Time-Varying Exposure to Long-Run Consumption Risk

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Abstract

This paper develops a model of time-varying expected returns and shows that, when investors care about the long-run consumption risk, they also care about the persistence of an asset's exposure to this risk, and demand substantially higher compensation for more persistent exposure. The model also implies a negative sensitivity of price-dividend ratios to expected excess returns, and the magnitude of the sensitivity is substantially larger for more persistent exposure. In an application of the model, I specify individual stocks' dividend growth as containing two time-varying components of exposure to the long-run consumption risk — a fast mean-reverting component whose shocks are positively correlated with the independent dividend growth shocks, and a slow mean-reverting component whose shocks are negatively correlated with the independent dividend growth shocks. Firm level simulations from this model produce short-run momentum and long-run reversal quantitatively comparable to empirically documented patterns in the cross section as well as along the time dimension. The simulations also show that the value premium across price-dividend ratio sorted portfolios is driven by a spread in the slow mean-reverting risk exposure. Together, these results propose potential interpretations of the value and momentum factors as representing time-varying loadings of different persistence on the long-run consumption risk factor.

1 Introduction

This paper develops a model of time-varying expected returns for individual stocks in an economy where a representative agent with Epstein-Zin preferences cares about the risk associated with a small and persistent expected component in the aggregate consumption growth. In the model, an individual asset is only exposed to the risk associated with the shocks to the expected consumption growth, which is assumed to carry a constant price of risk.[1] Consequently, time-varying expected returns on the asset arise as a result of time-varying loadings (or "betas") on the long-run consumption risk.[2]

There is a growing literature on consumption based asset pricing that suggests that exposure to consumption growth over a long horizon provides an improved measure of risk over the covariance between returns and contemporaneous consumption growth.[3] In particular, a consumption growth model featuring a small and persistent expected component is consistent with the time series properties of the U.S. aggregate consumption data, and, when combined with Epstein-Zin preferences, demonstrates promising capabilities in explaining the risk premia of the aggregate market and benchmark portfolios (See, for example, Bansal and Yaron (2004) and Kiku (2006)).[4] Typical empirical and theoretical specifications in the literature assume a constant exposure to the long-run consumption growth, resulting in a permanent loading on the long-run consumption risk.[5]

The purpose of this paper is to bring the focus to time-varying loadings on the long-run consumption risk, while holding constant the price of risk. More precisely, building on the Bansal and Yaron (2004) framework of the long-run consumption growth and Epstein-Zin preferences, I specify individual stocks' expected dividend growth as linked to the expected consumption growth through a leverage (or exposure) driven by mean-reverting state variables. The resulting model of time-varying loadings on the long-run consumption risk and consequently, time-varying expected returns, is particularly suited for the study of the cross-section and time-series behaviors of returns at the individual firm level.

A key result of the model is that when the exposure to the long-run consumption risk is time varying, investors care about how persistent the exposure is, and demand substantially higher compensation for a slow mean-reverting leverage than that driven by a fast mean-reverting state variable. As an example, for monthly intervals, a leverage with a persistence parameter of 0.91 (which implies that the impact of a shock attenuates by 50% after about 7 months) yields an expected excess return about only 1/4 of that resulting from a leverage with the same magnitude but a persistence of 0.99 (for which the half-life of shocks is more

[1]Price of risk is defined as the expected excess return on an asset with a unit exposure to the risk. In the language of factor pricing models, it is the factor risk premium.

[2]In other words, this is a conditional one-factor model.

[3]For measuring risk as exposure to long-run consumption growth, see Parker and Julliard (2005), Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2005), and Malloy, Moskowitz, and Vissing-Jorgensen (2005). For measuring risk as covariance between returns and contemporaneous consumption growth, see Hansen and Singleton (1982), Hansen and Singleton (1983), and Breeden, Gibbons, and Litzenberger (1989).

[4]Also see Bansal (2006) for a survey.

[5]The price of risk can be either constant or time-varying. See Bansal and Yaron (2004) and Kiku (2006).

than 5 years), which, in turn, generates less than 3/4 of the expected excess return associated with a constant, thus permanent, leverage.

In the model, a rise in the risk exposure increases the expected return, but decreases the price-dividend ratio. This results in a negative sensitivity of price-dividend ratios to expected returns — changes in expected returns are amplified into variations in price-dividend ratios in the opposite direction. The magnitude of the sensitivity increases dramatically with the persistence of the risk exposure. As an example, for the same variations in expected returns, when the underlying risk exposure persistence is 0.91, the magnitude of the corresponding variations in logarithm price-dividend ratios is only 1/8 of the magnitude of the variations when the persistence is 0.99. Consequently, even if a fast mean-reverting risk exposure generates considerable swings in the expected return, the corresponding variations in the price-dividend ratio may still be small.

These results suggest that the persistence of the risk exposure bears critical asset pricing implications. A highly persistent risk exposure not only implies that the risk premium sustains longer along the time dimension. It also implies a larger magnitude of the risk premium. Moreover, it implies that variations in the risk premium are more strongly reflected in the valuation ratio. These relations, borne out of the long-run consumption risk framework, are likely missing if investors are assumed to care only about contemporaneous consumption growth.

As an application, I supply additional assumptions and commit the model to an attempt to reconcile short-run momentum and long-run reversal returns in portfolios sorted on past returns.[6] These two phenomena provide exemplary testing grounds, given their divergent patterns of returns in both cross section and time series. More precisely, in addition to independent shocks and a constant exposure to the long-run consumption risk, I specify individual stocks' dividend growth as containing two components of time-varying risk exposure — a fast mean-reverting component whose shocks are positively correlated with the dividend shocks, and a slow-mean reverting component whose shocks are negatively correlated with the dividend shocks. For parsimony, all stocks share the same model parameters and thus only differ in the history of shocks. In particular, all firms have the same level of the constant exposure; consequently, patterns in expected returns must result from the cross-section distributions and the time-series evolutions of the time-varying loadings.

The positive and negative correlations between the dividend shocks and the state variable shocks are respectively responsible for generating the positive and negative correlations between past realized returns and future expected returns. All else being equal, positive (negative) returns are more likely associated with positive (negative) dividend growth shocks. Consequently, when firms are sorted on realized returns, a winner portfolio tends to pick up stocks with positive dividend growth shocks, which tend to associate with positive shocks to the fast mean-reverting component of risk exposure, and negative shocks to the slow mean-reverting component. Conversely, for a loser portfolio, the fast mean-reverting component of risk exposure decreases, and the slow mean-reverting component increases. At the

[6] See Jegadeesh and Titman (1993) and Jegadeesh and Titman (2001) for momentum, and De Bondt and Thaler (1985) and De Bondt and Thaler (1987) for reversal.

end of the portfolio formation period, there are always both a positive spread in the fast mean-reverting exposure and a negative spread in the slow-mean reverting exposure between winner and loser portfolios.

The low and high levels of persistence determine how fast the spreads in the state variables build up, and how long they sustain afterwards. A spread in the fast mean-reverting exposure is quick to grow, but it also dissipates fast. A spread in the slow mean-reverting exposure takes time to accumulate, but it also lasts longer. Consequently, a short formation period (e.g., from month –12 to month –2) leads the positive spread in the fast mean-reverting exposure to dominate, giving rise to the positive momentum spread. On the other hand, after a long formation period (e.g., from month –60 to month –13) plus a one-year waiting period, the negative spread in the slow mean-reverting exposure becomes dominant, giving rise to the negative reversal spread. The momentum spread, driven by the fast mean-reverting component, is doomed to be short-lived. The reversal spread, sustained by the slow mean-reverting exposure, is long-lasting.

In a calibration of the model for monthly intervals, I set the low and high persistence parameters at 0.91 and 0.99, and the associated correlations at 0.80 and –0.70, respectively. I adopt a leverage function so that unconditionally the fast mean-reverting leverage is distributed uniformly between –28 and 28, while the slow mean-reverting leverage between –7 and 7. In simulations of 2000 firms over a 40-year period, I find the model capable of matching the empirically documented momentum and reversal patterns in the cross section as well as along the time dimension. The model generates, after a portfolio formation period of month –12 to month –2, a short-lived momentum spread that starts with a value of about 1% (in monthly returns) between extreme deciles at month 0, narrows down and turns negative after month 12, and remains at about –0.3% per month into year 5. With a portfolio formation period of month –60 to month –13 and a one-year waiting period, the model records a persistent reversal spread, varying between –0.4% and –0.6% per month across extreme deciles from month 0 to the end of year 5.

Simulations also confirm a value premium across price-dividend ratio sorted portfolios.[7] More importantly, underlying the value premium is a large spread in the slow mean-reverting state variable. This is consistent with the model implication that changes in the high persistence risk exposure generate much larger variations in the price-dividend ratio. The magnitude of the value premium — about 1% per month between extreme price-dividend ratio deciles — is broadly consistent with empirical documentations, such as in Fama and French (1992).

As additional support to the model, the portfolio averages of the total exposure (or leverage) of the dividend growth to the long-run consumption growth obtained in the simulations are largely consistent with the empirical results documented in Bansal, Dittmar, and Lundblad (2005). In simulations, accompanying the momentum spread is a large dispersion in total leverage — about 12 for the top momentum quintile, and about –4 for the bottom

[7]Book value is not a part of the valuation model in this paper. I use the price-dividend ratio as the classifier of value versus growth stocks. This approach is also used, for example, in Santos and Veronesi (2005) and Lettau and Wachter (2006).

quintile. For the lowest and the highest price-dividend ratio quintiles, the leverage values are about 6 and 2.[8]

In spite of this seemingly monotonic relation between the leverage and portfolio returns, the model in this paper does not imply a linear, or even a monotonic correspondence between the total leverage and the expected returns. Rather, the expected returns are driven more fundamentally by the underlying state variables. Therefore, for example, the simulations show a small *positive* spread in total leverage accompanying the substantially *negative* spread in reversal returns at month 0. The lack of a clear leverage-return relation results from the competition between the two state variables that exhibit different speeds of mean reversion and thus carry different weights in contributing to total leverage versus to expected returns.

The results also point to a potential connection between persistent firm characteristics such as book-to-market and the slow mean-reverting state variable. On the other hand, the fast mean-reverting exposure, apparently being mimicked by the momentum factor, seems less likely to connect with any persistent firm characteristics. All together, these observations suggest that value and momentum factors in empirical models may potentially represent different components of risk exposure to the long-run consumption risk factor. They differ in the levels of persistence and the responses to cash flow news.

This paper presents an attempt to account for both short-term momentum and long-run reversal within a rational asset pricing framework.[9] In explaining momentum, this paper shares with Johnson (2002) the intuition that sorting on realized returns tends to allocate stocks by realized dividend growth shocks. Two studies subsequently diverge in terms of the mechanism relating dividend growth shocks to changes in risk.[10]

In accounting for the value premium, this paper also relates to a large literature that correlates variations in returns with variations in systematic risks.[11] A strand of this literature investigates the separate roles of cash flow risks and discount rate risks (See, for example, Campbell and Vuolteenaho (2004) and Santos and Veronesi (2005)). In this regard, the model of this paper assumes a constant aggregate discount rate, while cash flow risks give rise to time-varying expected returns. A number of other studies suggest that cash flows with different maturities, or durations give rise to different returns to value and growth stocks (Cornell (1999), Dechow, Sloan, and Soliman (2004), Santos and Veronesi (2005), and Lettau and Wachter (2006)). In contrast to their focus on the time-varying patterns of cash flows, the model of this paper emphasizes the time-varying exposure of cash flows to uncertainties in the long-run consumption growth.

In the rest of the paper, I first present and obtain approximate analytical solutions to a

[8]Note the caveat that stocks are sorted by price-dividend ratio in the simulations, but by market-to-book in empirical results. In addition, Bansal, Dittmar, and Lundblad (2005) assumes that the stock portfolios' leverage on the long-run consumption risk is constant.

[9]The behavior based explanations are surveyed in Barberis and Thaler (2003).

[10]In Johnson (2002), which employs a reduced-form pricing kernel, positive shocks raise the dividend growth rates, which entail higher expected returns.

[11]See Daniel and Titman (2006) and Lewellen, Nagel, and Shanken (2006) for the lists of the studies, and critiques of the econometric methodology. See Campbell (2003) and Cochrane (2006) for general surveys of the consumption based asset pricing literature.

general model in Section 2. This is followed by the construction and the simulation of two models separately addressing momentum and reversal, and finally a unified model resolving both phenomena in Sections 3.1 through 3.3. I verify in Section 3.4 that simulated economies also exhibit a value premium, and propose interpretations of empirical asset pricing factors. The concluding section addresses the limitations of the paper, and the appendices collect details of the model construction, solution, and simulation.

2 Model

2.1 Pricing kernel

I follow Bansal and Yaron (2004) in the specification of the aggregate consumption process and the representative agent utility. More details are supplied in Appendix D, which recasts the key results in Bansal and Yaron (2004) using the notation of this paper.

The logarithm aggregate consumption growth is modeled as consisting of independent shocks plus a small and persistent expected component, as in

$$\log \frac{C_{t+1}}{C_t} \equiv \Delta c_{t+1} = \mu_c + \sigma_c \varepsilon_{c,t+1} + x_t, \tag{1}$$

$$x_{t+1} = \rho_x x_t + \sigma_x \varepsilon_{x,t+1}, \tag{2}$$

where the shocks $\varepsilon_{x,t+1}$ and $\varepsilon_{c,t+1}$ are standard normal random variables that are independent across time and of each other. In this model, the consumption growth has a mean of μ_c, and the independent shocks have a standard deviation of σ_c. The expected component x_t is a zero mean AR(1) process; it has a persistence parameter ρ_x very close to 1 and is driven by shocks with a small standard deviation σ_x. With carefully calibrated parameters, Bansal and Yaron (2004) show that this model well replicates the time series properties of the observed U.S. aggregate consumption growth data.

A representative agent is assumed to exhibit the Epstein and Zin (1989) and Weil (1989) preferences. This leads to a logarithm pricing kernel

$$\log M_{t+1} \equiv m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

where $r_{c,t+1} \equiv \log R_{c,t+1}$ is the logarithm gross return on the aggregate wealth, which pays the aggregate consumption stream as dividends. The risk aversion parameter γ and the intertemporal elasticity of substitution parameter ψ are separately specified. The parameter $0 < \delta < 1$ is the time discount factor, while $\theta = (1 - \gamma)/(1 - 1/\psi)$.

Log-linearization solution to the logarithm price-consumption ratio produces the following result for the pricing kernel innovations

$$\begin{aligned} m_{t+1} - E_t[m_{t+1}] &\approx -\gamma \sigma_c \varepsilon_{c,t+1} - (\gamma - \frac{1}{\psi}) \frac{K_{1c}}{1 - K_{1c}\rho_x} \sigma_x \varepsilon_{x,t+1} \\ &= -\lambda_c \sigma_c \varepsilon_{c,t+1} - \lambda_x \sigma_x \varepsilon_{x,t+1}, \end{aligned} \tag{3}$$

where $0 < K_{1c} < 1$ is a log-linearization constant very close to 1.

This expression demonstrates the two sources of consumption risks. For the risk associated with the independent shocks $\varepsilon_{c,t+1}$, entering into its price of risk $\lambda_c \sigma_c$ is the risk aversion parameter γ. For the expected consumption growth shocks $\varepsilon_{x,t+1}$, the price of risk is $\lambda_x \sigma_x$. Clearly, even if the long-run consumption shocks have a very small volatility σ_x, investors may still demand a substantial compensation if such shocks have long-lasting implications, as reflected in λ_x as an increasing function of the persistence parameter ρ_x. In this specification, the price of the long-run consumption risk is constant.

The parameters are calibrated for monthly intervals and largely follow Bansal and Yaron (2004), as listed in Table 1. In particular, the expected consumption growth has a persistence of $\rho_x = 0.98$ and a volatility of $\sigma_x = 0.00034$; the representative agent exhibits a risk aversion of $\gamma = 10$ and an intertemporal elasticity of substitution of $\psi = 1.5$.

2.2 Dividend

I model an individual stock as characterized by a dividend growth process consisting of independent shocks and a time-varying leverage on the expected consumption growth x_t, as in

$$\log \frac{D_{t+1}}{D_t} \equiv \Delta d_{t+1} = \mu_d + \sigma_d \varepsilon_{d,t+1} + L(s_{1,t}, \ldots, s_{N,t}) x_t, \tag{4}$$

$$L(s_{1,t}, \ldots, s_{N,t}) = \sum_{n=1}^{N} L_n(s_{n,t}), \tag{5}$$

$$s_{n,t+1} = \rho_{s,n} s_{n,t} + \sqrt{1 - \rho_{s,n}^2}\, \varepsilon_{s,n,t+1}, \quad n = 1, \ldots, N. \tag{6}$$

Here, the shocks $\varepsilon_{d,t+1}$ and $\varepsilon_{s,n,t+1}$ are standard normal random variables that are independent across time. However, they may be correlated with each other contemporaneously. In this specification, the logarithm dividend growth has a mean of μ_d and the independent innovations have a standard deviation of σ_d. In addition, there is an expected dividend growth component, linked to the expected consumption growth x_t through a time-varying leverage $L(s_{1,t}, \ldots, s_{N,t})$. For tractability, I assume L is a sum of functions L_n, separately driven by underlying state variables $s_{n,t}$. Each $s_{n,t}$ is a zero mean AR(1) process with a different persistence parameter $\rho_{s,n}$, and unconditional variance is normalized to be 1.

I also assume that the shocks $\varepsilon_{d,t+1}$ and $\varepsilon_{s,n,t+1}$ exhibit zero correlations with the shocks $\varepsilon_{c,t+1}$ and $\varepsilon_{x,t+1}$ in the consumption growth. Hence, risk premiums are solely driven by the time-varying exposure L to shocks to the expected consumption growth, $\varepsilon_{x,t+1}$. The dividend growth model is also calibrated for monthly intervals.

2.3 Solution

As will be shown in a moment, with a very mild regularity condition, a generally nonlinear leverage function can be transformed into a linear function of multiple underlying state variables. Hence, I will first proceed with the approximate analytical solution of the model for a linear leverage function, which also serves to illustrate the intuition underlying the asset pricing implications of the persistence of a time-varying exposure to the long-run consumption risk.

2.3.1 A linear leverage function

Consider that the leverage is a linear function of the underlying state variables, as in

$$L(s_{1,t}, \ldots, s_{N,t}) = \alpha_0 + \sum_{n=1}^{N} \alpha_n s_{n,t},$$
$$s_{n,t+1} = \rho_{s,n} s_{n,t} + \sqrt{1-\rho_{s,n}^2}\,\varepsilon_{s,n,t+1}, \quad n = 1, \ldots, N.$$

The constant α_0 introduces a permanent component, while the slopes α_n translate the variations in s_n to the leverage.

I proceed to solve the model using log-linearization and leave the details to Appendix E. In the solution, the logarithm price-dividend ratio is

$$\log \frac{P_t}{D_t} \equiv z_t \approx \bar{z} + \left[(\alpha_0 - \frac{1}{\psi}) \frac{1}{1 - K_1 \rho_x} + \sum_{n=1}^{N} \alpha_n s_{n,t} \frac{1}{1 - K_1 \rho_{s,n} \rho_x} \right] x_t$$
$$- \sum_{n=1}^{N} \frac{1}{1 - K_1 \rho_{s,n}} \alpha_n s_{n,t} \frac{K_1 \rho_{s,n}}{1 - K_1 \rho_{s,n} \rho_x} \lambda_x \sigma_x^2, \tag{7}$$

where $0 < K_1 < 1$ is a log-linearization constant very close to 1.

The return innovation is

$$r_{t+1} - E_t[r_{t+1}]$$
$$\approx \sigma_d \varepsilon_{d,t+1} + \left[(\alpha_0 - \frac{1}{\psi}) \frac{K_1}{1 - K_1 \rho_x} + \sum_{n=1}^{N} \alpha_n s_{n,t} \frac{K_1 \rho_{s,n}}{1 - K_1 \rho_{s,n} \rho_x} \right] \sigma_x \varepsilon_{x,t+1}. \tag{8}$$

In the expected return,

$$E_t[r_{t+1}] - r_{f,t} + \frac{1}{2} \mathrm{var}_t[r_{t+1}] = -\mathrm{cov}_t[m_{t+1}, r_{t+1}]$$
$$\approx \left[(\alpha_0 - \frac{1}{\psi}) \frac{K_1}{1 - K_1 \rho_x} + \sum_{n=1}^{N} \alpha_n s_{n,t} \frac{K_1 \rho_{s,n}}{1 - K_1 \rho_{s,n} \rho_x} \right] \sigma_x \cdot \lambda_x \sigma_x. \tag{9}$$

The results show that the permanent component α_0 and the mean-reverting component $\alpha_n s_{n,t}$ exhibit very different asset pricing implications. In the valuation (the terms involving x_t), the return innovation, and the expected return, the coefficients for the constant component are determined only by the persistence of the expected consumption growth. On the other hand, the coefficients for the time-varying component depend on both the persistence of the expected consumption growth and the persistence of the s_n process. Moreover, the coefficients for the time-varying component are always smaller, and can be substantially smaller when the persistence parameter $\rho_{s,n}$ is small. In other words, when investors cares about the long-run consumption risk, they also care deeply how persistent the risk exposure is. They demand substantially smaller compensation if the exposure is likely to be transitory.

Examining together the expected return in Eq. (9), and the valuation equation (7) suggests that high values of the mean-reverting component $\alpha_n s_{n,t}$ not only induce high expected excess returns, but also imply low price-dividend ratios. The variations in expected returns are amplified in logarithm price-dividend ratios by a negative sensitivity of the magnitude

$$\frac{1}{1 - K_1 \rho_{s,n}}. \tag{10}$$

Hence, a high persistence of the mean-reverting component implies that the price-dividend ratio varies more dramatically when the expected return varies.

The ratio between the risk premia for a permanent component and a time-varying component of leverage is

$$\frac{K_1}{1 - K_1 \rho_x} : \frac{K_1 \rho_s}{1 - K_1 \rho_s \rho_x} = \beta_0 : \beta_s. \tag{11}$$

With $\rho_s < 1$, the permanent exposure always demands a higher premium than the time-varying exposure. Table 2 shows the dramatic increase in the value of β_s in comparison to β_0 as the persistence parameter ρ_s increases. The table also tabulates the half-life of a shock to the s process, so defined that, when measured in months,

$$\rho_s^{\text{half-life}} = 0.5.$$

For example, when $\rho_s = 0.91$, the half-life is less than 8 months, and β_s is about 1/5 of β_0. This implies that a mean-reverting component that momentarily exhibits the same magnitude as that of a permanent component only yields an expected excess return less than 20% of that associated with the latter. Even when $\rho_s = 0.99$, which is rather persistent as indicated by a half-life of nearly 6 years, β_s is still not up to 75% of β_0.

Table 2 also shows the dramatic increase in the sensitivity of the logarithm price-dividend ratio with respect to the expected return, as defined in Eq. (10). For the same variations in expected returns, with a low persistence of $\rho_s = 0.91$, the magnitude of the corresponding variations in logarithm price-dividend ratios is only 1/8 of the magnitude of the variations when the persistence is $\rho_s = 0.99$.

Overall, a recurring theme in the results is that, while the leverage L measures the total sensitivity of the expected dividend growth to the long-run consumption growth, the model ends up with a decomposition of the leverage and the asset pricing implications being driven more fundamentally by the multiple underlying state variables of different degrees of persistence. As a result, there is no explicit relation between the leverage and the expected returns.

It is also worth noting that, in spite of the superficial resemblance of Eq. (9) to a multi-factor model, the only risk factor in the model is the long-run consumption risk with a constant factor risk premium, and the multiple terms represent different components of time-varying loadings. The time-varying expected returns arise because these loadings, or "betas" change as the underlying state variables evolve.

2.3.2 A general time-varying leverage function

For a general leverage function, an Hermite polynomial expansion proves to be a convenient tool. Since the leverage function is additively separable, without loss of generality, I focus on a general function of only one state variable s,

$$s_{t+1} = \rho_s s_t + \sqrt{1-\rho_s^2}\,\varepsilon_{s,t+1}.$$

An Hermite expansion exists for $L(s)$ as long as

$$\int_{-\infty}^{\infty} L(s) e^{-\frac{s^2}{2}} ds < \infty,$$

which admits a very broad range of functional forms. From the expansion (generally an infinite series), one can derive an approximation using the first $J+1$ terms

$$L(s) \approx \sum_{j=0}^{J} \alpha_j H_j(s),$$

where $H_j(s)$ is the j-th degree Hermite polynomial and J is sufficiently large. A summary of the properties of Hermite polynomials is provided in Appendix B.

Given s_t as a mean-reverting process with persistence $\rho_s < 1$, how persistent is the transformed series $L(s_t)$, which in general involves a nonlinear leverage function? Appendix C, following Granger and Newbold (1976), shows that $H_j(s_t)$ has an equivalent persistence of ρ_s^j, in that

$$\text{corr}[H_j(s_{t+\tau}), H_j(s_t)] = \rho_s^{j\tau}. \tag{12}$$

That is, the persistence declines geometrically with the degree j of Hermite polynomials. Consequently, for a general $L(s_t)$

$$\text{corr}[L(s_{t+\tau}), L(s_t)] \le \rho_s^{\tau}.$$

In a word, no transformation could generate a leverage process that is more persistent than the underlying state variable.

More importantly, the geometric dependence of the autocorrelation on the lag τ in Eq. (12) implies that $H_j(s_t)$ may be viewed as an AR(1) process with a persistence parameter ρ_s^j. Consequently, the approximation

$$L(s_t) \approx \sum_{j=0}^{J} \alpha_j H_j(s_t)$$

essentially transforms a nonlinear L into a linear function of multiple state variables $H_j(s_t)$, each with a different persistence ρ_s^j. A simple application of Eqs. (7) through (9) yields

$$
\begin{aligned}
& E_t[r_{t+1}] - r_{f,t} + \frac{1}{2}\operatorname{var}_t[r_{t+1}] \\
\approx & \left[(\alpha_0 - \frac{1}{\psi}) \frac{K_1}{1 - K_1 \rho_x} + \sum_{j=1}^{J} \alpha_j H_j(s_t) \frac{K_1 \rho_s^j}{1 - K_1 \rho_s^j \rho_x} \right] \lambda_x \sigma_x^2 \\
= & \left[(\alpha_0 - \frac{1}{\psi}) \beta_0 + \sum_{j=1}^{J} \alpha_j H_j(s_t) \beta_j \right] \lambda_x \sigma_x^2 .
\end{aligned}
$$

Note that in

$$\beta_j = \frac{K_1 \rho_s^j}{1 - K_1 \rho_s^j \rho_x},$$

the persistence ρ_s^j (of the $\alpha_j H_j(s_t)$ component) enters into both the nominator and the denominator. With $\rho_s < 1$, β_j decays faster with j than does ρ_s^j. In other words, while the persistence declines geometrically with j, the contribution to the risk premium from the $\alpha_j H_j(s_t)$ component decreases even more dramatically. Hence, the expected returns are most likely primarily determined by the leading terms of the Hermite polynomial approximation.

3 Models for momentum and reversal

In the following subsections I explore the potentials of the model in explaining the short-run momentum and long-run reversal in returns when stocks are sorted on past returns. These two phenomena, particularly the momentum, have remained notable challenges to rational asset pricing theory.

To facilitate my investigations, I augment the generic model presented in the preceding section with a few additional assumptions. I adopt the following functional form for the leverage function,

$$L(s) = \bar{L} + \Omega(2\Phi(s) - 1).$$

Here, $\bar{L}$ is the permanent component, Ω is a positive constant, and $\Phi(\cdot)$ is the standard normal cumulative distribution function.

This parameterization exhibits several interesting properties. The functional form results in the leverage being bounded between $\bar{L} - \Omega$ and $\bar{L} + \Omega$. The mean is $\bar{L}$, and the amplitude of the variation is Ω. Moreover, the unconditional distribution of L over this range is uniform.[12] A symmetric leverage function, when applied subsequently to the study

[12] For a standard normal random variable X and the standard normal cumulative distribution function $\Phi(\cdot)$, $\Phi(X)$ is uniformly distributed on the interval $[0, 1]$. This follows from

$$\Pr(\Phi(X) \leq u) = \Pr(X \leq \Phi^{-1}(u)) = \Phi(\Phi^{-1}(u)) = u.$$

of momentum, implies largely equal contributions from winner and loser portfolios to the momentum profits, which accords well with the empirical evidence.

Additionally, the Hermite polynomial expansion of this function can be computed analytically. The odd symmetry of $(2\Phi(s) - 1)$ implies that the coefficients are zero for all $H_{2j}, j > 1$. The leading terms of the expansion are

$$L(s) = \bar{L} + \frac{\Omega}{\sqrt{\pi}} s - \frac{\Omega}{12\sqrt{\pi}}(s^3 - 3s) + \frac{\Omega}{160\sqrt{\pi}}(s^5 - 10s^3 + 15s) + \ldots$$

With fast diminishing expansion coefficients and the even faster decaying β_j, the risk premium is essentially determined by the first 3 terms for a fairly wide range of s values.

Several of the parameters governing the dividend growth process will remain unchanged while I explore the effects of the other parameters on the model performance. I set $\bar{L} = 4$, which in the simulation generates an unconditional average return of about 8% per year, broadly consistent with the equity premium on the aggregate market. I set the volatility of independent dividend growth shocks $\sigma_d = 0.1$. This roughly implies an annual dividend growth volatility of 35%, which for individual stocks appears reasonable, particularly when viewed in the broader context of cash flow news. The mean dividend growth is $\mu_d = 0.0015$.

I solve the models numerically, and then use simulations to evaluate the performance of the models. The solution and simulation details are left to Appendices A.2 and A.3.

3.1 A model for momentum

Momentum refers to the short-run return differentials between the portfolios of stocks sorted on returns over the past 6 to 12 months (Jegadeesh and Titman (1993)). Past winners continue to outperform past losers. The spread in monthly returns between the top and bottom deciles starts at about 1%, and gradually declines to zero over a horizon of about a year (Jegadeesh and Titman (2001)).

To generate momentum returns, therefore, a model needs to supply several key ingredients: (1) a positive correlation between past realized returns and future expected returns; (2) the short-lived nature of the return differentials along the time dimension; (3) substantial cross-sectional spreads in expected returns immediately after the portfolio formation. As will be shown subsequently, a model of dividend growth for individual stocks containing a mean-reverting exposure to the long-run consumption risk can potentially meet these requirements. In the model, (1) the risk exposure is driven by shocks that are positively correlated with the independent dividend growth shocks; (2) the risk exposure mean-reverts quickly; (3) sufficiently large spreads in the risk exposure emerge across portfolios following the formation.

More precisely, for firm i,

$$\Delta d^i_{t+1} = \mu_d + \sigma_d \varepsilon^i_{d,t+1} + L(m^i_t) x_t, \tag{13}$$

$$L(m^i_t) = \bar{L} + \Omega_m (2\Phi(m^i_t) - 1), \tag{14}$$

$$m^i_{t+1} = \rho_m m^i_t + \sqrt{1-\rho_m^2}\, \varepsilon^i_{m,t+1}, \tag{15}$$

$$\varepsilon^i_{m,t+1} = \chi_{dm} \varepsilon^i_{d,t+1} + \sqrt{1-\chi_{dm}^2}\, \eta^i_{m,t+1}, \quad \chi_{dm} > 0. \tag{16}$$

Here, the state variable m drives the leverage L on the long-run consumption growth. The shocks to m^i, $\varepsilon^i_{m,t+1}$, are positively correlated with the same firm's dividend growth shocks $\varepsilon^i_{d,t+1}$, while $\eta^i_{m,t+1}$ represents other sources of innovations to m^i. The shocks $\varepsilon^i_{d,t+1}$ and $\eta^i_{m,t+1}$ are independent across firms, and for each firm, independent across time and of each other. For parsimony, I use the same parameters for all firms. In particular, the correlation χ_{dm} is a constant and the same for all firms. Firms in this model differ only in their histories of the shocks $\varepsilon^i_{d,t+1}$ and $\eta^i_{m,t+1}$.

The model relies on the positive correlation between the independent dividend growth shocks and the m shocks to generate the positive correlation between past realized returns and future expected returns. It is seen from the logarithm return

$$r_{t+1} = \log(e^{z_{t+1}} + 1) + \Delta d_{t+1} - z_t$$

that, all else being equal, positive returns are more likely associated with contemporaneous positive shocks to the dividend growth rate, while negative returns with negative dividend shocks. Sorting by past realized returns from low to high, therefore, tends to allocate firms into deciles of increasing realized dividend growth shocks. The positive correlation between the dividend shocks and the m shocks subsequently translates this ordering to that in m, and thus in the risk exposure and expected returns.

To match the fast decay of the momentum spread over the one-year horizon following the portfolio formation, I rely on a fast mean-reverting process for the underlying state variable m. I set the persistence parameter $\rho_m = 0.91$. This corresponds to a half-life of slightly over 7 months and implies that after 1 year, an initial spread of 1% in the expected returns will decline to $1\% \times 0.91^{12} = 0.32\%$. The unconditional standard deviation of m is normalized to be 1, and thus the volatility of the m shocks is $\sqrt{1-\rho_m^2} = 0.41$.

Earlier discussions have revealed that risk exposure with a fast mean reversion, or equivalently, a low persistence, implies a small risk premium. Therefore, to generate large return spreads across momentum deciles immediately following the portfolio formation, I rely on sufficiently large values of the correlation χ_{dm} and the amplitude Ω_m. A large χ_{dm} results in a wide spread in m. However, this is not enough to warrant a large spread in expected returns since Ω_m is also a determinant of the risk loadings. Consequently, the model needs to allow the leverage to vary with a large amplitude.

Panel A of Table 3 presents simulation averages of the results for the portfolio formation period of month –12 to month –2 using $\chi_{dm} = 0.8$ and $\Omega_m = 24$. The portfolio returns increase largely uniformly from decile 1 to decile 10. The monthly return spread between the extreme deciles, $R10 - R1$, is 1.04% at month 0, and drops very quickly to 0.33% at month 12. The t-statistic starts at 2.92, and also declines over time. Underlying the return differential and its quick decline are the fast decaying spreads in the state variable m and the leverage L. The m spread is 1.46 at month 0 with an accompanying leverage spread of almost 19. Both decrease fast over time, as dictated by the low persistence parameter $\rho_m = 0.91$.

Panel A of Table 3 also shows that there are small positive m and L spreads across the deciles at month –12, the beginning month of the portfolio formation period. The spreads widen very quickly, particularly during the first 6 months of portfolio formation. The pace of widening appears to slow down with time. Such fast divergence in m and L is likely the combined result of the high correlation χ_{dm} and the low persistence ρ_m and the associated large volatility of m shocks. Additional summary statistics provided in Panel B of Table 3 suggest that fairly broad ranges of the results may be obtained when the economy is simulated for many times. The distributions of the results across simulations appear to exhibit a small amount of skewness.

Table 4 explores the complementary effects of the amplitude Ω_m and the correlation χ_{dm} on the results. When the amplitude is fixed, the return spread and the associated t-statistic, and the m and L spreads all increase as the correlation becomes more positive. Similarly, when the correlation is fixed, the results turn out stronger as the amplitude becomes larger.

3.2 A model for reversal

Reversal refers to the long-run return differentials between the portfolios of stocks sorted on returns over the past 3 to 5 years (De Bondt and Thaler (1985) and De Bondt and Thaler (1987)). After a one-year waiting period, past winners are found to have low returns while past losers have high returns. For the portfolio period of month –60 to month –13, the spread in monthly returns between the top and bottom deciles is about –0.7% at month 0 (Fama and French (1996)), and the differential persists well into year 5.

Reversal suggests a negative association between past realized returns and future expected returns, and the return spreads are fairly persistent. These characteristics stand at stark contrasts to those associated with momentum. To generate reversal returns, I model the dividend growth of individual stocks as containing a slow mean-reverting exposure to the long-run consumption risk. The risk exposure is driven by shocks that are negatively correlated with the independent dividend growth shocks.

More precisely, for firm i

$$\Delta d^i_{t+1} = \mu_d + \sigma_d \varepsilon^i_{d,t+1} + L(v^i_t) x_t, \tag{17}$$

$$L(v^i_t) = \bar{L} + \Omega(2\Phi(v^i_t) - 1), \tag{18}$$

$$v^i_{t+1} = \rho_v v^i_t + \sqrt{1-\rho_v^2}\, \varepsilon^i_{v,t+1}, \tag{19}$$

$$\varepsilon^i_{v,t+1} = \chi_{dv} \varepsilon^i_{d,t+1} + \sqrt{1-\chi_{dv}^2}\, \eta^i_{v,t+1}, \quad \chi_{dv} < 0. \tag{20}$$

Here, the state variable v drives the leverage L on the long-run consumption growth. The shocks to v^i, $\varepsilon^i_{v,t+1}$, are negatively correlated with the same firm's dividend growth shocks $\varepsilon^i_{d,t+1}$, while $\eta^i_{v,t+1}$ represents other sources of innovations to v^i. The shocks $\varepsilon^i_{d,t+1}$ and $\eta^i_{v,t+1}$ are independent across firms, and for each firm, independent across time and of each other.

The negative correlation between the independent dividend shocks and the state variable shocks gives rise to the negative association between past realized returns and future expected returns. When firms are sorted on past returns, winner portfolios tend to pick up stocks that have absorbed positive dividend growth shocks. The negative correlation implies that these stocks also tend to experience negative v shocks, resulting in a decrease in the risk exposure. Conversely, loser portfolios experience increases in the risk exposure.

The persistence of the reversal returns suggests a slow mean reversion for the state variable v. I set $\rho_v = 0.99$, corresponding to a half-life of 5 years 9 months. In other words, the reversal spread will only shrink to about half of its starting value over a horizon of 5 years. This also implies that the volatility of the v shocks is $\sqrt{1-\rho_v^2} = 0.14$.

Since a highly persistent risk exposure commands a large risk premium, a moderate amplitude Ω_v may be enough to generate the reversal return spreads. On the other hand, a persistent v driven by shocks of a small volatility suggests that a correlation parameter χ_{dv} of a sufficiently large magnitude is still necessary to achieve the spread in the v variable across deciles. In addition, it may also take a long portfolio formation period to obtain the needed separation.

Panel A of Table 5 presents the simulation averages of the results for the portfolio formation period of month –60 to month –13 using $\Omega_v = 5$ and $\chi_{dv} = -0.7$. The decile returns are fairly evenly distributed and the reversal spread between the extreme winner and loser portfolios is –0.56% (per month) at month 0. The spread narrows slowly over time to –0.35% at the end of year 5. The t-statistics, and the spreads in the underlying state variable v and the leverage L also decline slowly, all as a result of the large persistence $\rho_v = 0.99$. Panel B supplies additional statistics, suggesting wide distributions of the results across simulations.

Panel A of Table 5 also displays the evolution of the spreads $v10 - v1$ and $L10 - L1$ during the portfolio formation period. Initially, the winner portfolio has a slightly higher risk exposure than the loser portfolio. This positive spread then diminishes and becomes increasingly negative at an almost linear pace over time. The steady appearance of the speed of widening is attributable to the high persistence of v and the small volatility of its shocks.[13]

Table 6 shows that a rather wide range of reversal spreads can be obtained in the model when the amplitude Ω_v and the correlation χ_{dv} are properly adjusted. As expected, the larger the magnitudes of the two parameters, the wider the reversal spreads.

3.3 A unified model of momentum and reversal

The preceding subsections have explored two models to separately obtain momentum and reversal in stock returns. In the following, I put the ingredients from the two models together in a unified framework to generate both short-run momentum and long-run reversal.

[13]If the portfolio formation period is long enough, the leveling off of the widening speed will become more evident, as is the case for the state variable m.

Building on the earlier results, I model a firm's dividend growth rate as containing two components of the exposure to the long-run consumption risk. The two components are driven by separate underlying state variables — a fast mean-reverting m and a slow mean-reverting v. The two state variables also differ in their correlation with the independent dividend growth shocks. A positive dividend growth shock is more likely associated with a positive shock to m, whereas a negative shock to v.

More precisely, for firm i,

$$\Delta d^i_{t+1} = \mu_d + \sigma_d \varepsilon^i_{d,t+1} + L(m^i_t, v^i_t) x_t, \tag{21}$$

$$L(m^i_t, v^i_t) = \bar{L} + \Omega_m(2\Phi(m^i_t) - 1) + \Omega_v(2\Phi(v^i_t) - 1), \tag{22}$$

$$m^i_{t+1} = \rho_m m^i_t + \sqrt{1-\rho_m^2}\, \varepsilon^i_{m,t+1}, \tag{23}$$

$$\varepsilon^i_{m,t+1} = \chi_{dm}\varepsilon^i_{d,t+1} + \sqrt{1-\chi_{dm}^2}\, \eta^i_{m,t+1}, \quad \chi_{dm} > 0, \tag{24}$$

$$v^i_{t+1} = \rho_v v^i_t + \sqrt{1-\rho_v^2}\, \varepsilon^i_{v,t+1}, \tag{25}$$

$$\varepsilon^i_{v,t+1} = \chi_{dv}\varepsilon^i_{d,t+1} + \sqrt{1-\chi_{dv}^2}\, \eta^i_{v,t+1}, \quad \chi_{dv} < 0. \tag{26}$$

As before, all the shocks, $\varepsilon^i_{d,t+1}, \eta^i_{m,t+1}$, and $\eta^i_{v,t+1}$, are independent across firms, and for each firm, independent across time and of each other. The shocks to the state variables, $\varepsilon^i_{m,t+1}$ and $\varepsilon^i_{v,t+1}$, are correlated with the dividend growth shocks, and as the model is constructed, they are also negatively correlated with each other,[14]

$$\text{corr}[\varepsilon^i_{m,t+1}, \varepsilon^i_{v,t+1}] = \chi_{dm}\chi_{dv} < 0.$$

Based on the results from the previous sections, I expect the unified model to work with the following mechanism. When firms are sorted by past realized returns, a winner portfolio tends to pick up stocks with positive dividend growth shocks, which tend to associate with positive shocks to m and negative shocks to v. Therefore, during the portfolio formation period, the m value of the portfolio continually increases, while the v value continually decreases. The m variable has low persistence (and thus shocks of a large volatility), and so it grows very fast with time. The v variable is strongly persistent (and with shocks of a small volatility), and drops only slowly over time. As a result, while one year is sufficient for a dramatic m value to build up, a few years would be needed before the v variable declines substantially. After the portfolio formation, the m value will only sustain for about a year,

[14] In general,

$$\text{corr}[\varepsilon^i_{m,t+1}, \varepsilon^i_{v,t+1}] = \chi_{dm}\chi_{dv} + \sqrt{(1-\chi_{dm}^2)(1-\chi_{dv}^2)}\,\text{corr}[\eta^i_{m,t+1}, \eta^i_{v,t+1}].$$

Hence, the admissible range is

$$\chi_{dm}\chi_{dv} - \sqrt{(1-\chi_{dm}^2)(1-\chi_{dv}^2)} \le \text{corr}[\varepsilon^i_{m,t+1}, \varepsilon^i_{v,t+1}] \le \chi_{dm}\chi_{dv} + \sqrt{(1-\chi_{dm}^2)(1-\chi_{dv}^2)},$$

as $\left|\text{corr}[\eta^i_{m,t+1}, \eta^i_{v,t+1}]\right| \le 1$. Simulations confirm that the results are qualitatively insensitive to specific choices of $\text{corr}[\eta^i_{m,t+1}, \eta^i_{v,t+1}]$.

owing to its fast mean reversion. The v value, on the other hand, is able to persist well over 5 years.

Conversely, for a loser portfolio, the m value decreases while the v value increases over time. Since the m variable and v variable have opposite implications on the risk premia, they compete with each other in deciding the ultimate distribution of expected returns across the portfolios. Which one of them wins out depends on their respective levels of persistence, associated amplitudes, the correlations of their shocks with the dividend shocks. These dependencies are reflected in the relation between the observed return spreads and the length of the portfolio formation period, the waiting period between the formation period and the return measurement period, and the horizon of the measurement period.

Therefore, in order to form portfolios that demonstrate momentum, a relatively short portfolio formation period is desirable, so that the m spread has sufficiently widened and will dominate over the v spread that has not yet built up. In addition, one needs to start to measure the momentum spreads immediately after the formation period, since beyond one year, the m spread severely recedes, and the much more persistent v spread begins to dominate and generate the reversal returns.

On the other hand, in order to form reversal portfolios, a long formation period is needed so that the v spread can take time to grow. Immediately after the portfolio formation (that is, month –12 if the formation period is month –60 to month –13), however, there is also a substantial m spread that may cancel or even reverse the effect of the v spread. Therefore, it is advisable to wait a year for the m spread to go away, while the v spread sustains largely unabated. As a result, large reversal spreads will be observed at month 0, after the one-year waiting period.

These observations are largely borne out in the simulation results presented in Table 7 for momentum portfolios sorted on returns over month –12 to month –2, and in Table 8, for reversal portfolios sorted on returns over month –60 to month –13. The previous sections have provided valuable guidance on parameter calibration. The only adjustment is the increase in both amplitudes Ω_m and Ω_v. Larger values are needed for both of them, now that the effects of the m and v spreads cancel each other. The complementarity between the amplitude and the correlation suggests that another alternative is to raise the magnitudes of the correlation parameters χ_{dm} and χ_{dv}.

In particular, both Tables 7 and 8 present a pattern where momentum is observed immediately after the portfolio formation, and then taken over by reversals. In Table 7, after the month –12 to month –2 formation, the monthly return spread between the extreme deciles starts with a value of about 1% at month 0, turns negative after month 12, and then stays negative all the way beyond year 5. Similarly, Table 8 suggests that, at month –12 (immediately after the formation period of month –60 to month –13), the return spread also demonstrates momentum. The reversal becomes prominent after the one-year waiting period, and measure between –0.4% to –0.6% per month from month 0 to month 60.

Table 8 also reveals a non-monotonic pattern of reversal spreads from month 0 to month 60. The return spread continues to widen during the first two years and then begins to narrow down. This is the result of the interaction between the v spread and the remnant m

spread. Over the one-year waiting period, the m spread only drop to 32% of its initial value. As the m spread continues to decay, it decays faster than the v spread. Consequently, from month 0 to month 24, the return spread actually increases. Beyond month 24, the v spread decay becomes dominant and the spread narrows.

To provide additional support to the model, I compare the leverage of momentum portfolios obtained from the simulations with the empirical results documented in Bansal, Dittmar, and Lundblad (2005). In the simulations, the average values of the leverage are 12 for the top momentum quintile, and –4 for the bottom quintile. These values are largely consistent with the empirical findings.[15]

In contrast to a linear relation implied in Bansal, Dittmar, and Lundblad (2005) between expected returns and the leverage, earlier discussions in this paper point out that expected returns are fundamentally driven by the underlying state variables, and therefore in general there is no linear or even monotonous relation between the expected returns and the leverage. This is confirmed in the simulation results for the current model. Most notable example is for reversal returns following the month –60 to month –13 sorting. As shown in Table 8, at month 0 the leverage spread is slightly positive, while the return spread is strong negative. Moreover, while the return spreads subsequently display a non-monotonic pattern from month 0 to month 60, the leverage spread monotonically becomes more negative. In another example, Table 7 shows that at month 12 (following the month –12 to month –2 sorting) when the return spread is essentially zero, the leverage spread is still considerably positive. All in all, although there may appear to be a broad pattern that high (low) leverage spreads are associated with high (low) return spreads, the correspondence is not monotonic, let alone one-to-one.

Figure 1 plots the momentum and reversal spreads for various portfolio formation periods, and the results are consistent with the preceding discussions concerning the length of the formation period. In cases where the portfolio formation ends at month –2 and momentum is expected, the initial return spread declines as the portfolio formation period lengthens. In particular, a formation period of month –6 to month –2 can potentially yield a higher momentum spread than a formation period of month –12 to month –2. When the formation period stretches to month –60 to month –2, the initial momentum profit largely disappears. On the other hand, for portfolio formation periods ending at month –13, the longer the formation periods, the more negative the reversal return differentials at month 0. These results are consistent qualitatively and even quantitatively with the empirical evidence in Fama and French (1996).

3.4 Value premium

Having demonstrated the potential of the unified model in generating both short-run momentum and long-run reversal patterns quantitatively comparable to empirical documentations, I now turn to the model performance in producing the empirically documented value pre-

[15]For momentum portfolios, Bansal, Dittmar, and Lundblad (2005) reports a leverage of about 9 to 10 for the top quintile, and about –3 to –5 for the bottom quintile.

mium (Fama and French (1992)). Since book value is not a part of the model, I use the price-dividend ratio as an alternative classifier of value versus growth stocks. This is motivated by the use of earnings-price or cash flow-price ratios in Lakonishok, Shleifer, and Vishny (1994) and Fama and French (1996). The same approach is also used in Santos and Veronesi (2005) and Lettau and Wachter (2006).

As discussed earlier, the model implies a negative sensitivity of price-dividend ratios to expected excess returns. This effect arises regardless of any additional assumption on the correlation between dividend growth shocks and leverage shocks. The magnitude of the sensitivity is considerably larger for higher persistence of the underlying time-varying exposure to the long-run consumption risk.

For the unified model, this suggests that changes in the fast mean-reverting exposure only generate small variations in the price-dividend ratio, while the variations in the slow mean-reverting exposure generate much larger swings. This is consistent with the observation that price-dividend ratios tend to be slow moving. In addition, it implies that sorting stocks by this ratio from low to high is very much close to sorting by the value of the high persistence state variable from high to low.

Adopting a similar methodology to that of Fama and French (1992), I sort the stocks in simulated economies at each July by their price-dividend ratios at preceding December (see Appendix A.3 for details). Table 9 shows that in simulated economies, low price-dividend firms have high average returns. Between decile 1 and decile 10, the value premium is about 1% per month, quantitatively comparable to empirical documentations. More importantly, the results indicate that the return differential is driven by the wide spread in the slow mean-reverting v variable, at about –2. The spread in the fast mean-reverting m variable is much narrower at about 0.2; in addition, the sign of the m spread is opposite to that of the return differential. In other words, value firms exhibit high average returns because their slow mean-reverting risk exposure is high. Finally, the resulting total leverage spread is considerably large (about –5). The results also indicate that for the lowest and the highest price-dividend ratio quintiles, the leverage values are about 6 and 2. These values are broadly consistent with the empirical results in Bansal, Dittmar, and Lundblad (2005).[16]

The spread in the slow mean-reverting state variable underlying the value premium across price-dividend ratio sorted portfolios proposes an interpretation of the HML factor as mimicking a highly persistent exposure to the long-run consumption risk. A significant spread in this persistent risk exposure across stock portfolios implies a significant spread in loadings on the HML factor. In particular, this suggests that the return differentials in reversal portfolios, as driven by a negative spread in the slow mean-reverting risk exposure, can be explained by a negative spread in the coefficients on HML. This is consistent with the evidence in Fama and French (1996).

The proposed connection between the value characteristics, the slow mean-reverting risk loading, and the HML factor also helps explain the inability of HML in accounting for momentum. More precisely, the always negative spread in the slow mean-reverting exposure

[16]Bansal, Dittmar, and Lundblad (2005) reports leverage values of about 6 and –1 for extreme value and growth quintiles sorted by book-to-market.

between winner and loser portfolios, as implied by the model regardless of the portfolio formation period, suggests that the HML factor would always predict reversals, even for momentum portfolios based on a month –12 to month –2 sorting. This is indeed the case in Fama and French (1996).

In the model of this paper, momentum returns emerge because the positive spread in the fast mean-reverting exposure dominates the negative spread in the slow-mean reverting exposure. The fast mean-reverting exposure only makes small contributions to the price-dividend ratio. The low persistence also makes it unlikely to link to any other slow moving firm characteristics. Rather, this exposure is apparently mimicked by a momentum factor, as momentum portfolios differ more prominently in this fast mean-reverting exposure, as already demonstrated in Table 7.

Taken together, these observations suggest that value and momentum factors in empirical models may potentially represent different time-varying components of exposure to the long-run consumption factor. They differ in the level of persistence and the response to independent dividend shocks, or broadly, cash flow news.

4 Concluding remarks

In this paper, I show that in the long-run consumption risk framework, a time-varying risk exposure with high persistence commands a substantially higher risk premium than a risk exposure of low persistence. The resulting model of time-varying expected returns, when augmented with additional assumptions, demonstrates promising potentials in matching the momentum and reversal returns documented in the empirical literature. The model also yields the value premium, and suggests interpretations of empirical pricing factors as reflecting risk loadings of different persistence on the long-run consumption risk factor. In concluding the paper, I point out some limitations of this paper.

The model in this paper focuses on time-varying loadings on the long-run consumption risk. For that purpose, I have made simplifying assumptions that (1) assets do not load on the risk associated with independent consumption growth shocks, and (2) the long-run consumption risk carries a constant price of risk. Consequently, the model by construction only involves one consumption risk factor, and the absence of time-varying prices of consumption risk implies a constant premium on the aggregate market. Relaxation of these two restrictions may open up potentially interesting avenues for further research.

More compelling questions arise as to what are the firm-level, economically interpretable sources of these time-varying risk exposures, and why they demonstrate differential mean reversion and respond to cash flow news differently. Answers to these questions appear to be out of the scope of the current paper, which presents a model of time-varying expected returns within a consumption based asset pricing framework. Its partial equilibrium nature and the parsimonious specification precludes an exploration of the relations between risk, firm characteristics, and firm-level decisions. Such an exploration may lead to a more informed evaluation of the reduced-form assumptions made in the model. Presumably, a major source of time-varying risks is a firm's investment dynamics resulting in changes in the firm's portfolio of both assets in place and growth options. If dividend growth news is viewed as containing information about the expansion and contraction of a firm's investment opportunities, then the risk exposure state variables evolve with dividend news as a result of a firm's time-varying profile of and decisions over both existing and new projects. A fruitful approach to such investigations has been demonstrated in Berk, Green, and Naik (1999), Carlson, Fisher, and Giammarino (2004), Gomes, Kogan, and Zhang (2003), Zhang (2005), and Panageas and Yu (2006), among others.

Appendix

A Model solution and simulation

A.1 Analytical solution

The approximate analytical solution to the model involves log-linearization based on a Taylor series expansion (Campbell and Shiller (1988)). Let z_t denote the logarithm price-dividend ratio

$$z_t \equiv \log \frac{P_t}{D_t},$$

and $\bar{z}$ be the long-run mean. An expansion around $\bar{z}$ produces

$$\log(e^{z_t} + 1) \approx \log(e^{\bar{z}} + 1) + \frac{e^{\bar{z}}}{e^{\bar{z}} + 1}(z_t - \bar{z}) = K_0 + K_1 z_t.$$

For large $\bar{z}$, a typical value of K_1 is very close to 1.

The gross return, by definition, is

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1} + D_{t+1}}{D_{t+1}} \frac{D_{t+1}}{D_t} \frac{D_t}{P_t},$$

which implies a logarithm return of

$$\log R_{t+1} \equiv r_{t+1} = \log(e^{z_{t+1}} + 1) + \Delta d_{t+1} - z_t \approx K_0 + K_1 z_{t+1} + \Delta d_{t+1} - z_t.$$

The solution to the model ensues by substituting the linearized logarithm return into the pricing equation

$$1 = E_t \left[e^{m_{t+1} + r_{t+1}} \right].$$

To derive the expected returns, it is noted that if the conditional distribution of m_{t+1} and r_{t+1} is jointly normal, then

$$1 = E_t[e^{m_{t+1} + r_{f,t}}] = e^{E_t[m_{t+1}] + \frac{1}{2} \operatorname{var}_t[m_{t+1}] + r_{f,t}}$$

yields

$$r_{f,t} = -E_t[m_{t+1}] - \frac{1}{2}\operatorname{var}_t[m_{t+1}],$$

and

$$\begin{aligned} 1 = E_t[e^{m_{t+1}+r_{t+1}}] &= e^{E_t[m_{t+1}+r_{t+1}]+\frac{1}{2}\operatorname{var}_t[m_{t+1}+r_{t+1}]} \\ &= e^{E_t[m_{t+1}]+E_t[r_{t+1}]+\frac{1}{2}\operatorname{var}_t[m_{t+1}]+\frac{1}{2}\operatorname{var}_t[r_{t+1}]+\operatorname{cov}_t[m_{t+1},r_{t+1}]} \end{aligned}$$

yields

$$E_t[r_{t+1}] - r_{f,t} + \frac{1}{2}\operatorname{var}_t[r_{t+1}] = -\operatorname{cov}_t[m_{t+1}, r_{t+1}].$$

A.2 Numerical solution

The numerical solution of the price-consumption and price-dividend ratios employs the projection method detailed in Judd (1998). In particular, I use the Galerkin method and approximate the logarithm price-consumption and price-dividend ratios using Hermite polynomials.

A.3 Simulation

To start the simulation, I assign to the relevant variables random values drawn from their respective unconditional distributions, except that the initial monthly dividend is normalized to be $1 for all firms. The number of shares is normalized to be 1 so the stock price is also the market capitalization. I then discard the initial 480 months of simulated data and confirm that, at this point, the simulated economy has reached steady state distributions across firms. Ultimately, each simulated economy contains 2000 firms over a period of 480 months. The actual simulation period is longer due to the lead and lag months.

Momentum or reversal portfolios are formed as follows. For each month t, firms are sorted by compounded returns from month $t-j$ to month $t-k$ ($t-12$ to $t-2$, $t-60$ to $t-13$, and so on). Decile 1 contains the 10% stocks with the lowest past returns, while decile 10 consists of the 10% stocks with the highest past returns.

For each decile, equally-weighted returns of the stocks in the portfolio are calculated for months $t-60$ to $t+60$. Similarly, I calculate the equally weighted values of m, v, and the leverage L for months $t-60-1$ to $t+60-1$. Note the one-month lag. The monthly return spread $R10-R1$ is defined as the 480-month average of decile 10 portfolio monthly returns minus that of decile 1. The associated t statistic is then computed. The m spread $m10-m1$ is defined as the 480-month average of one-month lagged m values of decile 10 minus that of decile 1. The $v10-v1$ and $L10-L1$ spreads are computed similarly.

Price-dividend ratio portfolios are formed as follows. Each July of year y, I sort stocks by the price-dividend ratio at December of year $y-1$. Decile 1 contains the 10% stocks with the lowest price-dividend ratios, while decile 10 consists of the 10% stocks with the highest price-dividend ratios.

For each decile, equally-weighted returns of the stocks in the portfolio are calculated at each month from July of year 1 to June of year 40. Similarly, I calculate the equally weighted values of m, v, and the leverage L at each month from June of year 1 to May of year 40. Note the one-month lag. The monthly return spread $R10-R1$ is defined as the 480-month average of decile 10 portfolio monthly returns minus that of decile 1. The associated t statistic is then computed. The m spread $m10-m1$ is defined as the 480-month average of one-month lagged m values of decile 10 minus that of decile 1. The $v10-v1$ and $L10-L1$ spreads are computed similarly.

I conduct 1000 simulations, and report the average and distributions of the above quantities across the 1000 simulations.

Table 1 tabulates the parameters for the aggregate consumption growth and Epstein-Zin preferences, and the parameters for the dividend growth that remain fixed across models. The remaining dividend parameters are reported separately with the results.

B Hermite polynomials

The standard reference is Abramowitz and Stegun (1964). I follow Granger and Newbold (1976) and adopt the probabilists' convention to define Hermite polynomials using the standard normal distribution density function.

$$H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}, \quad j \geq 0.$$

Hence,

$$\begin{aligned} &H_0(x) = 1, && H_1(x) = x, && H_2(x) = x^2 - 1, \\ &H_3(x) = x^3 - 3x, && H_4(x) = x^4 - 6x^2 + 3, && H_5(x) = x^5 - 10x^3 + 15, \end{aligned}$$

and so on.

The orthogonality properties of the Hermite polynomials are illustrated using expectations with respect to standard normal random variables X and Y, where $\text{corr}(X,Y) = \chi$. It is straightforward to rewrite the results as integrals involving normal distribution density functions.

$$\begin{aligned} E[H_0(X)] &= 1, & E[H_j(X)] &= 0, \quad j > 0, \\ E[H_j(X)^2] &= j!, \quad \forall j, & E[H_j(X)H_k(X)] &= 0, \quad j \neq k, \\ E[H_j(X)H_j(Y)] &= \chi^j j!, \quad \forall j, & E[H_j(X)H_k(Y)] &= 0, \quad j \neq k. \end{aligned}$$

The Hermite polynomial expansion for a function $L(x)$,

$$L(x) = \sum_{j=0}^{\infty} \alpha_j H_j(x),$$

exists if

$$E[L(X)] < \infty.$$

The expansion coefficients are evaluated from

$$\alpha_j = \frac{1}{j!} E[L(X) H_j(X)] = \frac{1}{j!} E[L^{(j)}(X)],$$

where $L^{(j)}(\cdot)$ is the j-th derivative. In particular,

$$E[L(X)] = \alpha_0.$$

A good approximation for $L(x)$ can then be derived by retaining only the first $J+1$ terms in the expansion, where J is a large enough integer.

C Autocorrelation of leverage process

The following results are largely reproductions of Granger and Newbold (1976). Suppose s_t follows an AR(1) process,

$$s_{t+1} = \rho_s s_t + \sqrt{1-\rho_s^2} \varepsilon_{t+1}, \quad \varepsilon_t \sim i.i.d.\ N(0,1),$$

with $\rho_s < 1$ and an unconditional variance of 1. The autocorrelations decay geometrically with the lag, as in

$$\text{corr}[s_{t+\tau}, s_t] = \rho_s^\tau.$$

One is interested in the time series properties of the leverage process $L(s_t)$. Suppose $L(s)$ expands into a finite series of Hermite polynomials,

$$L(s) = \sum_{j=0}^{J} \alpha_j H_j(s),$$

then

$$\text{cov}[L(s_{t+\tau}), L(s_t)] = E\left[\sum_{j=1}^{J} \alpha_j H_j(s_{t+\tau}) \sum_{k=1}^{J} \alpha_k H_k(s_t)\right] = \sum_{j=1}^{J} \alpha_j^2 j! \rho_s^{j\tau},$$

$$\text{var}[L(s_t)] = \sum_{j=1}^{J} \alpha_j^2 j!.$$

Note the summations start from $j=1$ in the above.

It then follows that the autocorrelations

$$\text{corr}[L(s_{t+\tau}), L(s_t)] < \rho_s^\tau = \text{corr}[s_{t+\tau}, s_t], \quad \text{for } J > 1.$$

In other words, except for a linear function, the series $L(s_t)$ is less persistent, or mean-reverts faster than the underlying s process. In particular,

$$\text{corr}[H_j(s_{t+\tau}), H_j(s_t)] = \rho_s^{j\tau},$$

suggesting that $H_j(s_t)$ may be viewed as an AR(1) process of a persistence parameter ρ_s^j.

D Pricing kernel

The specification of the aggregate consumption growth and the representative agent preferences follows Bansal and Yaron (2004). I replicate the solutions using the notation in this paper.

The aggregate consumption growth is

$$\log \frac{C_{t+1}}{C_t} \equiv \Delta c_{t+1} = \mu_c + \sigma_c \varepsilon_{c,t+1} + x_t,$$
$$x_{t+1} = \rho_x x_t + \sigma_x \varepsilon_{x,t+1}.$$

The logarithm pricing kernel is

$$\log M_{t+1} \equiv m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

The approximate analytical solution for the price-consumption ratio is

$$\log \frac{P_{c,t}}{C_t} \equiv z_{c,t} \approx \bar{z}_c + \frac{1 - \frac{1}{\psi}}{1 - K_{1c}\rho_x} x_t.$$

This implies (suppressing constant terms)

$$\begin{aligned} \log R_{c,t+1} \equiv r_{c,t+1} &= \log(e^{z_{c,t+1}} + 1) + \Delta c_{t+1} - z_{c,t} \\ &\approx \frac{1}{\psi} x_t + (1 - \frac{1}{\psi}) \frac{K_{1c}}{1 - K_{1c}\rho_x} \sigma_x \varepsilon_{x,t+1} + \sigma_c \varepsilon_{c,t+1}, \end{aligned}$$

and

$$\begin{aligned} m_{t+1} &\approx -\frac{1}{\psi} x_t - \gamma \sigma_c \varepsilon_{c,t+1} - (\gamma - \frac{1}{\psi}) \frac{K_{1c}}{1 - K_{1c}\rho_x} \sigma_x \varepsilon_{x,t+1} \\ &= -\frac{1}{\psi} x_t - \lambda_c \sigma_c \varepsilon_{c,t+1} - \lambda_x \sigma_x \varepsilon_{x,t+1}. \end{aligned}$$

E Price-dividend ratio and expected return

The logarithm dividend growth is

$$\log \frac{D_{t+1}}{D_t} \equiv \Delta d_{t+1} = \mu_d + \sigma_d \varepsilon_{d,t+1} + L(s_{1,t}, \ldots, s_{N,t}) x_t,$$
$$L(s_{1,t}, \ldots, s_{N,t}) = \alpha_0 + \sum_{n=1}^{N} \alpha_n s_{n,t},$$
$$s_{n,t+1} = \rho_{s,n} s_{n,t} + \sqrt{1 - \rho_{s,n}^2}\, \varepsilon_{s,n,t+1}, \quad n = 1, \ldots, N.$$

Conjecture that the logarithm price-dividend ratio is

$$\log \frac{P_t}{D_t} \equiv z_t \approx \bar{z} + \left[A_0 + \sum_{n=1}^{N} A_n s_{n,t} \right] x_t + \sum_{n=1}^{N} B_n s_{n,t}.$$

This leads to (suppressing constants),

$$\begin{aligned} r_{t+1} &= \log(e^{z_{t+1}} + 1) + \Delta d_{t+1} - z_t \approx K_0 + K_1 z_{t+1} + \Delta d_{t+1} - z_t \\ &= \left[K_1 A_0 + \sum_{n=1}^{N} K_1 A_n s_{n,t+1} \right] x_{t+1} + \sum_{n=1}^{N} K_1 B_n s_{n,t+1} \\ &\quad + \sigma_d \varepsilon_{d,t+1} + \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n s_{n,t} \right] x_t - z_t. \end{aligned}$$

Hence, the return innovation is

$$r_{t+1} - E_t[r_{t+1}] \approx \sigma_d \varepsilon_{d,t+1} + \left[K_1 A_0 + \sum_{n=1}^{N} K_1 A_n \rho_{s,n} s_{n,t} \right] \sigma_x \varepsilon_{x,t+1},$$

where only the leading terms associated with $\varepsilon_{d,t+1}$ and $\varepsilon_{x,t+1}$ are retained.

The pricing equation

$$1 = E_t[e^{m_{t+1} + r_{t+1}}]$$

and the conditional normal distribution imply that

$$0 = E_t[m_{t+1}] + E_t[r_{t+1}] + \frac{1}{2}\operatorname{var}_t[m_{t+1}] + \frac{1}{2}\operatorname{var}_t[r_{t+1}] + \operatorname{cov}_t[m_{t+1}, r_{t+1}].$$

Apply the autocorrelation properties of x_{t+1} and $s_{n,t+1}$ and track only the x_t, $s_{n,t}x_t$, and $s_{n,t}$ terms in

$$E_t[m_{t+1}] \approx -\frac{1}{\psi} x_t,$$

$$\begin{aligned} E_t[r_{t+1}] \approx& \left[K_1 A_0 + \sum_{n=1}^{N} K_1 A_n \rho_{s,n} s_{n,t} \right] \rho_x x_t + \sum_{n=1}^{N} K_1 B_n \rho_{s,n} s_{n,t} \\ &+ \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n s_{n,t} \right] x_t \\ &- \left[A_0 + \sum_{n=1}^{N} A_n s_{n,t} \right] x_t - \sum_{n=1}^{N} B_n s_{n,t}, \end{aligned}$$

and

$$- \operatorname{cov}_t[m_{t+1}, r_{t+1}] \approx \left[K_1 A_0 + \sum_{n=1}^{N} K_1 A_n \rho_{s,n} s_{n,t} \right] \sigma_x \cdot \lambda_x \sigma_x.$$

Comparison of the two sides of the pricing equation yields

$$\begin{aligned} A_0 &= \frac{\alpha_0 - \frac{1}{\psi}}{1 - K_1 \rho_x}, \\ A_n &= \frac{\alpha_n}{1 - K_1 \rho_{s,n} \rho_x}, \quad n = 1, \ldots, N, \end{aligned}$$

and

$$\begin{aligned} B_n &= -\frac{1}{1 - K_1 \rho_{s,n}} K_1 A_n \rho_{s,n} \lambda_x \sigma_x^2 \\ &= -\frac{1}{1 - K_1 \rho_{s,n}} \frac{\alpha_n K_1 \rho_{s,n}}{1 - K_1 \rho_{s,n} \rho_x} \lambda_x \sigma_x^2, \quad n = 1, \ldots, N. \end{aligned}$$

Back to the expected excess return,

$$\begin{aligned} & E_t[r_{t+1}] - r_{f,t} + \frac{1}{2} \operatorname{var}_t[r_{t+1}] = - \operatorname{cov}_t[m_{t+1}, r_{t+1}] \\ & \approx \left[K_1 A_0 + \sum_{n=1}^{N} K_1 A_n \rho_{s,n} s_{n,t} \right] \lambda_x \sigma_x^2 \\ & = \left[(\alpha_0 - \frac{1}{\psi}) \frac{K_1}{1 - K_1 \rho_x} + \sum_{n=1}^{N} \alpha_n s_{n,t} \frac{K_1 \rho_{s,n}}{1 - K_1 \rho_{s,n} \rho_x} \right] \lambda_x \sigma_x^2. \end{aligned}$$

Now turn to the case of a general L function driven by a scalar state variable s,

$$\begin{aligned} \log \frac{D_{t+1}}{D_t} \equiv \Delta d_{t+1} &= \mu_d + \sigma_d \varepsilon_{d,t+1} + L(s_t) x_t, \\ s_{t+1} &= \rho_s s_t + \sqrt{1 - \rho_s^2} \, \varepsilon_{s,t+1}, \end{aligned}$$

and the leverage function is well approximated by an Hermite polynomial series

$$L(s_t) \approx \sum_{j=0}^{J} \alpha_j H_j(s_t).$$

Note that

$$\operatorname{corr}[H_j(s_{t+\tau}), H_j(s_t)] = \rho_s^{j\tau}$$

suggests that $H_j(s_t)$ may viewed as an AR(1) process with a persistence parameter of ρ_s^j. Consequently, the Hermite polynomial expansion essentially transforms a generally nonlinear

$L(s_t)$ into a linear leverage function driven by multiple state variables $H_j(s_t)$ of persistence ρ_s^j. Simple substitution obtains

$$
\begin{aligned}
& E_t[r_{t+1}] - r_{f,t} + \frac{1}{2}\operatorname{var}_t[r_{t+1}] \\
\approx & \left[(\alpha_0 - \frac{1}{\psi}) \frac{K_1}{1 - K_1 \rho_x} + \sum_{j=1}^{J} \alpha_j H_j(s_t) \frac{K_1 \rho_s^j}{1 - K_1 \rho_s^j \rho_x} \right] \lambda_x \sigma_x^2.
\end{aligned}
$$

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Table 1: Fixed simulation parameters

Consumption				Preferences			Dividend		
ρ_x	σ_x	μ_c	σ_c	δ	γ	ψ	μ_d	σ_d	$\bar{L}$
0.98	0.00034	0.0015	0.008	0.998	10	1.5	0.0015	0.1	4

These parameters imply that the unconditional standard deviation for x_t is

$$\frac{\sigma_x}{\sqrt{1-\rho_x^2}} = 0.0017,$$

and

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}} = -27.$$

Table 2: Effect of state variable persistence

ρ_s	0.91	0.95	0.98	0.99	0.995	
half-life (month)	7.35	13.5	34.3	69.0	138.3	$\beta_0 = \frac{K_1}{1-K_1\rho_x}$
half-life (year)	0.61	1.13	2.86	5.75	11.52	
$\beta_s = \frac{K_1\rho_s}{1-K_1\rho_s\rho_x}$	8.18	13.19	23.00	30.17	35.65	43.46
$\frac{1}{1-K_1\rho_s}$	10.78	18.92	43.59	77.10	125.23	

See Table 1 for simulation parameters. The log-linearization constants K_{1c} and K_1 are set to 0.997, roughly corresponding to a value of 5.8 for the long-run means of both the logarithm price-consumption and price-dividend ratios.

Table 3: Simulation of the momentum model

Panel A: Average across simulations

Time (Month)	Decile return (%) 1	4	7	10	$R10 - R1$ (%)	t-stat	$m10 - m1$	$L10 - L1$
–12							0.08	1.03
–6							1.13	14.74
0	0.23	0.52	0.87	1.27	1.04	2.92	1.46	18.78
6	0.42	0.63	0.78	1.02	0.60	2.83	0.83	11.05
12	0.55	0.68	0.75	0.88	0.33	2.49	0.47	6.33
18	0.63	0.69	0.74	0.80	0.18	1.97	0.27	3.61
24	0.66	0.70	0.73	0.76	0.10	1.34	0.15	2.05

Panel B: Distribution across simulations (t = month 0)

	$R10 - R1$ (%)	t-stat	$m10 - m1$	$L10 - L1$
Mean	1.04	2.92	1.46	18.78
5%	0.08	0.27	0.25	2.72
Median	0.99	2.89	1.53	19.72
95%	2.21	5.55	2.37	30.85

The portfolio formation period is from month -12 to month -2. In Panel A, the "Decile return" columns report the simulation averages of the portfolio monthly returns. The remaining columns report the simulation averages of the monthly return spreads $R10 - R1$ and the associated t-statistics, the m spreads $m10 - m1$, and the leverage spreads $L10 - L1$. Panel B presents additional summary statistics for month 0.

See Appendix A.3 for simulation details and variable definitions. See Table 1 for fixed simulation parameters.

The other dividend parameters are: $\rho_m = 0.91, \chi_{dm} = 0.80, \Omega_m = 24$.

Table 4: Momentum with different parameters

Ω_m	χ_{dm}	$R10 - R1$ (%)	t-stat	$m10 - m1$	$L10 - L1$
22	0.8	0.95	2.83	1.23	16.09
24	0.7	0.94	2.80	1.18	15.32
24	0.8	1.04	2.92	1.46	18.78
24	0.9	1.12	2.94	1.71	21.95
26	0.8	1.11	2.94	1.70	21.51

The portfolio formation period is from month -12 to month -2. The table reports, for different values of Ω_m and χ_{dm}, the simulation averages of the monthly return spreads $R10-R1$ and the associated t-statistics, the m spreads $m10-m1$, and the leverage spreads $L10 - L1$.

See Appendix A.3 for simulation details and variable definitions. See Table 1 for fixed simulation parameters.

The other dividend parameters are: $\rho_m = 0.91$.

Table 5: Simulation of the reversal model

Panel A: Average across simulations

Time	Decile return (%)				$R10 - R1$			
(Month)	1	4	7	10	(%)	t-stat	$v10 - v1$	$L10 - L1$
–60							0.53	1.49
–48							–0.10	–0.26
–36							–0.67	–1.88
–24							–1.21	–3.37
–12							–1.72	–4.75
–6							–1.62	–4.48
0	0.93	0.70	0.57	0.37	–0.56	–2.71	–1.53	–4.23
12	0.90	0.70	0.58	0.40	–0.50	–2.69	–1.35	–3.77
24	0.87	0.69	0.58	0.43	–0.44	–2.63	–1.20	–3.35
36	0.84	0.68	0.59	0.45	–0.39	–2.58	–1.06	–2.98
48	0.81	0.67	0.60	0.47	–0.35	–2.53	–0.94	–2.64
60	0.79	0.67	0.60	0.49	–0.31	–2.48	–0.84	–2.35

Panel B: Distribution across simulations (t = month 0)

	$R10 - R1$ (%)	t-stat	$v10 - v1$	$L10 - L1$
Mean	–0.56	–2.71	–1.53	–4.23
5%	–0.79	–4.20	–1.87	–5.22
Median	–0.57	–2.72	–1.55	–4.32
95%	–0.25	–1.12	–1.08	–2.96

The portfolio formation period is from month −60 to month −13. In Panel A, the "Decile return" columns report the simulation averages of the portfolio monthly returns. The remaining columns report the simulation averages of the monthly return spreads $R10 - R1$ and the associated t-statistics, the v spreads $v10-v1$, and the leverage spreads $L10-L1$. Panel B presents additional summary statistics for month 0.

See Appendix A.3 for simulation details and variable definitions. See Table 1 for fixed simulation parameters.

The other dividend parameters are: $\rho_v = 0.99, \chi_{dv} = -0.70, \Omega_v = 5$.

Table 6: Reversal and different parameters

Ω_v	χ_{dv}	$R10 - R1$ (%)	t-stat	$v10 - v1$	$L10 - L1$
4	–0.7	–0.43	–2.53	–1.31	–3.44
5	–0.6	–0.48	–2.60	–1.34	–3.72
5	–0.7	–0.56	–2.71	–1.53	–4.23
5	–0.8	–0.63	–2.78	–1.71	–4.73
6	–0.7	–0.68	–2.85	–1.81	–5.01

The portfolio formation period is from month -60 to month -13. The table reports, for different value of Ω_v and χ_{dv}, the simulation averages of the monthly return spreads $R10 - R1$ and the associated t-statistics, the v spreads $v10 - v1$, and the leverage spreads $L10 - L1$.

See Appendix A.3 for simulation details and variable definitions. See Table 1 for fixed simulation parameters.

The other dividend parameters are: $\rho_v = 0.99$.

Table 7: Momentum in the unified model

Time (Month)	$R10 - R1$ (%)	t-stat	$m10 - m1$	$v10 - v1$	$L10 - L1$
–12			0.12	0.16	2.49
–6			1.21	–0.46	16.81
0	1.04	3.18	1.54	–0.94	19.60
3	0.67	2.72	1.16	–0.91	14.32
6	0.39	1.99	0.87	–0.88	10.14
9	0.17	1.07	0.66	–0.85	6.94
12	0.01	0.12	0.49	–0.83	4.52
24	–0.27	–1.84	0.16	–0.73	–0.39
36	–0.33	–2.29	0.05	–0.65	–1.76
48	–0.33	–2.39	0.02	–0.58	–2.02
60	–0.30	–2.36	0.01	–0.51	–1.94

The portfolio formation period is from month -12 to month -2. The table reports the simulation averages of the monthly return spreads $R10 - R1$ and the associated t-statistics, the m spreads $m10 - m1$, the v spreads $v10 - v1$, and the leverage spreads $L10 - L1$.

See Appendix A.3 for simulation details and variable definitions. See Table 1 for fixed simulation parameters.

The other dividend parameters are: $\rho_m = 0.91, \chi_{dm} = 0.80, \Omega_m = 28; \rho_v = 0.99, \chi_{dv} = -0.70, \Omega_v = 7$.

Table 8: Reversal in the unified model

Time (Month)	$R10-R1$ (%)	t-stat	$m10-m1$	$v10-v1$	$L10-L1$
–60			0.00	0.39	1.62
–48			0.88	–0.17	13.03
–36			1.16	–0.68	15.26
–24			1.26	–1.15	14.87
–12	0.38	1.68	1.24	–1.58	13.05
–6	–0.14	–0.67	0.71	–1.49	5.28
0	–0.42	–2.04	0.40	–1.40	0.84
3	–0.49	–2.37	0.30	–1.36	–0.55
6	–0.55	–2.58	0.23	–1.32	–1.57
9	–0.58	–2.73	0.17	–1.28	–2.31
12	–0.60	–2.82	0.13	–1.24	–2.83
24	–0.61	–2.98	0.04	–1.10	–3.66
36	–0.57	–2.99	0.01	–0.97	–3.62
48	–0.51	–2.95	0.00	–0.86	–3.33
60	–0.45	–2.88	0.00	–0.77	–2.99

The portfolio formation period is from month -60 to month -13. The table reports the simulation averages of the monthly return spreads $R10-R1$ and the associated t-statistics, the m spreads $m10-m1$, the v spreads $v10-v1$, and the leverage spreads $L10-L1$.

See Appendix A.3 for simulation details and variable definitions. See Table 1 for fixed simulation parameters.

The other dividend parameters are: $\rho_m = 0.91, \chi_{dm} = 0.80, \Omega_m = 28; \rho_v = 0.99, \chi_{dv} = -0.70, \Omega_v = 7$.

Table 9: Returns spreads across price-dividend ratio sorted portfolios

$R10 - R1$ (%)	$m10 - m1$	$v10 - v1$	$L10 - L1$
–1.03	0.24	–2.36	–5.21

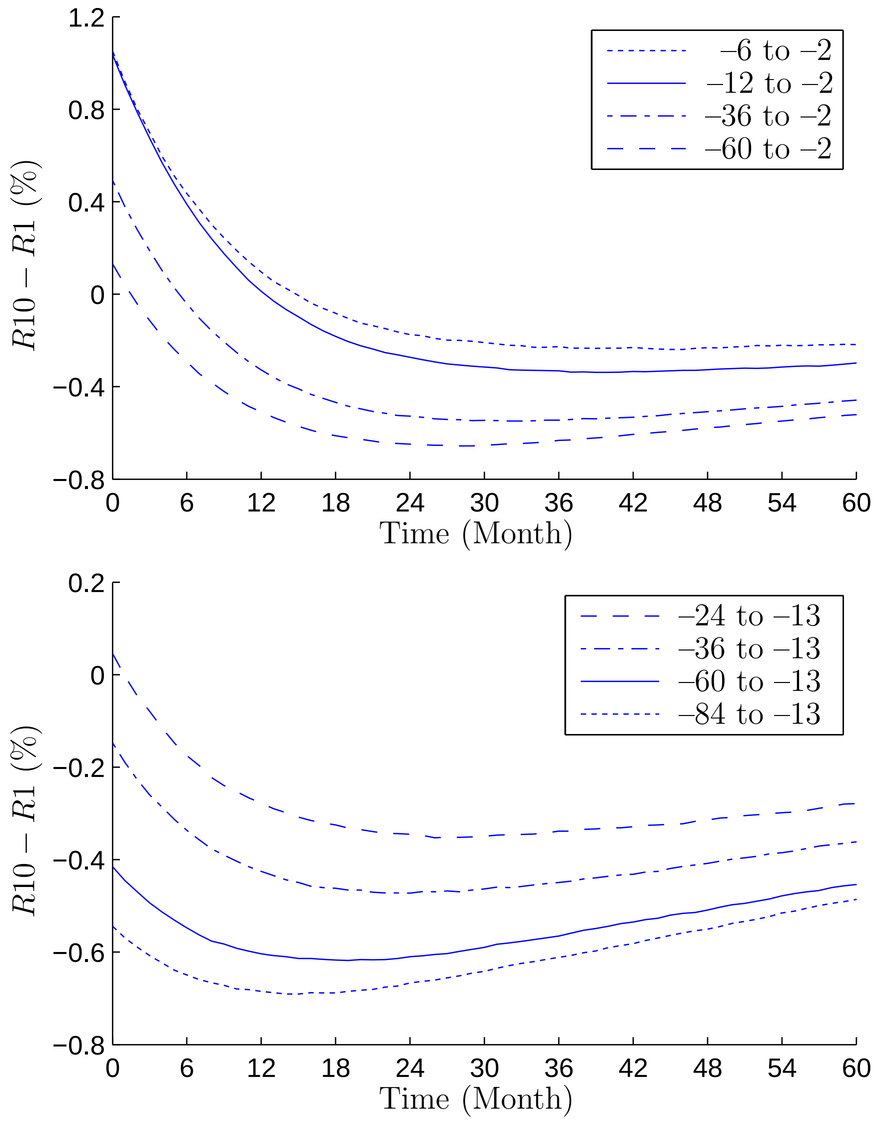
Firms are sorted every July using the dividend-price ratios of preceding December.

The table reports the simulation averages of the monthly return spreads $R10-R1$, the m spreads $m10 - m1$, the v spreads $v10 - v1$, and the leverage spreads $L10 - L1$.

See Appendix A.3 for simulation details and variable definitions. See Table 1 for fixed simulation parameters.

The other dividend parameters are: $\rho_m = 0.91, \chi_{dm} = 0.80, \Omega_m = 28; \rho_v = 0.99, \chi_{dv} = -0.70, \Omega_v = 7$.

Figure 1: Return spreads and portfolio formation period



This figure plots, for different portfolio formation periods, month $-j$ to month $-k$, the simulation averages of the monthly return spreads $R10 - R1$.

See Appendix A.3 for simulation details and variable definitions. See Table 1 for fixed simulation parameters.

The other dividend parameters are: $\rho_m = 0.91, \chi_{dm} = 0.80, \Omega_m = 28; \rho_v = 0.99, \chi_{dv} = -0.70, \Omega_v = 7$.