Rare Disasters and Exchange Rates: A Theory of the Forward Premium Puzzle

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Abstract

We propose a new model of exchange rates, which yields a theory of the forward premium puzzle. Our explanation combines two ingredients: the possibility of rare economic disasters, and an asset view of the exchange rate. Our model is frictionless, has complete markets, and works for an arbitrary number of countries. In the model, rare worldwide disasters can occur and affect each country's productivity. Each country's exposure to disaster risk varies over time according to a mean-reverting process. Risky countries command high risk premia: they feature a depreciated exchange rate and a high interest rate. As their risk premium reverts to the mean, their exchange rate appreciates. Therefore, the currencies of high interest rate countries appreciate on average. This provides an explanation for the forward premium puzzle (a.k.a. uncovered interest rate parity puzzle). We then extend the framework to incorporate two factors: a disaster risk factor, and a business cycle factor. We calibrate the model and obtain quantitatively realistic values for the volatility of the exchange rate, the forward premium puzzle regression coefficients, and near-random walk exchange rate dynamics. Finally, we work out a model of the stock market, which allows us to make a series of predictions about

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the joint behavior of exchange rates, bonds, options and stocks across countries. The evidence from the options market appears to be supportive of the model. (JEL: E43, E44, F31, G12, G15)

1 Introduction

According to the uncovered interest parity (UIP) equation, the expected depreciation of a currency should be equal to the interest rate differential between that country and the reference region. A regression of exchange rate changes on interest rate differentials should yield a coefficient of 1. Instead, empirical work starting with Hansen and Hodrick (1980) and Fama (1984) consistently produces a regression coefficient that is less than 1, and often negative.

This invalidation of UIP has been termed the forward premium puzzle: currencies with high interest rates tend to appreciate. In other words, currencies with high interest rate feature positive predictable excess returns. There are three possible explanations: time-varying risk premia, expectational errors, and illiquid markets.

Our paper provides a theory of time-varying risk premia, in a complete markets, frictionless and rational framework. In our model, the exchange rate is both a relative price of non-traded and traded goods, and an asset price: it is the net present value of the export sector's productivity.

We take up on the idea championed by Rietz (1988), Barro (2006) and Weitzman (2007) that the possibility of rare but extreme events is a major determinant of risk premia in asset markets. In our model, rare world crises can happen. In those episodes, the productivity of each country drops. The country-specific exposure to disaster risk is a mean-reverting process.

Risky countries command high risk premia: they feature a depreciated exchange rate and a high interest rate. As their risk premium reverts to the mean, their exchange rate appreciates. Therefore, the currencies of high interest rate countries appreciate on average. This provides an explanation for the forward premium puzzle.

The model is consistent with a forward premium puzzle, both in sample with and without disasters. Therefore it does not suffer from a peso problem. The driving force of our result is that the risk premium covaries positively with interest rate. In other words, our theory does not rely on mismeasurement of expectations.

The model is very tractable, and expressions for the exchange rate, interest rate, risk premia, and forward premium puzzle coefficients obtain in closed forms. The framework is very flexible. In a second part of the paper, we extend it to incorporate two factors: a slow moving productivity factor, and a fast mean-reverting disaster risk factor. We calibrate the model and obtain quantitatively realistic values for the volatility of the exchange rate, the forward premium puzzle regression coefficients, and near-random walk exchange rate dynamics.

Moreover, the model offers a number of additional predictions. First, there should be a clear link between equity and currency risk premia through interest rates. High domestic interest rates imply high currency risk premia – an expected appreciation of the domestic currency – and low equity risk premia in the form of low Sharpe ratios. Fama and Schwert (1977) and Campbell and Yogo (2006) provide evidence of the link between equity excess returns and nominal interest rates. Hau and Rey (2004) find that for Japan, France,Germany and Switzerland, a negative shock to the foreign stock market – relative to the US– lead to s foreign currency appreciation.

Second, the model has rich implications for the relation between the relative shape of the yield curves between two countries and the expected change in the bilateral exchange rate. Boudoukh, Richardson Whitelaw (2006) propose to regress the exchange rate movement on the $T-$period forward rate from T periods ago, and find that the regression coefficient increases towards 1 with the horizon T. Indeed, our theory is consistent with this empirical finding in a context where risk-premia are fast mean-reverting, and productivity is slowly mean reverting.

Curency option prices potentially contain a lot of information on currency risk premia. Indeed, according to our model, a risky country will feature relatively more expensive out of the money puts than out of the money calls. Carr and Wu (2007) and Farhi, Gabaix, Ranciere and Verdelhan (2007) provide evidence that, as predicted by the model, when out of the money put prices increase relative to out of the money call prices, the corresponding currency simultaneously depreciates.

Time varying disasters are inherently difficult to assess, and as such might be especially amenable to expectational errors. Hence, our model can interpreted along behavioral lines as a consistent way to analyze the impact of investor sentiment on international asset prices.

Relation to the literature. This paper adds to a large body of empirical and theoretical work on the UIP condition. To the best of our knowledge, we are the first to adapt the Rietz-

Barro paradigm to exchange rates. Guo (2007), subsequently, also adopts this paradigm, in the context of a monetary model.

On the empirical side, most papers test the UIP condition on nominal variables. Two recent studies cast the puzzle in terms of real variables. Hollifield and Yaron (2003) decompose the currency risk premium into conditional inflation risk, real risk, and the interaction between inflation and real risk. They find evidence that real factors, not nominal ones, drive virtually all of the predictable variation in currency risk premia. Lustig and Verdelhan (2007a) find that real aggregate consumption growth risk is priced on currency markets. This provides support for a model which – like ours – focuses on real risk, abstracting from money and inflation. However, Burnside, Eichenbaum, Kleschelski and Rebelo (2007) document that forward premium strategies yield very high Sharpe ratios, but argue that the payoffs of such strategies are not correlated with traditional risk factors. This disagreement spurred a debate on whether or not consumption growth risk explains excess returns on currency speculation (Burnside 2007, Lustig and Verdelhan 2007b).

On the theory side, numerous studies have attempted to explain the UIP puzzle in rational expectations settings. Few models, however, are able to reproduce the negative UIP slope coefficient. Here we concentrate on some of the most successful studies. We start by reviewing arguments that rely on counter-cyclical risk premia. We then go over the literature that departs from rational expectations and introduces behavioral biases.

Frachot (1996) shows that a two-country Cox, Ingersoll, and Ross (1985) framework can account for the UIP puzzle but it does not provide an economic interpretation of the currency risk premium. Alvarez, Atkeson, and Kehoe (2005) rely on a model with endogenously segmented markets to generate qualitatively the forward premium anomaly. In their model, higher money growth leads to higher inflation. This induces more agents to enter the asset market because the cost of non-participation is higher. This, in turn, decreases risk premia. Most recently, Verdelhan (2007) generates counter-cyclical risk premia via the varying habit formation models pioneered by Abel (1990) and Campbell and Cochrane (1999). In his model, the domestic investor expects to receive a positive foreign currency excess return in bad times when he is more risk-averse than his foreign counterpart. Times of high risk-aversion correspond to low interest rates at home. Thus domestic investors expect positive currency excess returns when domestic interest rates are low and foreign interest rates are high. Finally, Colacito (2006) and Colacito and Croce (2006) apply Bansal and Yaron (2004)'s model with Epstein-Zin-Weil preferences to international economics. Bansal and Shaliastovich (2007) have two-country setting, rely on a perfect cross-country correlation among shocks to the long run components of consumption growth rates to reproduce the UIP puzzle.

Bacchetta and van Wincoop (2006) develop a model where information is costly to acquire and to process. Because of these costs, many investors optimally choose to assess available information and revise their portfolios infrequently. This rational inattention mechanism produces a negative UIP coefficient along the lines suggested by Froot and Thaler (1990) and Lyons (2001): if investors are slow to respond to news of higher domestic interest rates, there will be a continued reallocation of portfolios towards domestic bonds and a appreciation of the currency subsequent to the shock. Finally, another strand of the literature departs from the assumption of frictionless markets. Using microstructure frictions, Burnside, Eichenbaum and Rebelo (2007) rely on asymmetric information and behavioral biases to explain the forward premium puzzle.

Finally, the closed forms in this paper are made possibly by the "linearity-generating" processes developed in Gabaix (2007a), and the modelling of environment with stochastic rare disasters proposed in Gabaix (2007b).

2 Model setup

2.1 Macroeconomic environment: The stock view of the exchange rate

We consider a stochastic infinite horizon open economy model. There are N countries indexed by i. Each country i is endowed with two goods, a traded good, called a, and a non-traded good, called b_i. The traded good is common to all countries, the non-traded good is country-specific.

Preferences. In country i, agents value consumption streams $\left(C_{it}^{a}, C_{it}^{b}\right)_{t\geq 0}$ according to

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} e^{-\delta t}\frac{\left(C_{it}^{a}\right)^{1-\gamma}+\left(C_{it}^{b_i}\right)^{1-\gamma}}{1-\gamma}\right]$$

Note that the two goods enter separably in the utility function. Together with the assumption of complete markets, this will allow us to derive a simple expression for the pricing kernel.

Numéraires. Our choice of numéraires follows the Harberger convention: we choose the traded good a to be the international numéraire, and the non-traded good b_i to be the numéraire in country i. We will sometimes call the traded good the "international good" or the "world currency".

We call e_{it} the exchange rate of country i in terms of the international good, with the convention that a high e_{it} means a "high value" domestic currency (when e_{it} increases, the domestic currency appreciates).[1] Hence, if a good has a price p_{it} in the currency of country i, it has price $p_t^* = e_{it}p_{it}$ in terms of the world currency. Stars (*) denote values in terms of the international good.

As the non-traded good b_i is the numéraire in country i, its price in country i is $p_{it}^{b_i} = 1$. Hence, its price in terms of the traded good is $p_t^{b_i*} = e_{it}p_{it}^{b_i}$, so that

$$e_{it} = p_t^{b_i*} \tag{1}$$

The exchange e_{it} rate of country i, in terms of the international currency (i.e., in terms of the traded good), is simply the price of the non-traded good of country i in terms of the traded good.

So, the exchange rate between country i and country j is the ratio of the e's of the two countries, e_{it}/e_{jt}.

Markets. Markets are complete: there is perfect risk sharing across countries in the consumption of international goods. Let C_t^{a*} be the world consumption of the traded good. The pricing kernel in terms of the traded good can therefore be expressed as

$$M_t^* = e^{-\delta t} \left(C_t^{a*}\right)^{-\gamma}.$$

The pricing kernel means that an asset producing a stochastic stream $\left(D_{t+s}\right)_{s\geq 0}$ of the traded good, has a price: $\mathbb{E}_t \left[\sum_{s=0}^{\infty} M_{t+s}^* D_{t+s}\right] / M_t^*$.

[1] This choice of numéraire, although it does not follow the tradition which is to define the numéraire as a basket of goods in the country, brings tractability to the analysis.

Technology. There is a linear technology to convert the non-traded good of country i into the traded good. By investing one unit of the non-traded good at time t, one obtains $e^{-\lambda s}\omega_{i,t+s}$ units of the international good, at all periods $s \geq t$. The interpretation is that ω_{it} is the productivity of the export technology, and the initial investment depreciates at a rate λ.

Hence, the non-traded good is a capital good that produces dividends $D_{t+s} = e^{-\lambda s}\omega_{i,t+s}$. So, in terms of the traded good, the price of the non-traded good b_i of country i is:

$$p_t^{b_i *} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} M_{t+s}^* e^{-\lambda s} \omega_{i,t+s} \right] / M_t^*$$

Given that $e_{it} = p_t^{b_i *}$ (Eq. 1), the following obtains.

Proposition 1 *(Stock view of the exchange rate) In terms of the "international currency," the exchange rate e_{it} of country i is the discounted present value of its future export productivity:*

$$e_{it} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} M_{t+s}^* e^{-\lambda s} \omega_{i,t+s} \right] / M_t^* \tag{2}$$

with the convention that an increase in e_{it} means an appreciation of country i's currency.

In Eq. 2, $\omega_{i,t+s}$ is the productivity of country i's export sector at time $t+s$. M_{t+s}^* is the international pricing kernel, and is independent of country i.

To our knowledge, the above formulation is novel, complete-market microfoundation for the "asset view" of the exchange rate (Engel and West 2005 survey earlier "asset view" models, that feature incomplete markets). The exchange rate is the relative price of two goods, the traded and the non-traded good. At the same time, Eq. 2 gives us a stock view of the exchange rate: the exchange is a present value of future levels of productivity in the country. The above formulation could be used for many other models of the exchange rate. For instance, the stochastic discount factor M_{t+s}^* could come from a model with habit formation (Abel 1990, Campbell Cochrane 1999) or long run risk (Bansal and Yaron 2004). We choose to study disasters, in part because they have been less studied.

2.2 Macroeconomic environment: Disaster risk

World consumption of the traded good. We will study equilibria where the world consumption of the traded good C_t^{a*} follows the following stochastic process. As Rietz (1988) and Barro (2006), we assume that in each period $t+1$, a disaster may happen, with a probability p_t. If a disaster does not happen, $C_{t+1}^{a*}/C_t^{a*} = e^g$, where g is the normal-times growth rate of the economy. If a disaster happens, then $C_{t+1}^{a*}/C_t^{a*} = e^g B$, with $B > 0$.[2] For instance, if $B = 0.7$, consumption falls by 30%. To sum up:

$$\frac{C_{t+1}^{a*}}{C_t^{a*}} = \begin{cases} e^g & \text{if there is no disaster at } t+1 \\ e^g B_{t+1} & \text{if there is a disaster at } t+1 \end{cases} \tag{3}$$

Hence the pricing kernel is given by

$$\frac{M_{t+1}^*}{M_t^*} = \begin{cases} e^{-R} & \text{if there is no disaster at } t+1 \\ e^{-R} B_{t+1}^{-\gamma} & \text{if there is a disaster at } t+1 \end{cases} \tag{4}$$

where

$$R = \delta + \gamma g_c$$

is the risk-free rate in an economy that would have a zero probability of disasters. For future reference, we refer to it as the Ramsey interest rate.

Process (3) can be rationalized as the general equilibrium outcome in a model with a finite number of countries, provided the endowments of those countries satisfy some conditions spelled out in Lemma 1 of Appendix B.

Productivity. We assume that productivity of country i follows:

$$\frac{\omega_{i,t+1}}{\omega_{i,t}} = \begin{cases} e^{g_{\omega_i}} & \text{if there is no disaster at } t+1 \\ e^{g_{\omega_i}} F_{i,t+1} & \text{if there is a disaster at } t+1 \end{cases}$$

i.e. during disaster, the relative productivity of the traded good is multiplied by $F_{i,t+1}$. For instance, if productivity falls by 20%, then $F_{i,t+1} = 0.8$. We define the "resilience" of country

[2] Typically, extra i.i.d. noise is added, but given that it never materially affects the asset prices, it is omitted here.

i as:

$$H_{it} = p_{it} \left(\mathbb{E}_t \left[B_{t+1}^{-\gamma} F_{it+1} \mid \text{Disaster at } t+1 \right] - 1 \right) = H_{i*} + \widehat{H}_{it}. \tag{5}$$

where H_{i*} and $\widehat{H}_{it}$ are respectively the constant and variable part of the resilience. This is a measure of how well productivity is insulated from world disaster.[3] In (5), the probability p_t and world intensity of disasters B_{t+1} are common to all countries, but the recovery rate $F_{i,t+1}$ is country-specific. Of course, the recovery rates could be correlated across countries. In order to facilitate taking the continuous time limit, it is useful to write write $H_{i*} = e^{h_{i*}} - 1$.

To obtain tractability, we postulate a Linearity-Generating process (Appendix A) for $M_t^* e^{-\lambda t} (1, \omega_{it})$. The law of motion for $\widehat{H}_{it}$ is:

$$\widehat{H}_{it+1} = \frac{1 + H_{i*}}{1 + H_{it}} e^{-\phi_{H_i}} \widehat{H}_{it} + \varepsilon_{i,t+1}^{H}, \tag{6}$$

where $\mathbb{E}_t \left[\varepsilon_{i,t+1}^{H} \right] = \mathbb{E}_t \left[\varepsilon_{i,t+1}^{H} \mid \text{Disaster at } t+1 \right] = 0$.

Eq. 6 means that $\widehat{H}_t$ mean-reverts to 0, but as a "twisted" autoregressive process. As H_{it} hovers around H_{i*}, $\frac{1+H_{i*}}{1+H_{it}}$ is close to 1, so that the process behaves much like a regular AR(1): $\widehat{H}_{it+1} \simeq e^{-\phi_{H_i}} \widehat{H}_{it} + \varepsilon_{i,t+1}^{H}$, an equation that holds up to second order terms. The $\frac{1+H_*}{1+H_t}$ term is a "twist" term that makes the process very tractable. It is best thought as economically innocuous, and simply an analytical convenience. Gabaix (2007, Technical Appendix) shows that the process, physically, behaves indeed like an AR(1).

Its continuous time analogue is:

$$\widehat{H}_{it} = - \left(\phi_{H_i} + \widehat{H}_{it} \right) \widehat{H}_{it} dt + dN_{it}^{H}, \tag{7}$$

where where N_t^H is a martingale, $E_t \left[dN_t^H \right] = \mathbb{E}_t \left[dN_t^H \mid \text{Disaster at } t+1 \right] = 0$.

This assumption allows us to derive the equilibrium exchange rate in closed form.

Proposition 2 *(Level of the exchange rate) In terms of the "international currency," the exchange rate of country i is:*

$$e_{it} = \frac{\omega_{it}}{1 - e^{-r_{ei}}} \left(1 + \frac{e^{-r_{ei} - h_{i*}}}{1 - e^{-r_{ei} - \phi_H}} \widehat{H}_{it} \right) \tag{8}$$

[3] This model addresses the concern of Brandt, Cochrane and Santa-Clara (2006), who note that discount factors must be highly correlated across countries. They are in this model, because the crisis affect all countries.

where ω_t is the current productivity of the country. In the limit of small time intervals, the exchange rate is:

$$e_{it} = \frac{\omega_{it}}{r_{ei}} \left(1 + \frac{\widehat{H}_{it}}{r_{ei} + \phi_{H_i}} \right) \tag{9}$$

with

$$r_{ei} \equiv R + \lambda - g_{\omega_i} - h_{i*}. \tag{10}$$

Formula (9) is a modified version of Gordon's formula. It can be verified that e_{it} is decreasing in r_{ei}: the exchange rate is decreasing in the Ramsey interest rate R, decreasing in the depreciation rate of capital λ, increasing in the growth of productivity g_ω. Formula (9) implicitly exhibits a Balassa-Samuelson effect: more productive countries – countries with a higher ω_t – have a higher real exchange rate.[4] Countries with a high expected productivity growth also have a high exchange rates.

Importantly, e_t is increasing in h_* and $\widehat{H}_t$: Risky countries have a low exchange rate. Finally, at this stage, the volatility of the exchange rate comes from the volatility of its resilience $\widehat{H}_t$. Later, we generalize the setup and introduce other factors.

In Section (6), we explain how to infer a country's resilience from currency options data and provide evidence that riskier countries have depreciated real exchange rates.

2.3 The forward premium puzzle

Consider a one period domestic bond in country i, that yields 1 unit of the currency of country i at time $t+1$. It will be worth $e_{i,t+1}$ of the international currency. Hence the domestic price of that bond is given by:[5]

$$\frac{1}{1 + r_{it}} = \mathbb{E}_t \left[\frac{M^*_{t+1} e_{i,t+1}}{M^*_t e_{i,t}} \right] \tag{11}$$

[4] Our formula holds for more general specifications of the utility function. For example, we could allow utility to be defined by

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} e^{-\delta t} \frac{(C^a_t)^{1-\gamma}}{1-\gamma} \right] + V(\{C^b_t\}_{t \geq 0})$$

where V is any consistent utility function over non-traded goods consumption processes $\{C^b_t\}_{t \geq 0}$. Were we to follow this route, our model would not generate a perfect correlation between total consumption and real exchange rates, which Backus and Smith (1993) have demonstrated doesn't hold in the data.

[5] The derivation is standard. In the international currency, the payoff of the bond is e_{t+1}, so its price is $\mathbb{E}_t \left[\frac{M^*_{t+1} e_{t+1}}{M^*_t} \right]$, and its domestic price is (11).

where r_t is the domestic interest rate – the nominal interest rate in domestic currency.

Proposition 3 *(Level of the domestic short term interest rate, when there is no inflation on the home goods). The value of the domestic short term rate is*

$$r_{it} = e^{r_{ei}-\lambda}\left[1 - \frac{\left(1-e^{-r_{ei}}\right)e^{-h_{i*}}\widehat{H}_{it}}{1-e^{-r_{ei}-\phi_H}+e^{-h_{i*}}\widehat{H}_{it}}\right] - 1 \tag{12}$$

In the limit of small time intervals, the interest rate is:

$$r_{it} = r_{ei} - \lambda - \frac{r_{ei}\widehat{H}_{it}}{r_{ei}+\phi_H+\widehat{H}_{it}} \tag{13}$$

When a country is very "risky", ($\widehat{H}_{it}$ low), its interest rate is high (13), because its currency has a high risk of depreciating in bad states of the world. Note that this risk is a risk of depreciation, not a default risk.

Hence, countries with high interest rates will see their exchange rate appreciate – that's the "forward exchange rate premium puzzle" or "uncovered interest rate parity puzzle" highlighted by Hansen and Hodrick (1980) and Fama (1984), and replicated for various countries and time periods many times since (Engel 1996, Lewis 1995 provide surveys).

We analyze the predictions of our model for Fama regressions in two different types of samples: with and with no disaster. We consider countries with identical constant parameters, but possibly different $\widehat{H}_t$ and ω_t.

Fama regressions conditional on no disaster. In the continuous time limit, the expected growth rate of the exchange rate, conditional on no disasters is, dropping the index i for country i,

$$\mathbb{E}_t\left[\frac{1}{e_t}\frac{de_t}{dt}\right] = g_\omega + \frac{\mathbb{E}_t\left[\frac{d\widehat{H}_t}{dt}\right]}{r_e+\phi+\widehat{H}_t} = g_\omega - \frac{\left(\phi+\widehat{H}_t\right)\widehat{H}_t}{r_e+\phi+\widehat{H}_t}.$$

In a first order approximation in $\widehat{H}_t$:

$$\mathbb{E}_t\left[\frac{1}{e_t}\frac{de_t}{dt}\right] = g_\omega - \frac{\phi}{r_e+\phi}\widehat{H}_t$$

When the country is very risky, $\widehat{H}_t$ is high, and its exchange rate is low (9); as the exchange rate mean-reverts, its exchange rate will appreciate, so that $\mathbb{E}_t\left[\frac{de_t}{dt}\right]/e_t > 0$.

Similary, in a first order approximation in $\widehat{H}_t$:

$$r_t = r_e - \lambda - \frac{r_e \widehat{H}_t}{r_e + \phi}$$

Hence

$$\mathbb{E}_t\left[\frac{1}{e_t}\frac{de_t}{dt}\right] = \frac{\phi}{r_e} r_t + g_\omega - \frac{\phi\,(r_e - \lambda)}{r_e}$$

Consider the Fama (1984) regression of the changes in the exchange rate between countries A and B regressed on the difference in interest rates, in a sample with no disasters:

$$\text{Fama regression: } \mathbb{E}_t\left[\frac{e^A_{t+1} - e^A_t}{e^A_t} - \frac{e^B_{t+1} - e^B_t}{e^B_t}\right] = \alpha - \beta(r^A_t - r^B_t) \tag{14}$$

The expectation hypothesis predicts $\beta = 1$. The present model however predicts a negative coefficient. For simplicity, we consider the case where the two countries, A and B, have the same r_e.

Proposition 4 *(Coefficient in the Fama regression, conditionally on no disasters). In the Fama regression (14), in a sample with no disasters, the coefficient is:*

$$\beta = -\frac{\phi}{r_e}. \tag{15}$$

With the calibrated numbers with $\phi = 20\%/\text{year}$, $r_e = 10\%/\text{year}$, the coefficient in a yearly regression should be $\beta = -2$, which is in the order of magnitude of the results of the literature. We conclude that even quantitatively, the UIP puzzle seems accounted for by the framework.

Unconditional Fama regressions. We next turn to the unconditional Fama regression. Using Eq. 11, we have

$$\frac{1 + r^B_t}{1 + r^A_t} = \frac{\mathbb{E}_t\left[\frac{M^*_{t+1} e^A_{t+1}}{M^*_t e^A_t}\right]}{\mathbb{E}_t\left[\frac{M^*_{t+1} e^B_{t+1}}{M^*_t e^B_t}\right]}$$

which in the continuous time limit can be expressed as

$$r_t^B - r_t^A = \mathbb{E}_t\left[\frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right] + Cov_t\left(\frac{M_{t+1}^*}{M_t^*}, \frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right)$$

i.e.

$$\mathbb{E}_t\left[\frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right] = r_t^B - r_t^A - Cov_t\left(\frac{M_{t+1}^*}{M_t^*}, \frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right)$$

This expression highlights the role of the risk premium $\pi_t^{A,B}$:

$$\pi_t^{A,B} = -Cov_t\left(\frac{M_{t+1}^*}{M_t^*}, \frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right)$$

Consider now the Fama (1984) regression of the changes in the exchange rate between countries A and B regressed on the difference in interest rates in a full sample:

$$\text{Fama regression: } \mathbb{E}_t\left[\frac{e_{t+1}^A - e_t^A}{e_t^A} - \frac{e_{t+1}^B - e_t^B}{e_t^B}\right] = \alpha^{Full} - \beta^{Full}(r_t^A - r_t^B) \tag{16}$$

The coefficient β^{Full} is now given by

$$\beta^{Full} = 1 - \frac{Cov(\pi_t^{A,B}, r_t^A - r_t^B)}{Var\left(r_t^A - r_t^B\right)}$$

Therefore, we can have $\beta^{Full} < 0$ if and only if the risk premium covaries positively enough with the interest rate differential. It is easy to compute

$$\pi_t^{A,B} = (1-\beta)(r_t^A - r_t^B) + p_t \mathbb{E}_t\left[F_{t+1}^A - F_{t+1}^B\right]$$

which leads to

$$\begin{aligned}\beta^{Full} &= \beta - \frac{Cov\left(p_t\mathbb{E}_t\left[F_{t+1}^A - F_{t+1}^B\right], r_t^A - r_t^B\right)}{Var\left(r_t^A - r_t^B\right)} \\ \beta^{Full} &= \beta + (1-\beta)\frac{Cov\left(p_t\mathbb{E}_t\left[F_{t+1}^A - F_{t+1}^B\right], \widehat{H}_t^A - \widehat{H}_t^B\right)}{Var\left(\widehat{H}_t^A - \widehat{H}_t^B\right)} \qquad (17)\end{aligned}$$

In the case where B_{t+1} is constant and equal to B, and $p_t \mathbb{E}_t \left[F_{t+1}^A - F_{t+1}^B \right] = \left(\widehat{H}_t^A - \widehat{H}_t^B \right) B^\gamma$

$$\beta^{Full} = \beta + (1-\beta) B^\gamma = -\frac{\phi}{r_e} + \left(1 + \frac{\phi}{r_e} \right) B^\gamma$$

Proposition 5 *(Coefficient in the Fama regression, unconditionally). In the Fama regression (14), in a full sample, in the case where $B_t \equiv B$, the coefficient is:*

$$\beta^{Full} = 1 - \frac{\phi}{r_e} \left(1 - B^\gamma \right) \tag{18}$$

In particular, for $B = 1$, there is no disaster risk (consumption doesn't fall during disasters), so that $\beta^{Full} = 1$. Hence, the Fama regression yields a negative coefficient only if disaster risk is high enough. We note that the negative β^{Full} does not come from a peso problem explanation, in the sense that, in the model, even in a sample that includes disasters, there can a negative coefficient in the Fama regression.

3 A setup with a risk factor and a business cycle factor

The above setup gave the essence of the disaster mechanism, but it has only one factor, so that, controlling for current productivity, exchange rate and risk premia are perfectly correlated, which in a variety of context is not a desirable feature. Accordingly, we extend the framework to a two-factor model, a risk factor, and a business cycle factor.

3.1 Setup with a risk factor and a business cycle factor

In the baseline model, the real rate varies only because of the risk premium. We can easily extend the model to business cycle movements in the interest rates. For ease of notations, we typically drop the index i for country i. We say that the country's productivity is $\omega_t = \overline{\omega}_t (1 + y_t)$, where $\overline{\omega}_t$ is the "permanent" component of productivity, and y_t is a "business cycle" fluctuation or "deviation of productivity from trend". We model:

$$\frac{\overline{\omega}_{t+1}}{\overline{\omega}_t} = \begin{cases} e^{g_\omega} & \text{in normal times} \\ e^{g_\omega} F_{t+1} & \text{if disaster} \end{cases}$$

and LG-twisted process for y_t:

$$\mathbb{E}_t\left[y_{t+1}\right] = \frac{1+H_*}{1+H_t}e^{-\phi_y}y_t$$

with innovation uncorrelated to the ones of ω_t and M_t. This allows to calculate the exchange rate.

Proposition 6 *(Exchange rate with a business cycle factor) The exchange rate is*

$$e_t = \frac{\overline{\omega}_t}{1-e^{-r_e}}\left(1+\frac{e^{-r_e-h_*}}{1-e^{-r_e-\phi_H}}\widehat{H}_t+\frac{1-e^{-r_e}}{1-e^{-r_e-\phi_y}}y_t\right) \tag{19}$$

and in the continuous time limit:

$$e_t = \frac{\overline{\omega}_t}{r_e}\left(1+\frac{\widehat{H}_t}{r_e+\phi_H}+\frac{r_e y_t}{r_e+\phi_y}\right) \tag{20}$$

and the interest rate is:

$$r_t = r_e - \lambda + \frac{-\frac{r_e}{r_e+\phi_H}\widehat{H}_t+\frac{r_e\phi_y}{r_e+\phi_y}y_t}{1+\frac{\widehat{H}_t}{r_e+\phi_H}+\frac{r_e y_t}{r_e+\phi_y}} \tag{21}$$

In this setup, the resilience $\widehat{H}_t$ has the same effect as before. But there is an additional factor, the deviation of productivity from trend y_t, which is not associated with any risk premium. As would be expected, when productivity is high, the exchange rate is high, and is expected to depreciate, so that the interest rate rate is high.

3.2 Fama regression with two factors

Let us revisit the Fama regression (14):

$$\mathbb{E}_t\left[\frac{e^A_{t+1}-e^A_t}{e^A_t}-\frac{e^B_{t+1}-e^B_t}{e^B_t}\right] = \alpha' - \beta'(r^A_t - r^B_t).$$

The next Proposition relates the coefficient β' in a sample with no disaster, and the coefficient β'^{Full} in full sample, to their corresponding values β and β^{Full} previously derived for the one-factor model.

Proposition 7 *(Value of the β coefficient in the Fama regression, with two factors). Up to second order terms, in the Fama regression, the coefficients are:*

$$\beta' = \nu\beta + 1 - \nu \tag{22}$$

$$\beta'^{Full} = \nu\beta^{Full} + 1 - \nu \tag{23}$$

where β and β^{Full} are given in Eqs. 15 and 18, and ν is the share of variance in the interest rate due to $\widehat{H}_t$,

$$\nu = \frac{\left(\frac{r_e}{r_e+\phi_H}\right)^2 var\left(\widehat{H}_t\right)}{\left(\frac{r_e}{r_e+\phi_H}\right)^2 var\left(\widehat{H}_t\right) + \left(\frac{r_e\phi_y}{r_e+\phi_y}\right)^2 var\left(y_t\right)}. \tag{24}$$

In Eq. 22, β' is the weighted average of two Fama coefficients. One, β, comes from the variations in the risk premium. The second, 1, comes from the cyclical variations in productivity, and is the value predicted by the expectation hypothesis. The weight ν is the relative share of the two factors in the variance of the interest rate.

4 Yield Curve, Forward Rates, and Exchange Rates, Real and Nominal

4.1 Exchange rates and long term real rates

To study the forward premium puzzle for long term rates, we first derive the price of long term bonds. The price of a bond yield one unit of the currency at time $t+T$ is: $Z_t(T) = \mathbb{E}_t\left[\frac{M^*_{t+T}e_{t+T}}{M^*_t e_t}\right]$.

The yield at maturity T, $Y_t(T)$, and the forward rates $f_t(T)$ are defined by $Z_t(T) = e^{-Y_t(T)T} = e^{-\sum_{T'=1}^{T} f_t(T')}$.

Proposition 8 *(Price of a domestic bond, when there is no inflation on the home goods) The domestic price of a domestic bond of maturity T, in the continuous time limit:*

$$Z_t(T) = e^{-(r_e-\lambda)T}\frac{1 + \frac{r_e\left(1-e^{-\phi T}\right)+\phi}{(r_e+\phi)\phi}\widehat{H}_t + e^{-\phi_y T}\frac{r_e y_t}{r_e+\phi_y}}{1 + \frac{\widehat{H}_t}{r_e+\phi} + \frac{r_e y_t}{r_e+\phi_y}} \tag{25}$$

and the forward rate is, up to second order terms in $\widehat{H}_t$ and y_t,

$$f_t(T) = r_e - \lambda - \frac{r_e}{r_e + \phi_H} e^{-\phi_H T} \widehat{H}_t + \frac{r_e \phi_y}{r_e + \phi_y} e^{-\phi_y T} y_t \tag{26}$$

The proof to this Proposition also calculates the expressions for bonds, yields, forward rates, in discrete and continuous time.

The proof is in Appendix B.

To illustrate the economics, suppose that the country has a very high $\widehat{H}_t$, i.e. is very safe. Future $\widehat{H}_t$ will, on average, mean-revert to 0. Hence, by (9), the exchange rate (which is high now) will depreciate. The short terms rates are low (Eq. 13), which is the forward premium puzzle. Eq. 25 says that long term rates are low (the bond price is high because $\widehat{H}_t$.is high). Hence, perhaps paradoxically at first, investors expect the exchange rate to depreciate in the long term, and also, long term rates are low. In the model, this is because investors perceive the country as very "safe", and require a small risk premium on it.

4.1.1 Fama regression with forward rates

Boudoukh, Richardson Whitelaw (BRW, 2006) propose to regress the exchange rate movement on the $T-$period forward rate from T periods ago:

$$\text{BRW regression: } \mathbb{E}_t \left[\frac{e^A_{t+1} - e^A_t}{e^A_t} - \frac{e^B_{t+1} - e^B_t}{e^B_t} \right] = \alpha^{Fwd}(T) - \beta^{Fwd}(T) \left(f^A_{t-T}(T+1) - f^B_{t-T}(T+1) \right) \tag{27}$$

Our model's prediction is in the next Proposition.

Proposition 9 *(Value of the β coefficient in the Fama regression, with two factors, with forward rates). Up to second order terms, in the BRW (27) regression with forward rates, the coefficients are:*

$$\beta^{Fwd}(T) = \nu(T)\beta + 1 - \nu(T) \tag{28}$$

and

$$\beta^{Fwd,Full}(T) = \nu(T)\beta^{Full} + 1 - \nu(T) \tag{29}$$

where β *and* β^{Full} *are given in Eqs. 15 and 18, and*

$$\nu(T) = \frac{\left(\frac{r_e}{r_e+\phi_H}\right)^2 var\left(\widehat{H}_t\right) e^{-2\phi_H T}}{\left(\frac{r_e}{r_e+\phi_H}\right)^2 var\left(\widehat{H}_t\right) e^{-2\phi_H T} + \left(\frac{r_e \phi_y}{r_e+\phi_y}\right)^2 var\left(y_t\right) e^{-2\phi_y T}} \tag{30}$$

is the share of variance in the forward rate due to $\widehat{H}_t$. *In particular, when* $\phi_H > \phi_y$, *the long horizon regression have coefficient going to 1:* $\lim_{T\to\infty} \widetilde{\beta}(T) = \lim_{T\to\infty} \widetilde{\beta}^{Full}(T) = 1$.

BRW (2006) find that $\beta^{fwd}(T)$ increases toward 1 with the horizon. Our theory is consistent with this empirical finding. Indeed, to interpret Proposition 9, consider the case where risk-premia are fast mean-reverting, and productivity is slowly mean reverting, $\phi_H > \phi_y$. Then, large T, $\nu(T)$ tends to 0, which means that, at long horizons, the forward rate is mostly determined by the level of y_t, not of the risk premium. Hence, at long horizon the model behaves like a model without risk premia, hence generates a coefficient β close to 1.

4.2 A simple model of exchange rates and nominal yield curves

Until recently, forward real interest rates were not available. Only their nominal counterparts were the support of actively traded securities. Even today, most bonds are nominal bonds.

To model nominal bonds, we build on the real two factor model developed above. Let $Q_t = Q_0 \prod_{s=1}^{t} (1 - i_s)$ be the value of money (the inverse of the price level). The nominal interest rate $\widetilde{r}_t$ satisfies $\frac{1}{1+\widetilde{r}_t} = \mathbb{E}_t\left[\frac{M^*_{t+1} e_{t+1}}{M^*_t e_t}(1 - i_t)\right]$, so that, in the continuous time limit,

$$\widetilde{r}_t = r_t + i_t, \tag{31}$$

the nominal interest rate is the real interest rate, plus inflation. The Fisher neutrality applies: there is no burst of inflation during disasters. With a burst of inflation, even short term bonds would command a risk premium.

Inflation hovers around i_*, according to the LG process:

$$i_{t+1} = i_* + \frac{1 - i_*}{1 - i_t} e^{-\phi_i}(i_t - i_*) + \varepsilon^i_{t+1} \tag{32}$$

where ε_{t+1}^{i} has mean 0, and is uncorrelated with innovations in M_{t+1}, in particular with disasters. One could correlate this, but the analysis is a bit more complicated (the analysis is available upon request). The expected value of 1 unit of currency T period later is:

$$\mathbb{E}_t\left[\frac{Q_{t+T}}{Q_t}\right] = (1-i_*)^T\left(1-\frac{1-e^{-\phi_i T}}{1-e^{-\phi_i}}\frac{i_t-i_*}{1-i_*}\right) \tag{33}$$

or $\mathbb{E}_t\left[\frac{Q_{t+T}}{Q_t}\right] = e^{-i_* T}\left(1-\frac{1-e^{-\phi_i T}}{\phi_i}(i_t-i_*)\right)$ in the continuous time limit.

To fix notations, we denote nominal variables with a tilde. The price of long term nominal bonds yielding one unit of the currency at time $t+T$ is $\widetilde{Z}_t(T) = \mathbb{E}_t\left[\frac{M_{t+T}^* e_{t+T} Q_{t+T}}{M_t^* e_t Q_t}\right]$. Because we assume that shocks to inflation are uncorrelated with disasters, the value present value of one nominal unit of the currency is

$$\widetilde{Z}_t(T) = \mathbb{E}_t\left[\frac{M_{t+T}^* e_{t+T} Q_{t+T}}{M_t^* e_t Q_t}\right] = \mathbb{E}_t\left[\frac{M_{t+T}^* e_{t+T}}{M_t^* e_t}\right]\mathbb{E}_t\left[\frac{Q_{t+T}}{Q_t}\right]$$

Hence, the value of the zero coupon bond is:

Proposition 10 *(Price of a nominal domestic bond, with no inflation risk premia) The domestic price of a domestic nominal bond of maturity T, in the continuous time limit:*

$$\widetilde{Z}_t(T) = e^{-(r_e-\lambda)T}\frac{1+\frac{r_e\left(1-e^{-\phi T}\right)+\phi}{(r_e+\phi)\phi}\widehat{H}_t + e^{-\phi_y T}\frac{r_e y_t}{r_e+\phi_y}}{1+\frac{\widehat{H}_t}{r_e+\phi}+\frac{r_e y_t}{r_e+\phi_y}}\cdot e^{-i_* T}\left(1-\frac{1-e^{-\phi_i T}}{\phi_i}(i_t-i_*)\right) \tag{34}$$

and the nominal forward rate is, up to second order terms in $\widehat{H}_t$ and y_t, $i_t - i_*$:

$$f_t(T) = r_e - \lambda - \frac{r_e}{r_e+\phi_H}e^{-\phi_H T}\widehat{H}_t + \frac{r_e\phi_y}{r_e+\phi_y}e^{-\phi_y T}y_t + i_* + e^{-\phi_i T}(i_t-i_*) \tag{35}$$

The proof to this Proposition also calculates the expressions for bonds, yields, forward rates, in discrete and continuous time.

The nominal forward rate in (35) depends on real and nominal factors. The real factors are the resilience of the economy (the $\widehat{H}_t$) term, the expected growth rate of productivity ($-\phi_y y_t$). The nominal factor is inflation i_t.

Each of the three terms is multiplied by a term of the type $e^{-\phi_H T}$. For small speeds of mean reversion ϕ's, it means that the forward curve is fairly flat.

With Q_t the value of money, the nominal exchange rate is: $\widetilde{e_t} = e_t Q_t$. The expected depreciation of the nominal exchange rate is, up to second order terms, and conditionally on no disasters:

$$\mathbb{E}_t \left[\frac{d\widetilde{e}_t}{\widetilde{e}_t} \right] / dt = g_\omega - \frac{\phi_H \widehat{H}_t}{r_e + \phi_H} - \frac{r_e \phi_y y_t}{r_e + \phi_y} - i_t \tag{36}$$

We can derive the implications of our model for a Fama regression in nominal terms:

$$\mathbb{E}_t \left[\frac{\tilde{e}^A_{t+1} - \tilde{e}^A_t}{\tilde{e}^A_t} - \frac{\tilde{e}^B_{t+1} - \tilde{e}^B_t}{\tilde{e}^B_t} \right] = \tilde{\alpha}^{nom} - \tilde{\beta}^{nom} (\tilde{r}^A_t - \tilde{r}^B_t) \tag{37}$$

where $\tilde{r}^A_t$ and $\tilde{r}^B_t$ are now, with some a slight abuse of notational, the nominal interest rates in countries A and B. Our model's prediction is in the next Proposition.

Proposition 11 *(Value of the β coefficient in the Fama regression in nominal terms). Up to second order terms, in the nominal Fama regression (37) regression with forward rates, the coefficients are:*

$$\widetilde{\beta}^{nom} = \nu^{nom} \beta + 1 - \nu^{nom} \text{ and } \widetilde{\beta}^{nom,Full} = \nu^{nom} \beta^{Full} + 1 - \nu^{nom} \tag{38}$$

where β and β^{Full} are the coefficients in the Fama regression defined in propositions (4) and (5), and

$$\nu^{nom} = \frac{\left(\frac{r_e}{r_e + \phi_H}\right)^2 var\left(\widehat{H}_t\right)}{\left(\frac{r_e}{r_e + \phi_H}\right)^2 var\left(\widehat{H}_t\right) + \left(\frac{r_e \phi_y}{r_e + \phi_y}\right)^2 var\left(y_t\right) + var(i_t)} \tag{39}$$

is the share of variance in the forward rate due to $\widehat{H}_t$. In particular, when $\phi_H > \phi_y$, the long horizon regression have coefficient going to 1: $\lim_{T\to\infty} \widetilde{\beta}(T) = \lim_{T\to\infty} \widetilde{\beta}^{Full}(T) = 1$.

In this simple model with no inflation risk premia, the higher the variance of inflation , the closer to 1 is β^{nom}. Hence, countries with very variable inflation (typically, those are also countries with high average inflation) satisfy approximately the uncovered interest rate parity conditions. When disaster risks are very variables —and the real exchange rate is very variable – then β^{nom} is more negative.

4.3 A richer model with nominal risk premia

We now develop a richer model with an inflation-specific risk premium. We extend the framework developed in the previous section by incorporating inflation risk along the lines of Gabaix (2007).

The variable part of inflation now follows the process:

$$\widehat{i}_{t+1} = \frac{1-i_*}{1-i_t} \cdot \left(e^{-\phi_i} \widehat{i}_t + 1_{\{\text{Disaster at } t+1\}} \left(j_* + \widehat{j}_t \right) \right) + \varepsilon^i_{t+1} \tag{40}$$

In case of a disaster, inflation jumps by an amount $j_t = j_* + \widehat{j}_t$. This jump in inflation makes long term bonds particularly risky. j_* is the baseline jump in inflation, $\widehat{j}_t$ is the mean-reverting deviation from baseline. It follows a twisted auto-regressive process, and, for simplicity, does not jump during crises:

$$\widehat{j}_{t+1} = \frac{1-i_*}{1-i_t} \cdot e^{\phi_\pi} \widehat{j}_t + \varepsilon^j_{t+1} \tag{41}$$

We define $\pi^i_t \equiv \frac{pB^{-\gamma}F}{1+H} \widehat{j}_t$, which is the mean-reverting part of the "risk adjusted" expected increase in inflation if there is a disaster. We parametrize the typical jump in inflation j_* in terms of a number $\kappa \leq (1-\rho_i)/2$:

$$\frac{pB^{-\gamma}Fj_*}{1+H} = (1-i_*)^2 \kappa (1-\rho_i - \kappa).$$

κ represents a risk premium for the risk that inflation increases during disasters. Also, we define $i_{**} \equiv i_* + \kappa$ and $\psi_\pi \equiv \phi_\pi - \kappa$. They represent the "risk adjusted" trend and mean-reversion parameter in the inflation process.

To fix notations, we denote nominal variables with a tilde. The price of long term nominal bonds yielding one unit of the currency at time $t+T$ is

$$\widetilde{Z}_t(T) = \mathbb{E}_t \left[\frac{M^*_{t+T} e_{t+T} Q_{t+T}}{M^*_t e_t Q_t} \right]$$

The yield at maturity T, $\widetilde{Y}_t(T)$, and the forward rates $\widetilde{f}_t(T)$ are defined by $\widetilde{Z}_t(T) = e^{-\widetilde{Y}_t(T)T} = e^{-\sum_{T'=1}^{T} \widetilde{f}_t(T')}$. The next Proposition calculates the forward rate.

Proposition 12 *(Price of a domestic nominal bond, with inflation risk premia) In the con-*

tinuous time limit, in up to second order terms in $\left(\widehat{H}_t, y_t, i_t, \pi_t^i\right)$*:*

$$\widetilde{f}_t(T) = r_e - \lambda - \frac{r_e}{r_e + \phi_H} e^{-\phi_H T} \widehat{H}_t + \frac{r_e \phi_y}{r_e + \phi_y} e^{-\phi_y T} y_t + i_{**}\left(1 - e^{-\phi_i T}\right) + e^{-\phi_i T} i_t + \frac{e^{-\phi_i T} - e^{-\psi_\pi T}}{\psi_\pi - \phi_i} \pi_t^i \tag{42}$$

The nominal forward rate in (42) depends on real and nominal factors. The real factors are the resilience of the economy (the $\widehat{H}_t$) term, the expected growth rate of productivity ($-\phi_y y_t$). The nominal factors are inflation i_t, and the variable component of the the risk premium for inflation jump risk, π_t^i.

When a disaster occurs, inflation increases (on average). As very short term bills are essentially immune to inflation risk, while long term bonds lose value when inflation is higher, long term bonds are riskier, hence they get a higher risk premium. Hence, the yield curve slope up on average – as implied by the term $i_{**}\left(1 - e^{-\phi_i T}\right) \sim i_{**}\phi_i T$.

Each of the three terms is multiplied by a term of the type $e^{-\phi_H T}$. For small speeds of mean reversion ϕ's, it means that the forward curve is fairly flat. The last term, however, is close to T for small maturities ($\frac{e^{-\phi_i T} - e^{-\psi_\pi T}}{\psi_\pi - \phi_i} \sim T$). It creates a variable slope in the forward curve. Hence, we obtain a rich, potentially realistic, forward curve.

Nominal yield curves contain a lot of potentially information useful to predict exchange rates. We now explain how to best extract the relevant information to compute exchange rate risk premia. As above, the expected depreciation of the nominal exchange rate is, up to second order terms, and conditionally on no disasters:

$$\mathbb{E}_t\left[\frac{d\widetilde{e}_t}{\widetilde{e}_t}\right]/dt = g_\omega - \frac{\phi_H \widehat{H}_t}{r_e + \phi_H} - \frac{r_e \phi_y y_t}{r_e + \phi_y} - i_t \tag{43}$$

It involves three factors that are also reflected in the nominal forward curve. Note however, that it does not involve the inflation risk premium π_t^i. So, an optimal combination of forward rates should predict expected currency returns with more accuracy than the simple Fama regression. The next Proposition derive the minimal such combination.

Proposition 13 *The expected appreciation of the currency can be expressed:*

$$\mathbb{E}_t\left[\frac{d\widetilde{e}_t}{\widetilde{e}_t}\right]/dt = \alpha - \beta_i i_t - \beta_r r_t - \beta_{slope}\partial_T \widetilde{f}_t(0) - \beta_{Curvature}\partial_T^2 \widetilde{f}_t(0) \tag{44}$$

where i_t is the inflation rate, r_t is the short-term nominal rate, $\partial_T \widetilde{f}_t(0)$ is the slope of the yield curve, $\partial_T^2 \widetilde{f}_t(0)$ is the curvature of the yield curve, and

$$\begin{aligned}
\beta_i &= 1 - \phi_i \psi_\pi / K \\
\beta_r &= \left[\frac{1}{r_e} \phi_y \left(\phi_y - \phi_i - \psi_\pi \right) + \phi_H - \phi_i - \psi_\pi \right] \phi_H / K \\
\beta_{slope} &= - \left(\phi_i + \psi_\pi \right) \left(\frac{\phi_H}{r_e} + 1 \right) / K \\
\beta_{curvature} &= - \left(\frac{\phi_H}{r_e} + 1 \right) / K \\
K &= \left(\phi_i + \psi_\pi - \phi_y - \phi_H \right) \left(\phi_y - \phi_H \right)
\end{aligned}$$

For instance, in the calibration where productivity y_t is a near random walk ($\phi_y = 0$), $\beta_r = 1$. Hence, in the regression (44), the coefficient in the nominal interest rate is 1 – the one predicted by UIP – once we control for the risk premia terms encoded in the slope and curvature of the yield curve.[6]

5 Equity premia and exchange-rate risk premia

Our model allows to think in a tractable way about the joint determination of exchange rate and equity values.

5.1 Local market price of risk and local maximal Sharpe ratios

A clean way of getting at this question is to characterize the maximal Sharpe ratio and the market price of risk in local currency. The stochastic discount factor in local currency is $m_{t+1} = \frac{M_{t+1} e_{t+1}}{M_t e_t}$. The maximal Sharpe ratio is given by: $\mathcal{S}_t \equiv \frac{Var_t^{1/2}(m_{t+1})}{\mathbb{E}_t(m_{t+1})}$. It is given by the formula[7]

$$\mathcal{S}_t = \sqrt{\sigma_{e,t}^2 + (1 + \sigma_{e,t}^2) \frac{1 - p_t}{p_t} \frac{H_t^2}{\left(1 + H_t\right)^2}}$$

[6]For instance, the curvature can be approximated by the discrete formula: $f(1) - 2f(2) + f(3)$. Another discrete proxy is (the opposite of) the tent shape factor for Cochrane and Piazzesi (2006), that explains bond risk premia.

[7]$m_{t+1} = \mathbb{E}_t \left[\frac{e_{t+1}}{e_t} \mid \text{No disaster} \right] e^{-R} \left(1 + \varepsilon_{t+1}\right) \left(1 + B_t^{-\gamma} F_t J_{t+1}\right)$, where $var\left(\varepsilon_{t+1}\right) = \sigma_e^2$, and $J_{t+1} = 0$ with probability $1 - p_t$, and 1 with probability p_t.

where $\sigma_{e,t} = Var_t^{1/2}(e_{t+1} \, |\text{No disaster})/\mathbb{E}_t(e_{t+1} \, |\text{No disaster})$ is the standard deviation of the log exchange rate in normal times. In the continuous time, limit, we can derive a very simple expression

$$\mathcal{S}_t = \sqrt{\sigma_{e,t}^2 + \frac{H_t^2}{p_t}} \text{ with } \sigma_{e,t}^2 = Var_t\left(\frac{de_t}{e_t} \mid \text{No disaster}\right)$$

The only source of time variation in $\sigma_{e,t}^2$ comes from time variations in the variance and covariance of the structural shocks to H_t and y_t: ε_t^H and ε_t^y.

The maximum Sharpe ratio $\mathcal{S}_t$ is high when resiliency H_t is high. Therefore, countries that demand low currency risk premia will feature high local Sharpe ratio and high local equity premia.

5.2 Explicit stock values

Another way to proceed is to take a stand on what fraction of present and future endowments is capitalized in each stock market. A route commonly taken in Lucas-tree economies is to equate the market to a claim on the entirety of the present and future national endowments of goods. However, listed stocks only account for a very small and potentially non-representative fraction of future GDP. Hence, we model stocks without taking a specific stand on the link between the aggregate dividend of listed stocks and GDP.

5.2.1 Firm producing the international good

Consider first the case of a of domestic firm, that produces the international good. More precisely, the dividend D_t follows the following process

$$\frac{D_{t+1}}{D_t} = \begin{cases} e^{g_D}(1+\varepsilon_{D,t+1}) & \text{in normal times} \\ e^{g_D}(1+\varepsilon_{D,t+1})F_{D,t} & \text{if crisis} \end{cases}$$

where an idiosyncratic shock uncorrelated with the stochastic discount factors.

Define the resilience $H_{D,t}$ of the stock as

$$H_{D,t} = p_t\left(\mathbb{E}_t\left[B_{t+1}^{-\gamma}F_{D,t+1}\right] - 1\right) = H_{D*} + \widehat{H}_{D,t}.$$

It is convenient to define $H_{D*} = e^{h_{D*}} - 1$. The law of motion for $\widehat{H}_{D,t}$ is:

$$\widehat{H}_{D,t+1} = \frac{1 + H_{D*}}{1 + H_{D,t}} e^{-\phi_{H_D}} \widehat{H}_{Dt} + \varepsilon_{t+1}^{H_D},$$

where ϕ_{H_D} is the speed of mean reversion of the resilience of the stock.

Proposition 14 *(Domestic stocks producing international goods). The domestic price of the stock $P_{D,t}$ is*

$$P_{D,t} = \frac{D_t}{e_t} \frac{1 + \frac{e^{-r_D - h_{D*}}}{1 - e^{-r_D - \phi_{H_D}}} \widehat{H}_{D,t}}{1 - e^{-r_D}}$$

In the continuous time limit

$$P_{D,t} = \frac{D_t}{e_t} \frac{1 + \frac{\widehat{H}_{D,t}}{r_D + \phi_{H_D}}}{r_D} \tag{45}$$

A more resilient stock (high $\widehat{H}_{D,t}$) has a higher price-dividend and lower future returns. Controlling for this resilience, if the currency is strong (because the country as a whole is safe), then the stock price in domestic currency is low. As e_t is expected to depreciate, the expected return of the stock in local currency is high. In this sense, currency risk premia and local currency equity premia are negatively correlated. Hence, the theory provides an explanation for Hau and Rey (2006)'s evidence that the home-currency stock price is decreasing in the exchange rate.

5.2.2 Firm producing the domestic good

We now turn to a domestic producer producing D_s quantities of the domestic good. Its stock price, in the international currency, is $P_t^* = E_t \left[\sum_{s \geq t} M_s e_s D_s\right]$, so that its domestic price is $P_t = P_t^*/e_t$, hence:

$$P_t/D_t = \frac{\mathbb{E}_t \left[\sum_{s \geq t} M_s e_s D_s\right]}{e_t D_t} \tag{46}$$

We postulate the following process for D_t

$$\frac{D_{t+1}}{D_t} = \begin{cases} e^g & \text{if there is no disaster at } t+1 \\ e^g F_t^i & \text{if there is a disaster at } t+1 \end{cases}$$

F_t^i is the recovery rate in the dividend of that firm. We postulate the F_t^i also follows a LG process, hovering around F_*^i, and mean-reverts at a rate ϕ_F, with a twist spelled out in Eq. (63).

Proposition 15 *(Price of a domestic stock producing non-traded goods). To a first order approximation, the price of stock producing domestic goods is, in terms of the international currency:*

$$P_t^* = P_t e_t = \omega_t D_t \left[\frac{1}{r_D} + \frac{(H_* + p_*)\,\widehat{F}_{it}}{r_D\,(r_D + \phi_F)} + \left(\frac{F_*^i}{r_D} + \frac{1}{r_e + \phi_H} \right) \frac{\widehat{H}_t}{r_D + \phi_H} \right] \tag{47}$$

and in the domestic currency,

$$P_t = \frac{r_e}{r_D} D_t \left[1 + \frac{(H_* + p_*)\,\widehat{F}_{it}}{r_D + \phi_F} + \left(F_*^i - \frac{\phi_H}{r_e + \phi_H} \right) \frac{\widehat{H}_t}{r_D + \phi_H} \right] \tag{48}$$

where $r_D = R - g_D - g_\omega - (H_* + p_*)\,F_*^i$.

To analyze the above expression, we take the polar case where $\widehat{F}_{it}$ (the resiliency of the firm's technology) is uncorrelated with $\widehat{H}_t$ (the country's resilience). The international price of the stock (47) increases with $\widehat{H}_t$, hence with the exchange rate.

The domestic price (48) of the stock can decrease or decrease with the exchange rate, depending on the sign of $F_*^i - \frac{\phi_H}{r_e + \phi_H}$. The price of resilient stock increases with the exchange rate, while the price of non-resilient stocks decreases with the exchange rate. The reason for this ambiguous result can be see in Eq. 46, where an increase in e_t increases both the numerator and the denominator. Take a resilient stock, with F_*^i close to 1. A increase in the country's resilience, $\widehat{H}_t$, increases the present value of future dividends (the numerator of Eq. 46), because future resiliences are high, and the discount rate is lower. Hence, the numerator in (46) increases a lot. The denominator increases also, but not as much. The net effect is that the domestic stock price increases: The cash flows that the firm produces are more valuable, and less risky. However, take a stock with $F_*^i = 0$, i.e. a stock that will be bankrupt after a disaster. Then, there is no "discount rate effect" in the numerator of (46), as cash-flows always have maximal riskiness (they disappear in a disaster). So, the effect due to the rise in the denominator is stronger. Hence, the stock falls, when the exchange rate increases.

All in all, we see that the price of domestic stocks producing nontradables increases with the exchange rate, when it is expressed in international currency, but, expressed in domestic currency, it increases only for the most resilient stocks. Again, one might hope to test that prediction.

6 Option prices and exchange rate risk premia

6.1 Theory

Option prices incorporate direct information about the probability and severity of disasters. In particular, consider the implied volatility smile of a pair of currencies: a risky currency and a safe currency. The smile will be much steeper on the risky currency side. A high "smile-skew" should predict currency appreciation, high interest rate differential and high bond returns.

In order to gain in tractability, we make two simplifying assumptions. First we assume that if a disaster occurs in period $t+1$, ε_{t+1}^{H} is equal to zero. Second, we assume that the distribution of e_{t+1} conditional on date t information and no disaster occurring in period $t+1$ is lognormal around its mean with standard deviation $\sigma_t^{e,i}$ where i indexes countries.

Consider two countries A and B. The currency A price at date 0 of a call that gives the option to buy at date T one unit of currency A for $K=(1+k)\frac{e_0^A}{e_0^B}$ units of currency B is

$$V^{Call}=\frac{1}{e_0^A}\mathbb{E}_0\left[\frac{M_T^*}{M_0^*}\left(e_T^A-Ke_T^B\right)^+\right]=\mathbb{E}_0\left[\frac{M_T^*}{M_0^*}\left(\frac{e_T^A}{e_0^A}-(1+k)\frac{e_T^B}{e_0^B}\right)^+\right]$$

Likewise, the currency A price at date 0 of a put that gives the option to sell at date T one unit of currency A for $K'=(1-k)\frac{e_0^A}{e_0^B}$ units of currency B is

$$V^{Put}=\mathbb{E}_0\left[\frac{M_T^*}{M_0^*}\left((1-k)\frac{e_T^B}{e_0^B}-\frac{e_T^A}{e_0^A}\right)^+\right]$$

Consider the price of a "risk reversal" position $W=V^{Put}-V^{Call}$. Calculations show that it is approximately:

$$W=V^{Put}-V^{Call}=\Psi\left(\left(\widehat{H}_B-\widehat{H}_A\right),kH_*\right)T$$

where $\Psi(x,k) = (x-k)^+ - (-x-k)^+$. The risk reversal position is such that the Black-Scholes component of the put and call have the same price in the limit of short maturities. This allows us to extract the disaster intensities from option prices.

Hence, when k is small,and conditionally on no disasters:

$$\mathbb{E}_t^{ND}\left[\frac{e_{t+\Delta t}^A - e_t^A}{e_t^A} - \frac{e_{t+\Delta t}^B - e_t^B}{e_t^B}\right] = \frac{\phi_H}{r_e + \phi_H}\frac{\Delta t}{T}W \tag{49}$$

That is, a currency with a high put price should have a low price, and should subsequently appreciate. This is because it has a high risk premium, that affects both the put value, and a low value of the exchange rate. Eq. 49 expresses quantitatively the magnitude of the effect.

6.2 Evidence

Carr and Wu (2007) compute the risk-reversals for two pairs of countries: UK and Japan versus the US. They find a high correlation between changes in the price of risk-reversal options and changes in nominal exchange rates: currencies that become riskier – for which puts become relatively more expensive than calls – experience a simultaneous depreciation. Farhi, Gabaix, Ranciere and Verdelhan (2007) extends their analysis to a sample of 25 countries. Their analysis confirms the finding of Carr and Wu (2007) and shows that it also holds for real exchange rates, providing direct evidence in favor of our model.

7 A Calibration

7.1 Choice of Parameter Values

We use yearly units.

Preferences. The coefficient of relative risk aversion is $\gamma = 4$.

Macroeconomy. In normal times, consumption of nontradables grows at rate $g_c = 3\%$. We set $g_\omega = g_c$, but values are not really sensitive to that parameter.

To keep the calibration parsimonious, the probability and intensity of disasters are constant. The probability of disaster is $p = 1.7\%$, as estimated by Barro (2006). We present two calibrations. In disasters, the utility-weighted average recovery rate of consumption is

$\mathbb{E}[B^{-\gamma}]^{-1/\gamma} = 0.55$ (in Calibration 1), or $\mathbb{E}[B^{-\gamma}]^{-1/\gamma} = 0.45$ (in Calibration 2). We make sure that the riskless domestic short term rate is on average around 1%, which pins down the rate of time preference, δ.

Exchange rate. An initial investment depreciates at a usual rate $\lambda = 8\%$. To specify the volatility of the recovery rate F_t, we specify that it has a baseline value $F_* = 0.8$, and its the range is $F_t \in [F_{\min}, F_{\max}] = [0.2, 1.2]$. That means that the technology of transforming domestic goods into international goods could improve. This is because ω_t is really the ratio between two productivities – to produce domestic or international goods, so that relative ratio could increase or decrease. This possibility of a worst-case fall of productivity to 0.2 of its initial level may seem high. Perhaps it proxies for disruptions not directly linked to productivity, e.g. the introduction of taxes, regulation, or a loss of property rights (as in Barro 2006), though we do not not model those interpretations here.

The speed of mean-reversion is $\phi_H = 0.2$, which gives a high-life of $\ln 2/\phi_H = 3.5$ years, and is in line with typical estimates from the exchange rate predictability literature (Rogoff 1996).

This translate into a range for $\widehat{H}_t = p\left(B^{-\gamma}F_t - 1\right)$, $\left[\widehat{H}_{\min}, \widehat{H}_{\max}\right]$. We parametrize the volatility according to Appendix C, with

$$\sigma^2\left(\widehat{H}\right) = 2v\phi_H \left|\widehat{H}_{\min}\right| \widehat{H}_{\max} \left(1 - \widehat{H}/\widehat{H}_{\min}\right)^2 \left(1 - \widehat{H}/\widehat{H}_{\max}\right)^2 \tag{50}$$

which guaranties that $\widehat{H}$ remains within $\left[\widehat{H}_{\min}, \widehat{H}_{\max}\right]$, as the volatility dies down fast enough at the boundaries. The parameter v controls the volatility $\widehat{H}$ and F. For instance, a country with volatile riskiness will have a high v.

To calibrate the exchange rate fluctuations, we start from (9), and take the benchmark of a constant productivity ω_t during the "normal times" period under study. Then, the only changes in the real exchange rates are due to expectation about the "resilience" of a country if a disaster happens. Differentiation of (9) gives a bilateral exchange rate volatility between two uncorrelated exchange rates[8] $\sigma_{e_{ij}} \simeq \sqrt{2}\sigma_H / \left(R_e + \phi\right)$. If two countries are perfectly correlated, then $\sigma_{ei} = 0$, while if they have a correlation of -1, then $\sigma_{e_{ij}} \simeq 2\sigma_H / \left(R_e + \phi\right)$. We report

[8] This is because $\frac{de_t}{e_t} = \frac{\widehat{H}_t}{r+\phi+\widehat{H}_t} \simeq \frac{\widehat{H}_t}{r+\phi}$, and the bilateral exchange rate $e_{ij} = e_i/e_j$ has twice the variance of any of the exchange rates, if the $\widehat{H}_t$ shocks are uncorrelated.

the values for the uncorrelated case.

Default risk To keep the model parsimonious, we assume no default risk on debt. This is the cleanest assumption for developed countries. Of course, in many cases (e.g. to price sovereign debt), the It can be added without changing anything to the exchange rate.

7.2 Implications for levels and volatilities

Table 1 presents the result of the calibration. In Calibration 1, with $v = 0.2$, the volatility of the bilateral exchange rate is 11%, and with $v = 0.1$, 8%. In Calibration 2, the corresponding volatilities of the exchange rate are 25.7% and 18.7%, respectively. Hence, the model can reasonably easily generate a high volatility of the exchange rate. The reason is that disasters have a high importance: their importance is magnified by $\mathbb{E}\left[B^{-\gamma}\right]$, which is 10.9 (in calibration 1), and 24.4 (Calibration 2). A disaster is $\mathbb{E}\left[B^{-\gamma}\right] = 10.9$ times more important for a risk averse agent, that it would be for a risk-neutral agent.

For a high value of the volatility of F, with $v = 0.2$ (which might correspond to a country with a very volatile F, like Brazil) we obtain a volatility of the bilateral exchange rate equal to 19% / year, and a volatility of F equal to . For $v = 0.1$ (which might correspond to a more stable country, like Germany), the volatility is 13.6%. The model parameters give a volatility of the bilateral exchange rate equal to 8.2%, a value in line with the typical historical values of 10%.

The volatility of F_t (defined as $stdev\left(F_{t+1} - F_t\right)$) is, in the case, $v = 0.1$, 9% per year. That means that means that expectations about recovery rate vary pretty rapidly from year to year.

As $\widehat{H}_t$ is quite volatile, the exchange rate is hard to forecast (the same way stocks are hard to forecast). At short horizons, it behaves like a random walk (qualitatively consistent with Meese and Rogoff 1983).

We conclude that while the above numbers are somewhat speculative, the model may account for the magnitude of exchange rate volatility.

Table 1: Two calibrations of the model

	Calibration 1 Medium riskiness	Calibration 2 High riskiness
Postulated values		
Size of disasters $\mathbb{E}\left[B^{-\gamma}\right]^{-1/\gamma}$	0.55	0.45
Time preference δ (in %)	4	22
Implied values		
Range for domestic riskless short rate (in %)	$\{-1.0, 1.3, 4.9\}$	$\{-4.1, 1.0, 8.8\}$
Range for FX $e_t/\left(\omega_t/r_e\right)$	$\{0.62, 1, 1.25\}$	$\{0.14, 1, 1.57\}$
Volatility of F_t, when $v = 0.1, v = 0.2$	$0.090, 0.127$	$0.090, 0.127$
Simple Fama Regression coefficient, $-\phi/r_e$	-2.1	-2.2
Bilateral FX volatility σ_e (in %) when $v = 0.1, v = 0.2$	8.1, 11.4	18.7, 25.7

Explanation: Each of the two calibration has, we postulate a value of the utility-weighted average size of recovery in disasters, $E\left[B^{-\gamma}\right]^{-1/\gamma}$. We then fit the rate of time preference δ, to get a typical value of the interest rate close to 1%. We report the minimum, typical (corresponding to $\widehat{H}_t = 0$) and maximum range for the domestic short term interest rate; the minimum, typical and maximum value for the exchange rate over "steady state fundamentals" (ω_t/r_e). Finally, we report for volatility of the bilateral exchange rate for currencies with two uncorrelated fundamentals. Perfectly correlated currencies have 0 bilateral FX volatility, perfectly anticorrelated currencies, a volatility equal to the one reported in the table, times $\sqrt{2}$. The time unit is the year.

8 Conclusion

This paper proposes a simple, tractable model of exchange rates and interest rates, and offers a theory of the forward premium puzzle. Its main modelling contributions are, first, to develop an "exchange rate as a stock" view of the exchange rate, in a complete market setting (Proposition 1). Second, to work out the exchange rate in a stochastic disaster framework, and to obtain closed forms for the value of the exchange rate, and the forward premium puzzle coefficients.

The paper suggests several questions for future research. First, it would be good to examine new predictions that the model might generate, including the relationships between bonds, options and exchange rate premia and predictability. Second, it would be interesting to extend the model to stocks, so as to study the link between exchange rates and stock markets. Third, given that the model is very simple to state, and to solve (thanks to the modeling “tricks” allowed by linearity-generating processes), it can serve as a simple framework for various questions. This gives hope that a solution to more puzzles in international economics (Obstfeld and Rogoff 2001) may be within reach.

Appendix A. Some results on Linearity-Generating processes

The paper constantly uses the Linearity-Generating (LG) processes of Gabaix (2007). This Appendix gathers the main results. LG processes are given by $M_t D_t$, a pricing kernel M_t times a dividend D_t, and X_t, a n-dimensional vector of factors (that can be thought as stationary). For instance, for bonds, the dividend is $D_t = 1$.

Discrete time By definition, a process $M_t D_t (1, X_t)$ is LG if and only if it follows, for all t's:

$$\mathbb{E}_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \right] = \alpha + \delta' X_t \tag{51}$$

$$\mathbb{E}_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} X_{t+1} \right] = \gamma + \Gamma X_t \tag{52}$$

Those conditions write more compactly:

$$\mathbb{E}_t Y_{t+1} = \Omega Y_t \text{ with } Y_t = \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$$

Higher moments need not be specified.

The main result is that stocks and bonds have simple closed-form expressions. The price of a stock, $P_t = \mathbb{E}_t \left[\sum_{s \geq t} M_s D_s \right] / M_t$, is:

$$P_t / D_t = \frac{1 + \delta' (I_n - \Gamma)^{-1} X_t}{1 - \alpha - \delta' (I_n - \Gamma)^{-1} \gamma} \tag{53}$$

The price-dividend ratio of a "bond", $Z_t(T) = \mathbb{E}_t [M_{t+T} D_{t+T}] / (M_t D_t)$, is: (with 0_n a $n-$dimensional row of zeros):

$$\begin{aligned} Z_t(T) &= \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \Omega^T \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \\ &= \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \text{ when } \gamma = 0 \end{aligned}$$

Continuous time In continuous time, $M_t D_t (1, X_t)$ is LG if and if only it follows:

$$\mathbb{E}_t \left[\frac{d (M_t D_t)}{M_t D_t} \right] = - (a + \beta' X_t) \, dt \quad (54)$$

$$\mathbb{E}_t \left[\frac{d (M_t D_t X_t)}{M_t D_t} \right] = - (b + \Phi X_t) \, dt \quad (55)$$

i.e. more compactly

$$\mathbb{E}_t [dY_t] = -\omega Y_t dt \text{ with } Y_t = \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} \text{ and } \omega = \begin{pmatrix} a & \beta \\ b & \Phi \end{pmatrix}.$$

The price of a stock, $P_t/D_t = \mathbb{E}_t \left[\int_t^\infty M_s D_s ds \right] / (M_t D_t)$, is:

$$P_t/D_t = \frac{1 - \beta' \Phi^{-1} X_t}{a - \beta' \Phi^{-1} b}$$

and the price-dividend ratio of a "bond" is: $Z_t (T) = \mathbb{E}_t [M_{t+T} D_{t+T}] / (M_t D_t)$

$$\begin{aligned} Z_t (T) &= \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \exp \left[- \begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix} T \right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \quad (56) \\ &= e^{-aT} + \beta' \frac{e^{-\Phi T} - e^{-aT} I_n}{\Phi - a I_n} X_t \text{ when } b = 0 \end{aligned}$$

9 Appendix B. Proofs

For simplicity, we drop the country index i in most proofs.

Proof of Proposition 2 By proposition 1, we have

$$\frac{e_t}{e^{-\lambda t} \omega_t} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} M^*_{t+s} \frac{e^{-\lambda (t+s)} \omega_{t+s}}{e^{-\lambda t} \omega_t} \right] / M^*_t$$

Let $D_t = e^{-\lambda t}\omega_t$ and $X_t = \widehat{H}_t$. With this notation,

$$\begin{aligned}
\mathbb{E}_t\left[\frac{M^*_{t+1}D_{t+1}}{M^*_t D_t}\right] &= e^{-R-\lambda+g_\omega}\left\{(1-p_t)+p_t\mathbb{E}_t\left[B^{-\gamma}_{t+t}F_{t+1}\right]\right\} \\
&= e^{-R-\lambda+g_\omega}(1+H_t) = e^{-R-\lambda+g_\omega}(1+H_*)+e^{-R-\lambda+g_\omega}\widehat{H}_t \\
&= e^{-R-\lambda+g_\omega}(1+H_*)+e^{-R-\lambda+g_\omega}X_t = e^{-r_e}+e^{-r_e-h_*}X_t,
\end{aligned}$$

using $r_e = R+\lambda-g_\omega-h_*$. Also:

$$\begin{aligned}
\mathbb{E}_t\left[\frac{M^*_{t+1}D_{t+1}}{M^*_t D_t}X_{t+1}\right] &= \mathbb{E}_t\left[\frac{M^*_{t+1}D_{t+1}}{M^*_t D_t}\right]\mathbb{E}_t\left[X_{t+1}\right] \\
&= e^{-R-\lambda+g_\omega}(1+H_t)\frac{1+H_*}{1+H_t}e^{-\phi_H}\widehat{H}_t \\
&= e^{-R-\lambda+g_\omega-\phi_H}(1+H_*)\widehat{H}_t = e^{-r_e-\phi_H}X_t
\end{aligned}$$

Hence with the notations of Appendix A, we find that $Y_t = M^*_t D_t(1, X_t)$ is a LG process, with generator Ω:

$$\mathbb{E}_t Y_{t+1} = \Omega Y_t \text{ with } \Omega = \begin{pmatrix} e^{-r_e} & e^{-r_e-h_*} \\ 0 & e^{-r_e-\phi_H} \end{pmatrix}$$

Using equation 53, we find

$$e_t = \frac{\omega_t}{1-e^{-r_e}}\left(1+\frac{e^{-r_e-h_*}}{1-e^{-r_e-\phi_H}}\widehat{H}_t\right)$$

which proves the proposition.

The lower bound for $\widehat{H}_t$ is: $e^{-r_e}\widehat{H}_t > e^{-\phi_H}-1$, i.e., in the continuous time limit, $\widehat{H}_t > -\phi_H$.

A Lemma

Lemma 1 *(Existence of the Equilibrium) There are (an infinity of) endowment processes that generate the equilibrium described in the paper.*

Proof. Call $\eta^a_{i,t}$ and $\eta^b_{i,t}$ country i's endowment of the international good, and domestic good, respectively. We work out under which conditions they generate the announced equilibrium.

Say that the equilibrium is described by a social planner's maximization of $\sum_i \lambda_i^\gamma U_i$, where $U_i = \mathbb{E}_0\left[\sum_{t=0}^\infty e^{-\delta t}\frac{(C^a_t)^{1-\gamma}+\left(C^b_t\right)^{1-\gamma}}{1-\gamma}\right]$ is country i's utility, and λ_i^γ the Negishi weight on country

i. We normalize $\sum \lambda_i = 1$. Calling q_t the Arrow-Debreu price of 1 unit of the international good at date t, and Y_{ta} the world production of the international good. Amongst other things, the planner optimizes the consumptions of the domestic good, so solves:

$$\max_{C_{it}^a} \sum_i \lambda_i^\gamma \sum_{t=0}^{\infty} e^{-\delta t} \frac{\left(C_{it}^a\right)^{1-\gamma} + \left(C_{it}^b\right)^{1-\gamma}}{1-\gamma} + \sum_t q_t \left(Y_t^a - \sum_i C_{it}^a \right)$$

where so that $e^{-\delta t} \lambda_i^\gamma \left(C_{it}^a\right)^{-\gamma} - q_t = 0$, and $C_{it}^a = \lambda_i q_t^{-1/\gamma} e^{\delta t/\gamma}$. Using $Y_t^a = \sum_i C_{it}^a$, we get: $C_{it}^a = \lambda_i Y_t^a$.

Let us now study country i's consumption and investment decisions. Country i at time t, solves $\max_{C_{it}^a,\, C_{it}^b} \frac{\left(C_{it}^a\right)^{1-\gamma} + \left(C_{it}^b\right)^{1-\gamma}}{1-\gamma}$ s.t.$C_{it}^a + e_{it} C_{it}^b$ =expenditure at time t, so $\left(C_{it}^b\right)^{-\gamma} = e_{it} \left(C_{it}^a\right)^{-\gamma}$, hence $C_{it}^b = e_{it}^{-1/\gamma} \lambda_i Y_t^a$. The investment in the capital good is $\eta_{it}^b - C_{it}^b = \eta_{it}^b - e_{it}^{-1/\gamma} \lambda_i Y_t^a$, so that the accumulated quantity of the capital good is $K_{it} = \sum_{s=0}^{\infty} e^{-\lambda s} \left(\eta_{i,t-s}^b - e_{i,t-s}^{-1/\gamma} \lambda_i Y_{t-s}^a \right)$. As country i produces $K_{it} \omega_{it}$ of the world good, and also has an endowment η_{it}^a of it, the total available consumption of the world good at time t is:

$$Y_t^a = \sum_i \eta_{it}^a + \sum_i \omega_{it} \sum_{s=0}^{\infty} e^{-\lambda s} \left(\eta_{i,t-s}^b - e_{i,t-s}^{-1/\gamma} \lambda_i Y_{t-s}^a \right). \tag{57}$$

The first term is the endowment of the world good, and second is the production of it.

The equilibrium is described as in the paper, if the endowment processes $\eta_{i,t}^a$ and $\eta_{i,t}^b$ satisfy (57), with $Y_t^a = C_t^{a*}$. By inspection there is an infinity of such endowment processes.

Proof of Proposition 3 In this proof, it is useful to define $x_t = e^{-h_*} \widehat{H}_t$. Then, $\mathbb{E}_t \left[\frac{M_{t+1}^* \omega_{t+1}}{M_t^* \omega_t} \right] = e^{-R+g_\omega} \left(1 + H_t\right) = e^{-r_e+\lambda} \left(1 + x_t\right)$. Also, $\mathbb{E}_t \left[x_{t+1}\right] = e^{-\phi} \frac{x_t}{1+x_t}$, and $e_t = \omega_t A \left(1 + B x_t\right)$, with $A = 1/\left(1 - e^{-r_e}\right)$, $B = \frac{e^{-r_e}}{1-e^{-r_e-\phi_H}}$,

$$
\begin{aligned}
1+r_t &= \frac{M_t^* e_t}{\mathbb{E}_t\left[M_{t+1}^* e_{t+1}\right]} = \frac{A\left(1+Bx_t\right)}{\mathbb{E}_t\left[\frac{M_{t+1}^*\omega_{t+1}}{M_t^*\omega_t} A\left(1+Bx_{t+1}\right)\right]} = \frac{1+Bx_t}{\mathbb{E}_t\left[\frac{M_{t+1}^*\omega_{t+1}}{M_t^*\omega_t}\right]\mathbb{E}_t\left[1+Bx_{t+1}\right]} \\
&= \frac{1+Bx_t}{e^{-r_e+\lambda}\left(1+x_t\right)\left(1+B\frac{e^{-\phi_H}x_t}{1+x_t}\right)} = e^{r_e-\lambda}\frac{1+Bx_t}{1+x_t\left(1+Be^{-\phi_H}\right)} = e^{r_e-\lambda}\frac{1+\frac{e^{-r_e}}{1-e^{-r_e-\phi_H}}x_t}{1+\frac{1}{1-e^{-r_e-\phi_H}}x_t} \\
&= e^{r_e-\lambda}\left[1-\frac{\left(1-e^{-r_e}\right)e^{-h_*}\widehat{H}_t}{1-e^{-r_e-\phi_H}+e^{-h_*}\widehat{H}_t}\right].
\end{aligned}
$$

Proof of Proposition 6 *Derivation of the exchange rate.* Call $m_t = M_t^* e^{-\lambda t}\overline{\omega}_t$. We show that $Y_t = m_t\left(1, \widehat{H}_t, y_t\right)$ is a LG process. As in the Proof of Proposition 2:

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{m_{t+1}}{m_t}\right] &= e^{-r_e}\left(1+e^{-h_*}\widehat{H}_t\right) = e^{-r_e-h_*}\left(1+H_t\right) \\
\mathbb{E}_t\left[\frac{m_{t+1}}{m_t}\widehat{H}_t\right] &= e^{-r_e-\phi_H}\widehat{H}_t
\end{aligned}
$$

The new moment is:

$$
\mathbb{E}_t\left[\frac{m_{t+1}}{m_t}y_{t+1}\right] = \mathbb{E}_t\left[\frac{m_{t+1}}{m_t}\right]\mathbb{E}_t\left[y_{t+1}\right] = e^{-r_e-h_*}\left(1+H_t\right)\frac{1+H_*}{1+H_t}e^{-\phi_y}y_t = e^{-r_e-\phi_y}y_t
$$

So Y_t is a LG process, with generator:

$$
\Omega = e^{-r_e}\begin{pmatrix} 1 & e^{-h_*} & 0 \\ 0 & e^{-\phi_H} & 0 \\ 0 & 0 & e^{-\phi_y} \end{pmatrix}. \tag{58}
$$

The exchange rate follows:

$$
\begin{aligned}
\frac{e_t}{\overline{\omega}_t} &= \mathbb{E}_t\left[\sum_{s=0}^{\infty}\frac{M_{t+s}^*}{M_t^*}e^{-\lambda s}\overline{\omega}_{t+s}\left(1+g_t\right)\right] = \begin{pmatrix}1\\0\\1\end{pmatrix}' \cdot \left(I_3-\Omega\right)^{-1}\cdot\begin{pmatrix}1\\ \widehat{H}_t \\ y_t\end{pmatrix} \\
&= \frac{1}{1-e^{-r_e}}\left(1+\frac{e^{-r_e-h_*}}{1-e^{-r_e-\phi_H}}\widehat{H}_t\right)+\frac{1}{1-e^{-r_e-\phi_y}}y_t
\end{aligned}
$$

The last equation comes from the fact that $I_3 - \Omega$ is bloc-diagonal. This yields the announced expression.

Derivation of the interest rate. In the continuous time limit,

$$\mathbb{E}_t\left[d\widehat{H}_t\right] = -\left(\phi_H + \widehat{H}_t\right)\widehat{H}_t dt \tag{59}$$

$$\mathbb{E}_t\left[dy_t\right] = -\left(\phi_y + \widehat{H}_t\right) y_t dt \tag{60}$$

so the interest rate satisfies:

$$\begin{aligned}
-r_t &= \mathbb{E}_t\left[\frac{d\left(M_t^* e_t\right)}{M_t^* e_t}\right]/dt = \mathbb{E}_t\left[\frac{dM_t^*}{M_t^*} \mid \text{no disaster}\right] + \mathbb{E}_t\left[\frac{de_t}{e_t} \mid \text{no disaster}\right] \\
&\quad + p_t\left(\mathbb{E}_t\left[\frac{M_{t^+}e_{t^+}}{M_t e_t} - 1 \mid \ \text{disaster}\right]\right) \\
&= -R + g_\omega + \frac{\frac{\mathbb{E}_t\left[d\widehat{H}_t\right]/dt}{r_e+\phi_H} + \frac{r_e \mathbb{E}_t\left[dy_t\right]/dt}{r_e+\phi_y}}{1 + \frac{\widehat{H}_t}{r_e+\phi_H} + \frac{r_e y_t}{r_e+\phi_y}} + p_t\left(B_t^{-\gamma}F_t - 1\right) \\
&= -R + g_\omega + \frac{\frac{-\left(\phi_H+\widehat{H}_t\right)\widehat{H}_t}{r_e+\phi_H} + \frac{-r_e\left(\phi_y+\widehat{H}_t\right)y_t}{r_e+\phi_y}}{1 + \frac{\widehat{H}_t}{r_e+\phi_H} + \frac{r_e y_t}{r_e+\phi_y}} + H_* + \widehat{H}_t \\
&= -r_e + \lambda + \frac{\frac{r_e}{r_e+\phi_H}\widehat{H}_t - \frac{r_e\phi_y}{r_e+\phi_y}y_t}{1 + \frac{\widehat{H}_t}{r_e+\phi_H} + \frac{r_e y_t}{r_e+\phi_y}}.
\end{aligned}$$

Proof of Proposition 7 We start by the case of the regression in a sample that does not contain disasters. As in the proof of Proposition 6,

$$\mathbb{E}_t\left[\frac{de_t}{e_t}\right]/dt = g_\omega + \frac{\frac{-\left(\phi_H+\widehat{H}_t\right)\widehat{H}_t}{r_e+\phi_H} + \frac{-r_e\left(\phi_y+\widehat{H}_t\right)y_t}{r_e+\phi_y}}{1 + \frac{\widehat{H}_t}{r_e+\phi_H} + \frac{r_e y_t}{r_e+\phi_y}}$$

So, up to second order terms in $\widehat{H}_t$ and y_t,

$$\begin{aligned}
\mathbb{E}_t\left[\frac{de_t}{e_t}\right]/dt &= g_\omega + \frac{-\phi_H\widehat{H}_t}{r_e+\phi_H} + \frac{-r_e\phi_y y_t}{r_e+\phi_y} \equiv a\widehat{H}_t + by_t + c \\
r_t &= r_e - \lambda - \frac{r_e}{r_e+\phi_H}\widehat{H}_t + \frac{r_e\phi_y}{r_e+\phi_y}y_t \equiv A\widehat{H}_t + By_t + C
\end{aligned}$$

so

$$\begin{aligned}
\beta' &= -\frac{Cov\left(\mathbb{E}_t\left[\frac{de_t}{e_t}\right]/dt, r_t\right)}{Var\left(r_t\right)} = -\frac{aAVar\left(\widehat{H}_t\right) + bBVar\left(y_t\right)}{A^2Var\left(\widehat{H}_t\right) + B^2Var\left(y_t\right)} \\
&= -\nu\frac{a}{A} - (1-\nu)\frac{b}{B} = \nu\beta + 1 - \nu
\end{aligned}$$

where

$$\nu = \frac{A^2Var\left(\widehat{H}_t\right)}{A^2Var\left(\widehat{H}_t\right) + B^2Var\left(y_t\right)}.$$

The case of the full sample regression is proved similarly.

Proof of Proposition 8 The proof of Proposition 6 showed that $M_t^* e^{-\lambda t}\overline{\omega}_t\left(1, \widehat{H}_t, y_t\right)$ is a LG process, with generating matrix given by (58). Writing $e_t = \overline{\omega}_t\left(a + b\widehat{H}_t + cy_t\right)$, we have

$$\begin{aligned}
Z_t &= \mathbb{E}_t\left[\frac{M_{t+T}^* e_{t+T}}{M_t^* e_t}\right] = \frac{e^{\lambda T}}{a + b\widehat{H}_t + cy_t}\mathbb{E}_t\left[\frac{M_{t+T}^* e^{-\lambda(t+T)}\overline{\omega}_{t+T}\left(a + b\widehat{H}_t + cy_t\right)}{M_t^* e^{-\lambda t}\overline{\omega}_t}\right] \\
&= \frac{e^{\lambda T}}{a + b\widehat{H}_t + cy_t}\begin{pmatrix} a \\ b \\ c \end{pmatrix}' \Omega^T \begin{pmatrix} 1 \\ \widehat{H}_t \\ y_t \end{pmatrix} \text{ by the rules on LG processes} \\
&= \frac{e^{\lambda T}}{a + b\widehat{H}_t + cy_t}\begin{pmatrix} a \\ b \\ c \end{pmatrix}' e^{-r_e T}\begin{pmatrix} 1 & e^{-h_*}\frac{1-e^{-\phi_H T}}{1-e^{-\phi_H}} & 0 \\ 0 & e^{-\phi_H T} & 0 \\ 0 & 0 & e^{-\phi_y T} \end{pmatrix}\begin{pmatrix} 1 \\ \widehat{H}_t \\ y_t \end{pmatrix} \\
&= e^{-(r_e - \lambda)T}\frac{a + \left(ae^{-h_*}\frac{1-e^{-\phi_H T}}{1-e^{-\phi_H}} + b\right)\widehat{H}_t + ce^{-\phi_y T}y_t}{a + b\widehat{H}_t + cy_t} \\
&= e^{-(r_e - \lambda)T}\frac{1 + \left(ae^{-h_*}\frac{1-e^{-\phi_H T}}{1-e^{-\phi_H}} + b\right)\widehat{H}_t + ce^{-\phi_y T}y_t}{1 + \frac{e^{-r_e - h_*}}{1-e^{-r_e-\phi_H}}\widehat{H}_t + \frac{1-e^{-r_e}}{1-e^{-r_e-\phi_y}}y_t},
\end{aligned}$$

So the zero-coupon price is:

$$Z_t(T) = e^{-(r_e-\lambda)T}\frac{1+\frac{1-e^{-r_e-\phi_H}-\left(1-e^{-r_e}\right)e^{-\phi_H T}}{\left(1-e^{-\phi_H}\right)\left(1-e^{-r_e-\phi_H}\right)}e^{-h_*}\widehat{H}_t+e^{-\phi_y T}\frac{1-e^{-r_e}}{1-e^{-r_e-\phi_y}}y_t}{1+\frac{e^{-r_e-h_*}}{1-e^{-r_e-\phi_H}}\widehat{H}_t+\frac{1-e^{-r_e}}{1-e^{-r_e-\phi_y}}y_t}$$

Taking Taylor expansions,

$$Z_t(T) = e^{-(r_e-\lambda)T}\left[1+\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_H T}\right)}{\left(1-e^{-\phi_H}\right)\left(1-e^{-r_e-\phi_H}\right)}e^{-h_*}\widehat{H}_t-\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_y T}\right)}{1-e^{-r_e-\phi_y}}y_t\right]+o\left(\widehat{H}_t,y_t\right)$$

$$Y_t(T) = r_e-\lambda-\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_H T}\right)}{T\left(1-e^{-\phi_H}\right)\left(1-e^{-r_e-\phi_H}\right)}e^{-h_*}\widehat{H}_t+\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_y T}\right)}{\left(1-e^{-r_e-\phi_y}\right)T}y_t+o\left(\widehat{H}_t,y_t\right) \tag{61}$$

$$f_t(T) = r_e-\lambda-\frac{\left(1-e^{-r_e}\right)e^{-\phi_H(T-1)}}{1-e^{-r_e-\phi_H}}e^{-h_*}\widehat{H}_t+\frac{\left(1-e^{-r_e}\right)\left(1-e^{-\phi_y}\right)e^{-\phi_y(T-1)}}{\left(1-e^{-r_e-\phi_y}\right)}y_t+o\left(\widehat{H}_t,y_t\right) \tag{62}$$

and in the continuous time limit,

$$f_t(T) = r_e-\lambda-\frac{r_e}{r_e+\phi_H}e^{-\phi_H T}\widehat{H}_t+\frac{r_e\phi_y}{r_e+\phi_y}e^{-\phi_y T}y_t+o\left(\widehat{H}_t,y_t\right)$$

Proof of Proposition 12 The real part of the forward rate was calculated in Eq. 26. The nominal part is calculated in Gabaix (2007). The two expressions add up, because we do a Taylor expansions.

Proof of Proposition 13 By inspection of (42):

$$\begin{aligned}
r_t &= r_e-\lambda-\frac{r_e}{r_e+\phi_H}\widehat{H}_t+\frac{r_e\phi_y}{r_e+\phi_y}y_t+i_{**}+i_t-i_{**} \\
\partial_T\widetilde{f}_t(0) &= \frac{r_e}{r_e+\phi_H}\phi_H\widehat{H}_t-\frac{r_e\phi_y^2}{r_e+\phi_y}y_t-\phi_i e^{-\phi_i T}i_t+\pi_t^i \\
\partial_T^2\widetilde{f}_t(0) &= -\frac{r_e}{r_e+\phi_H}\phi_H^2\widehat{H}_t+\frac{r_e\phi_y^3}{r_e+\phi_y}y_t+\phi_i^2 e^{-\phi_i T}i_t-\left(\phi_i+\psi_\pi\right)\pi_t^i
\end{aligned}$$

Combining this with (43) yields the Proposition.

Proof of Proposition 15 $P_t = E_t\left[\sum_{s\geq t} M_s\omega_s D_s\left(1+\frac{\widehat{H}_s}{R_e+\phi}\right)\right]/\left(e_t D_t\right)$. We calculate the corresponding LG moments. We start with:

$$\frac{M_{t+1}\omega_{t+1}D_{t+1}}{M_t\omega_t D_t} = e^{-R+g_\omega+g_D}\times\begin{cases}1 & \text{if there is no disaster at } t+1\\ F_t F_t^i & \text{if there is a disaster at } t+1\end{cases}$$

where as before F_t is the reduction in the country productivity in producing the international good. We postulate that the process for F_t^i allows the decomposition:

$$p_t B_t^{-\gamma}F_t F_t^i - p_t = \left(H_*+p_*\right)F_*^i - p_* + F_*^i\widehat{H}_t + \left(H_*+p_*\right)\widehat{F}_t^i$$

This decomposition is the natural one, as the central value of $p_t B_t^{-\gamma}F_t$ is H_*+p_*, and the central value of F_t^i is called F_*^i. The process for F_{it} is a LG-twisted autoregressive process:

$$E_t\left[d\widehat{F}_{it}\right]/dt = -\left(\phi_{F_i}+F_*^i\widehat{H}_t+\left(H_*+p_*\right)\widehat{F}_t^i\right)\widehat{F}_{it} \tag{63}$$

We define $r_D = R-g_D-g_\omega-\left(H_*+p_*\right)F_*^i+p_*$ The LG moments are (normalizing $g_D=g_\omega=0$ in the derivations):

$$E_t\left[\frac{d\left(M\omega D\right)_t}{\left(M\omega D\right)_t}\right]/dt = -R+p_t\left(B_t^{-\gamma}F_tF_t^i-1\right) = -r_D+F_*^i\widehat{H}_t+\left(H_*+p_*\right)\widehat{F}_t^i$$

$$E_t\left[\frac{d\left(M\omega D\cdot\widehat{F}_{it}\right)_t}{\left(M\omega D\right)_t}\right]/dt = -R\widehat{F}_{it}-\left(\phi_{F_i}+F_*^i\widehat{H}_t+H_*\widehat{F}_t^i\right)\widehat{F}_{it}+p\left(B^{-\gamma}F_tF_t^i\cdot F_t^i-\widehat{F}_{it}\right) = -\left(r_D+\phi_F\right)\widehat{F}_{it}$$

$$E_t\left[\frac{d\left(M\omega D\widehat{H}_t\right)_t}{\left(M\omega D\right)_t}\right]/dt = \left(-r_D+F_*^i\widehat{H}_t+\left(H_*+p_*\right)\widehat{F}_t^i\right)\widehat{H}_t-\left(\phi+\widehat{H}_t\right)\widehat{H}_t = -\left(r_D+\phi\right)\widehat{H}_t+h.o.t.$$

Hence the last expression involves a linearization. So, to a first order, $M_t\omega_tD_t\left(1,\widehat{H}_t,\widehat{F}_{it}\right)$ is a LG process, with generating matrix $\omega = \begin{pmatrix} r_D & -F_*^i & -\left(H_*+p_*\right)\\ 0 & r_D+\phi_H & 0\\ 0 & 0 & r_D+\phi_F\end{pmatrix}$. So (46) gives, in virtue of the rule on LG processes (Gabaix 2007, Theorem 4 and Proposition 4):

$$P_t e_t = \begin{pmatrix} 1 \\ 1/\left(r_e + \phi_H\right) \\ 0 \end{pmatrix} \omega^{-1} \begin{pmatrix} 1 \\ \widehat{H}_t \\ \widehat{F}_{it} \end{pmatrix} D_t,$$

which yields (47). Eq. 48 comes from a Taylor expansion.

Appendix C. Variance processes

Suppose an LG process centered at 0, $dX_t = -\left(\phi + X_t\right) X_t dt + \sigma\left(X_t\right) dW_t$, where W_t is a standard Brownian motion. Because of economic considerations, the support of the X_t needs to be some $(X_{\min}, X_{\max})$, with $-\phi < X_{\min} < 0 < X_{\max}$. The following variance process makes that possible:

$$\sigma^2\left(X\right) = 2K\left(1 - X/X_{\min}\right)^2 \left(1 - X/X_{\max}\right)^2 \tag{64}$$

with $K > 0$. K is in units of $[\text{Time}]^{-3}$. The average variance of X is $\overline{\sigma}_X^2 = E\left[\sigma^2\left(X_t\right)\right] = \int_{X_{\min}}^{X_{\max}} \sigma\left(X\right)^2 p\left(X\right) dX$, where $p\left(X\right)$ is the steady state distribution of X_t. It can be calculated via the Forward Kolmogorov equation, which yields $d \ln p\left(X\right)/dX = 2X\left(\phi + X\right)/\sigma^2\left(X\right) - d \ln \sigma^2\left(X\right)/dX$.

Numerical simulations shows that the process volatility is fairly well-approximated by: $\overline{\sigma}_X \simeq K^{1/2}\xi$, with $\xi = 1.3$. Also, the standard deviation of X's steady state distribution is well-approximated by $\left(K/\phi\right)^{1/2}$.

Asset prices often require to analyze the standard deviation of expressions like $\ln\left(1 + aX_t\right)$. Numerical analysis shows that the Taylor expansion approximation is a good one: Average volatility of: $\ln\left(1 + aX_t\right) \simeq aK^{1/2}\xi$, which numerical simulations prove to be a good approximation too.

For the steady-state distribution to have a "nice" shape (e.g., be unimodal), the following restrictions appear to be useful: $K \leq 0.2 \cdot \phi \left|X_{\min}\right| X_{\max}$.

When the process is not centered at 0, one simply centers the values. For instance, in our calibration, the recovery rate of the country productivity, F_t, has support $[F_{\min}, F_{\max}]$, centered around F_*. The probability and intensity of disasters (p and B) are constant. Define

$H_t = p\left(B^{-\gamma}F_t - 1\right)$, and the associated $H_{\min}$, $H_{\max}$, H_*. The associated centered process is $X_t = \widehat{H}_t = H_t - H_*$. We take the volatility parameter to be: $K = v \cdot \phi_H \left|X_{\min}\right| X_{\max}$, with the volatility parameter $v \in [0, 0.2]$. This yields a volatility of $\widehat{H}_t$ equal to $\overline{\sigma}_{\widehat{H}_t} = \xi\left(v \cdot \phi_H \left|\widehat{H}_{\min}\right| \widehat{H}_{\max}\right)^{0.5}$, a volatility of F_t equal to $\overline{\sigma}_F = \overline{\sigma}_{\widehat{H}_t} / \left(pB^{-\gamma}\right)$, and a volatility of the bilateral exchange rate (between two uncorrelated countries) equal to $\sqrt{2}\overline{\sigma}_{\widehat{H}_t} / \left(r_e + \phi_H\right)$.

References

Alvarez, Fernando, Andrew Atkeson, and Patrick J. Kehoe, "Money, Interest Rates, and Exchange Rates with Endogenously Segmented Markets," *Journal of Political Economy*, 110 (2002), 73–112.

Alvarez, Fernando, Andrew Atkeson, and Patrick J. Kehoe, "Time-varying risk, interest rates, and exchange rates in general equilibrium ," Working Paper (2006).

Bacchetta, P. and E. van Wincoop, "Can Information Heterogeneity Explain the Exchange Rate Determination Puzzle?" *American Economic Review 96* (2006), 552-76.

Backus, David K., Patrick J. Kehoe, and Finn E. Kydland, "International Real Business Cycles," *Journal of Political Economy*, 100 (1992), 745-75.

Backus, David K., and Gregor W. Smith, "Consumption and Real Exchange Rates in Dynamic Economies with Non-Traded Goods," *Journal of International Economics*, 35 (1993), 297-316.

Baker, Malcolm, and Jeffrey Wurgler, "Investor Sentiment and the Cross-Section of Stock Returns," *Journal of Finance*, 61 (2006), 1645-80.

Bansal, Ravi, and Amir Yaron, "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59 (2004), 1481-1509.

Bansal, Ravi, and Ivan Shaliastovich "Risk and Return in Bond, Currency and Equity Markets," Working Paper, Duke University (2007).

Barro, Robert, "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics*, 121 (2006), 823-866.

Bekaert, Geert, "The time variation of risk and return in foreign exchange markets: a general equilibrium perspective," *Review of Financial Studies*, 9 (1996), 427-470

Bekaert, Geert, Eric Engstrom, and Steve Grenadier, "Stock and Bond Returns with Moody Investors," *Working Paper*, *Columbia University*, 2005.

Boudoukh, Jacob, Matthew Richardson, and Robert F. Whitelaw, "The Information in Long-

Maturity Forward Rates: Implications for Exchange Rates and the Forward Premium Anomaly," *Working Paper*, *New York University*, 2006.

Brandt, Michael W., John H. Cochrane, and Pedro Santa-Clara, "International Risk Sharing is Better Than You Think, or Exchange Rates are Too Smooth," *Journal of Monetary Economics* 53, 2006, 671-698

Burnside, Craig, Martin Eichenbaum, Isaac Kleshchelski, and Sergio Rebelo, "The Returns to Currency Speculation," National Bureau of Economic Research Working Paper No. 12489, 2007.

Campbell, John Y., and John Cochrane, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107 (1999), 205-251.

Carr, Peter and Liuren Wu, "Stochastic Skew in Currency Options," Journal of Financial Economics, 2007, 86(1), 213–247.

Cochrane, John, "New Facts in Finance," *Economic Perspectives* 23 (3) Third quarter 1999 (Federal Reserve Bank of Chicago),

Colacito, R. (2006), 'Six Anomalies looking for a model. A consumption based explanation of International Finance Puzzles', working paper UNC Chapel Hill.

Colacito, R. and M.M. Croce (2006), "Risks for the long run and the real exchange rate," working paper UNC Chapel Hill.

Engel, Charles, "The Forward Discount Anomaly and the Risk Premium: A Survey of Recent Evidence," *Journal of Empirical Finance*, 3 (1996), 123–192.

Engel, Charles, and Kenneth West, "Exchange Rates and Fundamentals," *Journal of Political Economy* 113(2) (2005), 485-517

Engel, Charles, Nelson Mark and Kenneth West, "Exchange Rate Models Are Not as Bad as You Think " *NBER Macroeconomics Annual*, 2007.

Farhi, Emmanuel, Xavier Gabaix, Romain Ranciere, Adrien Verdelhan, "Currency movements and options prices," in preparation, 2007.

Fama, Eugene F., "Forward and Spot Exchange Rates," *Journal of Monetary Economics*, 14 (1984), 319–338.

Gabaix, Xavier, "Linearity-Generating Processes: A Modelling Tool Yielding Closed Forms for Asset Prices," *Mimeo, New York University*, 2007a.

Gabaix, Xavier, "A Simple, Exactly Solved, Framework for Ten Puzzles in Macro-Finance," *Mimeo, New York University*, 2007b.

Gabaix, Xavier, Parameswaran Gopikrishnan, Vasiliki Plerou, and H. Eugene Stanley,"Institutional Investors and Stock Market Volatility," *Quarterly Journal of Economics*, 121 (2006), 461–504.

Gao, Kai, (2007) "Exchange Rates and Asset Prices in An Open Economy with Rare Disasters," Working Paper.

Gourinchas, Pierre-Olivier, and Helene Rey, "From World Banker to World Venture Capitalist: The US External Adjustment and The Exorbitant Privilege," in Richard Clarida, ed., *G7 Current Account Imbalances: Sustainability and Adjustment* (Chicago: NBER and University of Chicago Press, 2007).

Hansen, Lars Pete rand Robert J. Hodrick, "Forward Exchange Rates as Optimal Predictors of Future Spot Rates: An Econometric Analysis." Journal of Political Economy, 1980, 88, pp. 829-53.

Hau, Harald and Helene Rey, "Exchange Rates, Equity Prices and Capital Flows," *Review of Financial Studies*, 19 (2006), 273-317.

Lewis, Karen K., "Puzzles in International Financial Markets," in Gene Grossman and Kenneth Rogoff, eds., *Handbook of International Economics,* Volume 3 (Amsterdam; New York and Oxford: Elsevier, North-Holland, 1995).

Liu, Jun, Jun Pan and TanWang, "An Equilibrium Model of Rare Event Premia," *Review of Financial Studies*, v18, n1, 131-164, Spring, 2005.

Lustig, Hanno, and Adrien Verdelhan, "The Cross-Section of Foreign Currency Risk Premia and US Consumption Growth Risk," *American Economics Review*, forthcoming.

Meese, Richard, and Kenneth Rogoff, "Empirical Exchange Rate Models of the Seventies: Do They Fit Out of Sample?," *Journal of International Economics*, 14 (1983), 3-24.

Mehra, Rajnish and Edward C. Prescott, "The equity risk premium: A solution?," *Journal of Monetary Economics*, 22 (1988), 133-136.

Obstfeld, Maurice, and Kenneth Rogoff, "The Six Major Puzzles in International Macroeconomics: Is There a Common Cause?," in Ben Bernanke and Kenneth Rogoff, eds. *NBER Macroeconomics Annual Year 2000* (Cambridge and London: MIT Press, 2001).

Pavlova, Anna, and Roberto Rigobon, "Asset Prices and Exchange Rates," *Review of Financial Studies*, forthcoming.

Piazzesi, Monika, and Martin Schneider, "Equilibrium Yield Curves," in NBER Macroeconomics Annual 2006, forthcoming.

Rietz, Thomas A., "The Equity Risk Premium: A Solution," *Journal of Monetary Economics*, 22 (1988), 117-131.

Rogoff, Kenneth S., 1996, "The Purchasing Power Parity Puzzle." *Journal of Economic Literature*, 34, pp. 647-68.

Verdelhan, Adrien, "A Habit-Based Explanation of the Exchange Rate Risk Premium," *Mimeo, Boston University*, 2006.

Weitzman, Martin, "Prior-Sensitive Expectations and Asset-Return Puzzles," *Working Paper, Harvard University*, 2006.