

# Young, Old, Conservative and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing

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First Draft, June 2007

## Abstract

We present a general equilibrium model featuring a continuum of overlapping generations, as in Blanchard (1985). In addition, we assume that agents have standard utilities exhibiting constant relative risk aversion and can be born with differing risk aversions, discount factors, and inclinations to work. Once we aggregate, we find that equilibrium asset prices are determined as if the economy was populated by a single representative agent with time varying risk aversion that follows a stationary process. Because of this, our model is observationally similar to the model of Campbell and Cochrane (1999) and is therefore successful at addressing a number of stylized facts about asset prices. The time variation in the risk aversion of the representative agent arises endogenously as a result of aggregating standard life cycle consumption and portfolio choice problems.

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# 1 Introduction

A significant body of research over the last two decades has focused on uncovering the link between variations in asset prices and the fundamental macroeconomic risks to the economy. This has proven to be a challenging task. The baseline textbook consumption-smoothing model predicts too low equity premia, too high risk-free rates, and asset prices that are substantially less volatile than in the data, to name just a few of the widely documented failures. This has led many researchers to become pessimistic about the potential of rational consumption-based pricing models to explain observed asset valuations.

In this paper we introduce some realistic variations to the standard textbook model of consumption smoothing. With the extensions that we consider, we show that our model can account reasonably well for many of the perceived failures of the consumption-based asset pricing model.

In particular, we take the following four main departures from the textbook model: a) Instead of assuming that agents are infinitely lived, we acknowledge the fact that lives are finite and generations may not be altruistically linked through gifts or bequests; b) agents age, and their ability to work declines with age; c) agents need not have the same preferences, and d) consumption and dividends are not equal: dividends are more volatile than consumption, but the two quantities are cointegrated over the long run.

All four extensions appear realistic and plausible. Furthermore, despite the vast diversity in the population that is introduced by the continuous arrival and departure of agents, their changing age, and the differences in their preferences, we are able to obtain a fairly tractable model that addresses several asset pricing puzzles.

The model is an extension of the perpetual youth model of Blanchard (1985). As in Blanchard (1985), agents arrive and die according to independent Poisson processes with constant intensity. In contrast to Blanchard (1985) however, agents endogenously choose the amount of hours that they want to work, and are hired by a representative firm that is faced with a stochastic productivity process obeying a random walk. Hence, in contrast to Blanchard (1985), our model is stochastic, so that we can analyze equity premia.

An additional extension is that agents can have different risk aversions, discount factors, and inclinations to work. These differences (especially the differences in risk aversion between agents)

are important for producing time variation in the market price of risk: In the model, agents who are more risk averse will hold fewer stocks and will be less exposed to aggregate productivity shocks. By contrast, less risk-averse agents will be holding the majority of risk in the economy. Because of this, the wealth of the latter group will increase (decline) proportionately more than the wealth of the former group in response to positive (negative) economic shocks.

Since the economic importance of agents with high risk aversion increases in response to negative news, these agents will need to absorb a larger fraction of aggregate risk during bad times. Because of their high risk aversion, they will require a large compensation for absorbing that risk. Accordingly, the market price of risk (Sharpe ratio) will increase. The opposite will happen in response to good news. Hence the model produces countercyclical variation in the price of risk.

This basic mechanism is simple, transparent and has been studied in models with infinitely lived agents by Dumas (1989) and Wang (1996). The new feature of our paper is that the birth and death of agents in the absence of intergenerational gifts and bequests will imply a stationary wealth distribution, price to dividend ratio, interest rate, etc. This is unlike the earlier papers of Dumas (1989) and Wang (1996) where these quantities could be non-stationary. The important benefit of stationarity is that we can compare the model's quantitative performance to the data and examine whether it can explain asset-pricing puzzles.

Furthermore, the assumption of overlapping generations helps address the risk-free rate puzzle, despite the fact that agents have standard expected-utility preferences. The reason is similar to the insight of the Blanchard (1985) model: Since agents are faced with declining labor income over their life cycle, there is a constant pressure to save when agents are young. This increases savings and reduces the real rate. This simple and intuitive mechanism is absent in models where agents are infinitely lived because of the absence of life-cycle motivations for savings.

Another important feature of the model is that risk-less rates have low volatility. This is especially true when we calibrate the model in such a way that agents exhibit similar saving behavior, despite their differing attitudes to risk. In this case, the variation in the relative importance of the different types of agents will not affect aggregate savings and hence the real rate, but it will affect the market price of risk and the risk premia in the market.

By introducing an explicit labor choice and a production function with non-constant dividend and labor shares, we can reproduce the fact that dividends are more volatile than consumption,

even though they are cointegrated over the long run. The higher volatility of dividends compared to consumption, along with the countercyclical variation in discount rates due to changing risk aversion, makes the volatility of the stock market high, which helps us obtain a reasonably high equity premium. Furthermore, because of the time-varying price of risk, the model can produce substantial predictability of excess returns.

In summary, the model is able to reproduce a number of the stylized facts about asset prices. Importantly, these facts emerge in a framework where the economic mechanisms are transparent and the model assumptions seem standard and natural. Furthermore, despite the richness of the setup, we can construct an equilibrium that is characterized by a single state variable, and is therefore simple to analyze and compute.

The paper is related to various strands of the literature.

There exists a vast literature on asset pricing that explains some of the stylized asset-pricing facts by utilizing habit formation. Constantinides (1990) and Abel (1990) were early contributions in this literature. Campbell and Cochrane (1999), in a highly influential paper, pursued the idea of external habit formation further. They succeeded in engineering a utility function exhibiting external habit formation that addresses several asset-pricing puzzles simultaneously.

Despite the success of external habit formation in addressing asset pricing puzzles, it appears that the degree of “envy” for other people’s consumption that is required by such models is strong. To give a few examples, Ljungqvist and Uhlig (1998) show that in an economy populated by agents with Campbell and Cochrane (1999) preferences, it might be optimal to produce business cycles, instead of trying to avoid them. Furthermore, agents should welcome labor taxes as high as 50%.

These implications seem at odds with the observed reluctance of most citizens to vote for high taxes, and the multitude of economic institutions whose mandate is to promote growth and avoid fluctuations. Hence, even though external habit formation is an appealing idea, the extent of “envy” required to explain asset prices seems strong when viewed against the broader implications of these preferences, beyond asset pricing.

In the model that we propose, the state variable that governs time variation in asset prices resembles in many ways the “surplus” ratio of Campbell and Cochrane (1999). Hence, we are able to obtain a model that is observationally similar to Campbell and Cochrane (1999), but whose economic mechanisms and justification are different. Additionally, in our model dividends

and consumption are different, yet co-integrated, so that the model can provide a laboratory to investigate the net present values of consumption, dividends and labor income as separate quantities. These distinctions have attracted the attention of recent empirical literature in asset pricing<sup>1</sup>.

We also relate to Chan and Kogan (2002). Chan and Kogan (2002) presents an interesting approach to obtain a stationary wealth distribution in the presence of heterogeneity. An advantage of their approach is that they can allow for a continuum of risk aversions. However, their approach requires that agents only care about their consumption relative to some exogenous habit level that is co-integrated with aggregate consumption. Agents with such preferences should accordingly be indifferent between high and low aggregate growth rates, an implication that seems strong, since citizens typically vote for policies that promote aggregate growth. At a more practical level, the framework of Chan and Kogan (2002) produces substantial variability in interest rates, in contrast to our approach.

Several papers utilize variations in the cross-sectional wealth distribution due to some incompleteness to obtain implications for asset prices. This literature is vast and we do not attempt to summarize it. The papers that relate more closely to ours include Basak and Cuoco (1998), Guvenen (2005), Storesletten, Telmer, and Yaron (2007), and Michaelides and Gomes (2007). The first two of these papers assume infinitely lived agents, and the presence of limited participation allows the time variation of the wealth distribution to affect returns. A common implication of models with infinitely lived agents is that wealth eventually concentrates in the hands of agents who participate in markets. Even though ours is not a model of limited stock market participation, the presence of differing risk aversions has observationally similar implications. More importantly, the assumption of overlapping generations implies that all agents start and end life with zero wealth, so that the equity premium will not asymptotically reflect only the risk aversion of one group<sup>2</sup>. Furthermore, an improvement over Guvenen (2005) is that our interest-rate volatility is very low and our consumption process is practically unpredictable. Storesletten, Telmer, and Yaron (2007) and Michaelides and Gomes (2007) study overlapping generations models and introduce frictions. Storesletten, Telmer, and Yaron (2007) study changes in the cross sectional variation of consumption shocks as Constantinides and Duffie (1996). Michaelides and Gomes (2007) analyze a rich

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<sup>1</sup>See, for instance, the work of Lettau and Ludvigson (2005).

<sup>2</sup>Guvenen (2005) avoids this problem by assuming a non-growing economy. Because of overlapping generations we can allow growth and still obtain a stationary wealth distribution.

setup (limited participation, heterogeneity in both preferences and income, etc.) and focus on understanding individual portfolio holdings in general equilibrium. However, both frameworks do not assume an endowment economy, but instead a production economy with exogenous depreciation shocks. Hence their setup resembles more Cox, Ingersoll, and Ross (1985) rather than Lucas (1978). In such a framework, the volatility of the stock market is given exogenously. By contrast in an endowment economy, the volatility of the stock market is determined endogenously. As Storesletten, Telmer, and Yaron (2007) admit, “solving the analogous endowment economy is substantially more difficult”. Because volatility is both challenging and central for many other moments (such as the equity premium, the predictability of returns etc.), we believe that our framework allows us to address a broader set of asset pricing puzzles compared to previous literature.

We further relate to Santos and Veronesi (2006) and Menzly, Santos, and Veronesi (2004), since both these papers produce a dividend process that is not identical to consumption in the short run, but is cointegrated over the long run. The important difference is that in our paper this share process arises endogenously and *jointly* with the time variation in discount rates. We do not have to exogenously assume a structure for the joint dynamics of dividends and discount rates. Hence our approach complements Santos and Veronesi (2006) and Menzly, Santos, and Veronesi (2004) and lets the economic mechanisms of the model dictate this crucially important choice.

There is a vast literature on overlapping generations models. We do not attempt to summarize this literature. A very partial listing of interesting applications of OLG frameworks to asset pricing include Abel (2003), Constantinides, Donaldson, and Mehra (2002), and Heaton and Lucas (2000). Most models in the OLG tradition share the feature that the minimal time periods of the model correspond to decades. The advantage of using a Blanchard (1985) framework is that the model produces implications for any time interval of interest. Given that most empirical regressions are run with monthly, quarterly or yearly data, this makes it easier to relate the model to the empirical asset-pricing literature.

Finally, an attractive feature of the model is that it is tractable, parsimonious, easy to analyze and relate to existing leading asset-pricing models, and it matches asset-pricing data reasonably well. Hence, it can form a departure point for more complex exercises (that we do not pursue here), such as deriving the asset-pricing effects of changes in governmental policies that affect intergenerational transfers, demographic transitions, etc.

Section 2 describes the model. Section 3 presents the solution of the model. Section 4 presents a qualitative discussion and section 5 contains quantitative implications. Section 6 concludes. All proofs are contained in the appendix.

## 2 Model

### 2.1 Agents' Lives and Preferences

There is a continuum of agents whose mass we will normalize to 1. Existing agents face a constant hazard rate of death  $\pi > 0$  throughout their lives. Furthermore newly born agents also arrive at a rate of  $\pi$  per unit of time, so that the population remains constant. These demographic assumptions are identical to Blanchard (1985) and are key for the tractability of all the aggregation results.

As is standard in the literature, we will furthermore assume that agents have constant relative risk aversion and enjoy leisure. A key departure from prevailing representative-agent approaches is that we will explicitly allow for the possibility that agents have heterogenous preferences. The most parsimonious way to introduce heterogeneity is to follow Dumas (1989) and Wang (1996) and assume the presence of two types of agents, which we will label as “type-A” and “type-B” agents. In particular, normalizing the amount of hours that an agent can work at birth to  $\frac{\pi+\chi}{\pi}$ , we will assume that “type-A” agents have mass  $v$  and preferences of the form

$$E_s \int_s^\infty e^{-(\rho_A+\pi)(t-s)} \frac{\left( (c_{t,s}^A)^{\psi_A} \left( \frac{\pi+\chi}{\pi} e^{-\chi(t-s)} - h_{t,s}^A \right)^{1-\psi_A} \right)^{1-\gamma_A}}{\psi_A (1-\gamma_A)} dt, \quad (1)$$

where  $\rho_A > 0$  is a subjective discount rate and  $\gamma_A > 0$  is the relative risk aversion for agents of type A. As is well known in the literature, the agent’s effective discount rate is given by  $\rho_A + \pi$ , because of the probability of death. Throughout we will keep the notational convention that  $h_{t_1,t_2}^A$  denotes the hours worked at time  $t_1$  by an agent of type A who was born at time  $t_2 \leq t_1$ .  $c_{t_1,t_2}^A$  is defined similarly. The constant  $\psi_A \in (0, 1)$  controls the relative importance of leisure and consumption.

We follow Blanchard (1985) and assume that the agent’s endowment of hours declines exponentially over the life-cycle at the rate  $\chi$ . Blanchard (1985) argues that this simple assumption captures the idea that agents retire, so that their income over the life cycle is downward-sloping. The exponential nature of the decline facilitates aggregation<sup>3</sup>.

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<sup>3</sup>We remark here that our results would be unchanged if we assumed that agents have a constant endowment of

The second type of agents (“type- $B$  agents”) have mass  $1 - v$  and preferences of the same form as (1), but with potentially different coefficients. Therefore, their expected utility is given by

$$E_s \int_s^\infty e^{-(\rho_B + \pi)(t-s)} \frac{\left( (c_{t,s}^B)^{\psi_B} \left( \frac{\pi + \chi}{\pi} e^{-\chi(t-s)} - h_{t,s}^B \right)^{1-\psi_B} \right)^{1-\gamma_B}}{\psi_B (1 - \gamma_B)} dt. \quad (2)$$

## 2.2 Technology

The representative competitive firm owns a fixed capital stock that we will normalize to 1 and produces a stochastic output

$$Y_t = Z_t f(H_t), \quad (3)$$

where  $Z_t$  follows a geometric Brownian motion

$$\frac{dZ_t}{Z_t} = \mu_Z dt + \sigma_Z dB_t$$

for two positive constants  $\mu_Z$  and  $\sigma_Z$ .  $H_t$  denotes the aggregate hours worked at time  $t$  and is given by

$$H_t \equiv \int_{-\infty}^t \pi e^{-\pi(t-s)} (v h_{t,s}^A + (1-v) h_{t,s}^B) ds. \quad (4)$$

Note that the expression in (4) accounts for the age distribution in the population though the term  $\pi e^{-\pi(t-s)}$  inside the integral.

The function  $f(H_t)$  in (3) is an increasing and concave function of the aggregate hours worked ( $H_t$ ). In particular, we will assume that  $f(H_t)$  solves the following ordinary differential equation:

$$f'(H) = \frac{\alpha(H)f(H)}{H}, \quad f(0) = 0 \quad (5)$$

and that  $\alpha(H)$  is a continuous function satisfying

$$\alpha(H) \in (0, 1), \quad \alpha'(H) \leq 0. \quad (6)$$

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hours that they lose for ever at some random exponentially distributed time that arrives with intensity  $\chi$ . One could interpret such a situation as a health shock that leads to retirement. If we interpret the model in this way, we would need to also assume the existence of “health insurance” markets, in order to preserve tractability. In those markets agents would be able to enter contracts that deliver payoffs contingent on the arrival of the health shock.



First note that  $f'(H) > 0$ , given the above assumptions. Second, differentiating both sides of (5) and using (6) shows that  $f''(H) < 0$ . Furthermore, in the special case where  $\alpha(H)$  is constant and equal to  $\alpha$ , the resulting solution to (5) is the familiar Cobb-Douglas production function  $f(H) = H^\alpha$ . When  $\alpha(H)$  is chosen as  $\alpha(H) = \frac{(1-b)H^{-\nu}}{(1-b)H^{-\nu}+b}$  for some  $\nu > 0$  and some  $0 < b < 1$ , then  $f(H)$  specializes to the CES production function. Later we show that allowing  $\alpha(H)$  to vary will make it possible to match the empirical fact that the labor share is counter-cyclical and that dividends are more volatile than consumption.

### 2.3 Budget Constraints

An agent who supplies  $h_t$  hours of labor at time  $t$  earns a labor income of  $w_t h_t$ , where  $w_t$  is the prevailing wage. The agent can also trade in a risk-less bond and a stock. The rate of return on bonds is given by  $r_t$ . The stock is a claim that delivers a dividend flow given by

$$D_t \equiv Y_t - w_t H_t.$$

It is reasonable to conjecture that the stock-price process follows a diffusion:

$$dS_t = (\mu_t S_t - D_t)dt + \sigma_t S_t dB_t \tag{7}$$

for some processes  $\mu_t$  and  $\sigma_t$ . The processes for  $w_t, r_t, \mu_t$ , and  $\sigma_t$  will be jointly determined later so that markets clear. It will be convenient for future reference to define the stochastic discount factor process as

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \kappa_t dB_t, \tag{8}$$

where  $\kappa_t$  is the Sharpe ratio in the market defined as

$$\kappa_t = \frac{\mu_t - r_t}{\sigma_t}. \tag{9}$$

For now, we just note that the agent's financial wealth evolves as

$$dK_{t,s} = (K_{t,s}(r_t + \pi) - c_{t,s} + \theta_{t,s}(\mu_t - r_t) + w_t h_{t,s}) dt + \theta_{t,s} \sigma_t dB_t, \tag{10}$$

where  $K_{t,s}$  denotes the financial wealth at time  $t$  of an agent born at time  $s$  and  $\theta_{t,s}$  denotes the dollar investment in stocks. (Since this equation is the same for agents of both types, we drop the

superscript  $A, B$  for simplicity). Equation (10) is a standard dynamic budget constraint. The term  $\pi K_{t,s}$  captures the fact that the agent has no bequest motive and hence will choose to annuitize her entire wealth<sup>4</sup>. Furthermore, competition amongst competitive life insurers will drive the income promised by each annuity contract towards the actuarially fair flow  $\pi$ . To keep the presentation concise, we refer the reader to Blanchard (1985) who shows in detail that  $\pi$  is the market clearing price of such a market. For the rest of the paper, we will be concerned with clearing the remaining markets.

## 2.4 Markets and Equilibrium

There are four markets that must clear in equilibrium: 1) labor market; 2) current-consumption-good market; 3) bond market, where a bond available in zero net supply is traded, and 4) a unit positive supply market for trading a claim to dividends (the stock market).

The definition of equilibrium is standard:

**Definition 1** *An equilibrium is defined as a set of progressively measurable processes for each agent's consumption, labor, and portfolio  $c_{t,s}^i, h_{t,s}^i, \theta_{t,s}^i$  for  $i \in \{A, B\}$  and a set of progressively measurable processes for the rate of return in the bond market ( $r_t$ ), wages ( $w_t$ ) and an appropriate stock market process of the form (7) with progressively measurable coefficients  $\mu_t, \sigma_t$  such that:*

1. *Given the process for  $\{r_t, w_t, \mu_t, \sigma_t\}$  and for any  $s$  and all  $t \geq s$ , the processes  $c_{t,s}^i, h_{t,s}^i, \theta_{t,s}^i$  for  $i = A, B$  maximize (1) (objective [1] respectively) subject to (10), the initial condition  $K_{t,t}^i = 0$  and the transversality condition  $\lim_{t \rightarrow \infty} e^{-\pi t} \xi_t K_{t,s}^i = 0$ .*
2. *Given  $w_t$ , firms choose hours  $H_t^{opt}$  so as to maximize profits:*

$$H_t^{opt} = \arg \max_{H_t} D_t \tag{11}$$

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<sup>4</sup>To be more specific, annuities work as follows in this context. The agent signs an instantaneous contract that delivers competitive insurers a fraction  $\eta_t$  of her wealth upon death in exchange for an income of  $\hat{\pi}_t \eta_t K_t$  while the agent is alive. Since the agent has no bequest motives,  $\eta_t = 1$ . For details see Blanchard (1985).

3. Given  $c_{t,s}^i, h_{t,s}^i, \theta_{t,s}^i$  for  $i \in \{A, B\}$  all markets clear, i.e.,

$$\int_{-\infty}^t \pi e^{-\pi(t-s)} (v h_{t,s}^A + (1-v) h_{t,s}^B) ds = H_t^{opt} \quad (12)$$

$$\int_{-\infty}^t \pi e^{-\pi(t-s)} (v c_{t,s}^A + (1-v) c_{t,s}^B) ds = Y_t \quad (13)$$

$$\int_{-\infty}^t \pi e^{-\pi(t-s)} (v \theta_{t,s}^A + (1-v) \theta_{t,s}^B) ds = S_t \quad (14)$$

$$\int_{-\infty}^t \pi e^{-\pi(t-s)} (v (K_{t,s}^A - \theta_{t,s}^A) + (1-v) (K_{t,s}^B - \theta_{t,s}^B)) ds = 0. \quad (15)$$

Equation (12) states that aggregate hours supplied by all agents of either type who are alive at time  $t$  have to add up to the total hours demanded by firms. Equations (13), (14), and (15) capture the analogous requirements for the goods market, the stock market, and the bond market.

### 3 Solution

In this section we construct an equilibrium. We start by letting  $X_t$  denote the consumption share of type  $A$  agents, namely

$$X_t \equiv \frac{v \int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s}^A ds}{Y_t}. \quad (16)$$

Since the consumptions of both agents are non-negative, the goods-market clearing condition (13) implies that  $X_t \in [0, 1]$ . In the remainder of the section we will construct an equilibrium with the following properties: a)  $(X_t, Z_t)$  are jointly Markovian, b)  $r_t, \mu_t, \sigma_t$  are functions of  $X_t$  exclusively, whereas  $w_t$  will have the form  $w_t = Z_t \omega(X_t)$  for an appropriate function  $\omega$  that we will determine explicitly. In practical terms, this implies that a single variable, namely  $(X_t)$  will be sufficient to characterize the equilibrium interest rate, expected stock market returns and volatility, despite the heterogeneity created by overlapping generations and differences in preferences.

#### 3.1 Consumption, Labor, and Human Capital

To establish the claims above, we start by defining

$$u(c_{t,s}^i, h_{t,s}^i) = \frac{\left( (c_{t,s}^i)^{\psi_i} \left( \frac{\pi + \chi}{\pi} e^{-\chi(t-s)} - h_{t,s}^i \right)^{1-\psi_i} \right)^{1-\gamma_i}}{\psi_i (1-\gamma_i)} \quad \text{for } i \in \{A, B\}. \quad (17)$$

We adopt the notational convention that  $u_1$  denotes the first partial derivative of  $u$  with respect to its first argument and  $u_2$  the first partial derivative with respect to the second argument. With this convention, and assuming that there exists a stochastic discount factor  $\xi_t$ , an agent's optimal consumption and labor choice satisfy the first order conditions

$$e^{-(\pi+\rho_A)(t-s)} \frac{u_1(c_{t,s}^i, h_{t,s}^i)}{u_1(c_{s,s}^i, h_{s,s}^i)} = e^{-\pi(t-s)} \frac{\xi_t}{\xi_s} \quad (18)$$

$$-\frac{u_2(c_{t,s}^i, h_{t,s}^i)}{u_1(c_{t,s}^i, h_{t,s}^i)} = w_t. \quad (19)$$

Equation (18) captures the intertemporal aspect of an agent's problem. Roughly speaking, it states that the marginal benefit of an additional unit of consumption in a given state as measured by the marginal utility of consumption should be equal to the "cost" of a unit of consumption in that state. In turn this "cost" is measured by the product of the stochastic discount factor and the probability that the consumer will live until time  $t$  (namely  $e^{-\pi(t-s)}$ ). Equation (19) is the standard intratemporal first order condition. It states that the ratio of marginal utilities of leisure to consumption should be equalized to the opportunity cost of leisure, namely the real wage.

Using the functional-form assumption (17) and the intratemporal first order condition (19) one arrives at the following relationship between hours, consumption and wages:

$$h_{t,s}^i = \frac{\pi + \chi}{\pi} e^{-\chi(t-s)} - \frac{(1 - \psi_i) c_{t,s}^i}{\psi_i w_t} \quad \text{for } i \in \{A, B\} \quad (20)$$

Letting  $H_t$  denote the aggregate hours supplied in the economy and using (20) along with (16) gives

$$\begin{aligned} H_t &= \int_{-\infty}^t \pi e^{-\pi(t-s)} (v h_{t,s}^A + (1-v) h_{t,s}^B) ds \\ &= 1 - \frac{Y_t}{w_t} \left( \frac{(1-\psi_A)}{\psi_A} X_t + \frac{(1-\psi_B)}{\psi_B} (1-X_t) \right) \end{aligned} \quad (21)$$

This expression is the aggregate labor supply relation implied by the model.

To clear the labor market, it remains to determine the aggregate labor demand. To achieve that, we turn attention to the representative firm's optimization problem (11), which leads to the first order condition

$$Z_t f'(H_t) = w_t. \quad (22)$$

Using (5) and (3), equation (22) becomes

$$\frac{Y_t}{w_t} = \frac{H_t}{\alpha(H_t)}, \quad (23)$$

and using (23) inside (21) results in

$$H_t = 1 - \frac{H_t}{\alpha(H_t)} \left( \frac{(1 - \psi_A)}{\psi_A} X_t + \frac{(1 - \psi_B)}{\psi_B} (1 - X_t) \right). \quad (24)$$

Given a value of  $X_t$ , equation (24) determines the equilibrium quantity of hours implied by the model. We shall therefore write  $H_t = \mathcal{H}(X_t)$  to denote this dependence on  $X_t$ . Furthermore, equation (22) implies that the equilibrium wage can be written in the form  $w_t = Z_t f'(\mathcal{H}(X_t))$ . It will be useful to define

$$\omega(X_t) \equiv f'(\mathcal{H}(X_t)),$$

so that the resulting equilibrium wage can be expressed as  $w_t = Z_t \omega(X_t)$  as asserted at the beginning of this section.

Next we turn our attention to the determination of the interest rate  $r_t$  and the Sharpe ratio  $\kappa_t$ . Using the functional-form specification (17) and carrying out the differentiations leads to

$$\frac{c_{t,s}^i}{c_{s,s}^i} = e^{-\frac{\rho_i}{1-\psi_i(1-\gamma_i)}(t-s)} \left( \frac{\frac{\pi+\chi}{\pi} e^{-\chi(t-s)} - h_{t,s}^i}{\frac{\pi+\chi}{\pi} - h_{s,s}^i} \right)^{-\frac{(1-\psi_i)(\gamma_i-1)}{1-\psi_i(1-\gamma_i)}} \left( \frac{\xi_t}{\xi_s} \right)^{-\frac{1}{1-\psi_i(1-\gamma_i)}}. \quad (25)$$

We observe that, since (20) has to hold at all dates and states, it implies the following relation between consumption, hours, and wages between two different points in time:

$$\frac{\frac{\pi+\chi}{\pi} e^{-\chi(t-s)} - h_{t,s}^i}{\frac{\pi+\chi}{\pi} - h_{s,s}^i} = \frac{c_{t,s}^i w_s}{c_{s,s}^i w_t}. \quad (26)$$

Combining (25) with (26) and rearranging leads to

$$\frac{c_{t,s}^i}{c_{s,s}^i} = e^{-\frac{\rho_i}{\gamma_i}(t-s)} \left( \frac{w_t}{w_s} \right)^{\frac{(1-\psi_i)(\gamma_i-1)}{\gamma_i}} \left( \frac{\xi_t}{\xi_s} \right)^{-\frac{1}{\gamma_i}}. \quad (27)$$

Equation (27) gives the consumer's ratio of optimal consumption between birth and any other point in time as a function of wages and the stochastic discount factor. In order to determine the initial consumption ( $c_{s,s}^i$ ) we employ the inter-temporal budget constraint

$$E_s \int_s^\infty e^{-\pi(t-s)} c_{t,s}^i \xi_t dt = E_s \int_s^\infty e^{-\pi(t-s)} w_t h_{t,s}^i \xi_t dt. \quad (28)$$

This states that the consumer's net present value of consumption over the life cycle should be equal to the net present value of her labor income (since she is born with zero financial wealth). Using (20) inside the right-hand side of (28) and rearranging gives

$$E_s \int_s^\infty e^{-\pi(t-s)} c_{t,s} \xi_t dt = \psi_i \frac{\pi + \chi}{\pi} E_s \int_s^\infty e^{-(\pi+\chi)(t-s)} w_t \xi_t dt.$$

It is convenient at this point to make the following two conjectures, which we will verify subsequently. Namely, we conjecture first that the net present value of human capital, defined as

$$\Phi_s^i \equiv \psi_i \frac{\pi + \chi}{\pi} E_s \int_s^\infty e^{-(\pi+\chi)(t-s)} w_t \frac{\xi_t}{\xi_s} dt, \quad (29)$$

can be expressed as

$$\Phi_s^i = \phi^i(X_s) Y_s \quad (30)$$

for an appropriate function  $\phi^i(X_t)$ . Second, we conjecture that the initial consumption  $c_{s,s}^i$  for each agent  $i \in \{A, B\}$  is given by

$$c_{s,s}^i = \beta^i(X_s) Y_s \quad (31)$$

for an appropriate function  $\beta^i(X_s)$  that will be determined subsequently.

### 3.2 Dynamics of the Stochastic-Discount-Factor and the Consumption Share

Given these assumptions, it is possible to derive the dynamics of  $X_t$ . In particular, we will be interested in determining the drift and diffusion coefficients of the diffusion

$$dX_t = \mu_X dt + \sigma_X dB_t. \quad (32)$$

To achieve that, define the function  $g(X_t)$  as

$$g(X_t) \equiv \frac{Y_t}{Z_t} = f(\mathcal{H}(X_t)). \quad (33)$$

Combining (27), the fact that  $w_t = Z_t \omega(X_t)$ , the definition of  $g$  in equation (33) and the definition of  $X_t$  in equation (16) leads to

$$X_t Y_t = v \int_{-\infty}^t \pi e^{-\left(\pi + \frac{\rho_1}{\gamma_1}\right)(t-s)} \beta_s^A Z_s g_s \left( \frac{Z_t \omega_t}{Z_s \omega_s} \right)^{\frac{(1-\psi_1)(\gamma_1-1)}{\gamma_1}} \left( \frac{\xi_t}{\xi_s} \right)^{-\frac{1}{\gamma_1}} ds, \quad (34)$$

where we have used the shorthand notation  $g_s = g(X_s)$  and  $\omega_s = \omega(X_s)$ . Applying Ito's Lemma on both sides of (34) and matching the resulting diffusion coefficients on both sides shows that (34) implies

$$\frac{\sigma_X}{X_t} + \sigma_Z + \frac{g'}{g}\sigma_X = \frac{(1-\psi_A)(\gamma_A-1)}{\gamma_A} \left( \frac{\omega'}{\omega}\sigma_X + \sigma_Z \right) + \frac{\kappa_t}{\gamma_A}. \quad (35)$$

Similarly, matching the drift coefficients on both sides gives

$$\begin{aligned} \mu_X + \left( \mu_Z + \frac{g'}{g}(\mu_X + \sigma_X\sigma_Z) + \frac{\sigma_X^2 g''}{2g} \right) X_t + \left( \frac{\omega'}{\omega}\sigma_X + \sigma_Z \right) \sigma_X \\ = X_t \left[ \mathcal{D} \left( w_t^{\frac{(1-\psi_A)(\gamma_A-1)}{\gamma_A}} \xi_t^{-\frac{1}{\gamma_A}} \right) - \left( \pi + \frac{\rho_A}{\gamma_A} \right) \right] + v\pi\beta_t^A, \end{aligned} \quad (36)$$

where we have used the shorthand notation

$$\begin{aligned} \mathcal{D} \left( w^a \xi^b \right) \equiv a \left( \mu_Z + \frac{\sigma_Z^2}{2}(a-1) + \frac{\omega'}{\omega}(\mu_X + a\sigma_Z\sigma_X - \kappa_t b\sigma_X) - b\kappa_t\sigma_Z \right) \\ + \frac{\sigma_X^2}{2} \left( a(a-1) \left( \frac{\omega'}{\omega} \right)^2 + a \frac{\omega''}{\omega} \right) - b \left( r_t + \frac{\kappa_t^2}{2}(1-b) \right). \end{aligned} \quad (37)$$

To solve for  $\mu_X$  and  $\sigma_X$  from equations (35) and (36) we need to obtain expressions for  $r_t$  and  $\kappa_t$ , which is facilitated by the goods-market clearing condition (13). Specifically, combining (13) with (27) gives

$$\sum_{i \in \{A,B\}} \int_{-\infty}^t \pi e^{-\left(\pi + \frac{\rho_i}{\gamma_i}\right)(t-s)} v_i \beta_s^i Z_s g_s \left( \frac{Z_t \omega_t}{Z_s \omega_s} \right)^{\frac{(1-\psi_i)(\gamma_i-1)}{\gamma_i}} \left( \frac{\xi_t}{\xi_s} \right)^{-\frac{1}{\gamma_i}} ds = Z_t g(X_t), \quad (38)$$

where  $v_A = v$  and  $v_B = 1-v$ . Once again, applying Ito's Lemma to both sides of (38) and matching diffusion terms on both sides yields

$$\sigma_Z + \frac{g'}{g}\sigma_X = \sum_{i \in \{A,B\}} x_t^i \left[ \frac{\kappa_t}{\gamma_i} + \frac{(1-\psi_i)(\gamma_i-1)}{\gamma_i} \left( \frac{\omega'}{\omega}\sigma_X + \sigma_Z \right) \right], \quad (39)$$

where  $x_t^A = X_t$  and  $x_t^B = 1 - X_t$ . Similarly, by matching drift coefficients we obtain

$$\mu_Z + \frac{g'}{g}(\mu_X + \sigma_X\sigma_Z) + \frac{1}{2} \frac{g''}{g} \sigma_X^2 = \sum_{i \in \{A,B\}} v_i \pi \beta_t^i + x_t^i \left[ \mathcal{D} \left( w^{\frac{(1-\psi_i)(\gamma_i-1)}{\gamma_i}} \xi^{-\frac{1}{\gamma_i}} \right) - \left( \pi + \frac{\rho_i}{\gamma_i} \right) \right]. \quad (40)$$

Fixing a value of  $X_t$ , equations (35) and (39) form a linear system in  $\sigma_X$  and  $\kappa_t$  that can be solved explicitly. This yields  $\sigma_X$  and  $\kappa_t$  as functions of  $X_t$ . Having obtained  $\sigma_X$  and  $\kappa_t$ , equations (36) and (40) also form a linear system in  $r_t$  and  $\mu_X$  that can be solved explicitly, yielding  $r_t$  and

$\mu_X$  as functions of  $X_t$ . Since both  $\mu_X$  and  $\sigma_X$  are functions of  $X_t$ , the consumption share process  $X_t$  is a Markov process as asserted at the beginning of the section.

The last step in the construction of the equilibrium stochastic discount factor is the explicit determination of the functions  $\phi^i$  and  $\beta^i$ . The following Lemma shows how to obtain these functions:

**Lemma 1** *Let  $\sigma_X(X_t), \kappa(X_t), \mu_X(X_t)$ , and  $r(X_t)$  denote the solution to the system (35)-(40) and let*

$$\mu_Y \equiv \mu_Z + \frac{g'}{g} (\mu_X + \sigma_X \sigma_Z) + \frac{\sigma_X^2}{2} \frac{g''}{g} \quad (41)$$

$$\sigma_Y \equiv \sigma_Z + \frac{g'}{g} \sigma_X. \quad (42)$$

Then the function  $\phi^A(X_t)$  is the solution to the differential equation

$$0 = \frac{\sigma_X^2}{2} (\phi^A)'' + (\phi^A)' (\mu_X + \sigma_X (\sigma_Y - \kappa)) + \phi^A (\mu_Y - r - \sigma_Y \kappa - \pi - \chi) + \psi^A \frac{\pi + \chi}{\pi} \frac{\omega(X_t)}{g(X_t)}, \quad (43)$$

where we have used the simpler notation  $\sigma_X, \mu_X, r, \kappa$ , rather than  $\sigma_X(X_t), \mu_X(X_t)$ , etc.. The function  $\phi^B(X_t)$  is given by  $\phi^B(X_t) = \frac{\psi_B}{\psi_A} \phi^A(X_t)$ . Finally, the functions  $\beta^i$ ,  $i \in \{A, B\}$  are given as  $\beta^i(X_t) = \phi^i(X_t) / \zeta^i(X_t)$ , where  $\zeta^i(X_t)$  solves the differential equation

$$\begin{aligned} -1 &= \frac{\sigma_X^2}{2} (\zeta^i)'' + (\zeta^i)' \left( \mu_X + \sigma_X \frac{(1 - \psi_i)(\gamma_i - 1)}{\gamma_i} \left( \sigma_Z + \frac{\omega'(X_t)}{\omega(X_t)} \sigma_X \right) - \sigma_X \frac{\gamma_i - 1}{\gamma_i} \kappa \right) \\ &+ \zeta^i \left[ \mathcal{D} \left( w_t \frac{(1 - \psi_i)(\gamma_i - 1)}{\gamma_i} \xi_t^{1 - \frac{1}{\gamma_i}} \right) - \left( \pi + \frac{\rho_i}{\gamma_i} \right) \right]. \end{aligned} \quad (44)$$

Equations (43) and (44) form a system of three ordinary differential equations in  $\zeta^i(X_t), \phi^A(X_t)$ . By determining the solution to these three differential equations we can obtain  $r_t, \kappa_t, \mu_X$  and  $\sigma_X$  as functions of  $X_t$ , which in turn allows us to determine the dynamics of the stochastic discount factor  $\xi_t$  for this economy.

### 3.3 Stock Price

Given  $\xi_t$  it is possible to define the stock market value as follows

$$S_t \equiv E_t \int_t^\infty \left( \frac{\xi_u}{\xi_t} \right) D_u du. \quad (45)$$



We will assume throughout that  $S_t < \infty$ . To verify that the constructed allocation forms an equilibrium, it remains to verify conditions (14) and (15). Adding up these two equations, and using Walras' law it suffices to verify that the aggregate financial wealth is equal to the stock market value

$$\sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i K_{t,s}^i ds = S_t \quad (46)$$

The next Lemma asserts that equation (46) holds.

**Lemma 2** *Let  $S_t$  be defined as (45). Then equation (46) holds.*

It is also possible to give a simple expression for  $S_t$  in terms of the functions  $\zeta^i, \phi^i$ .

**Lemma 3** *The stock market value is given as*

$$S_t = Y_t \left[ \sum_{i \in \{A, B\}} \zeta^i(X_t) \frac{X_t^i}{\psi^i} - \frac{\pi}{\pi + \chi} \frac{\phi^A(X_t)}{\psi^A} \right] \quad (47)$$

From (47) we obtain the price dividend ratio as

$$p(X_t) = \frac{S_t}{D_t} = \frac{1}{1 - \alpha(\mathcal{H}(X_t))} \left[ \sum_{i \in \{A, B\}} \zeta^i(X_t) \frac{X_t^i}{\psi^i} - \frac{\pi}{\pi + \chi} \frac{\phi^A(X_t)}{\psi^A} \right]$$

Finally, the stock-market volatility is computed as

$$\sigma_t = \sigma(X_t) = \sigma_Z + \left( \frac{p'}{p} - \frac{\alpha' \mathcal{H}'}{1 - \alpha} + \frac{g'}{g} \right) \sigma_X \quad (48)$$

and the expected return on the stock market as

$$\mu_t = r_t + \kappa_t \sigma_t.$$

## 4 Qualitative Features of the Model

Before proceeding with an analysis of the quantitative implications of the model, it is easiest to start by examining some special cases that will illuminate the channels behind the model. Our aim in this section is simply to give intuition. The quantitative importance of these channels is illustrated in the next section.

## 4.1 Homogenous Preferences and the Role of Life-Cycle Savings

We will start our analysis with the special case where  $\psi_A = \psi_B = 1$ ,  $\gamma_A = \gamma_B = \gamma$ , and  $\rho_A = \rho_B = \rho$ , so that agents of type  $A$  and agents of type  $B$  have identical preferences and supply labor inelastically. In this case  $X_t = 1$ , hence  $\mu_X = \sigma_X = 0$ , and also  $H_t = 1$ . Furthermore, all functions of  $X_t$  are constants that we can determine in closed form. Specifically, equation (24) shows that aggregate hours are constant<sup>5</sup>, and hence  $g', g'', \omega', \omega''$  are all zero. Hence equations (39) and (40) become

$$\kappa = \sigma_Z \gamma \tag{49}$$

$$r = \rho + \gamma \mu_Z - \frac{\kappa^2}{2} \left( \frac{1 + \gamma}{\gamma} \right) - \gamma \pi (\beta - 1). \tag{50}$$

The above two equations are reminiscent of the equations that are obtained in standard textbook treatments of the Lucas (1978) tree model with a representative agent having CRRA preferences and facing an endowment that follows a geometric Brownian motion. In particular, when  $\pi = 0$  the above two equations coincide with the well known equations for the Sharpe ratio and the interest rate in a Lucas tree setup.

The only departure from the standard representative-agent model that is introduced by overlapping generations is the additional term  $-\gamma \pi (\beta - 1)$  in the expression of the interest rate. This term will tend to reduce the interest rate whenever  $\beta > 1$  — i.e., when the consumption of the agents entering the economy is larger than that of the agents exiting the economy, in this model equal to average consumption — and hence can help resolve the so-called low risk free rate puzzle.

To determine  $\beta$ , we use equations (61) and (44) recognizing that  $\zeta' = \zeta'' = \phi' = \phi'' = 0$  and the fact that  $\beta = \frac{\phi}{\zeta}$  to obtain

$$\beta = \frac{\pi + \chi}{\pi} \frac{\alpha(1)}{r + \pi + \chi + \sigma_Z \kappa - \mu_Z} \left[ \frac{\gamma - 1}{\gamma} \left( r + \frac{\kappa^2}{2} \frac{1}{\gamma} \right) + \pi + \frac{\rho}{\gamma} \right].$$

Plugging this expression for  $\beta$  into (50) leads to the following quadratic equation for the interest

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<sup>5</sup>This is reminiscent of the result in the seminal paper by King, Plosser, and Rebelo (1988), who show an analogous result in a model that has a representative agent with preferences given by (17).

rate:

$$\begin{aligned}
0 &= \frac{1}{\gamma} r^2 + r \left( -\mu_Z - \left( \pi + \frac{\rho}{\gamma} \right) + \frac{1}{\gamma} (\pi + \chi + \sigma_Z \kappa - \mu_Z) + (\pi + \chi) \alpha(1) \frac{\gamma - 1}{\gamma} + \frac{\kappa^2}{2} \frac{1 + \gamma}{\gamma^2} \right) \\
&\quad \left[ -\mu_Z + \frac{\kappa^2}{2} \left( \frac{1 + \gamma}{\gamma^2} \right) - \left( \pi + \frac{\rho}{\gamma} \right) \right] (\pi + \chi + \sigma_Z \kappa - \mu_Z) \\
&\quad + (\pi + \chi) \alpha(1) \left[ \frac{\gamma - 1}{\gamma^2} \frac{\kappa^2}{2} + \pi + \frac{\rho}{\gamma} \right].
\end{aligned}$$

Figure 1 illustrates the effects of changing  $\chi$  on the equilibrium interest rate  $r$ . The graph reconfirms (in a stochastic environment) the observations originally made by Blanchard (1985): An increase in  $\chi$  reduces the interest rate.

This is intuitive: A more steeply declining labor income forces agents to save early in life, thus raising savings and lowering the equilibrium interest rate. This helps in resolving the low risk free rate puzzle. Figure 1 illustrates how one can still obtain relatively low interest rates even for high levels of  $\gamma$ .

Even though the decline of labor income over the life cycle can help explain the low real rates that are observed in reality, a model with identical agents produces a constant price-to-dividend ratio and hence cannot explain why the stock market is more volatile than dividends, which in turn are more volatile than consumption. Next, we will utilize the heterogeneity of preferences to introduce variation in discount rates and hence the price-to-dividend ratio.

## 4.2 Heterogenous Agents

### 4.2.1 Sharpe Ratio

In what follows we will continue to assume that  $\psi_A = \psi_B = 1$  but  $\gamma_A \neq \gamma_B$  and  $\rho_A \neq \rho_B$ . Without loss of generality we will assume that  $\gamma_A < \gamma_B$ . This special case is particularly attractive, because it implies that aggregate hours worked will still be  $H_t = 1$ . However, now  $X_t$  will be a time varying process. Since hours are constant, this implies that the aggregate output  $Y_t$  satisfies

$$\frac{dY_t}{Y_t} = \frac{dZ_t}{Z_t},$$

so that the aggregate endowment follows a geometric Brownian motion, as is commonly assumed in the literature. Furthermore, since hours are not time varying, both functions  $g(X_t) = f(\mathcal{H}(X_t))$  and

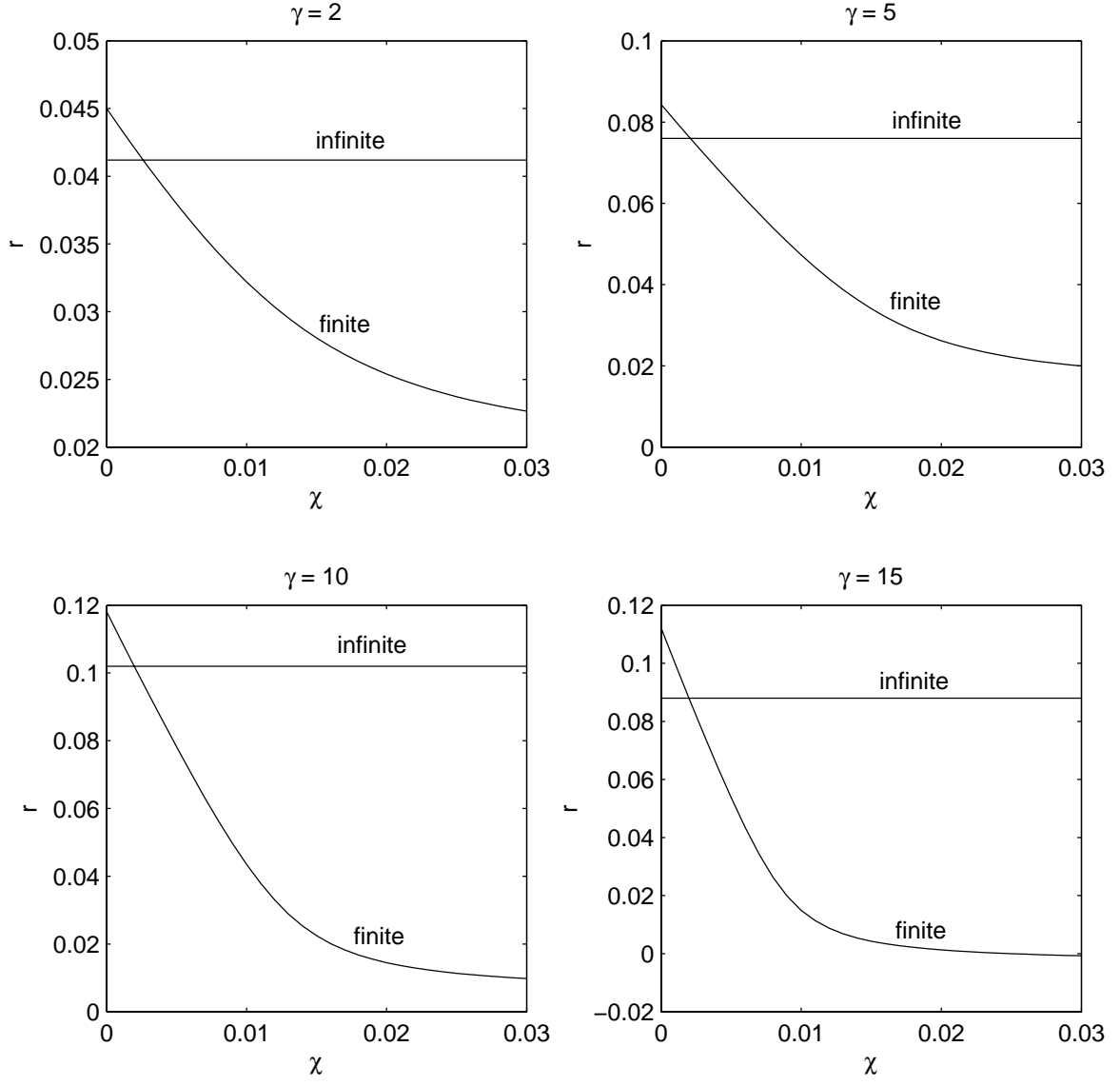


FIGURE 1: Interest rates as a function of  $\chi$  for various levels of risk aversion  $\gamma$ . The line “infinite” refers to the interest rate in the case where agents are infinitely lived and “finite” to the case where they are finitely lived and generations overlap. The rest of the parameters are  $\rho = 0.01$ ,  $\mu = 0.018$ ,  $\sigma = 0.04$ ,  $\pi = 0.01$ ,  $\alpha(1) = 0.8$ .

$\omega(X_t) = f'(\mathcal{H}(X_t))$  are constants. It will simplify the formulas to define the following expression that is a weighted harmonic average of agents' risk aversions:

$$\Gamma(X_t) \equiv \frac{1}{\frac{X_t}{\gamma_A} + \frac{(1-X_t)}{\gamma_B}}. \quad (51)$$

Using this definition and the fact that hours are constant, equation (39) simplifies to

$$\kappa_t = \Gamma(X_t) \sigma_Z. \quad (52)$$

Since  $\Gamma' < 0$ , it follows that  $\kappa_t$  is a declining function of  $X_t$ . Furthermore, equation (35) together with (52) leads to

$$\frac{\sigma_X}{X_t} = \sigma_Z \left( \frac{\Gamma(X_t)}{\gamma_A} - 1 \right). \quad (53)$$

Since  $X_t \in [0, 1]$ , both the numerator and the denominator are positive, so that  $\sigma_X \geq 0$ . Hence, the state variable  $X_t$  increases in response to positive innovations to the exogenous productivity process  $Z_t$  and hence to positive news about the aggregate endowment  $Y_t$ . Since  $\kappa_t$  is declining in  $X_t$ , this implies that the Sharpe ratio in the economy is countercyclical.

This property of the model is a first illustration of the forces of aggregation: Less risk-averse agents (type *A* agents) will have portfolios that are more tilted towards stocks, and hence their wealth is more exposed to aggregate productivity risks. As a result, their wealth increases more than the wealth of more risk-averse agents (type *B* agents) in response to positive economic news. This increases the relative importance of type-*A* agents in the economy, which is captured by  $X_t$ , i.e., the share of their consumption of the aggregate endowment. Furthermore, by equation (52), the Sharpe ratio is proportional to the (harmonic) weighted average of the risk aversions of the two agents, where the weights are given by  $X_t$  and  $1 - X_t$ . Accordingly, the Sharpe ratio declines when the less risk-averse agents become relatively more important.

It should be noted here that the interaction of heterogeneity with overlapping generations helps overcome a problem of models where agents have heterogeneous preferences, but are infinitely lived. In these models, the less risk-averse agent will typically drive out the more risk-averse agent asymptotically. This absence of stationarity makes it difficult to calibrate the model to the data. For instance, in such models the P/D ratio will asymptotically converge to a constant, as will the interest rate and the Sharpe ratio. This is in contrast to the data, where the P/D ratio follows a

stationary process. Overlapping generations help overcome this problem: Because agents have no bequests, the wealth distribution has a stationary distribution, as do the P/D ratio, the interest rate, and the Sharpe ratio.

#### 4.2.2 Interest Rate

The implications of the model for the interest rate can be seen by examining (40). Given the specific assumptions we have made in this section, this equation becomes

$$r_t = \Gamma(X_t) [\bar{\rho}(X_t) + \mu_Z - \pi(\bar{\beta}(X_t) - 1)] - \frac{(\Gamma(X_t) \sigma_Z)^2}{2} \left\{ \frac{\frac{X_t}{\gamma_A} \left( \frac{\gamma_A+1}{\gamma_A} \right) + \frac{(1-X_t)}{\gamma_B} \left( \frac{\gamma_B+1}{\gamma_B} \right)}{\frac{X_t}{\gamma_A} + \frac{(1-X_t)}{\gamma_B}} \right\}, \quad (54)$$

where

$$\bar{\beta}(X_t) \equiv v\beta_t^A + (1-v)\beta_t^B \quad (55)$$

$$\bar{\rho}(X_t) \equiv X_t \frac{\rho_A}{\gamma_A} + (1-X_t) \frac{\rho_B}{\gamma_B}. \quad (56)$$

Equation (54) looks remarkably similar to equation (50). The main difference is that the homogenous risk aversion  $\gamma$  in equation (50) is replaced with  $\Gamma(X_t)$ , the discount rate  $\rho$  is replaced with  $\Gamma(X_t)\bar{\rho}(X_t)$  and the term  $\beta$  in (50) is replaced by an average of  $\beta_t^A$  and  $\beta_t^B$ , both of which are functions of  $X_t$ . As in section 4.1, the presence of the term  $-\pi(\bar{\beta}(X_t) - 1)$  has a dampening effect on the interest rate, which drives the relatively low level of interest rates that we obtain later, when we calibrate the model.

Equation (54) also helps illustrate under what assumptions the model can produce a small variability in the interest rate, despite time-varying risk aversion. Equation (54) decomposes the interest rate into two components. The first component, namely

$$\Gamma(X_t) [\bar{\rho}(X_t) + \mu_Z - \pi(\bar{\beta}(X_t) - 1)],$$

captures the usual intertemporal smoothing motives, while the second component, namely

$$\frac{(\Gamma(X_t) \sigma_Z)^2}{2} \left\{ \frac{\frac{X_t}{\gamma_A} \left( \frac{\gamma_A+1}{\gamma_A} \right) + \frac{(1-X_t)}{\gamma_B} \left( \frac{\gamma_B+1}{\gamma_B} \right)}{\frac{X_t}{\gamma_A} + \frac{(1-X_t)}{\gamma_B}} \right\}, \quad (57)$$

captures the precautionary-savings motive. When  $X_t$  declines,  $\Gamma(X_t)$  increases and hence aggregate precautionary savings in equation (57) increase<sup>6</sup>, as the importance of more risk-averse agents

<sup>6</sup> As long as  $\gamma_B > \gamma_A > 1$ .

(type- $B$  agents) increases. If  $\Gamma(X_t)$  is sufficiently high and  $\bar{\beta}(X_t)$  is a declining function<sup>7</sup> of  $X_t$ , the interest rate would start to decline when  $\Gamma(X_t)$  increases, since  $r_t$  in (54) is a quadratic and concave function of  $\Gamma(X_t)$ . This would happen in response to negative productivity shocks that would reduce  $X_t$ . Hence, one would expect procyclical interest rates that would “inherit” some of the variation of  $X_t$ .

In the data the real interest rate is by and large acyclical and substantially less volatile than stock returns. Our setup allows a simple way to reproduce this fact within the model, namely by assuming that  $\bar{\rho}(X_t)$  is a declining function of  $X_t$ . In light of equation (56) this amounts to assuming that  $\rho_B > \rho_A$ . Intuitively, if the more risk averse agents are also sufficiently impatient, their increased appetite for precautionary savings caused by their high risk aversion will be counteracted by their lack of savings due to their impatience. Hence, the agents’ attitudes towards savings are roughly similar, whereas their attitude towards risk can be substantially different. As a result, variations in the relative importance of type- $A$  and type- $B$  agents will end up affecting risk premia rather than interest rates.

### 4.3 Dividends and Labor

The standard assumption in several asset pricing models since the work of Lucas (1978) is to assume that dividends are equal to consumption, which implicitly means that the labor share is zero. This is a useful theoretical abstraction. It implies, however, that the volatility of the two series is the same. In the data the volatility of dividends is larger than the volatility of consumption. However, one would expect the two quantities to be cointegrated over longer-run horizons.

The model can capture these effects in a simple way by allowing agents’ preferences to be heterogeneous not only with respect to risk aversion and the subjective discount factor, but with respect to the inclination to work as well. In particular, if  $\psi_A \neq \psi_B$ , then equation (24) implies that hours worked will become a function of  $X_t$ . An application of the implicit function theorem to (24) implies

$$\frac{1}{H_t} \frac{dH_t}{dX_t} = \frac{\left(\frac{1}{\psi_A} - \frac{1}{\psi_B}\right)}{\alpha'(H_t)(1 - H_t) - \frac{\alpha(H_t)}{H_t}}. \quad (58)$$

Since  $\alpha'(H_t) \leq 0$ ,  $\alpha(H_t) \geq 0$  and  $H_t \leq 1$ , the denominator on the right hand side of (58) will be

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<sup>7</sup>This turns out to be the case when one solves the model for reasonable parameters.

negative. If  $\psi_A > \psi_B$  the numerator will be negative and hours will be an increasing function of  $X_t$ , whereas if  $\psi_A < \psi_B$ , hours will be a declining function of  $X_t$ . In the data, hours are procyclical and hence we will assume from this point onward that  $\psi_A > \psi_B$ . The existence of stationary variation in hours implies that the model will endogenously produce cyclical variation in output alongside the variation caused by shocks to productivity ( $Z_t$ ). Using the definition of  $g$  in equation (33), the fact that  $H_t$  is an increasing function of  $X_t$  also implies that output is increasing in  $X_t$ . Mathematically,  $g'(X_t) > 0$ .

Since we are interested in the asset-pricing implications of the model, we will not focus on these effects. Instead, we will calibrate the model so as to ensure that hours supplied are roughly constant and as a result consumption is roughly a random walk. To achieve this, we will choose  $1 = \psi_A \simeq \psi_B$ .

For our purposes, we will only need a small and procyclical variation in hours, which in conjunction with (5) and (6), will result in a countercyclical labor share and hence a procyclical dividend share of output. More specifically, the volatility of dividends  $\sigma_D$  is given by

$$\sigma_D \equiv \sigma_Z + \left( \frac{g'}{g} - \frac{\alpha' \mathcal{H}'}{(1-\alpha)} \right) \sigma_X \quad (59)$$

Given the assumption  $\psi_A > \psi_B$  all three terms in the above expression are positive, since  $g' > 0$ ,  $\alpha' \leq 0$ ,  $\sigma_X > 0$ , and  $\mathcal{H}' > 0$  by equation (58). This means that dividends are more volatile than productivity and also than output (and hence consumption), since  $\sigma_Y < \sigma_D$  by (42). However, over longer horizons (log) dividends and (log) output are cointegrated since the dividend-to-consumption ratio  $1 - \alpha(X_t)$  is stationary.

One important challenge for models that produce such realistic dynamics for consumption and dividends is that the volatility of stock prices as given by (48) may become smaller than  $\sigma_D$ . Alternatively put, such models may produce countercyclical variation in the price to dividend ratio, contrary to the data. To see the source of the potential problem, it is easiest to consider figure 2 and consider the following thought experiment: Suppose that a model can produce a volatility of dividends that is higher than the volatility of consumption, say by a factor of  $k > 1$ . Roughly speaking, if consumption increases (instantaneously) by 1 percent, dividends have to increase (instantaneously) by  $k$  percent. To simplify matters, we shall assume furthermore that consumption is a random walk in logs, so that this 1 percent increase is permanent. Finally, we shall also assume that the model also implies that (log) consumption and (log) dividends are cointegrated. This last



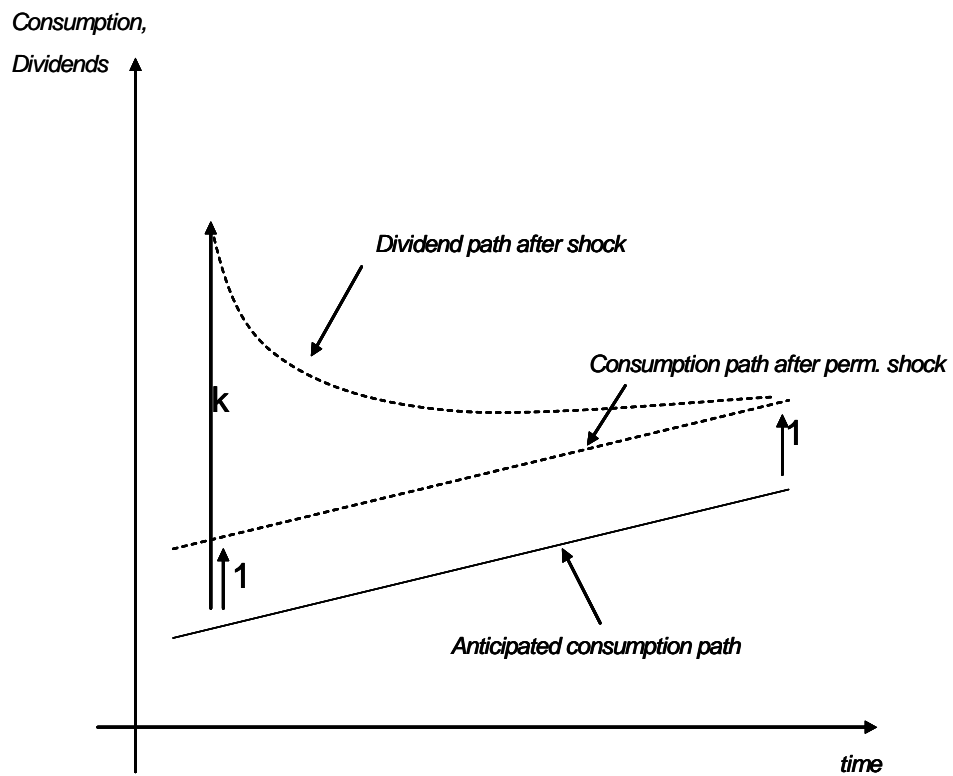


FIGURE 2: The implications of co-integration between dividends and consumption.

assumption implies that the long-run response of (log) dividends to a one percentage point change in (log) consumption must also be one percent. Else the two series would not be cointegrated. Since the short-run response of (log) dividends ( $k$  percent) is larger than their long-run response (1 percent), this means that the anticipated growth rate in dividends will be negative. Because of this, the well known Campbell-Shiller decompositions of the ratio of prices to dividends (P/D ratio) imply that the P/D ratio should be expected to *decline* in response to a positive consumption shock, if *discount rates are constant*.

However, in the present model discount rates will not be constant: Instead, they decline in response to positive shocks, as explained in section 4.2.1. If the decline in the anticipated dividend growth is smaller than the decline in discount rates, the price to dividend ratio is procyclical, as in the data.

As Lettau and Ludvigson (2005) point out, the comovement of discount rates with the anticipated growth rate in dividends can help account for the observed inability of the price-to-dividend ratio to predict dividend growth. If the variation in discount rates is sufficiently large, then the price to dividend ratio will be procyclical<sup>8</sup>.

Finally, because the labor share is countercyclical, the model is qualitatively consistent with the three observations about labor income growth reported in Lustig and Van Nieuwerburgh (2007). Specifically, dividend growth and labor income growth are negatively correlated in our framework. This is intuitive: When the labor share  $\alpha(X_t)$  is above its stationary mean, it can be expected to mean revert. Hence, dividends can be expected to increase as a fraction of the aggregate endowment, while labor income can be expected to decline. Furthermore, shocks to the productivity shock  $Z_t$  increase  $X_t$  (by equation [53]) and hence make  $\alpha(X_t)$  decline since  $\alpha'(X_t) \leq 0$ . Because  $\alpha(X_t)$  can be expected to mean revert after such a shock, anticipated labor income growth will be positively correlated with “current” shocks to the productivity process. Finally, when  $X_t$  is below its stationary mean,  $\alpha(X_t)$  is above its stationary mean and can be expected to mean revert. Hence, periods of high expected return will coincide with periods of low anticipated income growth. All these observations are consistent with the evidence reported in Lustig and Van Nieuwerburgh

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<sup>8</sup>It is even possible that the P/D ratio can predict dividend growth with a negative sign, as opposed to a positive sign. This is consistent with the data (see. e.g. the textbook of Cochrane (2005), p. 392) and when we calibrate the model we can reproduce this effect, as we explain below.

$\mu_Z$	0.018	$\gamma_A$	3	$\beta_1$	0.99
$\sigma_Z$	0.041	$\gamma_B$	18	$\beta_2$	0.81
$v$	0.1	$\psi_A$	1	$\beta_3$	0.5
$\pi$	0.01	$\psi_B$	0.95	$\beta_4$	0.3
$\chi$	0.018	$\rho_A$	0.001		
		$\rho_B$	0.3		

TABLE 1: Parameters used in the calibration.

(2007).

## 5 Quantitative Results

### 5.1 Parameter Choice and Calibration

To calibrate the model we need to choose eleven parameters along with a functional form for  $\alpha(H_t) = \alpha(H(X_t))$ .

The parameters that we use for the calibration are given in Table 1. The parameters  $\mu_Z$  and  $\sigma_Z$  are chosen so as to match the mean growth rate and the volatility of consumption growth respectively.

The parameters  $\pi$  and  $\chi$  are chosen so that the median agent dies at age 69, and half of her endowment of hours over her entire life cycle occurs before age 39. We note here that given the stylized assumptions of the model, the start of the work-life and natural life coincide. Clearly, both of these numbers are hard to calibrate exactly to the data. In real life, death rates are age dependent and the hours worked over the life cycle are not described by exponential decay. Nevertheless, given the tractability of aggregation that is allowed by these assumptions, we believe that our choices for  $\pi$  and  $\chi$  are reasonable quantitatively.

The parameter  $v$  controls the fraction of the population that is comprised by the less risk averse agents. Since these are the agents that are predominantly exposed to risk (holding stocks), we set that number to 10%, to reflect the order of magnitude of the average number of stockholders in the long historical sample of returns that we are interested in matching.

The parameters that pertain to agent preferences are given in the second column of Table 1.

These preferences are chosen so that the model can match asset-pricing data. To be able to match the joint empirical facts that equity premia are time varying, whereas interest rates are not very volatile, we need a joint assumption on the discount rates and the risk aversion of the agents. Roughly speaking, we need to keep the aggregate savings in the economy relatively unaffected by variations in  $X_t$ , while keeping the risk attitudes of agents very different. The latter is achieved by setting the risk aversion of type- $B$  agents substantially higher than the risk aversion of type- $A$  agents. In our model this implies that type  $A$ -agents hold more stock than type  $B$ -agents. Even though we choose these large differences in risk aversion so as to match aggregate asset pricing data, these differences do replicate a pattern in microeconomic data, namely that households that are wealthy and tend to hold more stock also tend to have a consumption that exhibits higher covariance with the stock market, especially over longer horizons. For instance Vissing-Jorgensen, Malloy, and Moskowitz (2007) argue that the “long-run” covariance between consumption and returns for wealthier, stock-holding households is 4 times larger than the equivalent covariance for the rest of the households. One can show that in our setup type- $A$  agents have a covariance between long run consumption growth and returns that is  $\gamma^B/\gamma^A = 6$  times higher than the equivalent quantity for type- $B$  agents.

To ensure that variations in  $X_t$  do not affect aggregate saving behavior, we need to set  $\rho^B$  higher than  $\rho^A$  for the reasons we gave in section 4.2.2. It should be noted here that despite these large differences in discount rates, equation (27) suggests that, since  $\frac{\rho_i}{\gamma_i}$  is similar for the two agents, the drift in their consumption path over the life cycle will not be affected considerably by the large difference in their preferences.

Finally, the parameters that control the agents’ disutility of work are intentionally chosen very close to each other. From equation (24) we know that, when  $\psi_A \approx \psi_B$ , hours do not vary considerably, and hence the predictable components of consumption growth become negligible. In particular, by combining values of  $\psi_A \approx \psi_B \approx 1$  with a steeply declining  $\alpha(H(X_t))$  we can ensure that the volatility in  $X_t$  will almost exclusively affect the share of dividends, and not the predictable components of consumption. To have enough flexibility to obtain these properties, we parameterize  $\alpha(H(X_t))$  as

$$\alpha(H(X_t)) = (\beta_1 - \beta_2) \mathcal{N}(\beta_3(X - \beta_4)) + \beta_2, \quad (60)$$

	Data	Model
Mean annual consumption growth	1.72	1.69
Volatility of annual consumption growth	3.28	3.41
Mean riskless rate	1.83	3.61
Standard deviation of riskless rate	3.00	0.52
Mean equity premium	4.18	2.73
Standard deviation of stock market returns	17.74	10.68
Sharpe ratio	0.24	0.25
Mean dividend share of output	17.6	13.6
Standard deviation of dividend share	4.3	6.5
Standard deviation of dividend growth rate	0.12	.09

TABLE 2: Unconditional annual moments of the data. The first seven rows in the data column are from Chan and Kogan (2002). In the data column, the volatility of the interest rate is the volatility of the ex-post real rate. Hence, it overstates the volatility of the ex-ante riskless rate, because it doesn't account for inflation surprises. The eighth and ninth row are from the Bureau of Economic Analysis (Table 1.10) and spans the years 1929-2005. We include both proprietors income and corporate profits in computing the "dividend share" in the data. The final row is based on data available on the website of R. Shiller (Real dividends between 1871-2005).

where  $\mathcal{N}$  is the cumulative normal distribution and  $\beta_1, \beta_2, \beta_3,$  and  $\beta_4$  are constants that we can choose to match certain properties of the data. Equation (60) implies that  $\alpha \in (\beta_1, \beta_2)$  for any value of  $X_t$ , so that  $\beta_1$  and  $\beta_2$  control the range of  $\alpha$ . The constants  $\beta_3$  and  $\beta_4$  control the steepness of the function and the point at which it achieves its maximum slope (in absolute value).

Our choices of  $\beta_1, \beta_2, \beta_3,$  and  $\beta_4$  control the production function of the economy and are chosen so as to imply a stationary dividend share and a volatility of that share close to the data.

## 5.2 Unconditional Moments

Table 2 compares the model's performance with some key moments in the data. The model's performance is not as good as Campbell and Cochrane (1999), but it does explain a significant fraction of some asset pricing facts. Most moments are within a reasonable distance from their

empirical counterparts. The main moment that is underpredicted by the model is the volatility of equity. The real rate and the equity premium are about 1.5 percent away from their target values.

The volatility of the interest rate is about 50 basis points in the model. We have intentionally kept this volatility low, in order to show the potential of our framework to address a common problem of most asset pricing models. The model fit could be improved further by increasing the volatility of the interest rate. This would raise the volatility in stock prices, which is the main moment that is missed by the model.

The main conclusion is that the model explains a significant fraction of certain unconditional asset pricing moments, despite the usage of standard expected utility specifications and without relying on excessive interest-rate volatility.

### 5.3 Conditional Moments

Figure 3 gives a depiction of the instantaneous Sharpe ratio, risk-free rate, conditional volatility, and equity premium as functions of  $X_t$ . The range of values of  $X_t$  correspond to  $\pm 3$  (stationary) standard deviations around its stationary mean. The range of values for the conditional equity premium is larger than the equivalent range for the riskless rate. Hence, most of the variation in discount rates is related to variations of the equity premium, not the interest rate. This presents an improvement over Chan and Kogan (2002) where the variability in interest rates is larger than the variability in excess returns.

Figure 4 addresses another feature of the model that is consistent with the data and presents a challenge for many models: The joint presence of a procyclical dividend share and a procyclical price-to-dividend ratio. Figure 4 presents the dividend share in the economy and the P/D ratio as a function of  $X_t$ . Note that both the dividend share and P/D are increasing in  $X_t$ . As we explained in section 4.3, this can only happen if the variation of discount rates is stronger than the variation in anticipated dividend growth rates. This is consistent with the evidence reported in Lettau and Ludvigson (2005) who find that the P/D ratio cannot predict dividends because of the offsetting effect of discount rates.

Table 3 gives a different perspective on these effects by showing the strong predictive ability of the P/D ratio for excess returns. The model overpredicts the absolute value of the coefficients in the predictive regressions for excess returns. This is partly driven by the fact that the model

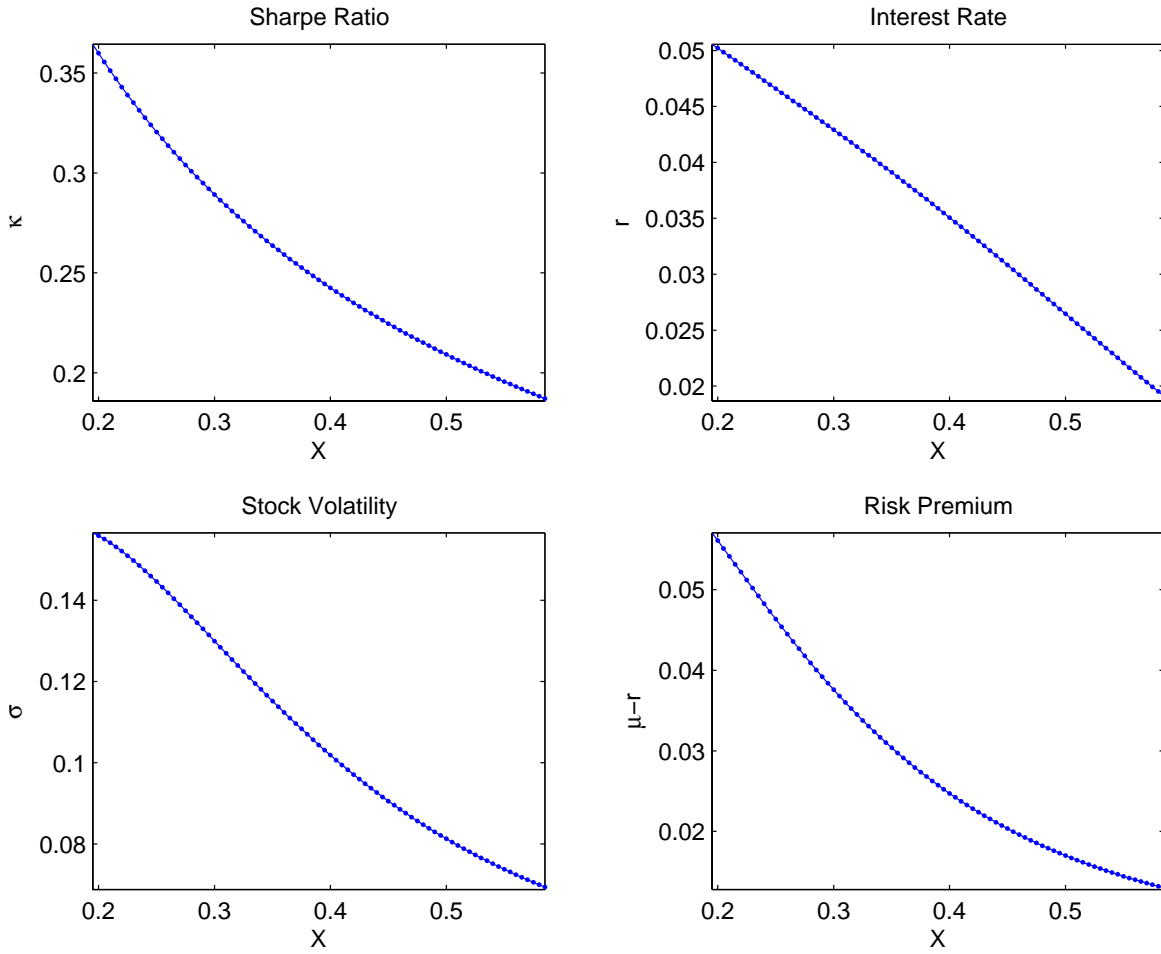


FIGURE 3: The top left panel depicts the Sharpe ratio as a function of the consumption share of type  $A$  agents (less risk averse agents), which is denoted as  $X$ . The top right panel depicts the interest rate as a function of  $X_t$ . The bottom left panel depicts the instantaneous volatility of the stock market and the bottom right panel the equity premium, both as functions of  $X_t$ .

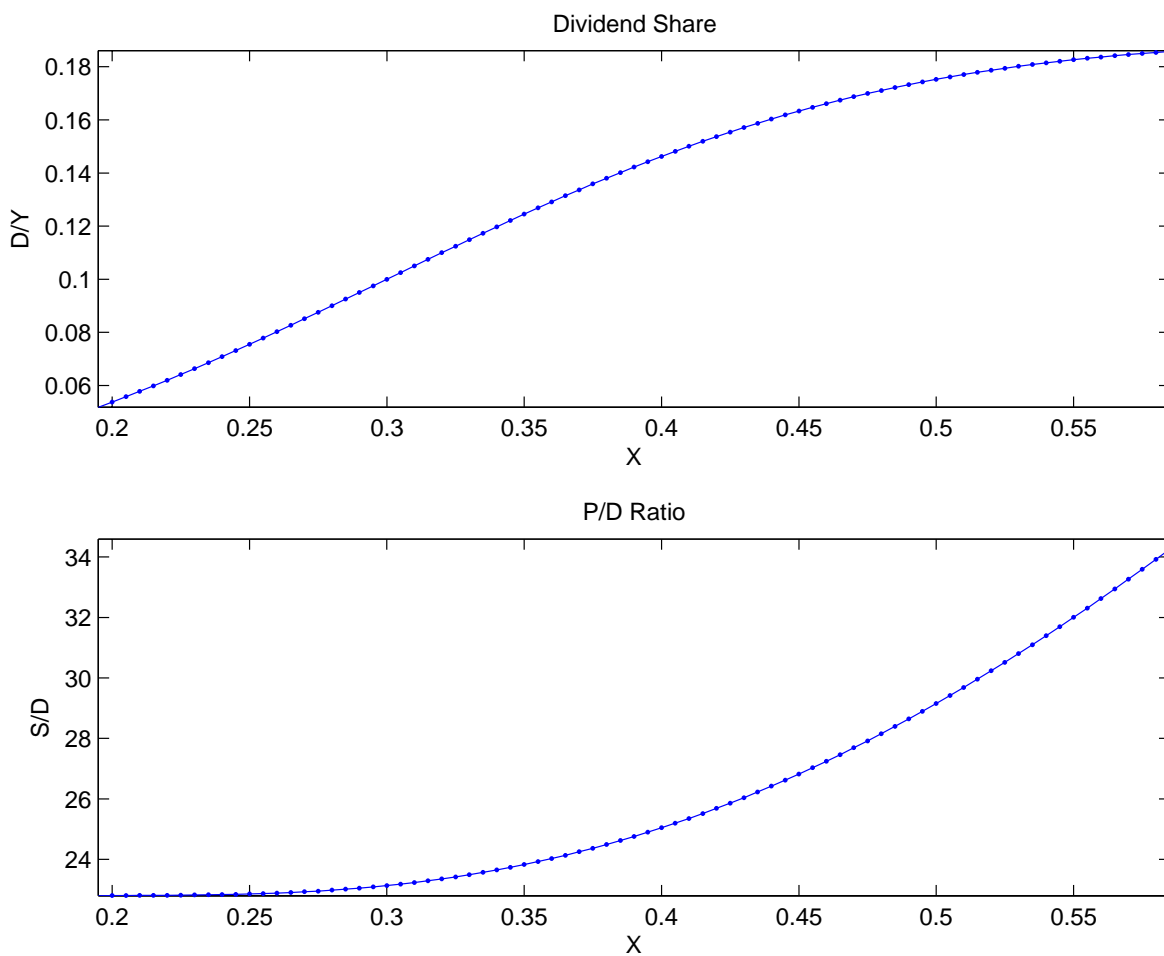


FIGURE 4: The top panel of the figure presents the dividend share of output as a function of the consumption share of type  $A$  agents (less risk averse agents), which is denoted as  $X$ . The bottom panel presents the price to dividend ratio as a function of  $X_t$ .



Horizon (Years)	Data		Model	
	Coefficient	$R^2$	Coefficient	$R^2$
1	-0.13	0.04	-0.31	0.02
2	-0.28	0.08	-0.74	0.04
3	-0.35	0.09	-1.15	0.07
5	-0.60	0.18	-1.91	0.12
7	-0.75	0.23	-2.58	0.17

TABLE 3: Long Horizon Regressions of excess returns on the log P/D ratio. The model column reports the mean of the regression results for 1000 simulated paths of length 80 years.

underpredicts the volatility of the (log) P/D ratio<sup>9</sup>. This is to be expected, because of the offsetting effects of dividend growth on the time variation in discount rates. The  $R^2$  of the regression, which is less affected by this issue, has the right order of magnitude when compared with the data.

#### 5.4 The Dynamics of Cross-Sectional Inequality

The model’s key mechanism is that the wealthier agents (type- $A$  agents) become comparatively richer when the stock market performs well and poorer agents become comparatively poorer when the stock market performs badly. Wolff (1992) provides some direct evidence to that effect. With data that go back to the twenties he shows that the wealth distribution becomes more uneven in response to positive excess stock market returns. This result is true even after controlling for changes in the income distribution. Furthermore, Vissing-Jorgensen, Malloy, and Moskowitz (2007) present evidence that the consumption share of shareholders is useful in predicting subsequent excess returns.

We next investigate whether the model relies quantitatively on “too much” high frequency variation in cross sectional inequality to explain returns. Figure 5 depicts the drift, diffusion and the stationary distribution of the share of consumption that accrues to type- $A$  agents ( $X_t$ ). The consumption share of type- $A$  agents exhibits small instantaneous volatility. The middle panel of 5 shows that the consumption share of type- $A$  agents changes by about  $\pm 0.015$  over the interval of

<sup>9</sup>The volatility of the (log) P/D ratio is about a third of its empirical counterpart.

a year. In simulations we find that this translates into a yearly change in the Gini coefficient of consumption inequality of about 0.0057. This is consistent with the numbers given in Cutler and Katz (1992) by looking at repeated CES samples in the seventies and eighties<sup>10</sup>.

The top panel of Figure 5 shows that changes in inequality are very persistent. The average slope of the curve in Figure 5 is about  $-0.025$ . This implies that if one regressed  $X_t$  on  $X_{t-1}$  over subsequent years, one would obtain a coefficient of about  $e^{-0.025} \approx 0.975$ . The persistence of changes in  $X_t$  implies that the stationary standard deviation of the Gini coefficient of consumption inequality is larger than the standard deviation of its year-to-year change, and this is how the model can produce sizable variations in asset prices despite small changes in  $X_t$  in the short run. More importantly, the persistence of changes in the cross sectional inequality seems to be supported by the data<sup>11</sup>.

## 6 Conclusion

In this paper we have presented a model that addresses a number of stylized facts about asset prices. The model combines four key ingredients: a) Agents are finitely lived, b) They can be heterogeneous in their preferences, c) They supply less labor as they age, and d) Consumption and dividends may differ.

These assumptions, which seem natural, help explain simultaneously several asset-pricing phenomena: a) Riskless rates are low, since life-cycle motivations enhance agents' incentive to save. b) The Sharpe ratio is volatile, since variations in the wealth distribution determine the relative importance of agents with differing risk aversion. This composition effect makes our model resemble an economy that is populated by a representative agents with time varying and countercyclical risk aversion. c) Since dividends are procyclical and more volatile than consumption, and discount rates vary countercyclically, stock market prices are volatile and the equity premium is reasonably high.

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<sup>10</sup>Cutler and Katz (1992) report the Gini coefficient for consumption inequality for the years 1960, 1972, 1980, 1984, and 1988. Computing the differences between those years and weighting them by the inverse of the square root of the time distance between these years (to account for heteroskedasticity in the observations) and then computing the standard deviation gives 0.0070.

<sup>11</sup>The Gini coefficient of income inequality in the CPS data follows almost a random walk. Data for consumption inequality are not available over such a long sample. However, as Cutler and Katz (1992) argue, consumption and income inequality share similar trends over longer horizons.

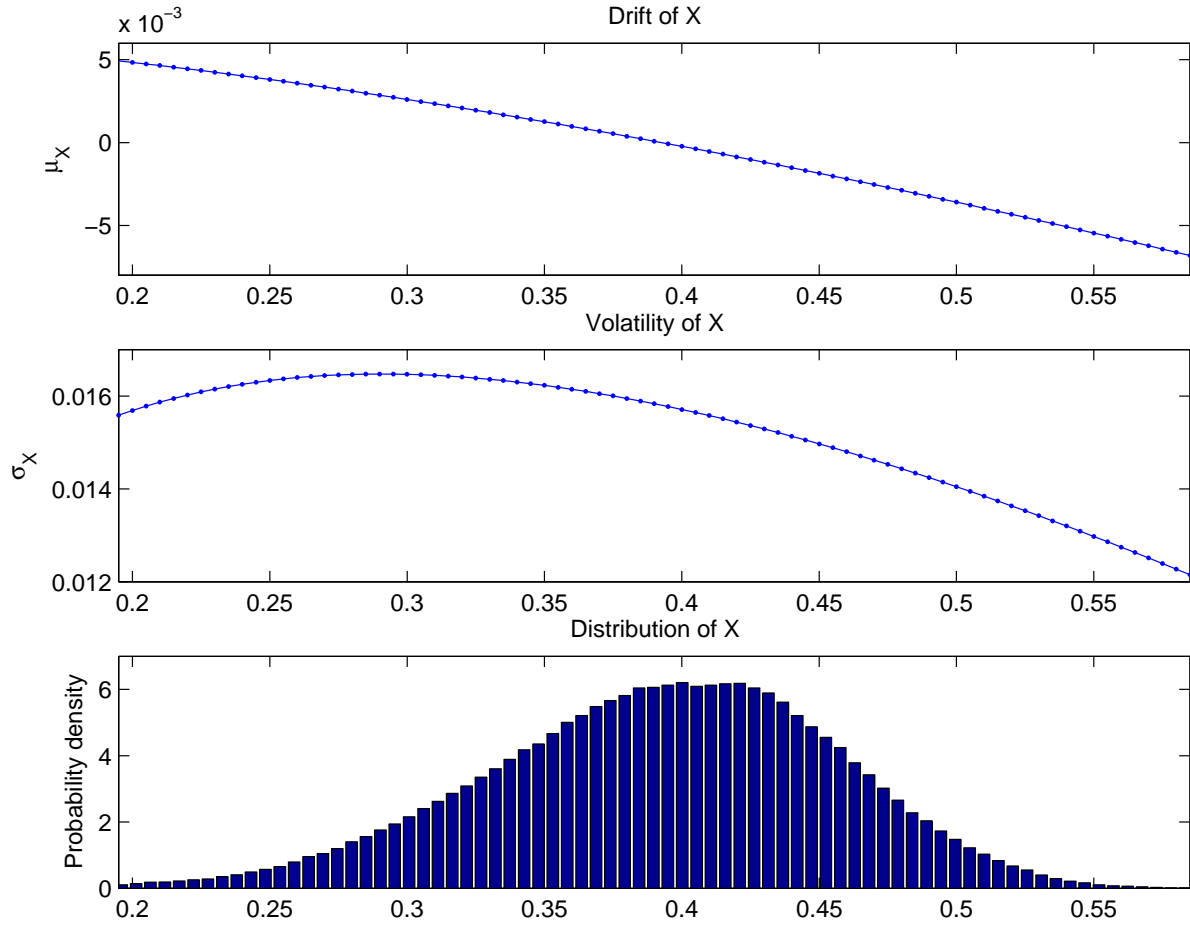


FIGURE 5: The top panel shows the drift of the consumption share of type- $A$  agents (less risk-averse agents), which is denoted by  $X$ . The middle panel depicts the conditional volatility and the bottom panel the stationary distribution of  $X_t$ .

d) Most of the variation in discount rates is due to changes in equity premia, not interest rates. e) The price-to-dividend ratio predicts excess returns. f) Even though dividends are predictable, the time variation in expected dividend growth is offset by changes in the stochastic discount factor. This makes the P/D ratio procyclical. g) Dividends are more volatile than consumption in the short run, but are cointegrated over the long run. h) Consumption is practically a random walk.

These facts are consistent with the data. Moreover, calibrated versions of the model produce a satisfactory but not perfect quantitative fit.

Accordingly, we believe that the broad conclusion of the model is that overlapping generations along with preference heterogeneity can go a long way towards explaining prevailing asset-pricing puzzles. Observationally, our framework resembles a model of exogenous habit formation of the type proposed by Campbell and Cochrane (1999). However, both the economic mechanisms and the broader implications of the models differ fundamentally.

Furthermore, our model allows us to draw a distinction between a claim to consumption and a claim to dividends in a framework where the joint dynamics of dividends and consumption are modelled realistically. This allows us to use our framework as a laboratory in order to understand the mechanisms that may be behind a recent empirical literature that exploited this distinction<sup>12</sup>.

Finally, an important advantage of the model is its analytic tractability. It provides us with a simple way of reproducing some key asset-pricing facts in a framework that can be used in various applications. For instance, the model could be expanded to investigate the effect of demographic shocks (such as a baby boom) on asset prices within a model that reproduces key asset-pricing facts. The conventional utilities that we use also facilitate policy experiments, such as the effects of a switch from pay as you go to a fully funded system. Such extensions and applications are left for future research.

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<sup>12</sup>See e.g. Lettau and Ludvigson (2005).

## A Proofs

**Proof of Lemma 1.** Combining (29) and (30) leads to

$$e^{-(\pi+\chi)s}\phi^i(X_s)Y_s\xi_s + \psi_i\frac{\pi+\chi}{\pi}\int_T^s e^{-(\pi+\chi)t}w_t\xi_t dt = \psi_i\frac{\pi+\chi}{\pi}E_s\int_T^\infty e^{-(\pi+\chi)t}w_t\xi_t dt \quad (61)$$

for any  $T < s$ . The right hand side of this expression is a martingale, since it is a conditional expectation. Accordingly, applying Ito's Lemma to the left hand side implies (43). Equation (29) implies that  $\phi^B(X_t) = \frac{\psi_B}{\psi_A}\phi^A(X_t)$ . To obtain the functions  $\beta^i$ , note that using equation (28), (30) and (31) gives

$$\zeta^i = E_s\int_s^\infty e^{-\pi(t-s)}\frac{c_{t,s}^i}{c_{s,s}^i}\frac{\xi_t}{\xi_s} dt. \quad (62)$$

Furthermore, equation (27) implies

$$\zeta^i = E_s\int_s^\infty e^{-\left(\pi+\frac{\rho_i}{\gamma_i}\right)(t-s)}\left(\frac{w_t}{w_s}\right)^{\frac{(1-\psi_i)(\gamma_i-1)}{\gamma_i}}\left(\frac{\xi_t}{\xi_s}\right)^{1-\frac{1}{\gamma_i}} dt.$$

A similar argument to the one given for  $\phi^i$  can now be used to arrive at (44). ■

**Proof of Lemma 2.** Applying Ito's lemma to compute  $d(e^{-\pi s}\xi_s K_s^i)$  and integrating leads to

$$K_{t,s}^i = E_t\int_t^\infty e^{-\pi(u-t)}\frac{\xi_u}{\xi_t}(c_{u,s}^i - w_u h_{u,s}^i) du \quad (63)$$

We next observe that generations that will be born at dates that are larger than  $t$  neither consume, nor supply hours, nor own any wealth at time  $t$ . This means that for any  $i \in \{A, B\}$

$$K_{t,s}^i = c_{t,s}^i = h_{t,s}^i = 0 \text{ if } s > t.$$

With this observation with (63) we obtain

$$\begin{aligned} \sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i K_{t,s}^i ds &= \sum_{i \in \{A, B\}} \int_{-\infty}^{+\infty} \pi e^{-\pi(t-s)} v^i K_{t,s}^i ds \\ &= \sum_{i \in \{A, B\}} \pi e^{-\pi(t-s)} v^i \int_{-\infty}^{+\infty} \left( E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} (c_{u,s}^i - w_u h_{u,s}^i) du \right) ds \end{aligned}$$

$$\begin{aligned}
&= E_t \int_t^\infty \frac{\xi_u}{\xi_t} \left( \sum_{i \in \{A,B\}} \int_{-\infty}^{+\infty} \pi e^{-\pi(u-s)} \nu^i (c_{u,s}^i - w_u h_{u,s}^i) ds \right) du \\
&= E_t \int_t^\infty \frac{\xi_u}{\xi_t} \left( \sum_{i \in \{A,B\}} \int_{-\infty}^u \pi e^{-\pi(u-s)} \nu^i (c_{u,s}^i - w_u h_{u,s}^i) ds \right) du \\
&= E_t \int_t^\infty \frac{\xi_u}{\xi_t} (Y_u - w_u H_u) du \\
&= E_t \int_t^\infty \frac{\xi_u}{\xi_t} D_u du.
\end{aligned}$$

■

**Proof of Lemma 3.** By Lemma 2 we know that

$$S_t = \sum_{i \in \{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \nu^i K_{t,s}^i ds \quad (64)$$

$$\begin{aligned}
&= \sum_{i \in \{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \nu^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds \\
&\quad - \sum_{i \in \{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \nu^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} (w_u h_{u,s}^i) du \right] ds
\end{aligned} \quad (65)$$

We can compute the first term in (65) as

$$\begin{aligned}
&\sum_{i \in \{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \nu^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds \\
&= \sum_{i \in \{A,B\}} \nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s}^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \frac{c_{u,s}^i}{c_{t,s}^i} du \right] ds
\end{aligned}$$

Now note that equation (27) implies that  $c_{u,s}^i/c_{t,s}^i$  is independent of  $s$ <sup>13</sup> i.e.,  $\frac{c_{u,s}^i}{c_{t,s}^i} = \frac{c_{u,t}^i}{c_{t,t}^i}$ . Using this observation together with (62) leads to

$$\begin{aligned}
\sum_{i \in \{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \nu^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds &= \sum_{i \in \{A,B\}} \nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s}^i \zeta^i(X_t) ds \\
&= Y_t \left[ \sum_{i \in \{A,B\}} \zeta^i(X_t) X_t^i \right]. \quad (66)
\end{aligned}$$

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<sup>13</sup>To see this, fix a time of birth  $s$ , apply equation (27) at two different points in time, say  $u$  and  $t$ , and then derive  $c_{u,s}^i/c_{t,s}^i$  which is independent of  $s$ .

Similarly we can compute the second term in (65) by using (20) as

$$\begin{aligned}
& \sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} (w_u h_{u,s}^i) du \right] ds \\
&= \sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \left( \frac{\pi + \chi}{\pi} e^{-\chi(u-s)} w_u - \frac{(1 - \psi_i)}{\psi_i} c_{u,s}^i \right) du \right] ds \\
&= \sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \left( \frac{\pi + \chi}{\pi} e^{-\chi(u-s)} \right) w_u du \right] ds \\
&\quad - \sum_{i \in \{A, B\}} \frac{(1 - \psi_i)}{\psi_i} \zeta^i(X_t) X_t^i.
\end{aligned} \tag{67}$$

The first term in (67) can be further rewritten as

$$\begin{aligned}
& \sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \left( \frac{\pi + \chi}{\pi} e^{-\chi(u-s)} \right) w_u du \right] ds \\
&= \sum_{i \in \{A, B\}} \int_{-\infty}^t e^{-(\pi + \chi)(t-s)} v^i \left[ E_t \int_t^\infty e^{-(\pi + \chi)(u-t)} \frac{\xi_u}{\xi_t} (\pi + \chi) w_u du \right] ds \\
&= Y_t \left[ \sum_{i \in \{A, B\}} \frac{v^i}{\psi_i} \phi^i(X_t) \pi \left( \int_{-\infty}^t e^{-(\pi + \chi)(t-s)} ds \right) \right] \\
&= Y_t \frac{\pi}{\pi + \chi} \left[ \sum_{i \in \{A, B\}} \frac{v^i}{\psi_i} \phi^i(X_t) \right]
\end{aligned} \tag{68}$$

Combining (68) with (67), (66), and the fact that  $\phi^B(X_t) = \frac{\psi_B}{\psi_A} \phi^A(X_t)$ , we arrive at (47). ■

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