})^{1/2}}$. In order to obtain the desired formula, divide both the numerator and the denominator by $\sqrt{\lambda} = \sqrt{2\lambda^U + 2\lambda^I}$. The ratio of adjustments to market orders and limit orders $\frac{\Delta_{BMO}}{\Delta_{BLO}}$ can be computed using Proposition 6. This ratio equals 3.912 for $i = 0.378$, and decreases in i .

Now we focus on the main case: $i < 0.378$. Since we approximated for example $a_{m,n} \approx 0$, we cannot take i to be too small (e.g., consider $i > 0.05$). Having $P_{BMO} = P_{SMO} = \frac{1}{4}$, makes the arrival rates of all types of order equal, which proves statement 1. To prove statement 2, compute $\Delta = \Delta_{BMO} = \frac{\sigma}{(\lambda^U + 4\lambda^I P_{BMO})^{1/2}} = \frac{\sigma}{(\lambda^U + \lambda^I)^{1/2}} = \frac{\sigma}{\sqrt{\lambda/2}}$. Δ_{BLO} can be computed similarly using Proposition 6, so one can check that $\frac{\Delta_{BMO}}{\Delta_{BLO}} = 3.912$.

Recall that the equation for $\tau = \frac{\sigma\varepsilon}{\Delta}$ is: $1 = \tau \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{h}{k\tau}\right)^2}}{\Phi\left(-\frac{h}{k\tau}\right) + \frac{1}{4i}}$, where $h = \frac{s'_{m,n}}{\Delta} = \frac{s_{m,n} - k\Delta}{\Delta}$. Since

$P_{BMO} = \Phi\left(-\frac{h}{k\tau}\right)$, it implies $\frac{h}{k\tau} = 0.675$. So the equation for τ becomes simple: $1 = \tau \frac{0.318}{0.25 + \frac{1}{4i}}$, from which we get $\tau = 0.787 \frac{i+1}{i}$. This proves statement 3. As in the proof of Proposition 8, the speed of convergence is the ratio of standard deviations: $\left(\frac{\text{Var}_t(v_{t+1}^e)}{\text{Var}_t(v_{t+1})}\right)^{1/2} = \frac{\sigma_0}{\sigma_e} = \frac{1}{\tau} = 1.271 \frac{i}{i+1}$, which proves statement 4.

The bid-ask spread is determined by the equation $\frac{h}{k\tau} = 0.675$. So $s_{m,n} = s'_{m,n} + k\Delta = h\Delta + k\Delta = 0.675k\tau\Delta + k\Delta = (0.675\tau + 1)k\Delta = \left(0.133\left(2 + \frac{3}{i}\right) + \frac{3+2i}{4+4i}\right)\Delta$. The bid-ask spread must be at least 2Δ , the component due to information asymmetry. So we require that $0.133\left(2 + \frac{3}{i}\right) + \frac{3+2i}{4+4i} > 2$, which is equivalent to $i < 0.378$. This concludes statement 5. To prove the last statement, consider the pricing error autocorrelation: $\rho = \text{Corr}(v_{t+1} - v_{t+1}^e, v_t - v_t^e) = \frac{a}{a + \frac{1 + \sqrt{1 + 4a}}{2}}$. The ratio $a = \frac{\sigma_e^2}{\sigma_0^2}$ can be obtained from equation (32): $a = \tau^2(\tau^2 - 1)$. So $\rho = \frac{\tau^2(\tau^2 - 1)}{\tau^2(\tau^2 - 1) + \tau^2} = \frac{\tau^2 - 1}{\tau^2} = 1 - \frac{1}{\tau^2} = 1 - 1.616 \left(\frac{i}{i+1}\right)^2$. To obtain the autocorrelation of order flow, compute $P(BMO_{t+1}, BMO_t | v_t^e) = \int \frac{\frac{\lambda^U}{2} + 2\lambda^I I(v_t > v_t^e + \alpha)}{2\lambda^U + 2\lambda^I} \cdot \frac{\lambda^U 2 + 2\lambda^I I(v_{t+1} > v_{t+1}^e + \alpha)}{2\lambda^U + 2\lambda^I}$, and notice that $\text{Corr}(v_{t+1} - v_{t+1}^e, v_t - v_t^e) = \rho$. \square

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