On a Connection Between Froot-Stein and the de Finetti Optimal Dividends Models

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ABSTRACT

Costly external capital is one of the frictions that violates the premises of the Modigliani and Miller irrelevance theorems. Froot et. al. developed a three-stage model to explain how costly external capital, along with other frictions, provides an opportunity for risk management to create value. This paper extends their model to analyze risk management (in particular, reinsurance) in the context of a going concern. It relates the extended Froot model to a fifty-year-old stochastic dividend optimization problem, introduced by Bruno de Finetti, which has received growing interest in the recent decade.

Keywords: friction, Modigliani-Miller, Froot-Stein, de Finetti, optimal dividends, risk management, reinsurance, shareholder value

JEL Classification: C61

The Modigliani and Miller (1958, Miller and Modigliani (1961)) irrelevance theorems state that in the absence of financial frictions and in a situation of fixed investment strategy, neither capital structure, dividend policy, nor (by implication) risk management affect the firm's market value. Since then, many researchers have developed models that illuminate the mechanisms by which certain market frictions can make risk management and capital structure relevant. Jensen and Meckling (1976) develop a one-period theory for minimizing agency costs which are represented as abstract functions of leverage. They state, "In the end the shape of these functions is a question of fact and can only be settled by empirical evidence." Some empirical data is provided by Altman (1984). Merton and Perold (1993) point out how financial frictions are particularly important to "opaque" financial intermediaries such as banks and insurance companies. They introduce the notion of risk capital as the cost of a guarantee, and analyze it and other components of the economic balance sheet. Frictional costs are seen as "spreads" on the market value of risk capital. Perold (2005) presents a closed-form version of this analysis, where distress costs are proportional to the default put option and tax and agency costs are proportional to the call option represented by distributable surplus. Gründl and Schmeiser (2001) focus on the high sensitivity of insurance policyholders' reservation prices to the threat of default and find that the optimal capital level maximizes the spread between shareholders' and policyholders' reservation prices. Hancock, Huber, and Koch (2001) analyze the components of the economic balance sheet. In particular, they view the bulk of distress cost as being the expected loss of franchise value (excess of market value over book value). Smith, Moran and Walczak (2003) present a dynamic analysis based on the risk of losing franchise value at ruin with immediate adjustment of capital (dividends or recapitalization) after normal (non-ruinous) fluctuations. Estrella (2004), and later, Chandra and Sherris (2005), model distress costs as being proportional to the deficit at ruin and agency costs as proportional to capital; adjustment costs are quadratic in the change in capital. Ostaszewski (2004) also considers costs quadratic in the dividend payout. Exley and Smith (2006) present a recursive equation for franchise value. The frictional cost of holding capital is the tax burden, and the cost of distress is a credit default spread times the default-free franchise value. Krvavych and Sherris (2006) present several alternatives for modeling distress costs, mostly given as proportional to a measure of distress such as terminal assets, terminal surplus, or surplus shortfall with respect to a regulatory boundary. O'Keeffe et. al. (2005) do not present a model per se, but rather report on the extensive variety of modeling efforts – including recent attempts to model frictions – going on in the life insurance industry under the heading of "embedded value." These typically utilize separate valuations of the various components of the economic balance sheet.

This paper focuses on the work of Froot, Scharfstein and Stein (1993), Froot and Stein (1998), and Froot (2003) in the context of insurance companies. In their work, a three-stage model illustrates how costly external capital interacts with other frictions to affect market value and how risk management can serve to increase market value.

Around the same time that Modigliani and Miller (M&M) wrote their seminal papers, de Finetti (1957) considered the question of value-maximizing dividend policy[1]

[1] While it might be more illuminating to think of it as "capital retention policy," the terminology is fixed by history.

as an alternative to actuarial science's then-reigning collective risk theory, whose focus made avoiding insolvency the primary objective of insurance company management. Subsequent papers on optimal dividends models refined the stochastic optimization approach to a high degree of mathematical sophistication. This is discussed further in section II.

Until very recently, there seemed to be a fundamental disconnect between the optimal dividends literature and financial economics' hedging literature. Optimal dividends papers of the past decade routinely invoke M&M, but only to justify the discounted cash flow valuation of a firm. It wasn't until Peura (2003) that the Froot model was acknowledged, and even there, the reference is *pro forma* and not related to the rest of the paper. The obvious question – which M&M assumptions are being violated and how do those violations relate to the optimal dividends model? – has yet to be asked.

This paper demonstrates a way in which the optimal dividends models can be related to the M&M framework. Specifically, it modifies the Froot model so that the investment opportunity, rather than being given exogenously, is *continuation of the going concern*, thereby realizing its franchise value. It is then shown that under the assumption of *no external financing* (infinite cost) this modified Froot model can be construed as a version of the early "classical" optimal dividends models. Recently, optimal dividends models allowing for finite cost of external financing have appeared in the literature. This refinement is discussed as well.

Moreover, this paper demonstrates that numerical methods may be applied to solve relatively sophisticated (and realistic) instances of the model, making this framework not only theoretically relevant to the concerns of value-maximizing insurance firms, but practical as well.

The paper proceeds as follows. Section I presents a version of the Froot model, then extends it to model the market value of the firm as a going concern. The extended model is then seen to be an optimal dividends model. Section II discusses optimal dividends models: their history and their use in modeling the value of risk management, particularly in the context of an insurance firm. Section III presents an example, which is solved numerically for a range of external cost factors. Section IV discusses avenues for further model development and section V concludes. An appendix examines a simplified version of the extended Froot model in detail.

I. A Going-Concern Froot Model

A. The Original Model of Froot et. al.

This section presents a somewhat abstract version of the Froot model. Figure 1 illustrates the situation. At time 0, the firm chooses how much liquid capital ($K \geq 0$) to hold. There is a cost of holding capital, however. Additional funds τK must be paid on the side. K is the firm's initial wealth,[2] W_0. Between time 0 and time 1, business operations result in the change of W by a random amount ΔW. The firm then has the

[2] This is variously known as net worth, capital, surplus, risk reserve, and book value.

option of investing an amount I in an opportunity with a gross return of $M(I)$, therefore a net value of $M(I) - I$.

The quantity I may equal W, or it may be greater than W, in which case the difference must be made up from external borrowing $e = I - W$ with associated borrowing cost $C(e)$. The net (of initial capital) final market value of the firm is therefore $M(I) - e - C(e) - (1+\tau)K$. There are two questions:

1. What is the optimal value of K to maximize the expectation of the net final value of the firm?
2. To what extent is it advantageous to trade off profit expectation for volatility in the operational step?

In this rendering of the model, we have not specified the precise nature of business operations nor how their properties may be altered by management control. Those details varied from paper to paper, but examples had generally involved normal random variables and the selection of hedge ratios.

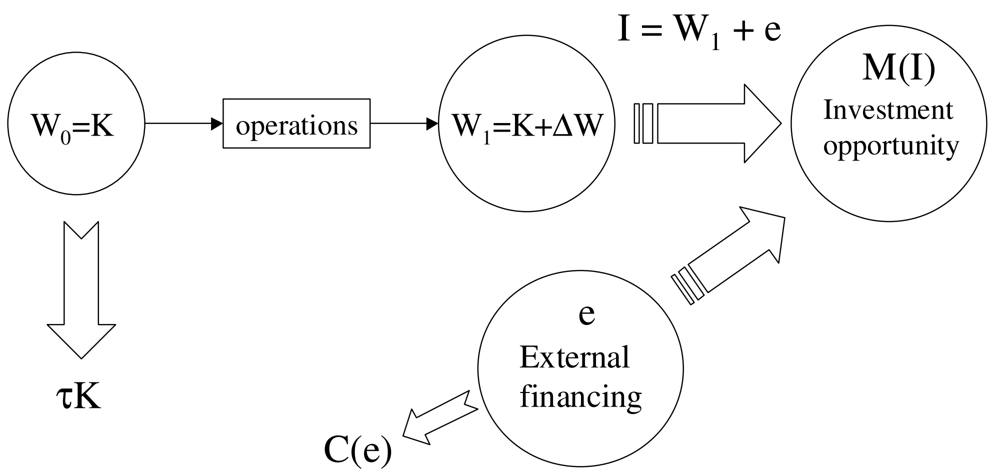

Figure 1: The model of Froot et. al.

Nonetheless, it is possible to draw conclusions when the cost of holding capital, τ, and the cost of borrowing, $C(e)$, are both zero. The net market value of the firm is then

$$M(I) - e - K = M(I) - I + W_1 - K = M(I) - I + K + \Delta W - K = M(I) - I + \Delta W. \qquad (1)$$

Therefore the answer to #1 is: subject to constraints, the choice of initial capital K does not matter – only the choice of I. The expectation of the final market value is given by $M(I) - I + \mathbf{E}[\Delta W]$, so the answer to #2 is: volatility (and therefore risk management) does not matter, either, to a risk-neutral owner.

This is a version of the Modigliani and Miller world, where neither capital structure nor risk management matter to the value of the firm. What is important is making the right investment decision to gain the maximum available NPV: $M(I) – I$.

In the case of costly capital, the analysis is not so simple. First, Froot et. al. assume that the investment gross return $M(I)$ is concave, giving the NPV of $M(I) – I$ a local and global maximum at some specific I. Furthermore, they assume $C(e)$ is convex.

In assuming continuous values for *W* and *e*, and smoothness of *M*(*I*) and *C*(*e*), they are able to derive first-order conditions for optimality and analyze the comparative statics.

B. Extending to a Going Concern

The original Froot, Scharfstein, and Stein (1993) model was formulated with the non-financial firm in mind. There, the assumption that the end-of-period target was a new investment opportunity conformed well to the industrial economics paradigm. In Froot and Stein (1998), the model was refocused on the financial firm, and the example of a new investment opportunity was extension of new loans. In Froot (2003), the focus was further narrowed to insurance and reinsurance firms, where examples of new investment opportunities were the opening of a new line of business or the acquisition of an existing business. For a mature industry like insurance, however, the bulk of shareholder value arguably lies in the operations themselves, and not in new opportunities the operations can finance.

This observation motivates the reconfiguration of the Froot model to a going-concern basis, as shown in figure 2. Rather than a time t+1 investment in some new opportunity with gross return $M(I)$, consider that the firm has the following options at time $t+\Delta t$:

1. It may elect to go out of business, returning all wealth to the shareholders,[3] or
2. It may make an adjustment to wealth (dividend or capital infusion) and continue operations for another period.

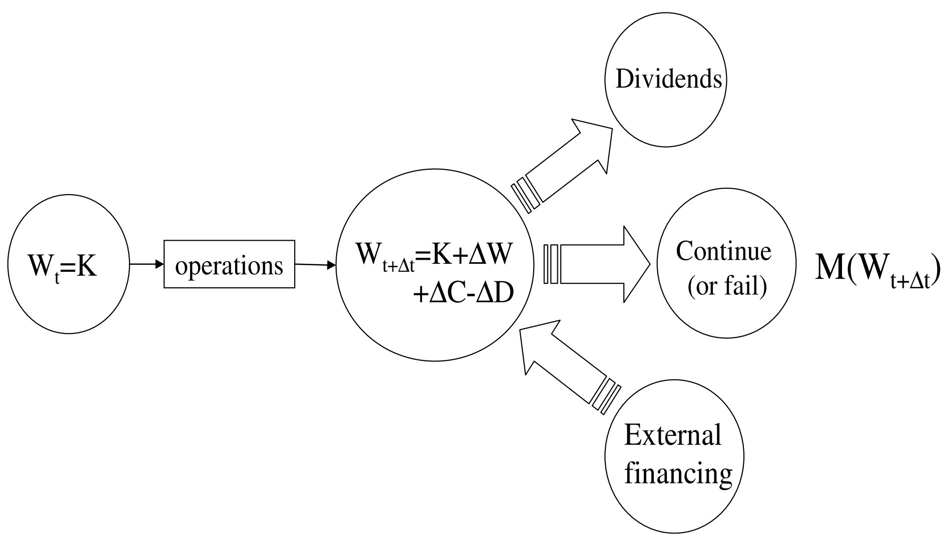

Figure 2: Going-concern version of the Froot model.

If the firm goes out of business, its value at that time is equal to the wealth on hand. If the firm stays in business, its value is the net[4] capital flow to the shareholders

[3] This assumes that all insurance claims are settled each period or that the setting of reserves for incurred-but-not-reported claims can be done without error.

[4] Here, "net" means both in the sense of algebraically netting positives and negatives, and also subtracting costs of raising capital.

plus the ex-dividend market value, $M(I)$, of the firm. Here, $I = W_{t+\Delta t} = W_t+\Delta W+\Delta C-\Delta D$ where C and D represent (nondecreasing) cumulative shareholder capital inflows and outflows (dividends), respectively.[5] With ΔC now playing the role that e played above, we will also assume that the cost of external capital, $C(e)$ above, is linear with factor κ. Thus, investors must spend (1+κ) dollars to increase the wealth of the firm by one dollar.[6]

The market value of the firm, $M(W)$, cannot be deduced without further assumptions. However, following standard valuation logic (see, e.g., Brealey and Myers (1996)), we can express the expected return between times t and $t+\Delta t$ as follows:

$$r\Delta t = E[return] = E\left[\frac{Divs + CapGains}{Value}\right] = E\left[\frac{-(1+\kappa)\Delta C_{t+\Delta t} + \Delta D_{t+\Delta t} + M_{t+\Delta t} - M_t}{M_t}\right] \quad (2)$$

where market values M are ex-dividends, i.e., they are measured just after capital flows occur. If we consider that M_t is known and that $M_{t+\Delta t}$ is a random variable, we may rewrite this as:

$$M_t = \frac{1}{1+r\Delta t} E\left[-(1+\kappa)\Delta C_{t+\Delta t} + \Delta D_{t+\Delta t} + M_{t+\Delta t}\right] \quad (3)$$

We may also treat this as a recursive valuation equation by taking r to be specified exogenously, as the risk-free rate, or by the Capital Asset Pricing Model (CAPM), or by Arbitrage Pricing Theory (APT), etc. Also note that equation (3) may be rewritten in terms of cum-dividends market values:

$$M_t^c = -(1+\kappa)\Delta C_t + \Delta D_t + \frac{1}{1+r\Delta t} E\left[M_{t+\Delta t}^c\right]. \quad (4)$$

Furthermore, we may "unwind" the recursion as follows:

$$M_t = E\left[\sum_{j=1}^{\infty} \frac{\Delta D_{t+j\Delta t} - (1+\kappa)\Delta C_{t+j\Delta t}}{\prod_{i=1}^{j}(1 + r_i \cdot \Delta t)}\right]. \quad (5)$$

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[5] Only equity flows will be considered here; any debt is held fixed. This is reasonable for insurance companies, whose primary financial concerns – solvency, regulation, and ratings – revolve around the sufficiency of book surplus and cannot be addressed by issuing debt. Capital outflows could consist of dividends or stock repurchases; the distinction is immaterial to the mathematics.

[6] This is consistent with investment banking fees and other costs associated with seasoned equity offerings. See Lee et. al. (1996). With this interpretation, however, values of κ greater than 0.5 may be difficult to justify in the model. An interpretation of κ related to asymmetric information will be discussed in section IV.C.

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Equation (5) expresses the market value of the firm as the discounted expectation of future cash flows to and from the shareholders. To maximize the value of the firm, it is management's task to devise and execute dividend and recapitalization strategies that maximize this expectation. Note this is consistent with the Miller and Modigliani (1961) formulation of share valuation.

A few comments are in order as to what this expectation encompasses. Since the ability to pay dividends and the need for external capital both depend on the fortunes of business operations (ΔW), risk management plays a role in the expectation. If the firm goes out of business, capital flows are zero thereafter. Since required market returns vary over time, the r_i are random variables. Moreover, since risk management and dividend policy may affect the beta (covariance between cash flows and market returns) of the firm, the r_i depend on strategy as well. If required returns are assumed constant (as we shall do from this point forward),[7] then the denominator in equation (5) may be replaced by $(1+r\Delta t)^j$.

We may state management's optimization problem more formally as follows. Consider the following stochastic process for wealth W:

$$\Delta W_t \equiv W_{t+\Delta t} - W_t = \mu(W_t, u_t)\cdot \Delta t + \Delta X(W_t, u_t) + \Delta C(u_t) - \Delta D(u_t) \tag{6}$$

where μ (drift) is a deterministic function, $u_t = U(W_{t^-})$ is a (possibly vector-valued) control parameter, and X (volatility) is a stochastic process whose probability law may depend functionally on its two arguments but is otherwise stationary (not explicitly dependent on time) with independent increments.[8] Let the instantaneous valuation rate r be a given constant. The problem is to determine the value of the firm

$$M(w) = \max_U E\left[\sum_{j=1}^{\infty} e^{-r\cdot j\cdot\Delta t}\cdot \Delta D_{j\cdot\Delta t} - (1+\kappa)\cdot \sum_{j=1}^{\infty} e^{-r\cdot j\cdot\Delta t}\cdot \Delta C_{j\cdot\Delta t} \,\middle|\, W_0 = w\right] \tag{7}$$

and the associated optimal strategy $U(W)$. An optimization problem of this form is known as an *optimal dividends* problem. We will refer to the market function $M(w)$ as the *M-curve*.

A few observations can be made. First, due to stationarity of X and constancy of r, the M-curve is independent of time. Second, the M-curve respects the distinction between cum-dividends and ex-dividends market values. For example, if the optimal strategy at $W=w$ is the release of a dividend ΔD, then we have $M(w) = \Delta D+M(w-\Delta D)$. Conversely, if the optimal strategy is to acquire capital ΔC, then $M(w) = M(w+\Delta C) - (1+\kappa)\Delta C$. Third, note there is no explicit provision for a cost of holding capital; this may be assumed as part of the specification of μ if desired. Because of discounting, there is a natural penalty for holding excess capital if it earns less than the required rate of return.

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[7] In section IV.A, the possibility of variable beta is discussed.

[8] Note that there is a redundant degree of freedom here because the behavior of μ could be absorbed in the specification of X. However, it is convenient to have both terms appear.

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II. Optimal Dividends Problems

A. History

The earliest version of the optimal dividends problem appears to be the binomial lattice model of de Finetti (1957). The drift term was zero and the volatility term consisted of a random $\Delta W = \pm 1$ with probability p or 1-p, respectively. Unit dividends were allowed, but there was no recapitalization. (A modified version of de Finetti's model, featuring a restriction to $W \leq 2$ but with a provision for recapitalization, is analyzed in the appendix.) This was presented as an alternative formulation[9] of the reigning paradigm of collective risk theory, which focused on the probability of ruin as the objective function. It was not so much an "optimal dividends" model as an "optimal capital buffer" model.[10]

In the 1960s, the discrete form of the optimal dividends problem gave way to the continuous version, with the random accumulation X_t specified as a *Brownian motion* B_t. Equation (6) was rendered as the following *stochastic differential equation* (SDE):

$$dW_t = \mu \cdot dt + \sigma \cdot dB_t + dC_t - dD_t \tag{8}$$

The continous-time equivalent of equation (7) is:

$$M(w) = E\left[\int_0^\infty e^{-rt} dD(W_t) - (1+\kappa) \cdot \int_0^\infty e^{-rt} dC(W_t) \,\middle|\, W_0 = w \right] \tag{9}$$

where $D(W)$ and $C(W)$ are written as functions of W to emphasize that they are strategies dependent on the current state of W, and not of time per se. Brownian motion versions of the model studied in the 20th century considered only dividends, not capital inflows; this corresponds to the Froot model when external capital is prohibitively costly.

Gerber and Shiu (2004) present a lucid exposition of the solution logic. They show that the optimal strategy is to distribute immediately all wealth W-β above a "barrier" point β whose value depends on μ and σ. When W is less than β, no dividends are distributed. The barrier point can be interpreted as the optimal level of capital for the firm: if wealth is above that, excess amounts should be returned to the shareholder immediately;[11] below that, all profits should be retained until the optimal level is reached.[12] The dividend barrier is the counterpart of the optimal initial capital K in the model of Froot et. al.

[9] "impostazione alternativa"

[10] The "optimal dividends" problem is sometimes referred to as the "cash management" problem, especially in the banking literature.

[11] Such a strategy might be implemented through stock repurchase.

[12] Since this strategy is counter to the typical one of paying a steady stream of dividends, it is reasonable to consider an alternative model with an exogenously given constraint on the minimum dividend payment per unit time. See Højgaard & Taksar (1999) and Asmussen et. al. (2000) for examples.

Bather (1969) took the step of adding risk transfer as a second form of risk management, introducing no-load quota-share (proportional) reinsurance as another decision variable. The SDE becomes:

$$dW_t = U(W_t)\cdot\mu\cdot dt + U(W_t)\cdot\sigma\cdot dB_t - dD_t \tag{10}$$

where $0 \le U(w) \le 1$ is the fraction of the risk retained by the firm. Notice how both the expected gain μ and the risk σ are modulated by the same factor. In addition to determining dividend strategy, the new part of the problem is to determine the optimal risk transfer strategy, represented by the function $U(W)$.

The optimal dividend strategy is essentially the same as before, with a slight downward shift in the location of the barrier β compared to when reinsurance is not available. The optimal reinsurance strategy involves a second barrier, ρ, above which all risk is retained ($U = 1$). For $W < \rho$, $U(W)$ is linear in W down to $U(0) = 0$. Being able to cede risk ($U<1$) is seen to increase the market value of the firm, M. As in the original Froot model, by comparing the two M curves (when the best strategy for $U(W)$ is used versus when $U = 1$ is enforced), the amount by which (this form of) reinsurance adds market value to the firm can be computed.

From the late 1990s this field of study grew rapidly, as more complicated situations were addressed. For more on the history of the models, see Gerber and Shiu (2004) and Major (2006, 2007).

B. Optimal Capital Inflows

It took until Sethi and Taksar (2002) for capital inflows (dC) to be addressed in this model; their formulation will be discussed below. Løkka and Zervos (2005) solved the constant-coefficient version of the continuous dividend and external capital model (with no reinsurance), i.e., equation (8). The optimal dividend strategy is essentially the same, an upper barrier β^+ above which all revenues should be dividended immediately. If μ is high enough, there is also a lower barrier β_- at zero[13] where just enough capital needs to be added to prevent bankruptcy.

In Sethi and Taksar (2002), the optimal dividend and recapitalization strategies (again, with no reinsurance) were derived for increasing functions $\mu(W)$ and $\sigma(W)$ satisfying certain properties guaranteeing that uncontrolled W would never reach zero nor grow infinitely. In particular, μ is concave with $\mu'(0) > r$ and eventually $\mu'(W) < r$. The optimal recapitalization strategy[14] again is a lower barrier β_- below which any W deficits should be made up immediately. This barrier is the point at which $M'(\beta_-) = (1+\kappa)$. Below that point, the market value function is linear with slope $(1+\kappa)$. The upper barrier β^+ occurs where $M'(\beta^+) = 1$ and the market value function is linear with slope 1 above that.

[13] They had to define bankruptcy as $W < 0$, rather than $W \le 0$, in order to achieve a unique solution; otherwise, there is a continuum of increasingly better strategies recapitalizing at $W = \varepsilon$ as $\varepsilon \to 0$, but no single best strategy.

[14] Their solution is really a meta-solution, as it requires solving a free boundary second order ordinary differential equation involving the particular functions for μ and σ.

In both the Løkka and Zervos and Sethi and Taksar models, if $\kappa = 0$, then the Modigliani and Miller world is obtained: the *M* curve is a straight line with slope 1 for all positive *W*. In Sethi and Taksar, the two barriers meet at the point where $\mu'(W) = r$. This has the classical interpretation that investors should fund all projects with returns greater than the cost of capital, but any funds beyond that should be dividended back to shareholders.[15] With *M* being linear in wealth, volatility of *W* (risk management) has no market value and becomes irrelevant.

Those papers, however, do not explain at length how their models relate to M&M violations; the ones discussed below do.

Peura (2003) addresses the optimal dividends model with the possibility of recapitalization. It discusses the Modigliani and Miller irrelevance theorems and the literature on their violations, including an explicit discussion of the Froot et. al. costly capital model. However, it does not relate the Froot model to the optimal dividends model.

Blazenko et. al. (2004) model insurance firms as being "regulated" in the sense that if $W < 0$, then shareholders must either add (frictionless) capital at a rate *k* per unit time, or abandon the business; hence they distinguish economic ruin (abandonment) from technical ruin (financial distress).[16] They derive the optimal abandonment barrier and the value of the abandonment option. In the limit as $k \to \infty$, corresponding to a requirement of instantaneous makeup of the capital deficit, the usual M&M linear equation for $M(W)$ is obtained and the value of the abandonment option goes to zero.

Rochet and Villeneuve (2004) analyze distinct and simultaneous possibilities for "hedging" (against Brownian motion) and "insuring" (against a Poisson risk with constant severity) along with two forms of costly external financing. They write:

> [W]hen liquidity management and risk management decisions are endogenized simultaneously, the theoretical impact of profitability and leverage is non monotonic…. Moreover when insurance decisions are explicitly modeled, we find that the optimal patterns of hedging and insurance decisions by firms are exactly opposite: cash poor firms should hedge but not insure, whereas the opposite is true for cash rich firms…. This may explain the mixed findings of empirical studies on corporate demand for hedging and insurance….

C. Solution of Jump-Diffusion Models

A more general, and from the perspective of modeling insurance firms, realistic, formulation of the problem is as follows. The SDE governing the evolution of wealth is given by:

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[15] Excess funds *could* be retained as long as they earned the required rate, making dividend policy irrelevant in the sense of M&M. That would correspond to $\mu' = r$ over a range of *W*, consistent with Sethi and Taksar.

[16] Without loss of generality, the capital constraint defining distress can be placed anywhere, e.g., at some barrier $\beta > 0$, which is more realistic in terms of how insurance regulation and ratings are conducted. Cf. the "ratings cliff" example of section III below.

$$dW_t = \mu(W_t, u_t) \cdot dt + dX(W_t, u_t) + dC(u_t) - dD(u_t) \quad (11)$$

where $u_t = U(W_{t^-})$ is a (possibly vector-valued) control parameter, X is a jump-diffusion stochastic process,[17] and both the drift and diffusion terms may be affected by both the state of W and the state of the controls u. Controls may now include the effects of excess of loss (XOL) reinsurance[18] on the jump portion of the risk. The objective is the same as before, to maximize M in equation (9).

While there is as yet no analytical solution to the model at this level of generality, important special cases have been solved, as noted above. In general, a dynamic programming strategy can be applied for numerical solution. This involves the so-called *optimality equation*, also known as *Bellman's equation* (Bellman (1954)):

$$M(w) = \max_U \left\{ -(1+\kappa) \cdot dC + dD + e^{-r\,dt} \cdot E\left[M(W_{t+dt}) \mid W_t = w \right] \right\}. \quad (12)$$

This equation must hold if M is the solution to the optimal control problem. Intuitively, it says that the value of the firm at time t is equal to the net of capital to be raised or dividends about to be given back to the shareholders at the beginning of the next infinitesimal period of time plus the discounted expected value of the firm at the end of that time, given that optimal control is always exercised. Note the similarity between this equation and the recursive valuation equation (4).

Numerical solution of the Bellman equation is addressed in Kushner and Dupuis (2001). For analytical solutions, the preferred technique is to solve its first-derivative counterpart, the Hamilton-Jacobi-Bellman equation (see, e.g., Yong and Zhou (1999), Øksendal and Sulem (2005)).

D. Representing Frictions

The optimal dividends jump-diffusion models allow for a range of stochastic risk processes relevant to insurance firms, enabling a realistic depiction of catastrophe (jump) risk and excess-of-loss reinsurance. This is not surprising, as they emerge from a fifty-year collaboration between the actuarial and stochastic process communities. In addition, however, these models can accommodate the representation of many of the most often cited financial frictions:

- Bankruptcy cost manifests itself in the restriction that ΔD and ΔC remain zero after W hits zero and in constraining $M(0) = 0$. Note this boundary could be set to some value other than zero, positive or negative.
- Sensitivity of customer demand to the risk profile – in particular, the effect of customer risk aversion – is a key addition to the early Froot model frictions, and plays a central role in Froot (2003). Customer risk aversion can be represented in the model through the specification of the probability law governing ΔW, either

[17] These have Brownian motion and compound Poisson processes as special cases.

[18] This is discussed in more detail in the example of section III.

through μ or X. Other non-bankruptcy distress costs (worsening terms of credit, employee turnover, etc.) can be represented the same way.

- Taxes and investment expenses at the firm level appear in the specification of μ or X (if profits are modeled as being after-tax). Taxes at the investor level, and in particular, the differential rate between dividend and capital gain taxation, are not supported in the model forms presented above, but may be amenable to a simple modification of the Bellman equation. Tax asymmetries (e.g. loss carryforwards versus immediate taxation of profits) cannot easily be represented due to the continuous-time and Markov nature of the model.
- Cost of external capital is represented above through the κ parameter. While the model does not contemplate recapitalization through debt issue,[19] sensitivity of the cost of servicing existing debt with respect to changing levels of book value can be represented in the specification of μ.
- Cost of holding capital, as mentioned before, has an implicit presence in the model due to discounting, but cost can be made explicit in the specification of μ.
- Regulatory capital costs are typically incurred as a result of holding "excess" capital (relative to "optimal" levels) or restrictions on investments. Capital quantity restrictions directly affect the definition of distress (see, e.g., the example in section III) and indirectly affect the cost of holding capital by affecting the quantity of capital held. Restrictions on investments affect the specification of μ and X.
- Agency effects, e.g., the risk aversion of managers, are only supported insofar as they can be represented as effects on μ or X. The potential for modeling the frictional effect of asymmetric information on equity issue, à la Myers and Majluf (1984), is discussed in section IV.

III. Ratings Cliff Example

A. No Recapitalization

A key innovation in Froot's (2003) model is the representation of customer risk aversion by having the expected profit depend on current wealth. This section presents the numerical solution of a jump-diffusion model where it does as well, and also illustrates how to handle constant background growth by changing to a growth-adjusted numéraire. Consider the following example:

> The economy is currently undergoing 4% inflation with a 5% required rate of return. The firm has book value (wealth, capital) of $9bn and expected one-year inflation-adjusted growth in book value of $1bn if it maintains its rating and experiences no natural catastrophes. Catastrophes occur at an average rate of 0.5 per year (Poisson distributed) and the magnitude of the loss is distributed as an exponential with mean $1bn. Catastrophes are assumed uncorrelated with the stock market.

[19] As mentioned earlier, issuing debt does not address adequacy of book equity.

Management estimates that if book value were to go below \$5bn, it would experience a ratings downgrade that would cause it to experience real per annum growth in book value of only \$250mm (less any catastrophe losses). Therefore, expected profits are \$500mm above the ratings boundary and -\$250mm (i.e., a loss) below the ratings boundary. Raising external capital, above or below the ratings boundary, is out of the question.

The firm has an opportunity to cede a portion of its catastrophe losses to a reinsurer by way of an *excess of loss* ("XOL") contract. If losses in a catastrophe exceed \$3bn, the reinsurer will reimburse a fraction U of loss amounts above that up to a maximum reimbursement of \$1bn times U. The firm may choose U between zero and one. The premium for the cover is \$70mm times U per annum.[20] Note that given the assumptions about the occurrence of catastrophes, the actuarial fair value of the coverage is only \$16mm times U per annum.

Consider the following questions:

1. What is the optimal level of capital; does the firm need more or is it overcapitalized?
2. Should it buy the XOL coverage, and if so, how much should it buy?
3. How does the market value of the firm respond to changes in its capital?
4. Does the availability of the XOL contract bring value to the firm, and if so, how much?

We model the situation as follows. The governing state equation is (11) with μ representing the catastrophe-free rate of profits and X (taking on negative values) representing the cumulative catastrophe losses net of reinsurance cover (positive contribution) and reinsurance premiums (negative contribution). The control variable u represents the fraction of coverage purchased.

Specifically, we may write:

$$\mu(W) = 10^9 \cdot \begin{cases} 1, & W \geq 5 \\ 0.25, & W < 5 \end{cases}$$

$$dX(u) = 10^9 \cdot \begin{pmatrix} J \cdot (S - \max(0, \min(u \cdot (S-3), u))) \\ -0.07 \cdot u \cdot dt \end{pmatrix}$$

$$J \sim Poisson(0.5 \cdot dt)$$

$$S \sim Exponential(1)$$

---

[20] In reality, catastrophe XOL cover demands *reinstatement premiums* after the first loss in a year. Here we simplify by assuming that the one premium, paid at a continuous rate through the year, suffices to cover as many catastrophes as might occur.

Note that μ is functionally dependent on *W* but not *u*, while the probability law for *X* is functionally dependent on *u* but not *W*. Since external capital is not to be used, we may consider the cost factor κ to be a very large number. The discount factor is the difference between the required return and the inflation rate: $r = 0.05\text{-}0.04 = 0.01$.

By solving the Bellman equation numerically, we discover that the resulting optimal dividend strategy consists of two zones where dividends should be paid. The "profit" zone extends above the optimal capital level of $W = 15.4$ ($bn), where all excess capital should be immediately distributed back to the shareholder. The other zone extends below another critical capital level of $W = 2.3$, where all capital should also be immediately distributed back to the shareholder, leaving the firm with zero, and, therefore, going out of business. In between those thresholds, dividends should not be paid, but rather the firm should retain earnings with the goal of increasing capital to the optimal level.

The answer to question 1, then, is that the firm, currently at $9bn, is undercapitalized. The optimal level is just over $15bn – in real terms. Next year, the target would be 4% higher in nominal dollars. Capital above this level should be dividended back, but for now, profits should be retained.

Figure 3 shows the optimal risk transfer strategy. There is a zone $8 < W < 10.4$ in which risk transfer should be conducted. Between book value of $8mm and $9.2mm, the firm should contract to buy 93% coverage. For higher book values, up to $10.4bn, progressively less cover should be purchased. Outside of this range, no cover should be purchased. We may rationalize this as follows: When the firm has relatively large amounts of capital, it can afford to bear a cat loss and the XOL cover is less valuable to it. On the other hand, for book value below $8bn, the coverage leaves an important gap – losses just short of the $3bn "retention," which would not be reimbursed, would push the firm over the ratings cliff. For that reason, the reinsurance is much less valuable to the firm and therefore its purchase is not indicated. Note that in *W* ranges where dividending takes place, the value of *U* is technically undefined, because the firm will immediately move to another state *W'*, making reinsurance inapplicable at *W*.

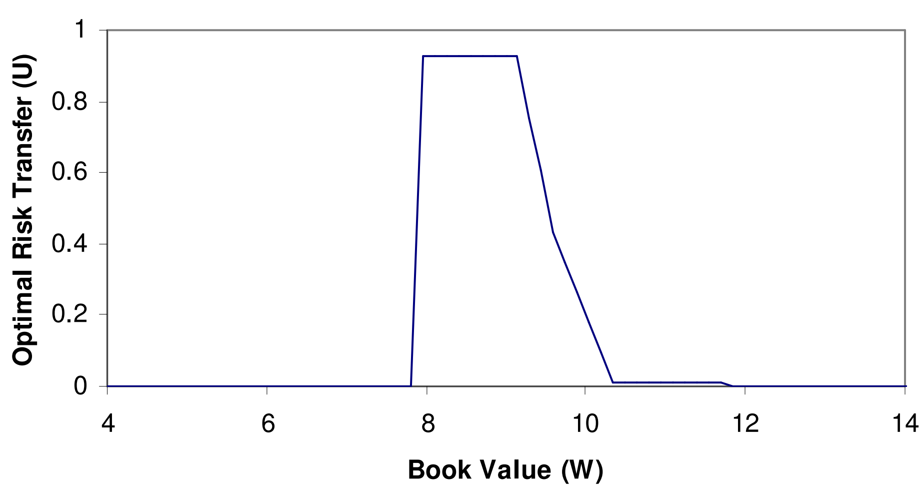

Figure 3: Optimal risk transfer strategy.

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The answer to question 2, then, is that the firm, being at W=9, should purchase 93% coverage.

Figure 4 shows the baseline *M*-curve, representing the market value without the availability of reinsurance. At this scale of resolution, there is no visible difference between the with- and without-reinsurance curves. For *W* greater than about 3, the curve rises steeply, then levels off to a 1-for-1 slope above the high dividend barrier of *W* = 15.4. For *W* between the "go out of business" barrier (2.3) and the ratings cliff (5), the *M*-curve is convex, representing increasing marginal value of capital additions as *W* approaches (from the left) a ratings upgrade.

The answer to question 3, then, is that around the current $9bn in capital, every new dollar in capital increases firm value by about $1.72. For lower levels of capital, the rate of market value change accelerates until it reaches a ratio of over 23:1 at the ratings cliff.

Computing the difference in M-curves answers question 4. The difference in M-curves is shown in figure 5. The risk-transfer strategy is overlaid for reference. Availability of risk transfer adds $75mm to the firm currently, but would be worth as much as $116mm if the firm were at $8bn book value. Note that the mere *availability* of reinsurance brings value, even when the optimal strategy is not to use it.

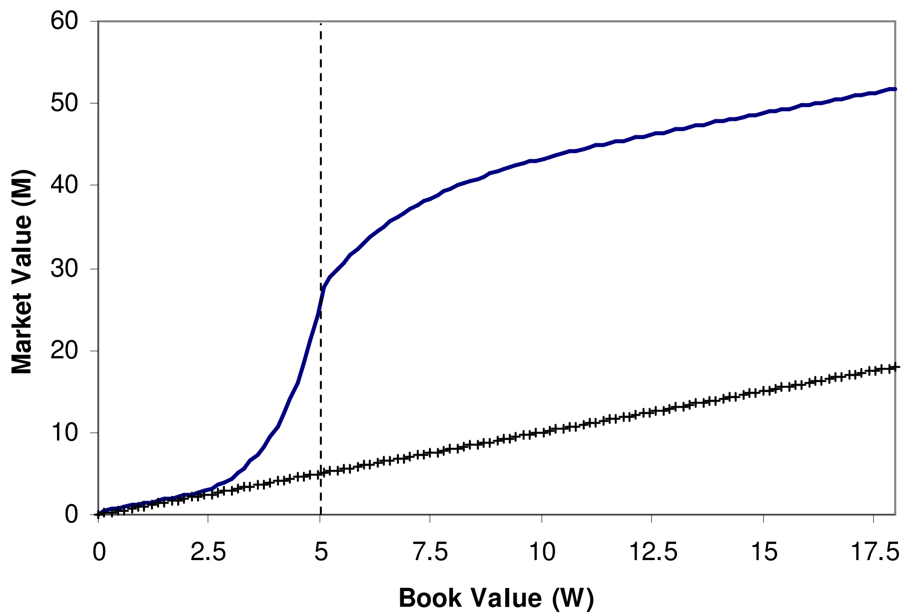

Figure 4: Market value of the firm.

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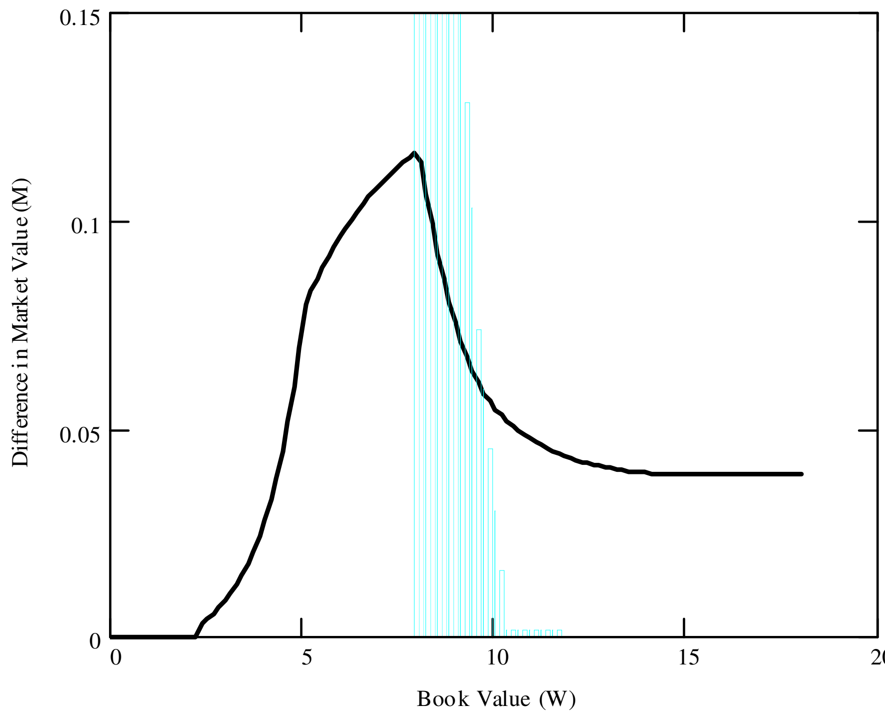


Figure 5: Value of risk transfer.

B. With Recapitalization

Preventing recapitalization is equivalent to assuming an arbitrarily large value for the external cost parameter κ. Because the maximum slope of the M-curve is around 23, any value of κ greater than 22 (=23-1) will prevent recapitalization from being feasible for any *W* value. However, as the cost parameter decreases, recapitalization becomes increasingly more applicable. Figure 6 shows the *M*-curves for seven values of κ. The dotted line identifies the locus of $M = W$. The solid curves, from highest to lowest, represent *M*-curves corresponding to κ values of 0.1, 0.7, 1.5, 3, 6, 9, and 30 (equivalent to anything over 22), respectively.

As κ approaches zero, the *M*-curve straightens out as shown, the β_{-} threshold for recapitalization increases and the β_{+} threshold for dividends decreases. Eventually, reinsurance is not valuable. In the limit, for costless external capital, the *M*-curve is a straight line[21] and $\beta_{-} = \beta_{+}$ around 7.5. A comprehensive view of the effect of different κ values on optimal strategies is given in figure 7.

[21] Note the parallels with the Sethi & Taksar (2002) and Blazenko et. al. (2004) analyses discussed in section II.B.

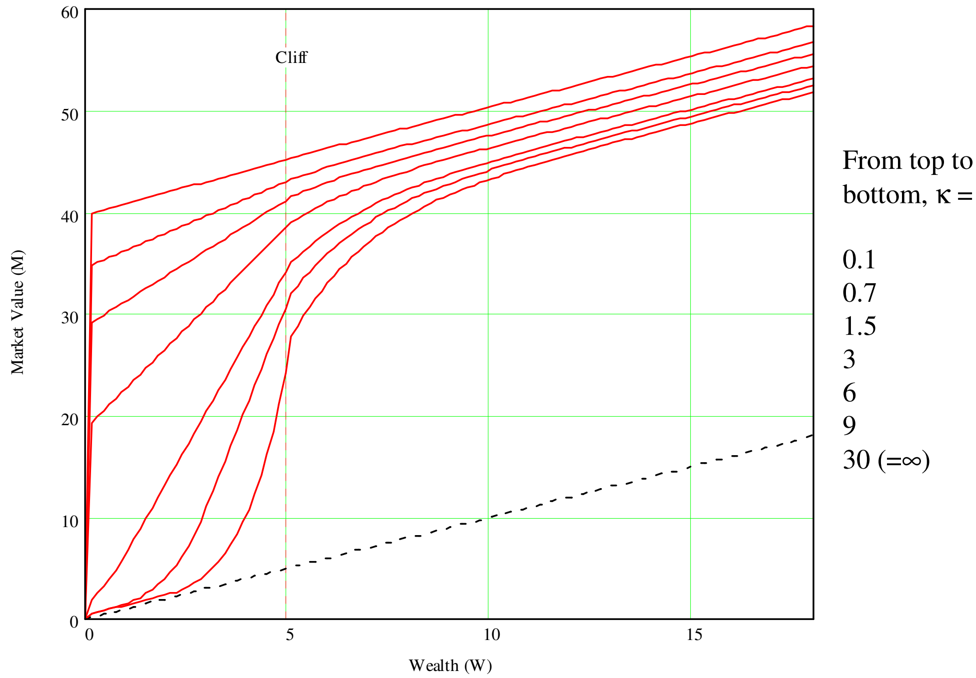
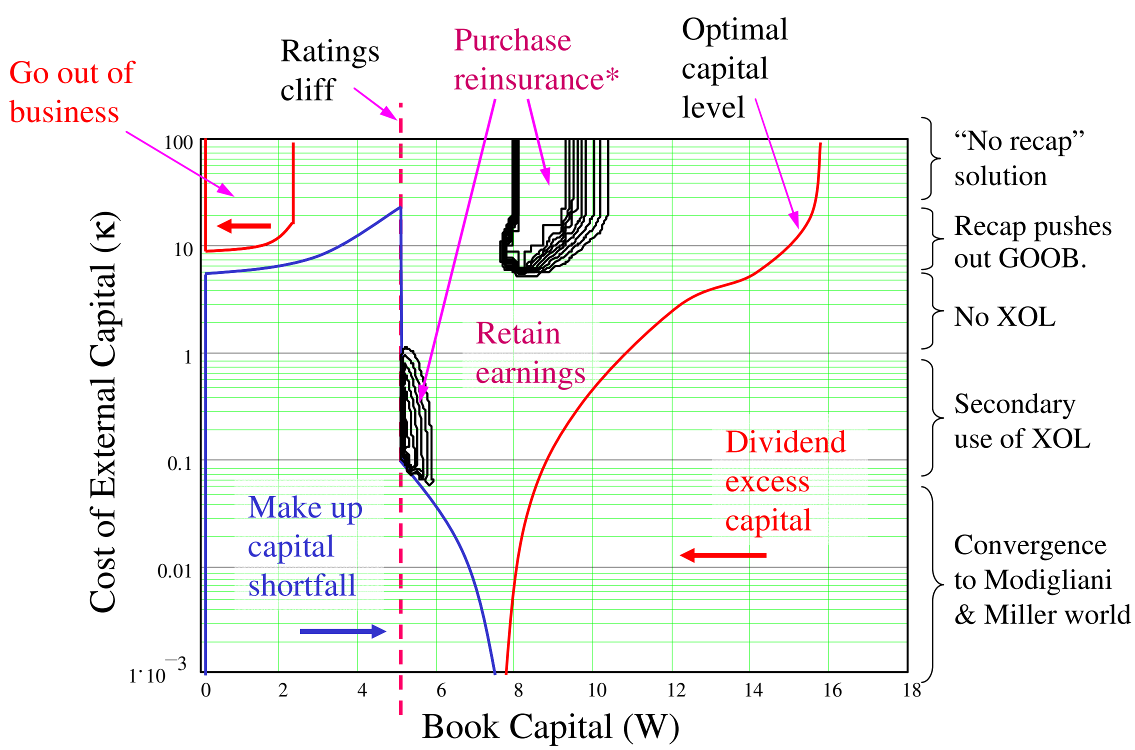


Figure 6: Market value of the firm as cost of external capital varies.



* Reinsurance purchase contours start at 0.1 on outside and increase by 0.1 at each inward step.

Figure 7: Optimal strategies as functions of the cost of external capital.

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Figure 7 presents *W* on the horizontal axis and κ on the (log-scale) vertical axis. The *W*=5 "ratings cliff" is identified by a vertical dashed line. Various zones are delineated inside the rectangle, indicating where capital flow strategies apply. Contours, indicating optimal reinsurance purchase regions, are also present.

The upper portion of figure 7 corresponds to high-κ values, where external capital is effectively unavailable. Proceeding from left to right, i.e., from low to high values of *W*, we can retrace some of the conclusions presented in the discussion of part III.A. For values of *W* less than 2.3, the optimal strategy is to go out of business. Between 2.3 and 8, the firm remains in business but does not buy reinsurance. Between 8 and 10.4, various levels of reinsurance cover are indicated. Between 10.4 and 15.4, no reinsurance is purchased. Above 15.4, surplus is given back to shareholders in order to maintain the optimal capital level at 15.4.

These conclusions are the same for all κ values above 22. As slightly lower values are contemplated, there are several changes. The use of recapitalization emerges for *W* values just below the cliff. The lower and upper dividend barriers shift downwards slightly. The zone for purchasing reinsurance also shifts downward. Note that purchasing reinsurance for *W* slightly below 8 now makes sense because falling just below the cliff (*W*=5) can be rectified by recapitalization.

As κ values decrease from 10 down to 6, the zone of recapitalization extends downward, eliminating the go-out-of-business zone, and the reinsurance purchase region shrinks and finally disappears. External capital has become cheap enough that reinsurance protection against falling below the cliff is no longer economical.

As κ values fall from 6 to 1, this status quo is maintained, with the only change being the upper dividend barrier (optimal capital level) falling rapidly.

At κ=1, a new reinsurance strategy emerges. Reinsurance purchase is indicated at *W* values just above the ratings cliff. This cover does not protect against falling below the cliff – it protects against bankruptcy. Recapitalization is not available after bankruptcy in this model. However, the franchise value of the firm has risen enough[22] to make it economical to purchase reinsurance for that purpose.

As κ falls from 1 to 0.1, this use of reinsurance continues and the upper dividend barrier continues to fall. However, below 0.1, capital edges out reinsurance yet again. The optimal recapitalization level increases, above the cliff. Now, rather than purchasing reinsurance to protect against bankruptcy, the firm simply makes sure to hold enough capital to withstand losses of a certain size.

As κ converges to zero, the Modigliani-Miller world obtains. The optimal capital level is a single point, where earnings that cause *W* to move above or below it are adjusted via capital flows to or from shareholders.

IV. Further Directions

This section presents three directions for further research and development.

[22] See figure 6. Shareholder value at the cliff has increased from $24bn to well over $40bn, approximately doubling franchise value.

A. Endogenous Beta

The general formulation of the optimal dividends model presented above assumed a constant valuation rate *r*, whether that be the risk-free rate or a market rate. A more realistic model would endogenize the market beta. If the average market return r_m and risk free rate r_f are assumed constant, then the Bellman equation (12) defining the *M*-curve needs only a single additional term. This is shown as follows.

Let the value ν of the market portfolio evolve according to a geometric Brownian motion:

$$\frac{d\nu_t}{\nu_t} = \mu_m \cdot dt + \sigma_m \cdot dB_t^m \tag{13}$$

Let the state equation for wealth, replacing equation (11), be:

$$\begin{aligned} dW_t &= \mu(W_t, u_t) \cdot dt + dJ(W_t, u_t) + dC(u_t) - dD(u_t) \\ &\quad + \sigma(W_t, u_t) \cdot \left\{ \rho(W_t, u_t) \cdot dB_t^m + \sqrt{1 - \rho(W_t, u_t)^2} \cdot dB_t^n \right\} \end{aligned} \tag{14}$$

where *J* is a cumulative jump process, B^n is a Brownian motion, and both are independent of B^m. The instantaneous correlation between the change in wealth and market returns is therefore $\rho(W,u)$. Let the required expected return on the firm, r_β, be given by:

$$r_\beta \cdot dt \equiv \left[r_f + \beta \cdot (r_m - r_f) \right] \cdot dt \equiv \left[r_f + \frac{\operatorname{cov}\left(\frac{\Delta + dM}{M}, \frac{d\nu}{\nu} \right)}{\sigma_m^2 \cdot dt} \cdot (r_m - r_f) \right] \cdot dt \tag{15}$$

where Δ represents the net capital flows, and capital gain *dM* is measured ex-dividends.

A derivation similar to that outlined in equations (2) to (4), yields the following version of the Bellman equation:

$$M(w) = \max_U \left\{ -(1+\kappa) \cdot dC + dD + e^{-r\,dt} \cdot E\left[M(W_{t+dt}) \mid W_t = w \right] - \left. \frac{\partial M}{\partial W} \right|_{W_t = w} \cdot \gamma(w) \cdot dt \right\} \tag{16}$$

where

$$\gamma(w) \equiv \rho(w, u(w)) \cdot \sigma(w, u(w)) \cdot \frac{r_m - r_f}{\sigma_m}. \tag{17}$$

B. Nonstationary Profitability

The optimal dividends models presented above are thoroughly stationary. As constructed, these models are not entirely consistent with certain realities facing the insurance industry. While growth can be accommodated by reference to a growing background numéraire (e.g., the inflation adjustment used in the ratings cliff example of section III), treatment of economic cycles, in particular the insurance "underwriting cycle," requires the introduction of another state variable.

We can model[23] the competitive price environment as a one-dimensional stochastic process π_t, which also serves as additional parameter in the state equation for wealth:

$$dW_t = \mu(W_t, \pi_t, u_t) \cdot dt + dX(W_t, \pi_t, u_t) + dC(u_t) - dD(u_t) \tag{18}$$

Now (W_t, π_t) jointly define a two-dimensional state variable for the *M*-curve (now an "*M*-surface"), $M(w,\pi)$. Expectations within the Bellman equation are taken conditional on both current wealth and the current state of the competitive environment, and contemplate changes in both. While conceptually simple, the introduction of another dimension in the state variable significantly increases the numerical computational effort.

C. Asymmetric Information

Myers and Majluf (1984) present a model that illustrates how asymmetric information between investors and firm management can lead to situations where a seasoned equity offering, while advantageous to the market value of the firm, will not be undertaken because the effect of dilution of ownership would make the transaction disadvantageous to existing shareholders. A similar feature can be incorporated into the optimal dividends analysis.

As in Myers and Majluf, while no closed-form solution is available, numerical solutions can be obtained. Given the investors' possibly erroneous beliefs about value *M* and slope M', and assumptions about the distributions from which these beliefs are drawn, one can determine the "threshold" capital gain rate M'_{crit}, and therefore whether the issue will be announced (and succeed). Furthermore, before these beliefs are known, one can compute the probability that issue will succeed by integrating over their joint probability density.

The correct application of the above logic must sidestep a potential paradox. In Myers and Majluf (as in Froot et. al.), the investment opportunity $M(I)$ is given exogenously, so a comparison of slope with theshold is straightforward. In the present model, the *M*-curve $M(W)$ is endogenous. The slope of the *M*-curve at a given $W = w$ is determined, in part, by the potential success or failure of recapitalization not only at w but at nearby points as well. In order to have a well-defined *M*-curve, we must interpret $M(w)$ as the value of the firm *to existing shareholders*. The capital gain demanded by new investors is not simply a hurdle, but a true cost to existing shareholders, making it economically sensible to contemplate values of κ substantially greater than the 5% to 30% typically attributed to underwriting fees. A management goal of maximizing the value of the firm to existing (passive) shareholders is consistent with empirical results cited in Myers and Majluf.

This analysis can be incorporated into an optimal dividends model in a number of ways, with the choice of alternatives being determined by one's view of the stochastic

[23] See Fung et. al. (1998) for an overview of models of the underwriting cycle.

process that generates investor valuation errors. If investor mispricing is regarded as persistent over time, the analysis may be ex-post or ex-ante. Ex post, when investor estimates are known, the induced "effective cost of external capital" will define the *M*-curve. Ex ante, mispricing and the true *M*-curve are unknown or undefined, so we only have a probability distribution on possible *M*-curves. Firm value, given that state of knowledge, is the average *M*-curve. If, on the other hand, mispricing is regarded as highly dynamic, then it must be incorporated into the state variable in much the same way that the competitive price environment was treated in the previous section.

V. Conclusion

This paper connects two as-yet disparate bodies literature: (1) hedging models within financial economics, articulating the failure modes of the Modigliani and Miller theorems, as exemplified by the models of Froot et. al., and (2) optimal dividends models within actuarial science and stochastic process theory, articulating a particular vision of the objective of risk management, as introduced by de Finetti and analyzed by scores of researchers over the past half century. It showed how the Froot model can be reconfigured to represent the market value of a going concern, and, in doing so, how it becomes an optimal dividends model.

Most of the optimal dividends models appearing in the 20th century literature do not provide for raising outside capital; this absence corresponds to prohibitively costly external capital in the Froot model. Versions of the model that do include external capital exhibit economically sensible behavior, and, in the limit where outside capital is frictionless, reproduce the Modigliani and Miller irrelevancy of risk management and dividend policy.

The paper also presented the jump-diffusion version of the model as a general form particularly well suited to modeling insurance firms and the impact of risk transfer, especially reinsurance, on market value. This suitability stems from the model's emphasis on book equity (usually referred to as "capital and surplus" in the insurance industry) as the measure of financial slack, as well as its roots in traditional risk-theoretic stochastic process formulations, thereby permitting "industrial strength" modeling of arbitrary risk distributions. Unlike static or "single point" models of firm value, the optimal dividends model is truly dynamic in that it – of necessity – solves for firm value and optimal management control over the entire possible range of the wealth variable. While analytical solutions to the more realistic cases are as yet unavailable, numerical solution has been found to be reasonably efficient.[24]

The approach is not a panacea, however; there are still many parameters to be fit or assumed, still difficult questions of what really matters and what can safely be ignored.

[24] The computations required for section III were completed in tens of minutes on a 1GHz pentium machine.

Appendix: A Discrete Going-Concern Froot Model

In this appendix, we follow de Finetti (1957) by presenting a simple binomial lattice model of the valuation of a going concern. The example is restrictive in that it is constrained to integer $W \leq 2$, but it extends de Finetti's discussion by considering external recapitalization.

A. Basic Model

Consider the following version of the problem:

> You have a magic coin box that can hold $W = 0$, 1, or 2 one-dollar coins. Every fixed time interval, one coin randomly appears (with probability p) or disappears (with probability 1-p). If the box already holds two coins and a third one appears, the new coin is ejected and you keep it. But once the box runs out of coins, it vanishes in a puff of smoke.
>
> Say also that these random transitions are uncorrelated with any financial markets and the risk-free rate is r. Currently, the box has two coins in it and will transition at the end of the next time interval. What is the fair market value[25] of this device?

We can organize our thinking about state transitions by drawing a lattice, as in figure A.1. Notice the similarities and differences between this and the Froot model. At a given point in time, we are at a particular W value, so do not have the luxury of "choosing" K; nonetheless, we can ask questions about the relative market value of starting at various levels of W. While there is no explicit cost of carrying capital, W is not earning interest, so in effect, there is an opportunity cost.[26] After the ΔW outcome, we cannot borrow money or otherwise influence the value of $W = I$ going into the "investment" of the next round. This corresponds to the Froot model with infinite cost of external capital.[27]

Since value is state-dependent, not time-dependent, we will write the market value function as $M(W)$ where W takes on the values 0, 1, or 2. Using the recursive form of the valuation equation (3), we can write the following three relationships:

$$M(0) = 0 \qquad \text{(A1)}$$

$$M(1) = \frac{p \cdot M(2) + (1-p) \cdot M(0)}{1+r} \qquad \text{(A2)}$$

$$M(2) = \frac{p \cdot (M(2)+1) + (1-p) \cdot M(1)}{1+r} \qquad \text{(A3)}$$

[25] Assume its sales value as a scientific or entertainment novelty is zero!

[26] Since the transition probabilities are independent of W, we cannot impute interest earnings, either.

[27] Since the W funds remain trapped inside the box, they are really just abstract states at this point; in the next subsection, they will be liberated and represent real money.

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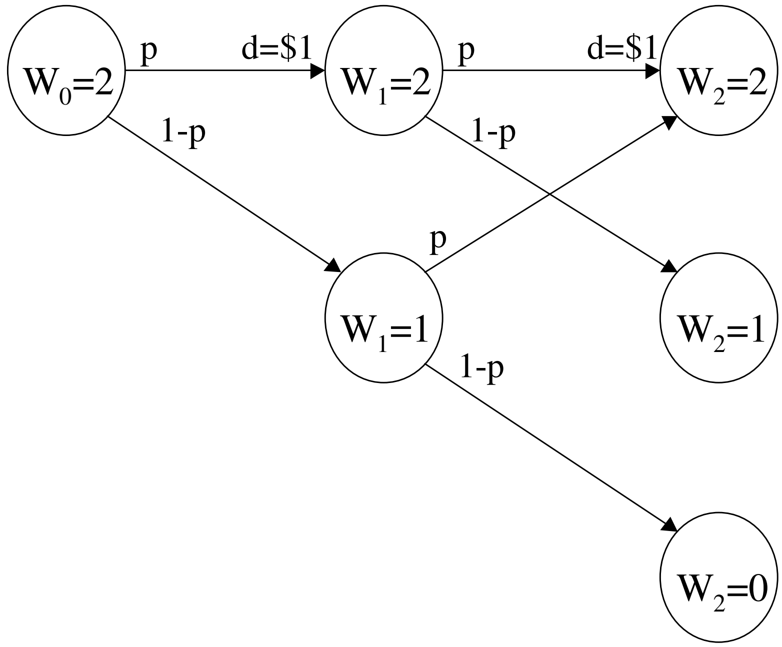

Figure A.1: A lattice of state transitions in the discrete Froot model.

The expressions for *M*(1) and *M*(2) are solved algebraically:

$$M(1) = \frac{p^2}{(1+r)\cdot(1+r-p) - p\cdot(1-p)} \tag{A4}$$

$$M(2) = \frac{(1+r)\cdot p}{(1+r)\cdot(1+r-p) - p\cdot(1-p)} \tag{A5}$$

The denominator in the expressions for *M*(1) and *M*(2) is always positive as long as r is nonnegative and p is between 0 and 1. (It is zero if $r = 0$ and $p = 1$.) Therefore, the box will have positive market value when $W = 1$ or 2 as long as the transition probability p is positive.

B. Risk Management

What if, instead of the random transition and possible dollar, the box dispensed the expected value of $2p$-1 (assuming $p > 1/2$) with certainty, every time? Would the box be worth more in that case? It depends. Such a perpetuity is worth $(2p-1)/r$, and this is greater than *M*(2) if and only if $p(1-p) > (1+r-p)^2$. Figure A.2 shows the region in $r \times p$ space where this occurs. For smaller r, there is a broader range of p where this "risk management" is worthwhile. For larger r, there is a narrower range (or no value at all) for p where it is worthwhile. Risk management matters.

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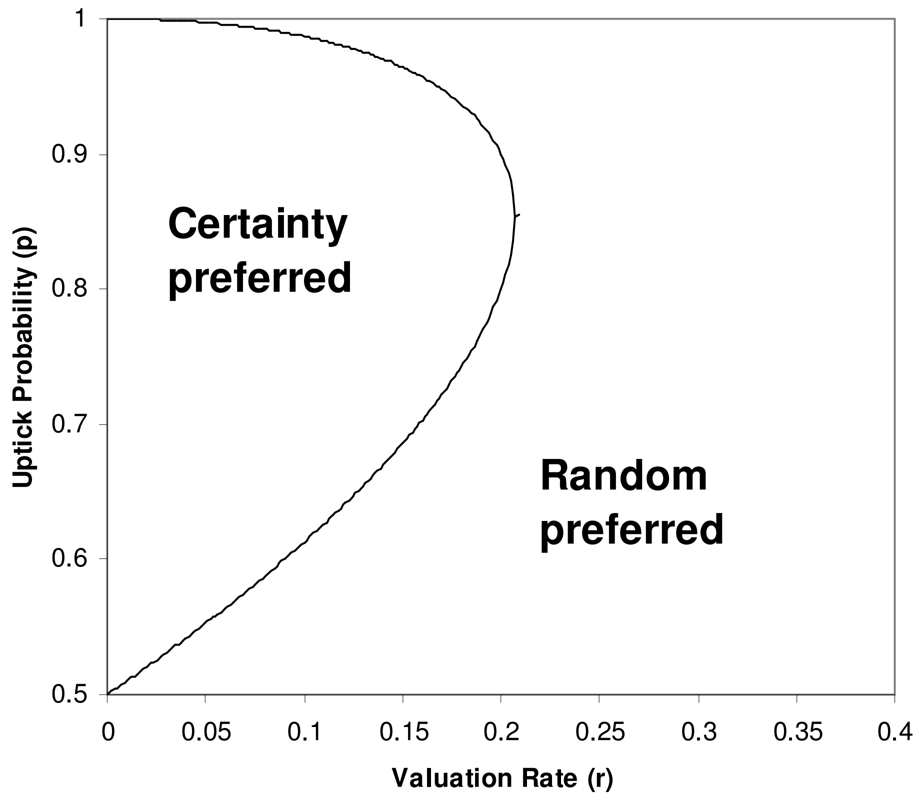

Figure A.2: Value of risk management.

C. Dividends

If the coins in the box are intended to represent the capital of a firm, we need to allow the box to be permeable to investor flows. What if, in the brief moment after a transition occurs, we could open the box and remove a dollar? Would it ever be to our advantage to do so?

Consider the option of going from W=1 to W=0. Clearly, the value obtained in doing so is an immediate dollar and there will be no more value to the box. For this to be advantageous, the M(1) of equation (A4) would have to be less than a dollar. This is the case if and only if $(1+r)^2 > p(2+r)$. The M(2) formula would no longer be correct, as it would reflect an incorrect value for M(1), but that does not alter the conclusion about when to take the dollar.

Similarly, exercising the option of going from W=2 to W=1 changes a putative value of M(2) into M(1)+1 and is advantageous if and only if $p(1\text{-}p) < (1+r\text{-}p)^2$. Note this latter condition is the opposite of the risk management condition derived above. Figure A.3 summarizes this "optimal dividend policy."

Given these optimal dividend policies, the value of the box needs to be recomputed. In the "Do not remove coins" region of the (r, p) space, the values $M(W)$ are as computed in equations (A1), (A4), and (A5) earlier. In the "Remove all coins" region, $M(W) = W$; in effect, we want to cease box operations immediately and liquidate the

-

assets. In the middle region, we still have $M(0) = 0$, of course, but now $M(1) = p/(1+r-p)$ and $M(2) = M(1)+1 = (1+r)/(1+r-p)$.

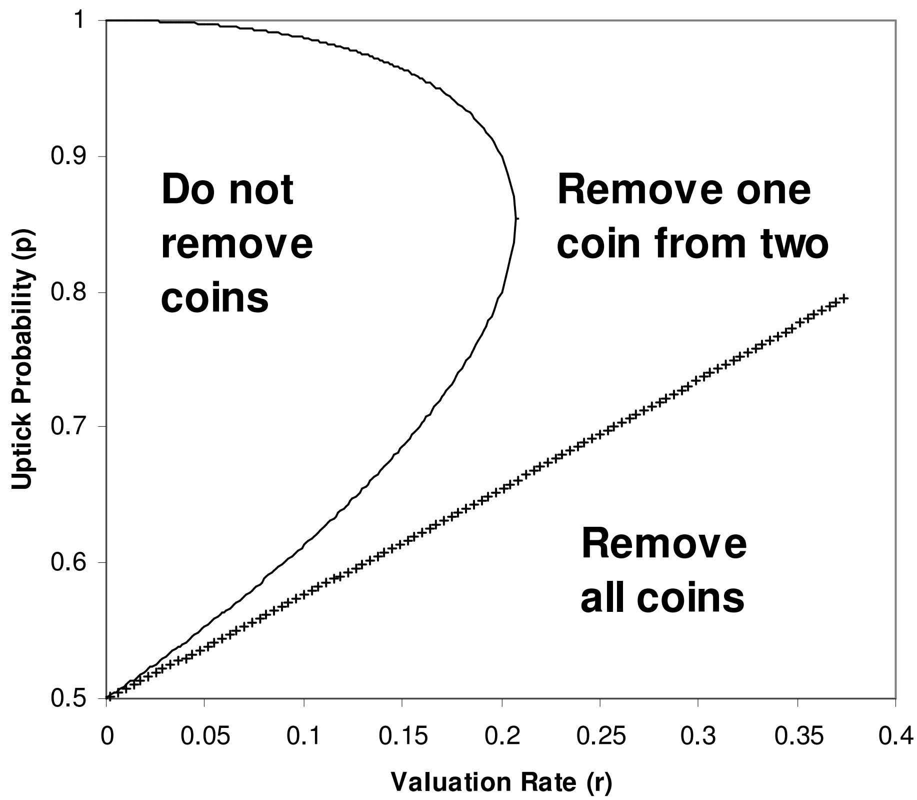

Figure A.3: Optimal dividend policy.

D. Costless External Capital

What if we had the option of adding a coin? Clearly, this option only applies if there is one coin in the box; with zero coins the box disappears, and with two coins an added third would just be disgorged immediately. It would be to our advantage to add one coin if $M(1)+1$ is less than $M(2)$, which is the opposite of the condition for removing one coin from two. The region in figure A.3 labeled "Do not remove coins" can now be relabeled "Add one coin to one."

The market values in this region of $r \times p$ space are now as follows: $M(0) = 0$, $M(1) = (2p-1-r)/r$, and $M(2) = M(1)+1 = (2p-1)/r$. Notice that this brings the value of the box when $W = 2$ to the same level that the "risk management" of section B did. Starting with two coins, we can keep the box running forever: if $\Delta W = 1$, we get to pocket a coin; if $\Delta W = -1$, we then insert a coin to bring the inventory back up to two coins. If the discount rate is correct, we should be indifferent to receiving our "capital stream" or its certainty equivalent stream, or selling the box for the capitalized value of those streams, $M(2)$. The magic box now lives in the Modigliani and Miller world where risk management does not matter.

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E. Costly External Capital and Bankruptcy Cost

What if an extra cost of $\kappa > 0$ were incurred every time we added a coin to the box? That is, we expend $1+\kappa$ in order to increase W by 1. How would this change our strategy?

Clearly, it would only be advantageous to do so if $M(1)+(1+\kappa) < M(2)$. This is true if and only if $(1+\kappa)^{-1}$ is greater than $(1+r)/p - (1-p)/(1+r-p)$. Contours of this equality for $\kappa = 0$, 0.25, and 1 are traced in figure A.4. The higher is κ, the higher p and the lower r need to be for this to be advantageous. As κ increases without bound, the limiting point $p = 1$, $r = 0$ is approached. (Note also that with increasing cost, the probability of incurring it decreases.)

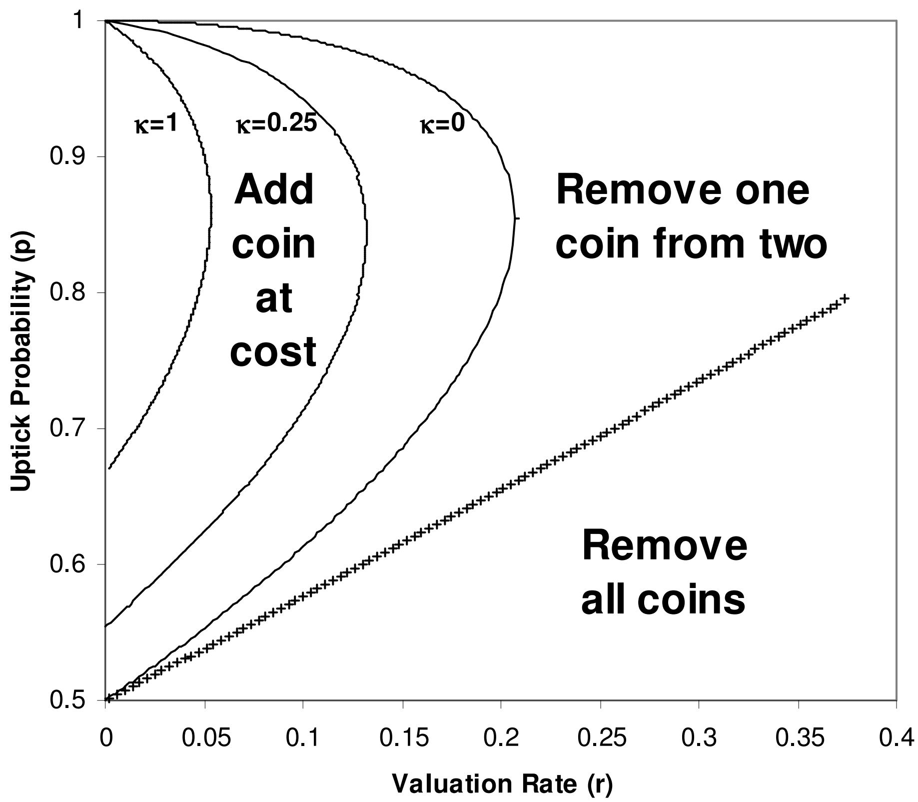

Figure A.4: The effect of costly external capital.

Inside the recapitalization region, the values are $M(1) = M(2) - (1+\kappa)$ and $M(2) = (2p-1-(1-p)\kappa)/r$. Outside that region, but still in the "Do Not Remove Coins" region, the values are as computed before.

Risk management adds value when there is costly external capital. This can be seen by considering the difference in value between the certainty-equivalent perpetuity, $(2p-1)/r$, and $M(2)$. The difference is just the expected net present value of the costs, $(1-p)\kappa/r$. Outside the recapitalization region (but still inside the "Do Not Remove" region) the value of risk management is less, but still positive. Thus costly external capital leaves a window of opportunity for risk management to add value.

-

It should be noted that this model also incorporates a severe degree of "bankruptcy cost," another source of capital market friction and violation of M&M assumptions. In the industrial economics paradigm, most of the assets of the firm consist of plant and equipment. When a firm goes bankrupt, it typically does not see its factories disappear in a puff of smoke! Bankruptcy cost is relevant in the case where costly external finance is not economical to obtain (that is, the best strategy is to use the coin box until it stops). If we were able to sell the empty coin box to someone else who could start it up again at $W = 2$, then we would have $M(0) \geq M(2)$-2.

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