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Island Matching*

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Abstract

A synthesis of the Lucas-Prescott island model and the Mortensen-Pissarides matching model of unemployment is studied. By assumption, all unmatched workers and jobs are randomly assigned to submarkets, islands, at the beginning of each period and the number of matches that form on a particular island is the minimum of the two realizations. When calibrated to the recently observed averages of U.S. unemployment and vacancy rates, the model fits the oberved vacancy-unemployment Beveridge relationship very well and implies an implicit log linear relationship between the job finding rate and the vacancy-unemployment relationship with an elasticity near 0.5. The socially efficient solution to the model, which obtains in market equilibrium when the wage on each island is the outcome of a modified auction, implies larger responses in the vacancy-unemployment ratio to productivity and job destruction shocks than the canonical model of equilibrium unemployment. Finally a variant of the model in which the wage is the solution to the strategic bargaining problem faced by worker and employer after they meet explains all the observed volatility of vacancies and unemployment in the U.S.

Key words: Matching function, Beveridge curve,labor market volatility.

JEL Codes: E24, J3, J64

*The comments and suggestions of Rob Shimer are gratefully acknowledged.

1 Introduction

The purpose of the paper is to study a matching model that incorporates features of both the Lucas and Prescott (1974) and the Mortensen and Pissarides (1994) equilibrium models of unemployment. That unmatched workers and jobs search submarkets, "islands" in the literature, at random and that the number of matches that form on any particular island is the minimum of the two realizations are the essential specification assumptions.[1] The solution to the associated social planner's problem is derived as well a decentralization in which the realized wage on each island is determined by a modified auction as in the Lucas-Prescott model. In addition, an alternative formulation in which wage on each island is the outcome of a strategic bilateral bargaining game played between worker and employer after they meet is considered.

The matching process is closely related to the original formulation of job-worker matching as summarized in the first chapter of Pissarides (2000). However, instead of supposing that some ad hoc matching function exists, the essential assumptions generate an endogenous positive relationship between the job finding rate and the ratio of vacancies to unemployment (a "reduced form" matching function) as well as a negative relationship between vacancies and unemployment (a Beveridge curve). Indeed, given parameters chosen to match the U.S. average unemployment and vacancy rates observed in the last six years in the U.S. and a match period length consistent with the average flows into and out of employment in the U.S., the implied Beveridge curve and observed unemployment rates explains 90% of the variation in vacancy rates observed over that same period. Furthermore, the implicit relationship between the job finding rate and the vacancy-unemployment ratio is essentially log linear over the relevant range with an elasticity of about 0.48, two facts consistent with the literature on the estimation of empirical matching functions reviewed by Petrongolo and Pissarides (2001).

When the wage is set at auction, equilibrium outcomes would be efficient if the matching process were characterized by constant returns to scale in the sense that the expected number of matches increases in proportion to the average numbers of unmatched workers and jobs per market holding their ratio constant. Although in fact the specified matching process exhibits increasing returns, the efficiency holds as an approximation when the average

[1] This quite old ideas has been fruitfully explored in a recent paper by Shimer (20060) in the case of limited mobility between island.

numbers of workers and jobs per island are large. In this case, the private and social incentives are (almost) aligned when agents on the short side of the market obtain all of match surplus for reasons anticipated by Mortensen (1982). Although the calibrated version of the model does not approximate the case of constant returns, an auction augmented by a relatively simple subsidy-tax system can decentralize the solution to the planner's problem in the general case.

The alternative wage mechanism considered is the strategic solution to the bilateral wage bargaining problem obtained when delay rather than continued search is the relevant default outcome in the game. Hall and Milgrom (2005) argue that this solution is more realistic than the so-called Nash solution with threat points equal to the value of continued search as usually assumed in matching models.[2] Although the marginal social and private values of participation are not equal in equilibrium, there is one and only one solution it has important implications for the volatility of unemployment and vacancies, an issue raised by Shimer (2005). Unlike the canonical matching model that he studied, the volatilities vacancies and unemployment implied by this modification are close to those observed in Shimer's time series data.

2 The Matching Process

In the standard search equilibrium framework, the matching function is a black box that relates the number of unemployed workers and vacant jobs to the flow of matches that form. As in Shimer (2006), the relationship between the match flow and the numbers of unmatched workers and jobs considered in this paper is the outcome of more primitive assumptions about how matching takes place. The specification follows.

The economy is composed of a continuum of workers and employers and a continuum of islands where exchange takes place. Time is divided into discrete periods of equal length denoted as $t = 1, 2, ...$ Let M and N represent measures of unmatched workers and unmatched jobs per island respectively at the beginning of any period. Under the assumption that participants are randomly assigned to islands, Shimer (2006) demonstrates that the joint probability that there are i workers and j jobs on any particular island are independent Poisson variables with means M and N respectively. Formally,

[2] This solution was also suggested by Binmore, Rubstein, and Wolinsky (1986) some time ago.

the probability is

$$\pi(i,j;M,N) = \frac{e^{-(M+N)}M^i N^{\,j}}{i!j!}. \tag{1}$$

As one can easily verify,

$$\begin{aligned} \frac{\partial \pi(i,j;M,N)}{\partial M} &= \pi(i-1,j;M,N) - \pi(i,j;M,N) \\ \frac{\partial \pi(i,j;M,N)}{\partial N} &= \pi(i,j-1;M,N) - \pi(i,j;M,N). \end{aligned} \tag{2}$$

In other words, the derivative of the probability that there are i workers (j jobs) in any market with respect to the average number of workers (jobs) per market is equal to the change in the probability induced by the marginal worker (job) in each market.

Given that the short side determines the match outcome on each island, the average number of matches created per island is given by

$$F(M,N) \equiv E\{\min(i,j)\} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \min(i,j)\pi(i,j;M,N) \tag{3}$$

As an implication of equations (1) and (2), Shimer (2006a) demonstrates that

$$\frac{\partial F}{\partial M} \equiv F_M(M,N) = \sum_{j=1}^{\infty}\sum_{i=0}^{j-1} \pi(i,j;M,N) = \Pr\{i<j\} > 0 \tag{4}$$

$$\frac{\partial F}{\partial N} \equiv F_N(M,N) = \sum_{i=1}^{\infty}\sum_{j=0}^{i-1} \pi(i,j;M,N) = \Pr\{j<i\} > 0. \tag{5}$$

In other words, the partial derivative, $F_N(M,N)$, is the share of the islands with unemployed workers at the end of a period while $F_M(M,N)$ is the fraction of islands with vacant jobs. In this paper, we refer to $F(M,N)$ as the structural matching function implied by the matching process. A characterization of its properties follow:

Proposition 1 *The matching function $F(M,N)$ is increasing and concave in M and N holding the other constant ($F_{MM} < 0$ and $F_{MM} < 0$). Furthermore,*

$$\begin{aligned} F_{MN}(M,N) &= 1 - F_N(M,N) - F_M(M,N) \\ &= \sum_{i=0}^{\infty} \pi(i,i;M,N) = \Pr\{i=j\} \le F_{MN}\left(\frac{M+N}{2}, \frac{M+N}{2}\right) \end{aligned} \tag{6}$$

and

$$\lim_{M+N\to\infty} F_{MN}(M,N) = 0. \tag{7}$$

Proof. See the Appendix. ■

The next result, a corollary of Shimer's (2006) Proposition 3, implies that the matching process exhibits increasing returns.

Proposition 2 *A one percent increase in both M and N increases the number of matches by more than one percent. Formally, the matching function exhibits increasing returns in the sense that*

$$\frac{MF_M}{F} + \frac{NF_N}{F} - 1 = F_{MN}(M,N)\frac{E\{i|i=j\}}{E\{\min(i,j)\}} > 0 \tag{8}$$

where

$$E\{i|i=j\} = \frac{\sum_{i=0}^{\infty} i\pi(i,i;M,N)}{\sum_{i=0}^{\infty} \pi(i,i;M,N)}.$$

Proof. See the Appendix. ■

Hence, equation (7) and (8) together imply that the matching function is approximately linearly homogenous when the sum of the number of workers and jobs per island is large. Still, the following fact and the graph in Figure 1 suggests that the speed of convergence can be slow:

$$\begin{aligned}
\Pr\{i &= j\} = F_{MN}(M,N) = \sum_{i=0}^{\infty} e^{-(M+N)}\frac{M^i N^i}{i!i!} \\
&\leq \max_{M,N\geq 0}\left\{\sum_{i=0}^{\infty} e^{-(x+y)}\frac{x^i y^i}{i!i!} s.t. x+y = M+N\right\} \\
&= F_{MN}\left(\frac{M+N}{2}, \frac{M+N}{2}\right).
\end{aligned}$$

3 The Beveridge Curve

The number of unemployed workers, U, and vacant jobs, V, are defined as those not matched during the period. That is $U = M - F(M,N)$ and

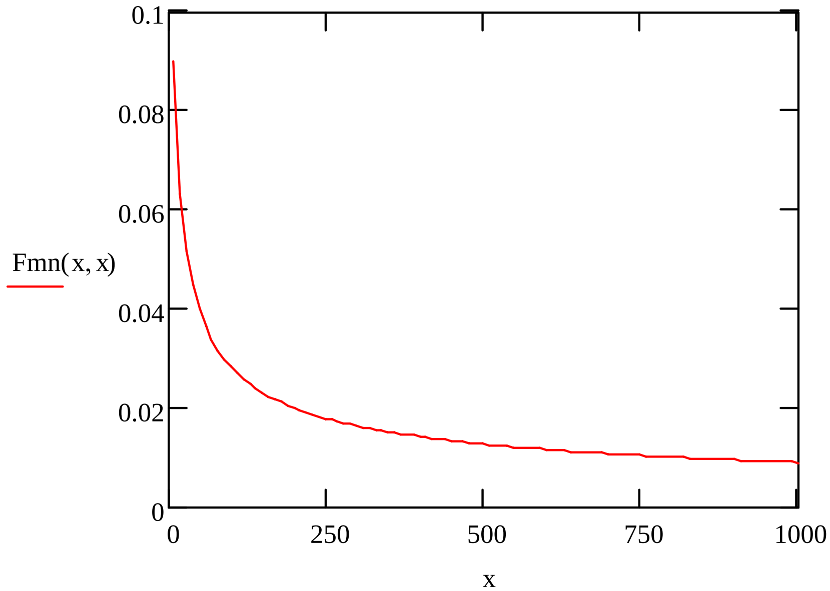


Figure 1: Convergence to Linear Homogeneity

$V = N - F(M, N)$. Hence, the corresponding unemployment and vacancy rates are

$$u = \frac{U}{L} = \frac{M - F(M,N)}{L} \tag{9}$$

and

$$v = \frac{V}{L - U + V} = \frac{N - F(M,N)}{L - M + N} \tag{10}$$

where $L \geq M$ is the average number of workers per market and $L - U + V$, the sum of the number of matched and vacant jobs, is the average number of jobs per market. Shimer (2006) shows that observations on the unemployment and vacancy rates tie down the number of unmatched workers and jobs conditional on the total average number of workers per market, L. Furthermore, variation in the number of unmatched jobs induce a negative relationship between the vacancy rate and the unemployment rate given the number of unmatched workers.

Although the number of unmatched jobs at the beginning of a period is a state variable fixed at a moment of time, it responds over time to changes in the number of unmatched jobs, induced say by shocks to match productivity. Under the assumption of random search,

$$M_{t+1} = U_t + s(L - U_t) = M_t + s(L - M_t) - (1 - s)F(M_t, N_t) \tag{11}$$

where s is the job separation rate. Indeed, in steady state the number of workers hired and jobs filled is equal to the separations flow, $(1 - s)F(M, N) = s(L - M)$. The assumption that all unmatched workers and jobs are randomly assigned is easily justified. Specifically, if all other workers (jobs) that find themselves not matched at the end of a period were to stay in their respective islands, then the chance of finding a job (worker) is the next period is higher at a randomly selected alternative island. Hence, staying is not a symmetric non-cooperative Nash equilibrium. However, if all search by selecting an island at random, then the likelihood of matching is the same in all islands. It is this assumption of continual reallocation that distinguishes the model studied in this paper from Shimer's (2006) mismatch model. In formulating the social planner's problem, we take the coordination problem implicit in the random assignment assumption as given by supposing that the planner cannot dictate the destination island for any individual worker or job.

The following strategy for calibrating the model suggests itself: First, choose the period length and set the value of the separation rate per period

s to its observed average value in the data accordingly. Given these numbers and the observed unemployment rate and vacancy rate averages, use equation (9), equation (10) and the steady state conditgionto determine M and N.

The choice of a matching period length is not totally arbitrary. Specifically, in the U.S. case the length must be consistent with the fact that the average duration of an unemployment spell is approximately one quarter and that the median spell length is considerably shorter. With these facts in mind, a period length of one month suggests itself as a base line case. Shimer's (2005) estimate of s is 10% per quarter or 3.33% per month. The average vacancy rate reported in the JOLTS data over the 72 month period from December 2000 to November 2006 inclusive was 2.51% while the monthly average of standard CPS measure of the (non-farm) unemployment rate over the same period was 5.29%. Given these choices, equations (9)-(11) imply $M = 1.174$, $N = 0.778$, and $L = 13.909$. These numbers suggest that a "island" might be interpreted as a firm of about median size which receives 1.17 applicants per month seeking jobs that become available with frequency 0.78 per month.

The raw data on vacancy and unemployment rates over the last six years, reported in the data Appendix, are plotted as the scatter of points illustrated in Figure 2. The plot represents a well defined empirical Beveridge curve. The vacancy-unemployment relationship obtained by varying N between 0.75 and 0.95 is illustrated as the solid curve in the figure. As anyone can see, the fit of the model is remarkable. Indeed, the percent of variance in the vacancy rate explained by the curve implied by the model and the unemployment rate is 91%.

The slope of the model's Beveridge curve is relatively invariant to choices of the length of the matching period within the range consistent with observed unemployment durations. This fact is illustrated in Figure 3 where the implied relationship between the vacancy and unemployment rate are drawn under the assumption that the period length is a quarter and a week as well as a month.

Of course, Shimer's (2006) mismatch model can also explain the recent time series data on unemployment and vacancy rates. In that model, the labor market is viewed as a collection of segmented markets for different occupations and regions. Given this interpretation, he argues that cross market mobility is quite small or non-existent. In this environment, the total number of workers per island (L in our notation) is M which he regards as fixed. He calibrates his model by setting M and N to match observed

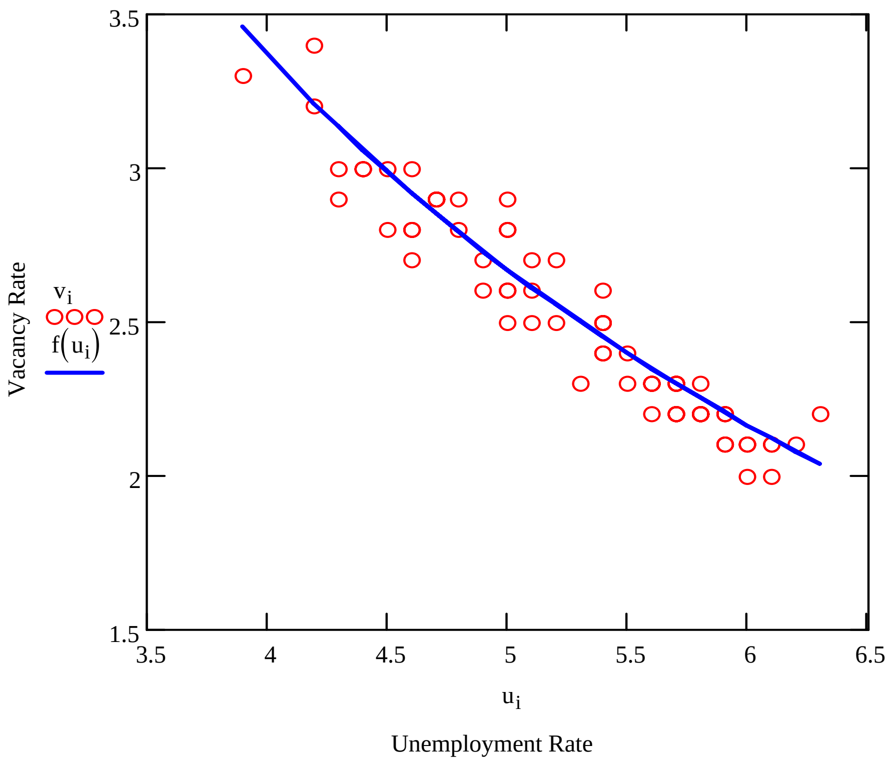


Figure 2: U.S. Beveridge Curve 12/2000-11/2006

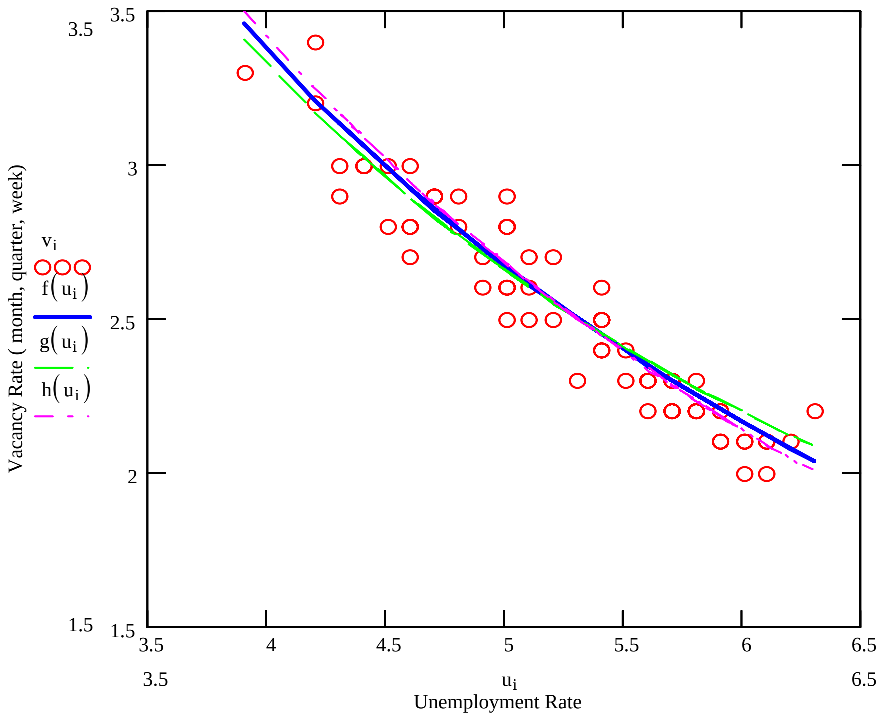


Figure 3: U.S. Beveridge Curve (f=month, g=quarter, h=week)

vacancy and unemployment rate averages and then varies N holding M fixed to generate a Beveridge curve.

4 The "Reduced Form" Matching Function

The job finding rate, defined as the ratio of the hires flow to the number of unemployed workers, is an empirical measure of unemployment spell hazard. As documented in Petrongolo and Pissarides (2001), the empirical literature on the matching function suggests that the job finding rate is well described as a log linear function of the vacancy-unemployment ratio with a elasticity in the range of 0.3 to 0.5.

In the model under study, the measured job finding rate per period is the ratio of the number matched to the number unemployed, $F(M,N)/U$, and the vacancy-unemployment ratio is V/U. Since both increase as N increases, a positive implicit relationship exists between the two variables, one that Shimer (2006) calls the "reduced form" matching function. Indeed, the log-log relationship obtained when N varies between 0.75 and 0.95, the same range used to generate the model's Beveridge curve, is illustrated in Figure 4. Obviously, the relationship is very close to linear over this range. Furthermore, the slope (elasticity) is 0.481, a number within the Petrongolo-Pissarides "plausible range".[3] Hence, the model provides a simple micro foundation for the empirical matching functions estimated in the literature.

5 The Social Planner's Problem

The planner posts a number of unmatched jobs in each period subject to a cost, c, the same cost that an employer would face. Assume that workers and employers are risk neural and discount future income by the factor $\beta \in (0,1)$ per period. A job-worker match produces market output of value p per period and the home production of any unmatched worker during a period has value z. Obviously, gain from trade require that $p > z$.

Consider the timing is as follows: Each period t is divided into three parts. In the first subperiod, matching takes place. In the second subperiod, all

[3] Although Shimer's mismatch model also implies a nearly log linear relationship, his implied elasticity is only about 0.2.

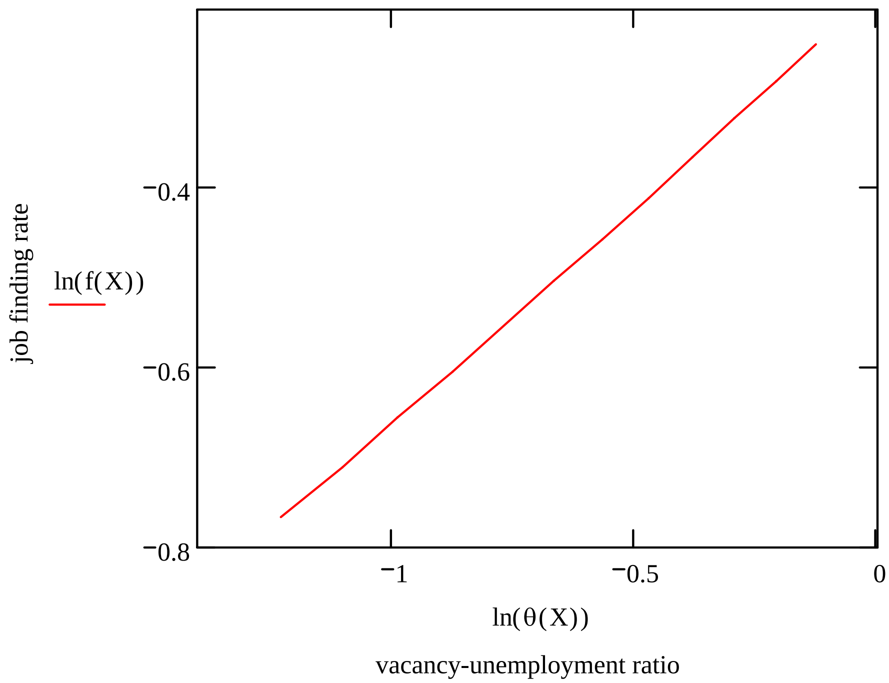


Figure 4: "Reduced Form" Matching Function

matches produce. Job seperations takes place and the number of unmatched jobs to post in period $t+1$ is determined in the final subperiod.

As employment is equal to $E_t = L - U_t$ and $M_{t+1} = U_t + sE_t = L - (1-s)E_t$ its law of motion is

$$\begin{aligned} E_{t+1} - E_t &= L - U_{t+1} - E_t = L - M_{t+1} + F(M_{t+1}, N_{t+1}) - E_t \\ &= F(L - (1-s)E_t, N_{t+1}) - sE_t \end{aligned} \tag{12}$$

by equation (11) where N_{t+1} is the number of jobs posted at the end of period t.In each period, the planner chooses a job entry strategy, a function of the state of the market as reflected in the current value of employment, that determines the number of unmatched jobs that will participate in the matching process at the beginning of the next period.

Given agent preferences, a benevolent planner chooses a strategy that maximizes the present value of aggregate match surplus net of recruiting costs. Since the match surplus flow is $p - z$, the Bellman equation associated with this dynamic programming problem is

$$V(E_t) = \max_{N \geq 0} \left\{ (p-z)E_t - cN + \beta V((1-s)E_t + F(L - (1-s)E_t, N)) \right\}.$$

The first order condition is

$$c \geq \beta F_N(L - (1-s)E_t, N)\lambda_{t+1} \text{ with equality holding if } N > 0 \tag{13}$$

where

$$\lambda_t \equiv V'(E_t) = p - z + \beta(1-s)\left[1 - F_M(L - (1-s)E_t, N)\right]\lambda_{t+1} \tag{14}$$

represents the present value of a job-worker match. In other words, the cost of posting an unmatched job is equal to the product of the job's marginal contribution to the total number of matches in the next period and the present value of a job-worker match. Because $F_{NN} < 0$, the second order necessary condition is satisfied. The solution to the planner's problem solves the first order condition, equation (13), the system of difference equations defined by equations (12) and (14), and the transversality condition $\lim_{\to\infty} \lambda_t (1+r)^{-t} = 0$.

The existence of at least one steady state solution to the problem can be demonstrated with the following argument. First, solve the free entry condition, equation (13) for N as a function of λ and E. Since $F_N(M, N)$

is increasing in M and decreasing in N from (6) and (5), the solution for the average number of unmatched jobs posted per island, denoted as $N = N(\lambda, E)$, is positive for any $\lambda > 0$, is increasing in λ and is decreasing in E. Since $F(M,N)$ is positive and increasing in both of its arguments, it follows from equation (12) and the properties of $N(\lambda, E)$ that the singular curve representing the steady state condition $\Delta E = E_{t+1} - E_t = 0$ can be represented by a strictly positvely sloped curve relating λ and E where $\lambda = \widehat{\lambda}$, the positive solution to $c = \beta F_N(L, N(\widehat{\lambda}, 0))$, at $E = 0$ as represented in phase diagrams illustrated in Figures 5. Finally, as the right hand side of (12) is decreasing in E_t, $E_{t+1} - E_t < (>)0$ to the right (left) of the singular curve as indicated by the direction arrow in the phase diagram.

Because $F_M(M,N)$ is a probability by (**??**), the curve defined by $\Delta\lambda = \lambda_{t+1} - \lambda_t = 0$, the solution to $\lambda = (p-z)/[1-\beta(1-s)\,[1 - F_M(L-(1-s)E, N(\lambda, E))]$, is bounded above by $(p-z)/[1-\beta(1-s)]$ and below by $p - z$. This fact and the properties of the $\Delta E = 0$ singular curve imply that the two singular curves must intersect at least once in the positive quadrant if $p - z > \widehat{\lambda}$. Because the coefficient on λ_{t+1} on the right side of equation (14) is strictly less than unity, $\Delta\lambda > (<)0$ at points above (below) the curve as indicated by the directional arrows in the phase diagrams portrayed in Figures 5. Finally, since $F(M,N)$ exhibits increasing returns, $F_{MM}F_{NN} - F_{MN}F_{NM} < 0$ can hold and

$$\frac{\partial F_M}{\partial E} = -(1-s)F_{MM} + F_{NN}\frac{\partial N}{\partial E} = (1-s)\left(\frac{F_{MN}F_{NM} - F_{MM}F_{NN}}{F_{NN}}\right),$$

the singular curve representing the condition $\Delta\lambda = 0$ can also has a positive slope as illustrated in Figures 5.

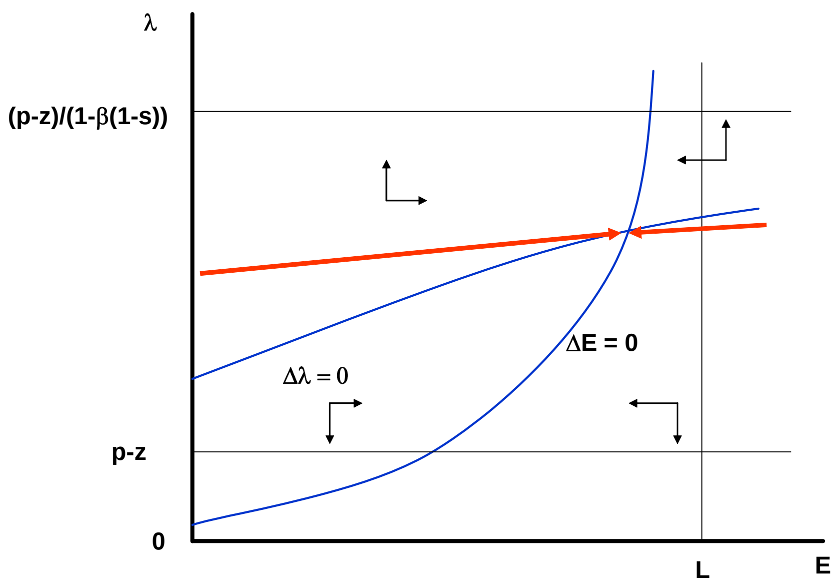
Although it might appear that multiple steady states can exist, in fact there is only one.[4] The assertion can be established by using the steady condition $\Delta\lambda = 0$ to eliminate λ in the first order condition. The result is

$$c = \frac{\beta F_N(L-(1-s)E, N)(p-z)}{1-\beta(1-s)[1 - F_M(L-(1-s)E, N(\lambda, E))]},$$

a condition that defines an downward sloping relationship between E and λ given the properties of the partial derivatives of $F(N,M)$ reported in Proposition 1. Of course, a steady state is the single solution pair at the intersection of this curve and the positively sloped relationship defined by $\Delta E = F(L-(1-s)E, N) - sE = 0$.

[4] I am indebted to Rob Shimer for pointing out this fact.

Finally, because the steady state is a saddle point, there is a unique path converging to it from any initial condition as illustrated in Figure 5. Because any other solution trajectory violates the transversality condition, the converging trajectory represents the unique solution to the planning problem associated with the initial stock of unmatched workers inherited from the past.



Phase Diagram: Planner's Problem

6 Auction Equilibrium

Suppose that the wage on each island are determined as the outcome of an auction as in Lucas and Prescott (1974) and Shimer (2006). Specifically, assume that the worker collects the entire match surplus if there are more jobs than workers on her island but receives only her reservation wage if the number of workers exceeds the number of jobs available. Although the auction outcome is indeterminate when the number of workers and jobs are equal, suppose for now that the employer obtains the surplus in this case. Shimer (2006) shows that these assumptions yield outcomes that are equivalent to the planner's solution in his "mismatch" model. In this model, the two solutions would be equivalent only under the counter factual condition that the expected number of matches that form is homogenous of degree one in the number of unmatched jobs and workers. Although this condition holds approximately when the average numbers of unmatched jobs and workers per market are large, the calibration presented above does not have this property, as illustrated by Figure 1. However, a relatively simple tax and subsidy system exists that will decentralize the planner's problem.

As the probability that there are i workers and j jobs in the same island from the point of view of any job on a particular island is equal to the probability that i workers and $j-1$ other jobs are also assigned to the island, an employer can expect to obtain the entire match surplus with probability equal to the fraction of markets that have strictly fewer jobs than workers. That is

$$Q(M,N) = \sum_{i=1}^{\infty}\sum_{j=1}^{i}\pi(i,j-1;M,N) = \sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\pi(i,j;M,N) = F_N(M,N)$$

by equation (5). An alternative way to obtain the same result is to realize that the relevant probability is equal to the expected number of jobs that are matched in markets with weakly fewer jobs than workers divided by the number of jobs to be matched. But that fraction is also

$$\frac{1}{N}\sum_{i=1}^{\infty}\sum_{j=1}^{i} j\pi(i,j;M,N) = \sum_{i=1}^{\infty}\sum_{j=1}^{i}\pi(i,j-1;M,N) = F_N(M,N)$$

by equation (1).

As a worker receives the surplus only if matched on an island with fewer workers than jobs, an analogous argument and equation (4) imply that the probability of such an event is

$$\begin{aligned} P(M,N) &= \frac{1}{M}\sum_{j=1}^{\infty}\sum_{i=1}^{j-1} i\pi(i,j;M,N) = \sum_{j=1}^{\infty}\sum_{i=1}^{j-1}\pi(i-1,j;M,N) \qquad (15) \\ &= F_M(M,N) - \sum_{j=1}^{\infty}\pi(j-1,j;M,N) \end{aligned}$$

where $\sum_{j=1}^{\infty}\pi(j-1,j;M,N)$ is the probability that there is one fewer other workers than jobs on the island.

The present value of employment to a worker originally matched on an island with fewer unmatched workers that unmatched jobs, denoted Y_t, is the solution to

$$Y_t = p + \beta\left[(1-s)Y_{t+1} + s\left[P_{t+1}Y_{t+1} + (1-P_{t+1})U_{t+1}\right]\right]$$

where $U_{t+1}$is the value of search and $P_{t+1} = P(M_{t+1}, N_t) = P(L - (1 - s)E_t, N_{t+1})$ is the probability of being matched on an island with fewer workers in the next period. In addition to the current flow of match product, the worker can expect to receive the total value of her match in the next period if either her current match continues or the match ends at the end of the current period but she is rematched on an island with few workers than jobs in the next period. This second possibility is a consequence of the fact that a worker can move directly from one job to another without a spell of unemployment. As the value of unemployed search solves

$$U_t = z + \beta[P_{t+1}V_{t+1} + (1 - P_{t+1})U_{t+1}]$$

and the value of an unmatched job is zero in equilibrium, the surplus value of a match, the difference $S_t = Y_t - U_t$, is the solution to the Bellman equation

$$S_t = p - z + \beta(1-s)[1 - P(L - (1-s)E_t, N_{t+1})]S_{t+1}. \qquad (16)$$

Finally, the free entry condition requires that the cost of posting a vacancy now equals its expected future return, the product of the present value of the match surplus and the employer's chance of receiving it in the next period. In other words, the number of unmatched jobs solves

$$c \geq \beta Q(L - (1-s)E_t, N_{t+1})S_{t+1} \text{ with equality holding if } N_{t+1} > 0. \qquad (17)$$

An equilibrium is a any solution to equations (12), (16) and (17) that satisfies the tranversality condition $\lim_{t\to\infty} \beta^t S_t = 0$.

A comparison of the equilibrium conditions, equations (17) and (16) , with the necessary conditions for a solution the planner's problem, equations (13) and (14) respectively, implies that the equilibrium solution is a candidate solution to the planner's problem if and only if the surplus value of a match is equal its shadow value in the efficient solution, i.e., $S_t = \lambda_t$ for all t. However, equations (14) and (16) together with equation (15) imply that $P(M_{t+1}, N_{t+1}) < F_M(M_{t+1}, N_{t+1})$. As a consequence, the private return to posting a vacancy generally exceeds the social return implying that that the market over invests in job creation relative to the efficient solution.

Alternative, if the worker receives the surplus instead, then

$$Q(M,N) = \sum_{i=1}^{\infty}\sum_{j=0}^{i-2}\pi(i,j;M,N) = F_N(M,N) - \sum_{i=1}^{\infty}\pi(i,i-1;M,N) < F_N(M,N)$$

is the probability that there are strictly few jobs than workers. In this case, the private and social surplus value of a match are equal since $P(M,N) = F_M(M,N)$, but too few unmatched jobs are created because $Q(M,N) < F_N(M,N)$.

These facts imply that the following tax and subsidy scheme will implement the efficient solution: Provide both parties with the full surplus value of a match when the number of unmatched workers and employers on an island are exactly equal and finance the subsidy with a lump sum tax levied on the workers, employed or not.[5] In this case, $P = F_M$ and $Q = F_N$ which implies $S_t = \lambda_t$ and $c \geq \beta F_M \lambda_{t+1}$. Of course, the transfer required,

$$T = F_{MN}(L - (1-s)E_t, N_{t+1})S_{t+1} \tag{18}$$

vanishes as the number of unmatched workers and jobs become large as a consequence of equation (7).

7 Bargaining Equilibrium

Suppose that the wage is determined as the outcome of a strategic non-cooperative bargaining game played after worker and employer meet along the lines outlined by Hall and Milgrom (2005). Hall and Milgrom argue that delay rather than search is the relevant default option so long as both sides receive at least the value of continued search. Assuming that the worker can generate value at the flow rate z while bargaining but neither can search for an alternative until next period, the unique perfect wage outcome of a symmetric alternating offer game is given by

$$w_t = \min\left(p, \max\left(z + \frac{1}{2}(p-z), R_t\right)\right) \tag{19}$$

[5] In other words, allocating the match surplus to the side responsible for forming the match, which is both when the numbers on the two sides of the market are equal, implements the efficient solution as pointed out in Mortensen(1983).

In other words, the pair split the flow surplus, $p - z$, equally provided that the result exceeds the flow value of the worker's search option, R_t, and is less than match output.

Because the value of a filled job to the employer is the present value of the profit flow and the ex ante probability of filling any job in the matching process is the ratio of the matching rate to the number of unmatched jobs at the beginning of the period, the free entry condition is

$$c \geq \frac{F(L-(1-s)E_t, N_{t+1})}{N_{t+1}} \beta J_{t+1} \text{ with strict equality holding if } N_t > 0. \tag{20}$$

where J_{t+1} is the employer's value of a match in the next period. The employer's value evolves according to the rule

$$J_t = p - w_t + \beta(1-s)J_{t+1}. \tag{21}$$

Provided that match output exceeds the opportunity cost of employment, $p \geq z$, and the worker's wage is no less then the reservation wage, $z + \frac{1}{2}(p - z) \geq R_t$, the wage is $w_t = z + \frac{1}{2}(p-z)$ and the employer's match value solves

$$J_t = \frac{p-z}{2} + \beta(1-s)J_{t+1}. \tag{22}$$

Obviously, the singular curve characterizing $\Delta J = 0$ is the horizontal line defined by $J = \frac{(p-z)/2}{1-\beta(1-s)}$ in Figure 7. Furthermore, because $0 < \beta(1-s) < 1$, the difference equation is unstable forward in time as indicated in figure by the directional arrows.

The solution to the free entry condition, equation (20), for the number of unmatched jobs, denoted as $N(J,E)$, increases with J and decreases with E $F(M,N)$ is increasing in both arguments and is concave in N. The locus of points for which $\Delta M = M_{t+1} - M_t = 0$ is defined by

$$s(L-M) - (1-s)F(L-(1-s)E, N(J,E)) = 0$$

is a postively sloped relationship between J and E as illustrated in Figue 7. Since $\Delta E > 0 (< 0)$ for all E to the left (right) of the singular curve characterizing $\Delta E = 0$, a unique saddle point steady state solution exists, coincident with the $\Delta J = 0$ curve, which is the only solution to the differential equation system defined by the necessary conditions and tranversality conditions. Furthermore, the state state value of employment is positive if $p - z$ is positive and sufficiently large.

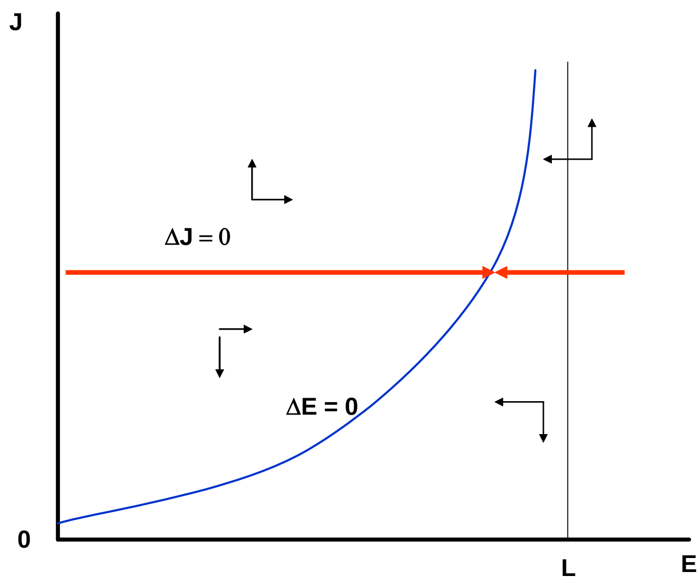


Figure 5: Phase Diagram: Bargaining Equilibrium

We conclude the section by verifying the supposition that the worker's participation condition, $w_t = \frac{p+z}{2} \geq R_t$, never binds. The value of worker's value of a match solves

$$W_{t+1} = w_t{+}\beta \Bigg[(1-s)W_{t+1} + s \Bigg[\Bigg(\frac{F(L-(1-s)E_t, N_{t+1})}{(L-(1-s)E_t}\Bigg) W_{t+1} + \Bigg(1 - \frac{F(L-(1-s)E_t, N_{t+1})}{(L-(1-s)E_t}\Bigg)$$

given that workers who lose their job at the end of period t can find a new one at the beginning of period $t{+}1$ with probability equal to $F(M_{t+1}, N_{t+1})/M_{t+1}$. As the worker's value of unemployment solves

$$U_t = z{+}\beta \left[\left(\frac{F(L-(1-s)E_t, N_{t+1})}{(L-(1-s)E_t}\right) W_{t+1} + \left(1 - \frac{F(L-(1-s)E_t, N_{t+1})}{(L-(1-s)E_t}\right) U_{t+1}\right],$$

the surplus value is

$$W_t - U_t = w_t - z + \beta(1-s)\left(1 - \frac{F(L-(1-s)E_t, N_{t+1})}{(L-(1-s)E_t}\right)(W_{t+1} - U_{t+1})$$

Since the reservation wage R_t is the value of w_t that equates the value of employment and unemployment, that is $W_t - U_t = w_t - R_t$, it follows that

$$\begin{aligned} R_t &= z + \beta(1-s)\left(1 - \frac{F(L-(1-s)E_t, N_{t+1})}{(L-(1-s)E_t}\right)(R_{t+1} - w_{t+1}) \\ &= z + \beta(1-s)\left(1 - \frac{F(L-(1-s)E_t, N_{t+1})}{(L-(1-s)E_t}\right)\left(R_{t+1} - \min\left(p, \max\left(\frac{p+z}{2}, R_{t+1}\right)\right)\right) \\ &\leq z < \frac{p+z}{2}. \end{aligned}$$

8 Labor Market Volatility

In a now famous paper, Shimer (2005) argues that the standard matching model can explain at most 10% of the observed volatility in the ratio of vacancies to unemployment. In this section, I show that the efficient solution to the island matching model can explain over 25% of the volatility given reasonable parameter values if productivity and separation shocks are sufficiently persistent.[6] However, if the wage is set as the outcome of the

[6] In his more recent paper, Shimer (2006) also shows that his model of mismatch unemployment does much better in this dimension.

symmetric non-cooperative bargaining game with delay as the default option then the volatility of the vacancy-unemployment ratio is virtually identical to that found in Shimer's (2005) post WWII time series data.

One implication of matching models that differentiate them from Shimer's (2006) mismatch model is that shocks to the match separation or job destruction rate, s, induce variation in both vacancies and unemployment as well as shocks to productivity, represented in the model by the parameter p. In the efficient solution case, the steady state values of the number of unmatched workers and jobs, M and N, are determined by the following equations:

$$c = \frac{\beta F_N(M,N)(p-z)}{1-\beta(1-s)[1-F_M(M,N)]} \text{ (free entry)}. \tag{23}$$

$$F(M,N) = s(L - M + F(M,N)) \text{ (steady state)}. \tag{24}$$

Without loss of generality, one can normalize the base line value of match productivity per period at $p = 1$. Given the normalization, c and z are expressed in units of output per period. Given a period length of one month, reasonable values of the interest rate and separation rate are $r = 0.004$ and $s = 0.033$. In his papers, Shimer (2005,2006) sets the opportunity cost of employment, z, equal to 0.4 (40% of market output). Hagadorn and Manovskii (2005), Hall (2006), and Mortensen and Nagypál (2006) argue for larger values. As in the last of these papers, I set $z = 0.7$. Because M , N, and L are determined by the steady state condition and the observed average values of the unemployment and vacancy rates over the 12/2000 to 11/2006 period as discussed above, the free entry condition evaluated at these bench mark values can be use to tie down the cost of vacancy posting. The implied value is $c = 0.499$, equal to about two week of match output. Given all these parameter values, one can now use equations (23) and (24) to compute the responses of all the endogenous variables to variation in both the productivity and separation rates.

The central variable of interest in the literature on labor market volatility is the vacancy-unemployment ratio. At the baseline parameter values, the elasticity of the steady state value of the ratio with respect to p, computed using equations (**??**) and (**??**), is 4.58.[7] This measure of the response to

[7] This is a comparative static result, and as such, is not the response one would see to a persistent but transitory shock under rational expectation. However, the evidence in Shimer(2005) suggest that productivity shocks are nearly permanent, a fact that justifies the use of the number as an approximation to the dynamic response.

productivity variation is somewhat larger than the value of 3.43 obtained using the version of the canonical matching model studied by Shimer (2005) and the same parameter values and much larger than the 1.72 number obtained when Shimer's choice for the opportunity cost of employment $z = 0.4$ is assumed.

As Mortensen and Nagypál (2006) argue, the elasticity of the vacancy-unemployment ratio with respect to p is not the only parameter needed to explain volatility given that separation shocks also occur and are known to be negatively correlated with productivity shocks as documented by Shimer (2005). The computed value of the elasticity of the vacancy-unemployment ratio with respect to s is only -0.135. Given the elasticities of the vacancy-unemployment ratio and the standard deviation of log productivity ($\sigma_p = 0.02$), the standard deviation of the log of the separation rate ($\sigma_s = 0.075$), and the correlation between the two ($\rho_{ps} = -0.524$) computed from U.S. post WWII data reported in Shimer (2005), the implied standard deviation of the vacancy-unemployment ratio is

$$\sigma_\theta = \left(\eta_{\theta p}^2 \sigma_p^2 + \eta_{\theta p}\eta_{\theta s}\rho_{ps}\sigma_p\sigma_s + \eta_{\theta s}^2\sigma_s\right)^{\frac{1}{2}} = 0.097$$

where $\theta = V/U$ represents the ratio of vacancies to unemployment and $\eta_{\theta x}$ is the elasticity of θ with respect to x. This number is about 25% of the standard deviation of the vacancy-unemployment ratio ($\sigma_\theta = 0.382$) in Shimer's data.

The worker's flow value of search, R, is procyclic because both the expected wage and the probability of becoming employed increase with productivity and decrease with the separation rate in the efficient solution to the model. To see this point, note that one can write the free entry condition in the efficient solution case as

$$c = \frac{F_N(M,N)(p-R)}{1-\beta(1-s)}$$

where

$$R = z + F_M(1-s)\left(\frac{p-R}{1-\beta(1-s)}\right)$$

is the worker's reservation wage. Given that $F_N(M,N)$, the probability of strictly fewer workers than jobs on an island, is decreasing in N and that M is fixed, at least in the short run, the direct effect of an increase in p is an increase in N. The reservation wage rises with p both directly and because the probability of being on the short side of the market increases with N. Similarly, the reservation wage falls with s. Hence, the response in the reservation wage dampens the effects of shocks on unemployment and vacancies as in the standard model.

When the wage is the outcome of a non-cooperative bargaining game as characterized by Hall and Milgrom (2005), I have shown above that the wage is independent of the reservation wage because $\frac{1}{2}(p+z) > R_t$ for all t. Hence, it follows from equations (19) - (21) that the equilibrium solution is demand determined. Indeed, because

$$w_t = z + \frac{1}{2}(p-z) \text{ and } S_f(t) = \frac{1}{2}\frac{p-z}{1-\beta(1-s)},$$

the number of unmatched jobs solves

$$c = \frac{F(M_t, N_t)}{N_t}\frac{\frac{1}{2}\beta(1-s)\,(p-z)}{1-\beta(1-s)} \text{ for all } t.$$

The response elasticities to shocks in both p and s are much different than those implied by the efficient solution. Indeed, at the baseline parameter values, the elasticity of the vacancy-unemployment ratio with respect to p is 15.11 and with respect to s is -2.77. Given these values and Shimer's (2005) statistics $\sigma_p = 0.02, \sigma_s = 0.075$, and $\rho_{ps} = -0.524$, the implied standard error of the log vacancy-unemployment ratio ($\theta = V/U$) is

$$\sigma_\theta = \left(\eta_{\theta p}^2\sigma_p^2 + 2\eta_{\theta p}\eta_{\theta s}\rho_{ps}\sigma_p\sigma_s + \eta_{\theta s}^2\sigma_s\right)^{\frac{1}{2}} = 0.447, \tag{25}$$

which actually exceeds the observed value of 0.382 reported by Shimer (2005).

Table 1 provides a complete summary of the models implications for its elasticities of each of the endogenous variables with respect to the forcing variables p and s as well as the standard deviations and correlations of the natural logs of the variable. For comparison, the corresponding statistics from Shimer's (2005) data are reported in parentheses. The standard deviations and the covariances used in the calculations are computed using formula implied by a log linear approximation of the relationships. Namely, for $x, y \in \{u, v, \theta, f\}$,

$$\sigma_x = \left(\eta_{xp}^2\sigma_p^2 + 2\eta_{xp}\eta_{xs}\rho_{ps}\sigma_p\sigma_s + \eta_{xs}^2\sigma_s\right)^{\frac{1}{2}}$$

and

$$\sigma_{xy} = \eta_{xp}\eta_{yp}\sigma_p^2 + (\eta_{xp}\eta_{ys} + \eta_{xs}\eta_{yp})\rho_{ps}\sigma_p\sigma_s + \eta_{xs}\eta_{ys}\sigma_s^2.$$

Clearly, given the parameter values, the model does a good job of matching the volatilities observed for all the labor market variables. Note that productivity and job destruction shock have opposite effects on each of the endogenous variables. As a consequence, the correlations among them are even higher than those found in the data even though there are two exogenous shocks.

Implied Elasticities, Standard Deviations and Correlations

	p	s	U	V	V/U	f
Standard Deviation			0.224	0.224	0.447	0.117
(Standard Deviation)			(0.190)	(0.202)	(0.380)	(0.118)
Elasticity Matrix			Correlation Matrix			
U	-6.71	1.64	1	-0.988	-0.997	-0.976
				(-0.894)	(-0.971)	(-0.949)
V	8.40	-1.13	-	1	0.997	0.998
					(0.975)	(0.897)
V/U	15.11	-2.77	-	-	1	0.990
						(0.948)
f	7.11	-.073	-	-	-	1

All variables are in logs.

Shimer's (2005) data statistics are reported in parentheses.

9 Conclusion

The implications of the aggregate matching function implied by assuming random assignment of unmatched worker and jobs to submarkets (islands) and that the match flow in each is equal to the minimum of the realized values of the numbers of worker and jobs that are assigned to the submarket is studied in the paper. When the model is calibrated to match the average unemployment and vacancy rates over the last six years in the U.S., the model fits well the negative relationship between all the vacancy and unemployment rates (Beveridge curve) observed over the same period. Furthermore, the implies relationship between the job finding rate and the vacancy-unemployment ratio is log linear with elasticity within the "reasonable range" reported in Petrongolo and Pissarides (2001). In other words, the matching process generates a "reduced form" matching function with the properties found in the empirical literature.

When the matching process is embedded in a standard model of equilibrium unemployment, I show that the solution to the planner's problem can be implemented by a modified auction. Namely, if the agents matched, workers or employers, are allocated the entire match surplus when their realized number on any island is less than or equal to the number on the other side of the market, then a search equilibrium is socially efficient. Although the efficient solution to the model implies too little volatility in the ratio of unemployment to vacancies for reasonable parameter values, an alternative variant in which wages are set as the outcome of a strategic bargaining game explains all the observed volatility of both unemployment and vacancies in the U.S. as reported in Shimer (2005).

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10 Appendix

10.1 Proof of Proposition 1

By applying equations (1) and (2), one obtains the following:

$$
\begin{aligned}
\frac{\partial F_N}{\partial N} &= F_{NN}(M,N) = \sum_{i=1}^{\infty}\sum_{j=1}^{i-1}\left(\pi(i,j-1;M,N) - \pi(i,j;M,N)\right) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i-1}\pi(i,j-1;M,N) - \sum_{i=1}^{\infty}\sum_{j'=1}^{i}\pi(i,j'-1;M,N) \\
&= -\sum_{i=1}^{\infty}\pi(i,i-1,M,N) < 0 \\
\frac{\partial F_M}{\partial M} &= F_{MM}(M,N) = \sum_{i=1}^{\infty}\sum_{j=0}^{i}\left(\pi(i,j;M,N) - \pi(i-1,j;M,N)\right) \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{i}\pi(i,j;M,N) - \sum_{i'=0}^{\infty}\sum_{j=0}^{i'+1}\pi(i',j;M,N) \\
&= -\sum_{i'=0}^{\infty}\pi(i',i'+1;M,N) = -\sum_{j=1}^{\infty}\pi(j-1,j,M,N) < 0 \\
\frac{\partial F_N}{\partial M} &= F_{NM}(M,N) = \sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\left(\pi(i-1,j;M,N) - \pi(i,j;M,N)\right) \\
&= \sum_{i'=0}^{\infty}\sum_{j=0}^{i'}\pi(i',j;M,N) - \sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\pi(i,j;M,N) \\
&= \sum_{i=0}^{\infty}\pi(i,i;M,N) = \sum_{j=0}^{\infty}\pi(j,j;M,N) = F_{MN}(M,N) = \frac{\partial F_M}{\partial N} > 0
\end{aligned}
$$

Note that

$$
\begin{aligned}
F_{MN}(M,N) &= \sum_{i=0}^{\infty}\frac{e^{-(M+N)M^iN^i}}{i!i!} \le \max_{(x_1,x_2)\ge 0}\left\{\sum_{i=0}^{\infty}\frac{e^{-(M+N)M^iN^i}}{i!i!}|x_1+x_2 = M+N\right\} \\
&= \sum_{i=0}^{\infty}\frac{e^{-(M+N)\left(\frac{M+N}{2}\right)^i\left(\frac{M+N}{2}\right)^i}}{i!i!} = F_{MN}\left(\frac{M+N}{2},\frac{M+N}{2}\right).
\end{aligned}
$$

An application of the limit operator in Mathcad yields $\lim_{x\to\infty} F_{MN}(x,x) = 0$.

10.2 Proof of Proposition 2

Define,

$$x \equiv \frac{M - F(M,N)}{M},$$

Shimer (2006a, footnote 7) claims that

$$\frac{\partial x}{\partial \ln M} + \frac{\partial x}{\partial \ln N} = -\frac{N}{M}\sum_{i=1}^{\infty} \pi(i, i-1; M, N).$$

Since the definition implies

$$\frac{\partial x}{\partial \ln M} = -\left(F_M(M,N) - \frac{F(M,N)}{M}\right)$$

and

$$\frac{\partial x}{\partial \ln N} = -\frac{N}{M} F_N(M,N),$$

it follows that

$$\begin{aligned} & \frac{MF_M(M,N)}{F(M,N)} + \frac{NF_N(M,N)}{F(M,N)} - 1 \\ = \; & \frac{N}{F(M,N)}\sum_{i=1}^{\infty} \pi(i, i-1; M, N) = \frac{N}{F(M,N)}\sum_{i=1}^{\infty} \frac{e^{-(M+N)} M^i N^{\ i-1}}{i!(i-1)!} \\ = \; & \frac{\sum_{i=1}^{\infty} \frac{e^{-(M+N)} i M^i N^{\ i}}{i! i!}}{\sum_{i=1}^{\infty} \frac{e^{-(M+N)} \min(i,j) M^i N^{\ i}}{i! i!}} = F_{MN}(M,N)\frac{E\{i|i=j\}}{E\{\min(i,j)\}}. \end{aligned}$$