Linearity-Generating Processes: A Modelling Tool Yielding Closed Forms for Asset Prices

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Abstract

This methodological paper proposes a new class of stochastic processes with appealing properties for theoretical or empirical work in finance and macroeconomics, the "linearity-generating" class. Its key property is that it yields simple exact closed-form expressions for stocks and bonds, with an arbitrary number of factors. It operates in discrete and continuous time. It has a number of economic modeling applications. These include macroeconomic situations with changing trend growth rates, or stochastic probability of disaster, asset pricing with stochastic risk premia or stochastic dividend growth rates, and yield curve analysis that allows flexibility and transparency. Many research questions may be addressed more simply and in closed form by using the linearity-generating class. (JEL: G12, G13)

Keywords: Modified Gordon growth model, Stochastic Discount Factor, Affine models, Long term risk, Growth rate risk, Interest rate processes, Yield curve, Bond premia, Equity Premium, Rare Disasters.

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1 Introduction

This methodological paper proposes a new class of stochastic processes that has a number of attractive properties for economics and finance, the "linearity-generating" (LG) processes. It is generates closed-form solutions for the prices of stocks and bonds. It is simple and flexible, applies to an arbitrary number of factors with a rich correlation structure, and works in discrete or continuous time. These features make it an easy-to-use new tool for pure and applied financial modelling.

The main advantage of the LG class is that it generates, with very little effort, multifactor stock and bond models, in a way that incorporates stochastic growth rates of dividend, and stochastic equity premium. Stock and bond prices are linear in the factors – hence the name "linearity-generating" processes.

Economically, a process is in the LG class if it satisfies two moment conditions: the expected growth rate of the stochastic discount factor (multiplied by the dividend, if one prices stocks), is linear in the factor. And, the expected growth rate of the stochastic discount factor, times the vector of factors next period, is also linear in the factors. Given only those moments, one can price assets. Higher order moments do not matter. In many applications, the variance of processes can be changed almost arbitrarily, the prices will not change. The fact that a few moments are enough to derive prices makes modelling easier.

Linearity-generating processes are meant to be a practical tool for several areas in economics. They are likely to be useful in: (i) macroeconomics, in models with stochastic trend growth rate or probability of disaster, (ii) asset pricing, for models with stochastic equity premium, interest rate, or earnings growth rate, and (iii) fixed-income analysis.

Several literatures motivate the need for a tool such as the LG process. Many recent studies investigates the importance of long-term risk for asset pricing and macroeconomics, e.g., Bansal and Yaron (2004), Barro (2006), Bekaert et al. (2005), Croce, Lettau and Ludvigson (2006), Gabaix and Laibson (2002), Hansen, Heaton and Li (2005), Hansen and Scheinkman (2006), Julliard and Parker (2004), Lettau and Wachter (2007), Parker (2001). The LG process offers a way to model long-term risk, while keeping a closed form for stock prices. In addition, there is debate about the existence and mechanism of the time-varying expected stock market returns, e.g., Campbell and Shiller (1988), Cochrane (2006), Goyal and Welch (2005) and many others. Because of the lack of closed forms, the literature relies on simulations and approximations. The LG process offers closed forms for stocks with time-varying equity premium, which is useful for thinking about those issues.

The motivation for the LG class is inspired by the broad applicability and empirical success of the affine class identified by Duffie and Kan (1996), and further developed by Dai and Singleton (2000) and Duffie, Pan and Singleton (2000), which includes the Vasicek (1978) and the Cox, Ingersoll, Ross (1985) process as special cases. Much theoretical and empirical work is done with the affine class. Some of this could be done with the LG class. Section 4.3 develops the link between the LG class and the affine class. The two classes give the same quantitative answers to a first order. The main advantage of the LG class is for stocks. The LG class gives a simple closed-form expressions for stocks, whereas the affine class needs to express stocks as an infinite sum. Hence, while the affine class can be expected to be remain for long the central model for options and bonds, one can think that the LG class with be most useful for stocks.

Closed forms for stocks, or perpetuities, are not available with the current popular processes, such as affine models those of Ornstein-Uhlenbeck / Vasicek (1977), Cox, Ingersoll, Ross (1985), or models in the affine class (Duffie and Kan 1996). Several papers have derived closed forms for stocks. Bhattacharya (1978) and, in another form, Menzly, Santos and Veronesi (2004), derive a closed form for asset prices, and their process turns out to be to belong to the LG class (see Example 11).[1] Bakshi and Chen (1996) derive a closed form, which is an exponential-affine function of a square root process. Mamayski (2002) derives another closed form, though in a non-stationary setting. Cochrane, Longstaff and Pedro Santa (2006) contains nice closed form solutions. Finally, we confirm results from Mele (2003, 2006), who obtains general results (particularly with one factor) for having bond and stock prices that are convex, concave, or linear in the factors. LG processes satisfy Mele's conditions for linearity. Mele, however, did not derive the closed forms for stocks and bonds in the linear case.

Finally, we contribute to the vast literature on interest rate processes, by presenting a new, flexible process. The main advantage is probably that, because the LG processes are so easy to analyze, they lend themselves easily to economic analysis. Gabaix (2007) develops a unified model of stocks and bonds, and many financial puzzles, using the LG class.

This paper stipulates ("reverse-engineers") a process for finding desirable properties for the pricing kernel. In this it follows a productive literature represented by, e.g., Abel (2007), Campbell and Cochrane (1999), Cox, Ingersoll, Ross (1985), Pastor and Veronesi (2005), Ross (1978), Sims (1990), and, particularly, Menzly, Santos and Veronesi (2004).

Section 2 is a gentle introduction to LG processes, with some simple examples of LG processes. Section 3 presents the discrete-time version of the process. Section 4 presents the continuous-time

[1] It is indeed the Menzly, Santos and Veronesi (2004) paper that alterted me to the possibility of a class with closed forms for stocks. On the economic side, this article originates from a lunch with Robert Barro, who was expressing the desirability of a model with stochastic probability of disaster. That conversation made me search for tractable ways to address this question, and led me to LG processes.

version of the LG process. Section 5 shows some extensions, one to option pricing, one to time-dependent coefficients. Section 6, which is more technical, studies the range of admissible initial conditions. Section 7 concludes.

2 A simple introduction to linearity-generating processes

This section presents some examples, from very simple to slightly more complex, that give a flavor for LG processes.

2.1 An elementary example: Generalized Gordon formula in discrete time

We start with a very simple, almost trivial example – the Gordon formula in discrete time.[2] We want to calculate the price:

$$P_t = E_t \left[\sum_{s=0}^{\infty} \frac{D_{t+s}}{(1+r)^s} \right]$$

of a stock with dividend growth:

$$\frac{D_{t+1}}{D_t} = 1 + g_t \tag{1}$$

g_t is the trend growth rate of the stock, and we want it to be autocorrelated (the i.i.d. case is trivial). This is a prototypical example of stock with stochastic trend growth. As the next example will perhaps make clearer, even in this example, simple processes for g_t typically give intractable expressions.

Let us reverse engineer the process for g_t, and see if we can find a way to obtain a linear ("affine" to be more formal) expression for the price-dividend ratio, i.e. if the P/D ratio can have the form:

$$\frac{P_t}{D_t} = A + Bg_t \tag{2}$$

for some A, B. The arbitrage equation for the stock is

$$P_t = D_t + \frac{1}{1+r} E_t \left[P_{t+1} \right] \tag{3}$$

i.e.

$$\frac{P_t}{D_t} = 1 + \frac{1}{1+r} E_t \left[\frac{D_{t+1}}{D_t} \frac{P_{t+1}}{D_{t+1}} \right]$$

[2] This example is so simple that it would not be surprising if it had already been done elsewhere, even though I did not find it in the previous literature. However, it is clear that LG processes (including the general structure with several factors, stocks bonds and continuous time) as an identified class presented in the present paper first.

Plugging in (1) and (2), the arbitrage equation reads:

$$A + Bg_t = 1 + \frac{1}{1+r} E_t \left[(1+g_t)(A + Bg_{t+1}) \right]$$

i.e.

$$A + Bg_t = 1 + \frac{A}{1+r}(1+g_t) + \frac{B}{1+r}(1+g_t) E_t[g_{t+1}] \tag{4}$$

If g_t is an AR(1), i.e. $E_t[g_{t+1}] = \rho g_t$, then (4) cannot hold: we have linear terms on the left-hand side, and non-linear terms on the right-hand side.

However, (4) can hold if we postulate that g_t follows the following "twisted" AR(1):

$$\text{Linearity-generating twist: } E_t[g_{t+1}] = \frac{\rho g_t}{1+g_t} \tag{5}$$

If g_t is close to 0, then to a first order, $E_t[g_{t+1}] \sim \rho g_t$, so that g_{t+1} behaves approximately like an AR(1). It's a twisted AR(1), because of the term $1+g_t$ in the denominator. However, in many applications, g_t will be say within a few percentage points from 0, so materially, the twist is small (more on this later).

If (5) holds, then (4) reads:

$$A + Bg_t = 1 + \frac{A}{1+r}(1+g_t) + \frac{B}{1+r} \rho g_t$$

which features only linear terms, and admits a solution. Indeed, we obtain $A = 1 + A/(1+r)$, i.e. $A = (1+r)/r$, and $B = A/(1+r) + B\rho/(1+r)$, i.e. $B = A/(1+r-\rho)$. Finally, plugging those values of A and B back in (2) gives:

$$\frac{P_t}{D_t} = \frac{1+r}{r}\left(1 + \frac{g_t}{1+r-\rho}\right) \tag{6}$$

Conclusion: (6) is the solution of (3), and by the usual arguments, the price-dividend ratio is given indeed by (6).

Example 1 *(Simple stock example with LG stochastic trend growth rate) Consider a stock with dividend growth rate g_t, with $D_{t+1}/D_t = 1+g_t$, and the linearity-generating "twist" for the growth rate:*

$$E_t[g_{t+1}] = \frac{\rho g_t}{1+g_t} \tag{7}$$

with price $P_t = E_t\left[\sum_{s=0}^{\infty} D_{t+s}/(1+r)^s\right]$. Suppose that, with probability 1, $\forall t, g_t > -1$. Then, the *price-dividend ratio, P_t/D_t is:*

$$\frac{P_t}{D_t} = \frac{1+r}{r}\left(1 + \frac{g_t}{1+r-\rho}\right). \tag{8}$$

Also:

$$E_t\left[D_{t+T}\right] = \left(1 + \frac{1-\rho^T}{1-\rho} g_t\right) D_t \tag{9}$$

In other terms, we can price finite maturity claims – "bonds". The rest of the paper develops this systematically.

A few remarks are called for. Eq. 7 imposes just one moment conditions. Higher order moments do not matter for the price. For instance, we could have a complicated nonlinear function for the variance of the growth rate, it would not affect the stock price.

For $g_t > -1$ to be possible for all t's, we need restrictions. Stability analysis of the process (and further analysis developed later in the paper) gives $g_t > \rho - 1$. In particular, the variance has to go to 0 near that boundary. [3]

We next turn to the continuous time version of the above process, before then turning to richer examples.

2.2 The generalized Gordon formula in continuous time

We extend the discrete-time process above to continuous time. Consider a stock with dividend $D_t = D_0 \exp\left(\int_0^t g_t du\right)$. g_t is the (stochastic) growth rate, and can be decomposed $g_t = g_* + \gamma_t$, where the constant g_* is a trend growth rate, and γ_t a fluctuation around the trend. The discount rate is r, and the value of a stock at time t is, assuming $R = r - g_* > 0$,

$$P_t = E_t\left[\int_t^\infty \exp\left(-r\left(s-t\right)\right) D_s ds\right] = E_t\left[\int_t^\infty \exp\left(-\int_t^s \left(r - g_* - \gamma_u\right) du\right) ds\right] D_t$$

so that the price-dividend ratio is:

$$P_t/D_t = E_t\left[\int_t^\infty \exp\left(-R\left(s-t\right) + \int_t^s \gamma_u du\right) ds\right]. \tag{10}$$

This paper proposes a process for γ_t that yields a closed-form for (10). Before doing this, it is useful to examine the most natural process, which is to take γ_t to be an Ornstein-Uhlenbeck,

[3] The reason is that the function $g \mapsto \rho g/\left(1+g\right)$ has two fixed points, 0 and $\rho - 1$, and the process needs to stay on the right side of the repelling fixed point, $\rho - 1$.

$d\gamma_t = -\phi\gamma_t dt + \sigma dB_t$. Calculating (10) yields:

$$P_t/D_t = \int_0^\infty \exp\left[-RT + \frac{1-e^{-\phi T}}{\phi}\gamma_t + \frac{\sigma^2}{2\phi^3}\left(\phi T + 2e^{-\phi T} - \frac{e^{-2\phi T}+3}{2}\right)\right] dT \qquad (11)$$

which is complicated and has no known closed-form expression. Likewise, a Cox, Ingersoll Ross (1985) process does not yield a closed form for the stock price.

However, a slight modification of the growth process makes prices completely tractable. Consider, the continuous-time version of the discrete-time process (5).[4]

Example 2 *(Generalized Gordon growth formula with LG stochastic trend growth rate) Consider a stock with dividend growth rate $g_t = g_* + \gamma_t$, with*

$$d\gamma_t = -\left(\phi\gamma_t + \gamma_t^2\right) dt + \sigma\left(\gamma_t\right) dB_t, \qquad (12)$$

where $\sigma(\gamma_t)$ is an essentially arbitrary function, but γ_t^2 must be -1. Consider the price $P_t = E_t\left[\int_t^\infty \exp(-rt) D_s ds\right]$. If the process is defined in $[t,\infty)$, the price-dividend ratio, P_t/D_t is:

$$P_t/D_t = \frac{1}{r-g_*}\left(1 + \frac{\gamma_t}{r-g_*+\phi}\right). \qquad (13)$$

The above example exhibits general traits of LG processes.[5]

As in (13), the price of assets are linear (affine) in the state variable – here, γ_t, which motivates the name "linearity-generating" process for (12).

Surprisingly perhaps, the volatility term $\sigma(\gamma_t)$ does not appear in the final expression of the stock price: $\sigma(\gamma_t)$ can by multiplied by any number without changing the stock price. This gives much modelling flexibility.

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[4] The limit comes from the following heuristic reasoning. Set $g_t = \gamma_t \Delta t$, where Δt will be small, and $\rho = 1 - \phi\Delta t$. Eq. (5) becomes:

$$E_t\left[g_{t+\Delta t}\right] - g_t = \frac{\rho g_t}{1+g_t} - g_t = \frac{(\rho-1)\, g_t - g_t^2}{1+g_t}$$

and dividing through by Δt,

$$E_t\left[\gamma_{t+\Delta t}\right] - \gamma_t = \frac{-\phi\gamma_t - \gamma_t^2}{1+\gamma_t\Delta t}\Delta t$$

so, taking the limit $\Delta t \to 0$, $E_t\left[d\gamma_t\right] = \left(-\phi\gamma_t - \gamma_t^2\right) dt$.

[5] The result in Example 2 appear new to the literature. The Fisher-Wright process (e.g., Karlin and Taylor 1982) does contain a quadratic term, but it has not been applied to the pricing bonds or stocks. Also, it is more special than the LG class, because it imposes a specific functional form on the variance. Driessen, Maenhout and Vilkov (2005) and Cochrane, Longstaff, and Santa-Clara (2006) apply the Fisher-Wright process. Other papers introduce different quadratic terms in stochastic process, for instance Ahn et al. (2002), Constantidines (1992), Lonstaff (1989), but they do not take the form of this paper.

There are drawbacks to having a stock price linear in the growth rate, and independent of volatility, as in some models (Johnson 2002; Pastor and Veronesi 2003) the link between volatility and stock price is important. Nonetheless, in many economic situations, this link is more an annoying side effect. Arguably, in many situations where LG models can be used, the gain in tractability in seeing the volatility terms drop out outweighs the cost. In any case, if when one thinks that the volatility effects are important, LG processes can be modified to incorporate them – see Example 12 below.

We need an extra term in the drift process, here $-\gamma_t^2 dt$, to get the LG properties. In many applications, the term is likely to be small quantitatively. For instance, if we think that the deviation from the mean ($|\gamma_t|$) is less than 5% per year typically (which is plausible for the predictable deviation from the trend growth rate, or the trend interest rate), then the extra drift term is less than $(5\%)^2 = 0.25\%$ per year. Hence, often, the extra drift term will not materially change the importance quantitative properties of the process. However, it confers a great tractability to asset prices.

Some care must be taken to make the process defined in $[t, \infty)$. This will be developed later in the paper, and is illustrated in Figure 1. In the context at hand, a sufficient condition is that $\sigma(\gamma)$ vanishes in a right neighborhood of $\gamma = -\phi$, and that the initial value of γ_t is above $-\phi$. This is analogous to the fact that the volatility must go to 0 as the interest rate goes to 0 in the Cox, Ingersoll and Ross (1985) process.

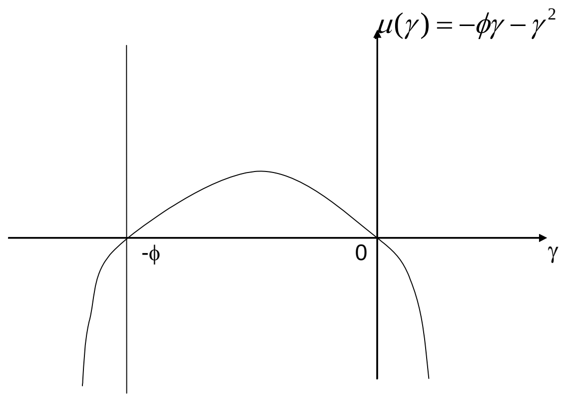

Figure 1: Illustration of the drift $\mu(\gamma) = -\phi\gamma - \gamma^2$ of the growth rate. If $\gamma > -\phi$, the process is stable, i.e. mean reverts to 0. However, if $\gamma < -\phi$, the process is unstable, and diverges away from 0. That is why we impose $\gamma_0 > -\phi$. To make sure that the process remains in $(-\phi, \infty)$, we impose that the volatility goes to 0 fast enough before at some $\underline{\gamma} \geq -\phi$. See Appendix A for details, and Section 6 for the generalization to several factors.

The economic interpretation of (13) is the following. When the deviation of the growth rate from its trend ($\gamma_t = g_t - g_*$) is 0, then $P_t/D_t = 1/(r - g_*)$, which is the traditional Gordon formula. When the growth rate is above trend ($\gamma_t > 0$), the P/D ratio is higher, as future dividends have superior growth. This initial superior growth γ_t decays at rate ϕ, and is discounted at rate $r - g_*$, so that its total duration is $1/(r - g_* + \phi)$. So the cumulative impact of the superior growth is the $\gamma_t/(r - g_* + \phi)$.

Let us now see why the price is a linear function of the initial growth rate.

A heuristic proof The proof of the result will be made fully rigorous in the rest of the paper, but a simple "plug and verify" derivation is instructive. Call the price-dividend ratio $V_t = P_t/D_t = E_t \int_t^\infty \exp\left(-\int_t^s (R - \gamma_u)\, du\right) ds$, with $R = r - g_*$. It is analogue to the price of a bond that gives 1 in every second, with an instantaneous interest rate of $R - \gamma_u$. Hence, the arbitrage equation for V_t is:

$$0 = 1 - (R - \gamma) V_t + E_t [dV_t]/dt.$$

As γ_t is the only state variable as far as V_t is concerned, we seek a solution of the form $V_t = V(\gamma_t)$. Call the drift of γ, where the drift of γ is

$$\mu(\gamma) = -\phi\gamma - \gamma^2$$

Ito's lemma gives: $E_t [dV_t]/dt = \mu(\gamma) V'(\gamma) + \frac{\sigma^2(\gamma)}{2} V''(\gamma)$, and the arbitrage equation is the classic equation:

$$0 = 1 - (R - \gamma) V(\gamma) + \mu(\gamma) V'(\gamma) + \frac{\sigma^2(\gamma)}{2} V''(\gamma) \tag{14}$$

We look for a solution affine in γ: $V(\gamma) = A + B\gamma$. The functional form implies $V''(\gamma) = 0$, so that, if the solution is correct, the $\sigma^2(\gamma)$ term will not matter. This explains why there are no σ terms in the final expression (13).

Substituting $V(\gamma) = A + B\gamma$ into (14), yields:

$$\begin{aligned} 0 &= 1 - (R - \gamma)(A + B\gamma) + \left(-\phi\gamma - \gamma^2\right) B + \frac{\sigma^2(\gamma)}{2} \cdot 0 \\ &= 1 - RA + \gamma(A - RB - \phi B) + \gamma^2 (B - B) \end{aligned} \tag{15}$$

The key simplification is that the terms in γ^2 cancel out – this is where the LG term γ_t^2 matters. To solve the last equation, we just set to 0 the constant and the γ term, which gives $A = 1/R$, and $B = A/(R+\phi)$, which gives.

$$V(\gamma) = \frac{1}{R}\left(1 + \frac{\gamma}{R+\phi}\right)$$

which is the announced result, as with $R = r - g_*$.

If the term γ^2 had been absent of the drift (as in an Ornstein-Uhlenbeck process), or been present with a coefficient different from -1, the cancellation of the γ^2 in (15) would not have occurred. $\square$

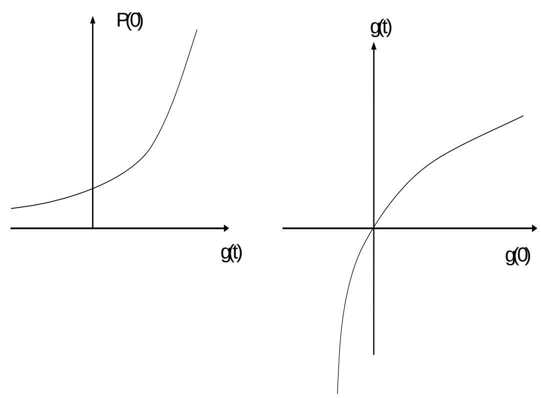

Figure 2: Why the price can be linear in the factor g_0. The price P_0, a sum of $\exp\left(\int_0^T g_t dt\right)$, is a convex function of future growth rates g_t. But, for instance in the deterministic version of the process, future growth rates are a *concave* function of the initial growth rate, $E_0[g_t \mid g_0]$ is concave in g_0. Hence the price is a composition of a convex function, composed with a concave function the initial growth rate. Hence, its concavity is undertermined. For the LG process, the price P_0 is precisely a linear function of the initial growth rate g_0.

Figure 2 illustrative an intuitive reason why the price can be linear in the initial growth rate g_0. The price, a sum of $\exp\left(\int_0^T g_t dt\right)$, is a convex function of future growth rates g_t . But, for instance in the deterministic version of the process, future growth rates are a *concave* function of the initial growth rate, $E_0[g_t \mid g_0]$ is concave in g_0.[6] Hence the price is a composition of a convex function (namely, $\exp\left(\int_0^T g_t dt\right)$), with a concave function, (namely, $g_t(g_0)$) the initial growth rate. Hence, its concavity is indeterminate. For the LG process, the price is precisely a linear function of the initial growth rate.[7]

[6] If the process is deterministic, then $\gamma_t = e^{-\phi t}\gamma_0/\left(1 + \gamma_0\left(1 - e^{-\phi t}\right)/\phi\right)$, a concave function. This can be shown directly, or by Proposition 1.

[7] However, Example 12 shows how to get convexity effect with the LG process. Mele (2003, and forth.) clarifies

The next example shows an example with several factors.

2.3 A richer example: A price-dividend ratio with time-varying growth rate and risk-premium

LG processes generalize to several factors. Suppose that the stochastic discount factor M_t and the dividend process D_t follow

$$\begin{aligned} dM_t/M_t &= -rdt - \frac{\pi_t}{\sigma} dz_t \\ dD_t/D_t &= g_t dt + \sigma dz_t \end{aligned}$$

The price of the stock is $P_t = E_t \left[\int_t^\infty M_s D_s ds \right] / M_t$. π_t is a the stochastic equity premium, and g_t is the stochastic growth rate of dividends.

We assume that π_t and g_t follow the following LG process, best expressed in terms of their deviation from trend, $\widehat{\pi}_t = \pi_t - \pi_*, \widehat{g}_t = g_t - g_*$,

$$\begin{aligned} d\widehat{g}_t &= -\phi_g \widehat{g}_t dt + \widehat{g}_t \left(\widehat{\pi}_t - \widehat{g}_t \right) dt + \sigma_\gamma \left(\widehat{g}_t, \widehat{\pi}_t \right) \cdot dB_t \\ d\widehat{\pi}_t &= -\phi_\pi \widehat{\pi}_t dt + \widehat{\pi}_t \left(\widehat{\pi}_t - \widehat{g}_t \right) dt + \sigma_\pi \left(\widehat{g}_t, \widehat{\pi}_t \right) \cdot dW_t \end{aligned}$$

where the (B_t, W_t) is a Wiener process independent of z_t, that can have arbitrary time- or state-dependent correlations. We suppose that the process is defined in $[t, \infty)$. Again the processes $d\widehat{g}_t$ and $d\widehat{\pi}_t$ are to a first order linear, but with quadratic "twist" terms added, $\widehat{g}_t \left(\widehat{\pi}_t - \widehat{g}_t \right) dt$ and $\widehat{\pi}_t \left(\widehat{\pi}_t - \widehat{g}_t \right) dt$ respectively.

Under the above assumptions, it is standard that $P_t/D_t = E_t \left[\int_t^\infty \exp \left(- \int_t^s \left(r + \pi_u - g_u \right) du \right) ds \right]$. The LG terms imply the following Proposition.

Example 3 *(Generalized Gordon formula, with stochastic trend in dividend growth, and stochastic equity premium) In the above setup, the stock price is*

$$P_t = \frac{D_t}{R} \left(1 + \frac{g_t - g_*}{R + \phi_g} - \frac{\pi_t - \pi_*}{R + \phi_\pi} \right). \tag{16}$$

with

$$R \equiv r + \pi_* - g_*$$

how prices can be concave or convex as a function of state variables.

In this expression the price-dividend ratio varies because of a stochastic equity premium (π_t), and a stochastic dividend growth rate (g_t).

It is a good and simple exercise to derive the above formula directly, from the arbitrage equation $1-(r+\pi_t-g_t)(P/D)_t+E[d(P/D)_t]/dt=0$. Otherwise, formula (16) comes from Theorem 4 below.

Equation 16 nests the three main sources of variations of stock prices in a simple and natural way. Stock prices can increase because the level of dividends increases (that's the D_t terms), because the expected future growth rate of dividend increases (the g_t-g_* term), or because the equity premium decreases (the $\pi_t-\pi_*$ terms). The two growth or discount factors (g_t and π_t) enter linearly, weighted by their duration (e.g., $1/(R+\phi_\pi)$), which depends of the speed of mean-reversion of the each process (parametrized by ϕ_π,ϕ_g), and the effective discount rate, R. As in the previous example, the volatility terms do not enter in (16), and the price does not change if one changes the correlation between the instantaneous innovation in g_t and π_t.

We now start our systematic treatment of LG processes.

3 Linearity-generating processes in discrete time

This section studies the discrete-time version of the LG process. As several factors are needed to capture the dynamics of stocks (Campbell and Shiller 1988, Fama and French 1996) and bonds (Litterman and Scheinkman 1991), we study it in the multifactor case. We want to price an asset with dividend D_t, given a discount factor M_t. The price at time t of a claim yielding a stochastic dividend D_s at date $S\geq t$ is:[8]

$$P_t=E\left[\sum_{T=0}^{\infty}M_{t+T}D_{t+T}\right]/M_t. \tag{17}$$

For instance, the price of a zero coupon bond of maturity T is, with $D_t=1$,

$$Z_t(T)=E_t[M_{t+T}D_{t+T}]/(M_tD_t). \tag{18}$$

[8] Some readers may not be familiar with the stochastic discount factor. The simplest example is $M_t=(1+r)^{-t}$, if the interest rate is constant. If the interest rate r_s is deterministic but not constant, $M_t=\prod_{s=1}^{t}(1+r_s)^{-1}$. If, in Lucas economy, a representative consumer with utility $\sum_t\delta^tU(C_t)$ prices assets, then $M_t=\delta^tU'(C_t)$. With the external habit of Cochrane and Cochrane (1999), one can define a habit level H_t such as $M_t=\delta^tU'(C_t-H_t)$. Absence of arbitrage guaranties that the price is a linear functional of future dividends, and under weak technical conditions this leads to the existence of factors M_{t+T} such that (17) holds.

We will also calculate the price-dividend of a stock:

$$P_t/D_t = E_t\left[\sum_{T=0}^{\infty}\frac{M_{t+T}D_{t+T}}{M_tD_t}\right] = \sum_{T=0}^{\infty} Z_t(T) \tag{19}$$

3.1 Definition and main properties

The state vector is $X_t \in \mathbb{R}^n$ ($n \in \mathbb{N}$)and can be generally thought of as stationary, while M_tD_t generally trends, and is not stationary. The definition of the LG process is the following.

Definition 1 *The process $M_tD_t(1, X_t)'_{t=0,1,2,...}$, with $M_tD_t \in \mathbb{R}$ and $X_t \in \mathbb{R}^n$, is a LG process if there are constants $\alpha \in \mathbb{R}, \gamma, \delta \in \mathbb{R}^n, \Gamma \in \mathbb{R}^{n^2}$, such that the following relations hold date $t = 0, 1, 2, ...$:*

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}\right] = \alpha + \delta' X_t \tag{20}$$

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}X_{t+1}\right] = \gamma + \Gamma X_t \tag{21}$$

The above conditions mean that the expected value of the (dividend augmented) stochastic discount factor is linear in the factors. As the examples below show, it is not difficult to write toy economic models satisfying conditions (20)-(21), e.g. in Lucas (1978) - Abel (1990) - Campbell Cochrane (1995) economies with exogenous consumption, dividend or marginal utility processes. Gabaix (2007) presents a fully worked-out economic model satisfying the conditions of Definition 1.

Also, models that to not directly fit into the conditions of Definition 1, could be approximated by projected linearly in (20)-(21). Also, by extending the state vector, equations (20)-(21) could hold to an arbitrary degree of precision. Appendix C illustrates how to approximate a non-LG process with an LG process, including to an arbitrary degree of precision.

One interpretation of (20)-(21) is that the specify the dynamics of the factors under the "risk-neutral measure" induced by M_tD_t.

There is a more compact way to summarize LG processes. Define the $(n+1)\times(n+1)$ matrix:

$$\Omega = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix} \tag{22}$$

and the process with values in $\mathbb{R}^{n+1}$

$$Y_t := \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} = \begin{pmatrix} M_t D_t \\ M_t D_t X_t^1 \\ ... \\ M_t D_t X_t^n \end{pmatrix}$$

so that with vector $\nu' = (1, 0, ..., 0)$,

$$M_t = \nu' Y_t \tag{23}$$

Y_t stacks all the information relevant to the prices of the claims derived below. [9] Conditions (20)-(21) can be written:

$$E_t [Y_{t+1}] = \Omega Y_t. \tag{24}$$

Hence, the (dividend-augmented) stochastic discount factor of a LG process is simply the projection (Eq. 23) of an autoregressive process, Y_t.

The basic pricing properties are the following.

Theorem 1 *(Bond prices, discrete Time) The price-dividend (18) of a zero-coupon equity or bond of maturity T is, with I_n the identity matrix of dimension n*

$$Z_t(T) = \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}^T \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{25}$$

When $\gamma = 0$, it can be expressed:

$$Z_t(T) = \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \tag{26}$$

Proof. The proof is very easy. Recall (24), $E_t [Y_{t+1}] = \Omega Y_t$. Iterating on T, it implies that for all $T \geq 0$,

$$E_t [Y_{t+T}] = \Omega^T Y_t \tag{27}$$

[9] Other assets, e.g. options, require of course to know more moments.

Hence, using the definition of the zero-coupon (18), and (23)

$$\begin{aligned} Z_t(T) &= (M_t D_t)^{-1} E_t[M_{t+T} D_{t+T}] = (M_t D_t)^{-1} E_t\left[\nu' Y_{t+T}\right] = (M_t D_t)^{-1} \nu' E_t[Y_{t+T}] \\ &= (M_t D_t)^{-1} \nu' \Omega^T Y_t = \nu' \Omega^T \left((M_t D_t)^{-1} Y_t \right) \\ &= \nu' \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & 0_n \end{pmatrix} \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} \end{aligned}$$

i.e. Eq. 25. The formula for $\gamma = 0$ comes from Lemma 3 in Appendix B. ■

For instance, when $D_t \equiv 1$, the above Theorem can price bonds, with n factors, in closed form.

In many applications (e.g., the examples in this paper), $\gamma = 0$, which means the state variables are re-centered around 0. For instance, the state variable is the deviation of the equity premium from its trend value.

The second main result is the most useful property of LG processes: the existence of a closed-form formula for stock prices.

Theorem 2 *(Stock prices, discrete time) Suppose that the process is defined from t on, and that all eigenvalues of Ω have a modulus less than 1 (finiteness of the price). Then, the price-dividend ratio of the stock (19) is:*

$$\begin{aligned} P_t/D_t &= \frac{1}{1-\alpha-\delta' (I_n - \Gamma)^{-1}\gamma} \left(1 + \delta' (I_n - \Gamma)^{-1} X_t\right) && (28) \\ &= \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \left(I_{n+1} - \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix} \right)^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix}. && (29) \end{aligned}$$

Proof. We use (25), which gives the perpetuity price:

$$P_t/D_t = \sum_{T=0}^{\infty} Z_t(T) = \nu' \left(\sum_{T=0}^{\infty} \Omega^T \right) \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \nu' (I_n - \Omega)^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$

$\sum_{T=0}^{\infty} \Omega^T$ is summable because all eigenvalues of Ω have a modulus less than 1. We use Lemma 2 to calculate $(I_n - \Omega)^{-1}$, and conclude.[10] ■

[10] There is a more elementary heuristic proof. We seek a solution of the type $P_t/D_t \equiv V_t = c - 1 + h' X_t$, which we know exists, by integration of (53). The arbitrage equation is: $V_t = 1 + E\left[\frac{M_{t+1}D_{t+1}}{M_t D_t} V_{t+1}\right]$, i.e.

$$c + h' X_t = 1 + E\left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \left(c + h' X_t\right)\right] = 1 + c\left(\alpha + \delta' X_t\right) + h' \left(\gamma + \Gamma Y_t\right) = \left[1 + c\alpha + h'\gamma\right] + \left[c\delta' + h'\Gamma\right] X_t$$

Theorem 2 allows to generate stock prices with an arbitrary number of factors, including time-varying growth rate, and risk premia.

To make formulas concrete, consider the case where Γ is a diagonal matrix: $\Gamma = \begin{pmatrix} \Gamma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Gamma_n \end{pmatrix} \equiv$ $Diag(\Gamma_1, ..., \Gamma_n)$. Then, $\frac{\alpha^T I_{n+1} - \Gamma^T}{\alpha I_{n+1} - \Gamma} = Diag\left(\left(\alpha^T - \Gamma_i^T\right) / (\alpha - \Gamma_i)\right)$,[11] so that (26) and (28) read:

$$Z_t(T) = \alpha^T + \sum_{i=1}^{n} \frac{\alpha^T - \Gamma_i^T}{\alpha - \Gamma_i} \delta_i X_t^i \qquad (30)$$

$$P_t / D_t = \frac{1 + \sum_{i=1}^{n} \frac{\delta_i X_i}{1 - \Gamma_i}}{1 - \alpha - \sum_{i=1}^{n} \frac{\delta_i \gamma_i}{1 - \Gamma_i}} \qquad (31)$$

3.2 Some examples

Example 4 *A Gordon growth formula with time-varying dividend growth.*

In this example, we generalize our introductory stock example. Suppose that the interest rate is constant at r, dividend D_t, and the growth rate of dividend is:

$$\frac{D_{t+1}}{D_t} = (1 + g_*)(1 + x_t)\left(1 + \eta_{t+1}\right) \qquad (32)$$

$$E_t[x_{t+1}] = \frac{\rho x_t}{1 + x_t} \qquad (33)$$

where η_t is some unimportant i.i.d. noise, greater than -1, independent of the innovation to x_{t+1}. x_t is the deviation from the trend growth rate. If x_t was an AR(1), it would follow $E_t[x_{t+1}] = \rho x_t$. Instead, the process is slightly modified, to (33), to make the process LG. Indeed, with $M_t = (1 + r)^{-t}$, and using the notation $1 + R = (1 + r) / (1 + g_*)$, we have:

$$E_t\left[\frac{M_{t+1} D_{t+1}}{M_t D_t}\right] = (1 + x_t) / (1 + R)$$

$$E_t\left[\frac{M_{t+1} D_{t+1}}{M_t D_t} x_{t+1}\right] = E_t\left[\frac{M_{t+1} D_{t+1}}{M_t D_t}\right] E_t[x_{t+1}] = \frac{(1 + x_t)}{1 + R} \frac{\rho x_t}{1 + x_t} = \frac{\rho x_t}{1 + R}$$

In the above equation, the $1 + x_t$ terms cancel out, because of the $1 + x_t$ term in the denominator of (33). We designed the process so that the LG equation (21) holds.

i.e. (i) $c = 1 + c\alpha + h'\gamma$ and (ii) $h' = c\delta' + h'\Gamma$. (ii) gives $h' = c\delta'(1 - \Gamma)^{-1}$, and plugging in (i) yields $c\left[1 - \alpha - \delta'(1 - \Gamma)^{-1}\gamma\right] = 1$, hence c and the announced result.

[11] If A matrix, and $f : \mathbb{R} \to \mathbb{R}$, is analytic with $f(x) = \sum_{n=0}^{\infty} f_n x^n$ then $f(A) = \sum_{n=0}^{\infty} f_n A^n$. If $A = Diag(a_1, .., a_n)$, $f(A) = Diag(f(a_1), ..., f(a_n))$

We have a LG process, for $M_t D_t (1, x_t)$, with:

$$\Omega = \begin{pmatrix} 1/(1+R) & 1/(1+R) \\ 0 & \rho/(1+R) \end{pmatrix} =: \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$$

Hence, we apply Theorem 2, with a dimension $n = 1$, $\gamma = 0$, $\delta = \Gamma = \alpha\rho$. We obtain, for the price, $P_t = E_t \sum_{s=0}^{\infty} D_{t+s}/(1+r)^s$, $P_t/D_t = \frac{1}{1-\alpha-\delta'(I_n-\Gamma)^{-1}\gamma}\left(1 + \delta'(I_n - \Gamma)^{-1} X_t\right)$, i.e.

$$P_t/D_t = \frac{1+R}{R}\left(1 + \frac{1}{1+R-\rho} x_t\right) \tag{34}$$

Formula (34) is the discrete-time analogue of (13), with very small r and g_*, and the substitutions $\rho = 1 - \phi$, for a small ϕ. The upshot of this example is that, in discrete time, LG processes take the form (33).

Example 5 *Flexible LG parametrization of state variables the stochastic discount factor*

Take an $n-$dimensional process X_t, such that

$$\frac{M_{t+1}D_{t+1}}{M_t D_t} = \alpha + \beta' X_t + \varepsilon_{t+1} \tag{35}$$

$$X_{t+1} = \frac{\gamma + \Gamma X_t}{\alpha + \beta' X_t} + \eta_{t+1} - \frac{E_t\left[\varepsilon_{t+1}\eta_{t+1}\right]}{\alpha + \beta' X_t} \tag{36}$$

with $E_t[\varepsilon_{t+1}] = 0, E_t\left[\eta_{t+1}\right] = 0$, but no other restrictions are necessary. Then, Eq. 20-21 are satisfied.

The above equations give the LG counterpart of the popular "affine" parametrization, $\frac{M_{t+1}D_{t+1}}{M_t D_t} = \exp(A + B' X_t)$, $X_{t+1} = \gamma + \Gamma X_t + u_{t+1}$, with u_{t+1} Gaussian. It is at least as flexible.

To interpret (36), consider the case $\gamma = E_t\left[\varepsilon_{t+1}\eta_{t+1}\right] = 0$. Eq. 36 expresses that, when X_t is small,

$$E_t[X_{t+1}] = \frac{\Gamma X_t}{\alpha + \beta' X_t} \sim \frac{\Gamma}{\alpha} x_t$$

which means that X_t follows approximately at AR(1). The corrective $1 + \beta'/\alpha \cdot X_t$ in the denominator is often small in practice, but ensures that the process is LG.

In many applications, there is no risk premium on the factor risk, so that $E_t\left[\varepsilon_{t+1}\eta_{t+1}\right] = 0$. However, when there is a risk-premium equation (36) means that it is enough to know that the process under the "risk-neutral" measure. Hence, in a first step, one can model the "risk-neutral" process for X_t, fit it to prices, and then later extract the risk-premium component, $cov\left(\frac{M_{t+1}D_{t+1}}{M_t D_t}, X_{t+1}\right)$.

Section 6 provides conditions to ensure $M_t > 0$ for all times.

Example 6 *A multifactor bond model with bond risk premia (in discrete time).*

There are n factors r_{it}. The stochastic discount factor is:

$$\frac{M_{t+1}}{M_t} = \frac{1}{1+r_*}\left(1 - \sum_{j=1}^{n} r_{jt}\right) + \varepsilon_{t+1} \tag{37}$$

where $E_t \varepsilon_{t+1} = 0$, and the process has to be defined for all t's, but otherwise ε_{t+1} is unspecified, and can be heteroskedastic. The short term rate is $r_t = 1/E_t\left[\frac{M_{t+1}}{M_t}\right] - 1 \simeq r_* + \sum r_{it}$ if the r's are small. Each factor r_{it} is postulated to evolve as:

$$r_{i,t+1} = \frac{\rho_i r_{i,t}}{1 - \sum r_{jt}} + \eta_{i,t+1} - \frac{E_t\left[\varepsilon_{t+1} \cdot \eta_{i,t+1}\right]}{E_t\left[M_{t+1}/M_t\right]} \tag{38}$$

where $E_t \eta_{i,t+1} = 0$, but the $\eta_{i,t+1}$ can otherwise have any correlation structure.

This is a LG process. Indeed, the last equation implies:

$$E_t\left[\frac{M_{t+1}}{M_t} r_{i,t+1}\right] = \frac{1}{1+r_*}\rho_i r_{i,t}$$

So the Ω matrix is for the process $M_t\left(1, r_{1,t}, ..., r_{n,t}\right)$ is:

$$\Omega = \frac{1}{1+r_*}\begin{pmatrix} 1 & -1 & \cdots & -1 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \rho_n \end{pmatrix}$$

so that by (22) and (26), the price of the bond of maturity T is:

$$Z_t\left(T\right) = \frac{1}{\left(1+r_*\right)^T}\left(1 - \sum_{i=1}^{n} \frac{1-\rho_i^T}{1-\rho_i} r_{it}\right) \tag{39}$$

This expression is quite simple, and accommodates a wide variety of specifications for the factors, Eq. 38.

The risk premium on the T maturity bond is:

$$\text{Risk premium} = \frac{cov\left(\varepsilon_{t+1}, Z_{t+1}\left(T-1\right)\right)}{Z_t\left(T\right)} = \frac{\sum \frac{1-\rho_i^{T-1}}{1-\rho_i} cov\left(\varepsilon_{t+1}, \eta_{i,t+1}\right)}{1 - \sum \frac{1-\rho_i^T}{1-\rho_i} r_{it}}\left(1+r_*\right) \tag{40}$$

Hence we easily generate an explicit yield curve. With a parametrization for $cov\left(\varepsilon_{t+1}, \eta_{i,t+1}\right)$, the above expression makes prediction for bond risk premia across maturities. It would be interesting to compare them with evidence, e.g. from Campbell and Shiller (1991), Cochrane and Piazzesi (2005, 2006), Fama and Bliss (1987). The next example sketches such an example.

Example 7 *A bond model that is consistent with the empirical findings of Fama-Bliss (1987), Campbell Shiller (1991), and Cochrane Piazzesi (2006).*

We normalize the central interest rate to 0. We postulate:

$$\begin{aligned} \frac{M_{t+1}}{M_t} &= 1 - r_t + \varepsilon_{t+1} \\ r_{t+1} &= \frac{\rho_r}{1-r_t} r_t + \frac{\pi_t \varepsilon_{t+1}}{var_t\left(\varepsilon_{t+1}\right)} + v_{t+1} \\ \pi_{t+1} &= \frac{\rho_\pi}{1-r_t} \pi_t + \eta_{t+1} \end{aligned}$$

where ε, v, η have mean 0, and ε_t is uncorrelated with (v_s, η_s). This means that the short term rate, r_t, mean reverts, but shocks to it carry a risk-premium, π_t. The size of the risk premium is itself mean-reverting, at rate ρ_π. We have $E_t\left[\frac{M_{t+1}}{M_t}\right] = 1 - r_t$,$E_t\left[\frac{M_{t+1}}{M_t} r_{t+1}\right] = \rho_r r_t + \pi_t$, and $E_t\left[\frac{M_{t+1}}{M_t}\pi_{t+1}\right] = \rho_\pi \pi_t$.So $M_t\left(1, r_t, \pi_t\right)$ is a LG process with matrix $\Omega = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \rho_r & 1 \\ 0 & 0 & \rho_\pi \end{pmatrix}$. Hence by (25), the price of the bond of maturity T, is

$$Z_t\left(T\right) = 1 - \frac{1-\rho_r^T}{1-\rho_r} r_t + \frac{\frac{1-\rho_r^T}{1-\rho_r} - \frac{1-\rho_\pi^T}{1-\rho_\pi}}{\rho_r - \rho_\pi} \pi_t \tag{41}$$

The forward rate is $f\left(T\right) = \left(Z\left(T\right) - Z\left(T+1\right)\right)/Z\left(T\right)$, i.e.

$$f_t\left(T\right) = \frac{1}{Z_t\left(T\right)}\left[\rho_r^T r_t + \left(\frac{\rho_r^T - \rho_\pi^T}{\rho_r - \rho_\pi}\right)\pi_t\right] \tag{42}$$

The risk premium on the bond is: $\Pi_t\left(T+1\right) = \frac{cov\left(\varepsilon_{t+1}, Z_{t+1}(T)\right)}{Z_t(T)}$, i.e.

$$\Pi_t\left(T\right) = \frac{\frac{1-\rho_r^T}{1-\rho_r}\pi_t}{Z_t\left(T+1\right)} \tag{43}$$

Take the limit where the short rate is very persistent $\rho_r \simeq 1$, while the risk premium is less persistent (e.g. $\rho_\pi = 0.7$), e.g. moves at business cycle frequency (see Cochrane and Piazzesi

(2006) for evidence supportive of this benchmark). Then

$$\Pi_t(T) = \frac{T\pi_t}{Z_t(T+1)} \simeq T\pi_t \tag{44}$$

e.g. we get the Cochrane and Piazzesi (2005) evidence that the risk premium on bonds grows linearly with the bond maturity.

Gabaix (2007) develops this example further, with an economic microfoundation, and reaches the following conclusions. We can explain the Fama-Bliss evidence, that forward rates predict bond premia. That is, because $f_t(T)$ contains a π_t term. Likewise, the model generates the Campbell-Shiller facts on the movement of yields. Finally, why the "tent shape" of Cochrane and Piazzesi (2005)? Look at equation (42), in the limit $\rho_r \simeq 1$ (persistent short term rate), and $\rho_\pi < 1$. As a function of maturity, the ρ_r^T is roughly linear in T, while the $-\rho_\pi^T$ term is very concave in T. Hence, a concave tent-shape average of forward rate will capture the π_t term, and eliminate the π_t terms. This is why the tent-shaped factor of Cochrane and Piazzesi (2005) approximate risk premia: this linear combination of the forward rates purges r_t, and still loads on π_t. Hence the above simple LG model is broadly consistent with the empirical findings of Fama Bliss, Campbell Shiller, and Cochrane Piazzesi.

Example 8 *Stock price with stochastic growth rate and stochastic equity premium*

Consider a dividend process:

$$\begin{aligned} \frac{D_{t+1}}{D_t} &= 1 + g_t + \eta_{t+1} \\ \frac{M_{t+1}}{M_t} &= \frac{1}{1+r}\left(1 - \frac{\pi_t}{var_t(\eta_{t+1})}\eta_{t+1}\right) \end{aligned}$$

so that

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right] = \frac{1}{1+r}(1 + g_t - \pi_t)$$

Postulate the following processes for $\widehat{g}_t$ and $\widehat{\pi}_t$:

$$\begin{aligned} \widehat{g}_{t+1} &= \frac{1+g_* - \pi_*}{1+g_t - \pi_t}\rho_g \widehat{g}_t + \varepsilon_{t+1}^g \\ \widehat{\pi}_{t+1} &= \frac{1+g_* - \pi_*}{1+g_t - \pi_t}\rho_\pi \widehat{\pi}_t + \varepsilon_{t+1}^\pi \end{aligned}$$

where at time t ε_{t+1}^g and ε_{t+1}^π have expected values 0 and are uncorrelated with η_{t+1}.The term $\frac{(1+g_*-\pi_*)}{1+g_t-\pi_t}$ will be close to 1 in many applications. Defining: $\alpha = (1+g_*-\pi_*)/(1+r)$, the

Gordon discount factor, and $\widehat{\pi}_t = \pi_t - \pi_*, \widehat{g}_t = g_t - g_*$,

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right] = \alpha + \frac{\widehat{g}_t - \widehat{\pi}_t}{1+r}$$

and

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\widehat{g}_{t+1}\right] = E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right]E_t\left[\widehat{g}_{t+1}\right] = \frac{1}{1+r}\left(1+g_t-\pi_t\right)\cdot\frac{1+g_*-\pi_*}{1+g_t-\pi_t}\rho_g\widehat{g}_t = \alpha\rho_g\widehat{g}_t$$

The analogue expression holds for $\widehat{\pi}_t$. The process $Y_t = M_t D_t \left(1, \widehat{\pi}_t, \widehat{g}_t\right)'$ is LG, with Ω matrix:

$$\Omega = \begin{pmatrix} \alpha & 1/\left(1+r\right) & -1/\left(1+r\right) \\ 0 & \alpha\rho_g & 0 \\ 0 & 0 & \alpha\rho_\pi \end{pmatrix}.$$

Applying (28) yields:

$$P_t/D_t = \frac{1+r}{r+\pi_*-g_*}\left(1+\frac{g_t-g_*}{1-\alpha\rho_g}+\frac{\pi_t-\pi_*}{1-\alpha\rho_\pi}\right) \quad (45)$$

In the limit of small times, with $\rho_g = 1-\phi_g$, $\rho_\pi = 1-\phi_\pi$, with r and ϕ small (ϕ_g is the speed of mean-reversion of g to its trend), we obtain:

$$P_t/D_t = \frac{1}{R}\left(1+\frac{g_t-g_*}{R+\phi_g}+\frac{\pi_t-\pi_*}{R+\phi_\pi}\right) \text{ with } R = r+\pi_*-g_* \quad (46)$$

which captures that the P/D ratio can change because of movements in the expected dividend growth rate (g_t)or the equity premium (π_t).

Example 9 *Markov chains*

There are n states. In state i the factor-augmented dividend grows at a rate G_i: $M_{t+1}D_{t+1}/\left(M_t D_t\right) = G_i$. Call $X_{it} \in \{0,1\}$, equal to 1 if the state is i, 0 otherwise. The probability of going from state j to state i is called p_{ij}. Then, $M_t D_t \left(1, X_1, ..., X_n\right)$ is a LG process. Indeed, $E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}\right] = \sum_i G_i X_{it}$, and

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}X_{i,t+1}\right] = E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}\right]E_t\left[X_{i,t+1}\right] = \left(\sum_k G_k X_{kt}\right)\left(\sum_j p_{ij}X_{jt}\right) = \sum_j p_{ij}G_j X_{jt}$$

as $X_{kt}X_{jt} = 0$ if $j \neq k$, and otherwise is equal to $X_{kt}X_{jt} = X_{jt}$, at exactly one of the X_{jt} is different from 0.

Hence, a Markov chain belongs to the LG class.[12] As many processes are (arbitrarily) well-approximated by discrete Markov chains, they are (arbitrarily) well-approximated by LG processes.

4 Linearity-generating processes in continuous time

We fix a probability space $\left(\Omega^P, \mathcal{F}, P\right)$ and an information filtration $\mathcal{F}_t$ satisfying the usual technical conditions (see, for example, Karatzas and Shreve 1991). The stochastic discount factor is M_t. For applications, we will express the results in terms of a dividend-augmented stochastic discount factor, $M_t D_t$. Often, it is better to imagine $D_t \equiv 1$.

4.1 Definition and main properties

The definition in continuous time is the limit of the definition in discrete time. The vector of factors is X_t.

Definition 2 *The process $(M_t D_t, X_t)_{t \in \mathbb{R}_+}$, with $M_t D_t \in \mathbb{R}$ and $X_t \in \mathbb{R}^n$, is a LG process if the following relations hold, for all $t \geq 0$,*

$$E_t\left[\frac{d\left(M_t D_t\right)}{M_t D_t}\right] = -\left(a + \beta' X_t\right) dt \tag{47}$$

$$E_t\left[\frac{d\left(M_t D_t X_t\right)}{M_t D_t}\right] = -\left(b + \Phi X_t\right) dt \tag{48}$$

with $a \in \mathbb{R}, b, \beta \in \mathbb{R}^n, \Phi \in \mathbb{R}^{n^2}$, and I_n the identity matrix of dimension $n \times n$.

The above equations describe the process for X_t under the "risk-neutral" measure induced by $M_t D_t$.

For instance, in the case $D_t = 1$ and $dM_t/M_t = -\left(a + \beta' X_t\right) dt$, Eq. 48 gives:

$$dX_t = -b - \left(\Phi - aI_n\right) X_t dt + \left(\beta' X_t\right) X_t dt + dN_t \tag{49}$$

with $N_t \in \mathbb{R}^n$ is a martingale. Hence, the process contains an AR(1) term, $-b - \left(\Phi - aI_n\right) X_t$, plus a "twist" quadratic term, $\left(\beta' X_t\right) X_t$. It is a "twisted" AR(1). In many applications, X_t represents

[12] Veronesi and Yared (2000) and David and Veronesi (2006) have already seen that this type of Markov chain yielded prices that are linear in the factors.

a small deviation from trend, and the quadratic term $\left(\beta' X_t\right) X_t$ is small. We are agnostic about how empirically relevant the "twist" is. It could be that it is absent in the physical probability, but present under the risk-neutral measure.

So $E_t [dN_t] = 0$, but its component dN_{it}, dN_{jt} *can* be correlated. The simplest type of martingale is $dN_t = \sigma(X_t) dB_t$, for B_t a Brownian motion, but richer structures, e.g. with jumps, are allowed. As in the one-factor process, the volatility of dN_t must go to zero in some limit regions for the process to be well-defined. We defer this more technical issue until section 6.

As in the discrete-time case, we define:

$$\omega = \begin{pmatrix} \alpha & \beta \\ b & \Phi \end{pmatrix} \tag{50}$$

and the process with values in $\mathbb{R}^{n+1}$

$$Y_t = \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix}$$

which encodes the information needed for prices. Conditions (47)-(48) write more compactly as:

$$E_t [dY_t] = -\omega Y_t dt. \tag{51}$$

which is the analogue of (24). The above process leads to a discrete-time process with time increments Δt, with a matrix $\Omega = e^{-\omega \Delta t}$. When Δt is small, $\Omega = 1 - \omega \Delta t + O\left((\Delta t)^2\right)$.

Hence, there is a $(n+1)$ dimensional process Y_t, and a vector $\nu' = (1, 0, ..., 0)$, such that (51) holds, and

$$M_t = \nu' Y_t \tag{52}$$

In other terms, there is a autoregressive process Y_t in the background, following (51). The (dividend-augmented) stochastic discount factor is the one-dimensional projection of it. LG processes are tractable, because they are the one-dimensional projection of an AR(1) process.

The next Theorem prices claims of finite maturity.

Theorem 3 *(Bond prices, continuous time). Given the LG process $(M_t D_t, X_t)$, the price of a claim on a dividend of maturity T, $P_t = E_t [M_{t+T} D_{t+T}]$, satisfies:*

$$Z_t(T) = P_t / D_t = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \exp\left[- \begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix} T \right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{53}$$

an expression which, when $b = 0$, simplifies to:

$$Z_t(T) = P_t/D_t = e^{-aT} + \beta' \frac{e^{-\Phi T} - e^{-aT} I_n}{\Phi - a I_n} X_t \tag{54}$$

Proof. Recall the definition of ω in (50), and $E_t[d(Y_t)]/dt = -\omega Y_t$. It is well-known that this implies:[13]

$$\forall T \geq 0, E_t[Y_{t+T}] = e^{-\omega T} Y_t. \tag{55}$$

Given (55) and $M_s = \nu' Y_s$,

$$\begin{aligned} Z_t(T) &= (M_t D_t)^{-1} E_t[M_{t+T} D_{t+T}] = (M_t D_t)^{-1} E_t\left[\nu' Y_{t+T}\right] = (M_t D_t)^{-1} \nu' E_t[Y_{t+T}] \\ &= (M_t D_t)^{-1} \nu' e^{-\omega T} Y_t = \nu' e^{-\omega T}\left((M_t D_t)^{-1} Y_t\right) = \nu' e^{-\omega T} \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & 0_n \end{pmatrix} e^{-\omega T} \begin{pmatrix} 1 \\ X_t \end{pmatrix}. \end{aligned}$$

i.e. Eq. 53. The formula for $b = 0$ comes from Lemma 3 in Appendix B. ■

As an example, bond prices come from $D_t = 1$. In many applications, $b = 0$, which can generically be obtained by re-centering the variables.

From this, we can now prove Theorem 4, which is probably the most useful of this section.

Theorem 4 *(Stock prices, continuous time). Given the LG process $(M_t D_t, X_t)$, suppose that all eigenvalues of ω have positive real part (finite stock price). Then, the price/dividend ratio, $P_t/D_t = E_t\left[\int_t^\infty M_s D_s ds\right] / (M_t D_t)$, is:*

$$P_t/D_t = \frac{1 - \beta' \Phi^{-1} X_t}{a - \beta' \Phi^{-1} b} \tag{56}$$

Proof. We use (53). The perpetuity price is:

$$P_t/D_t = \int_0^\infty Z_t(T)\, dT = \nu' \left(\int_0^\infty e^{-\omega T} dT\right) \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \nu' \omega^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$

[13] Indeed, to prove (55) in the case $t = 0$ (which is enough), set $T > 0$, and define $z_t = e^{\omega(t-T)} Y_t$. Then,

$$E_t[dz_t] = E_t d\left(e^{\omega(t-T)} Y_t\right) = E_t\left[d\left(e^{\omega(t-T)}\right)\right] Y_t + e^{\omega(t-T)} E_t[d(Y_t)] = \left[e^{\omega(t-T)} \omega Y_t dt + e^{\omega(t-T)}(-\omega Y_t)\, dt\right] = 0$$

Hence z_t is a martingale, and $E_0[z_T] = z_0$, i.e. $E_0[Y_T] = e^{-\omega T} Y_0$.

We use the Lemma 2 to calculate ω^{-1}, and conclude. [14] ■

To make things concrete, consider the case where Φ is a diagonal matrix: $\Phi = Diag(\Phi_1, ..., \Phi_n)$. Then, $e^{-\Phi T} = Diag\left(e^{-\Phi_i T}\right)$, and (26) and (28) read:

$$Z_t(T) = e^{-at} + \sum_{i=1}^{n} \frac{e^{-\Phi_i T} - e^{-aT}}{\Phi_i - a} \beta_i X_t^i \tag{57}$$

$$P_t/D_t = \frac{1 - \sum_{i=1}^{n} \frac{\beta_i X_i}{\Phi_i}}{a - \sum_{i=1}^{n} \frac{\beta_i b_i}{\Phi_i}} \tag{58}$$

Finally, the following Propositions show that one can price claims that have dividend a linear function of $D_t X_t$. The proofs are exactly identical to those of the previous two Theorems.

Proposition 1 *(Value of a single-maturity claim yielding $D_{t+T}\delta' X_{t+T}$). Given the LG process $M_t D_t(1, X_t)$, the price of a claim that yields $d_t := D_t\left(\delta_0 + \delta' X_t\right) = D_t \sum_{i=1}^{n} \delta_i X_{it}$, $P_t = E_t\left[M_{t+T} d_{t+T}\right]/M_t$, is:*

$$P_t = \begin{pmatrix} 0 \\ \delta \end{pmatrix}' \cdot \exp\left[-\begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix} T\right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} D_t \tag{59}$$

an expression which, when $b = 0$, simplifies to:

$$P_t = \delta' e^{-\Phi T} D_t X_t \tag{60}$$

Proposition 2 *(Value of an asset yielding $D_t \delta' X_t$ at each period) Under the conditions of Theorem 4, the price of a claim yielding $d_t := D_t \delta' X_t = D_t \sum_{i=1}^{n} \delta_i X_{it}$, $P_t = E_t\left[\int_t^\infty M_s d_t ds\right]/M_t$, satisfies,*

$$P_t = \begin{pmatrix} 0 \\ \delta \end{pmatrix}' \omega^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix} D_t = \frac{\delta' \Phi^{-1}\left(-b + a X_t\right)}{a - \beta' \Phi^{-1} b} D_t. \tag{61}$$

[14] The following elementary heuristic proof is useful to know. We seek a solution of the type $P_t/D_t \equiv V_t = c + h' X_t$, which we know exists, by integration of (53). The arbitrage equation is: $1 - r_t V_t + E\left[dV_t\right]/dt = 0$, i.e.

$$1 - \left(r_* + \beta' X_t\right)\left(c + h' X_t\right) + h'\left[b - \Phi X_t + \left(\beta' X_t\right) X_t\right] = 0$$

This is satisfies if and only if the constant and the term in X_t are zero, i.e. $r_* h' + \beta' c + h' \Phi = 0$ and $1 - r_* c + h' b = 0$. Hence $h' = -\beta' c \left(r_* + \Phi\right)^{-1}$ and $1 - c\left[r_* + \beta' \left(r_* + \Phi\right)^{-1} b\right]$, which gives $c = 1/\left[r_* + \beta' \left(r_* + \Phi\right)^{-1} b\right]$, and yields (56).

4.2 Some examples

We start with some stock-like examples.

Example 10 *Dividend growth rate as a sum of mean-reverting processes (e.g., a slow and a fast process).*

Suppose $M_T = e^{-rT}$, $D_T = D_0 \exp\left(\int_0^T g_t dt\right)$, with $g_t = g_* + \sum_{i=1}^n X_{it}$ and

$$E_t\left[dX_{it}\right]/dt = -\phi_i X_{it} - \left(g_t - g_*\right) X_{it}$$

The growth rate g_t is a steady state value g_*, plus the sum of mean-reverting processes X_{it}. Each X_{it} mean-reverts with speed ϕ_i, and also has the quadratic perturbation $\left(g_t - g_*\right) X_{it} dt$. The initial example of this paper, Example 1, is a particular case, with $n = 1$. We verify that it is LG.

$$\begin{aligned} E_t\left[\frac{d\left(M_t D_t\right)}{M_t D_t}\right]/dt &= -\left(r - g_*\right) + \sum_{i=1}^n X_{jt} \\ E_t\left[\frac{d\left(M_t D_t X_{it}\right)}{M_t D_t}\right]/dt &= \left[-\left(r - g_*\right) + \sum_{i=1}^n X_{jt}\right] X_{it} + \left(-\phi_i X_{it} - \left(\sum_{i=1}^n X_{it}\right) X_{it}\right) = -\left(r - g_* + \phi_i\right) X_{it} \end{aligned}$$

Hence $M_t D_t \left(1, X_{1t}, ..., X_{nt}\right)$ is a LG process, with

$$\omega = \begin{pmatrix} r - g_* & -1 & \cdots & -1 \\ 0 & r - g_* + \phi_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & r - g_* + \phi_n \end{pmatrix}$$

We apply the Theorem 3, with $a = r - g_*$, $\beta' = (-1, ..., -1)$, $\Phi = Diag\left(r - g_* + \phi_1, ..., r - g_* + \phi_n\right)$. The price-dividend ratio is:

$$P_t/D_t = \frac{1}{r - g_*}\left(1 + \sum_{i=1}^n \frac{X_{it}}{r - g_* + \phi_i}\right). \tag{62}$$

Each component X_{it} perturbs the baseline Gordon expression $1/\left(r - g_*\right)$. The perturbation is X_{it}, times the duration of X_i, discounted at rate $r - g_*$, which is the term $1/\left(r - g_* + \phi_i\right)$.[15]

[15] The formula suggests the following non-LG variant. Suppose we have a process with $d\psi_t = \left(r_t \psi_t + \alpha r_t - \beta\right) dt + dN_t$,where dN_t is an adapted martingale, and is essentially arbitrary except for technical conditions. Then: $V_t = \left(\psi_t + \alpha\right)/\beta$ is a solution of the perpetuity arbitrage equation: $1 - r_t V_t + E\left[dV_t\right]/dt = 0$. If the process well-

Also, the price of a claim paying a dividend at $t+T$ is:

$$E_t\left[M_{t+T}D_{t+T}\right]/M_t = e^{-(r-g_*)T}\left(1+\sum_{i=1}^{n}\frac{1-e^{-\phi_i T}}{\phi_i}X_{it}\right)D_t.$$

Example 11 *The aggregate model of Menzly, Santos and Veronesi (2004), and the Bhattacharya (1978) mean-reverting process, belong to the linearity-generating class.*

The following point is simple and formal. The Bhattacharya (1978) process is: , $dD_t = \phi\left(\overline{D}-D_t\right)dt + \sigma\left(D_t\right)dz_t$. It actually belongs to the LG class, with the state variable $X_t = 1/D_t$. Under another guise, it is used in the aggregate model of Menzly, Santos and Veronesi (2004), where S_t is their consumption-surplus ratio, which, defining $Y_t = 1/S_t$, satisfies $E_t\left[dY_t\right] = k\left(\overline{Y}-Y_t\right)dt$, with . The price-consumption ratio in their economy is $V_t = Y_t^{-1}E_t\left[\int_0^\infty e^{-\rho s}Y_{t+s}\right]$. In terms of the LG process, the state variable is $X_t := Y_t$, and $M_t = e^{-\rho t}$. We have $E_t\left[dM_t/dt\right]/M_t = -\rho dt$, and $E_t\left[d\left(M_tY_t\right)/dt\right]/\left(M_tD_t\right) = -\rho Y_t + k\left(\overline{Y}-Y_t\right)$. So $M_t\left(1,Y_t\right)$ is a LG process with matrix $\omega = \begin{pmatrix}\rho & 0\\ -k\overline{Y} & \rho+k\end{pmatrix}$. The Menzly, Santos and Veronesi pricing equation 17 comes directly from Proposition 2 of the present article, which yields $V_t = \left(k\overline{Y}+\rho Y_t\right)/\left[\rho\left(\rho+k\right)\right]$. Hence, in retrospect, the Menzly, Santos and Veronesi (2004) process is tractable because it belongs to the LG class.

Example 12 *A LG process where the stock price is convex (not linear) in the growth rate of dividends*

This example shows how one can obtain asset prices that are increasing in their variance, a case property that is important in some applications (Johnson 2002, Pastor and Veronesi 2003). Consider an economy with constant discount rate r (so that $M_t = e^{-rt}$), and a stock with dividend $D_t = D_0\exp\left(\int_0^t g_s ds\right)$, where[16]

$$dg_t = -\left(g_t^2/2+\phi g_t\right)dt + \sqrt{k\left(G^2-g^2\right)}dz_t$$

defined for $t \geq 0$, then V_t is the price of a perpetuity, $V_t = E_t\left[\int_t^\infty e^{-\int_t^s r_u du}ds\right]$. For instance, with the process $d\left(1/r_t\right) = \phi\left(r_t - r^*\right)dt + dN_t$, the price of a perpetuity is: $V_t = \left(1/r_t + \phi/r_*\right)/\left(1+\phi\right)$.

[16] We assume $0 < G < 2\left(\phi - k\right)$, and that the support of g_t is $\left(-G, G\right)$, with end points natural boundaries.

Direct computation shows that $Y_t = e^{-rT}\left(D_t, D_t g, D_t g_t^2\right)$ is a LG process, with generator $\omega = \begin{pmatrix} r & -1 & 0 \\ 0 & r+\phi & -1/2 \\ -kG^2 & -b & r+k+2\phi \end{pmatrix}$. By Theorem 4, the price-dividend ratio is:

$$P_t/D_t = \frac{2(\phi+r)(2\phi+k+r)+2(2\phi+k+r)g_t+g_t^2}{2r\,(\phi+r)\,(2\phi+k+r)-kG^2} \tag{63}$$

which is increasing in the parameter G of the volatility. In this example, the state vector is $\left(g_t, g_t^2\right)$, which makes the price quadratic and convex in g_t. More generally, by expanding the state vector, the price could be a polynomial of arbitrary order in g.

We next present some bond-like examples. The general canonical LG bond case is the following.

Example 13 *A multifactor bond model, with bond risk premia (continuous time).*

The following is Example 6 in continuous time. Suppose $dM_t/M_t = -r_t dt + dN_t$, where N_t is a martingale, and decompose the short rate in $r_t = r_* + \sum_{i=1}^n r_{it}$, with r_* a constant and:

$$E\left[dr_{it}\right] + \langle dr_{it}, dM_t/M_t\rangle = \left[-\phi_i r_{it} + (r_t - r_*)\, r_{it}\right] dt \tag{64}$$

Hence, it is enough to specify the the process "under the risk-neutral measure". One does not need to separately specify the dynamics of $E_t\left[dr_{it}\right]$ and its risk premium, the $\langle dr_{it}, dM_t/M_t\rangle$ term. Only the sum matters.

Then the process $M_t\left(1, r_{1t}, ..., r_{nt}\right)$ is LG, with:

$$\omega = \begin{pmatrix} r_* & 1 & \cdots & 1 \\ 0 & r+\phi_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & r+\phi_n \end{pmatrix}$$

and bond price is given by:[17]

$$Z_t(T) = e^{-r_* T}\left(1 - \sum_{i=1}^{n} \frac{1 - e^{-\phi_i T}}{\phi_i} r_{it}\right) \tag{65}$$

The risk-premium at t on the $T-$maturity zero coupon, $\pi(T) := -\left\langle \frac{dZ_t(T)}{Z_t}, \frac{dM_t}{M_t} \right\rangle / dt$, is:

$$\pi(T) = \frac{\sum_{i=1}^{n} \frac{1-e^{-\phi_i T}}{\phi_i} \langle dr_{it}, dM_t/M_t \rangle}{1 - \sum_{i=1}^{n} \frac{1-e^{-\phi_i T}}{\phi_i} r_{it}}. \tag{66}$$

Gabaix (2007) uses such an expression to think about models that fit the known facts on bond premia.

We study in more details the 1-factor process.

Example 14 *A one-factor bond model, with an always positive nominal rate.*

The following example is more here to illustrate LG process than a necessarily empirically relevant interest rate process – multifactor models are necessary to capture the yield curve. Suppose $M_t = \exp\left(-\int_0^t r_s ds\right)$, with $r_t = r_* + \widehat{r}_t$, with

$$d\widehat{r}_t = -(\phi - \widehat{r}_t)\widehat{r}_t dt + dN_t$$

where N_t is a martingale, and $\phi > 0$, and $\widehat{r}_t \leq \phi$. We examine the LG conditions for this process: $dM_t/M_t = -r_t dt = -(r_* + \widehat{r}_t) dt$, and:

$$\begin{aligned} d(M_t \widehat{r}_t) &= \widehat{r}_t dM_t + M_t dX_t = -\widehat{r}_t M_t (r_* + \widehat{r}_t) dt + M_t (-(\phi - \widehat{r}_t)\widehat{r}_t dt + \sigma_t dN_t) \\ &= M_t (-(r_* + \phi)\widehat{r}_t dt + \sigma_t dN_t) \end{aligned}$$

Importantly, the $\widehat{r}_t^2$ terms cancel out. So, using $E_t[dN_t] = 0$, we have a LG process:

$$\begin{aligned} E_t[dM_t/M_t] &= -r_* dt - \widehat{r}_t dt \\ E_t[d(M_t \widehat{r}_t)/M_t] &= -(r_* + \phi)\widehat{r}_t dt \end{aligned}$$

[17] As bond prices are independent of volatility, the process exhibits "unspanned volatility," a relevant feature of the data, as shown by Collin-Dufresne and Goldstein (2002). Of course, it could be the volatility depends on the factors directly, so that there would be a correlation between volatility and prices, but that would be an indirect correlation, rather than a direct one via the price formulas.

with $Y_t = M_t (1, \widehat{r}_t)$, and matrix $\omega = \begin{pmatrix} r_* & 1 \\ 0 & r_* + \phi \end{pmatrix}$. So, the bond price is:

$$Z_t (T) = e^{-r_* T} \left(1 + \frac{e^{-\phi T} - 1}{\phi} \widehat{r}_t \right). \tag{67}$$

The independence of bond prices from volatility greatly simplifies the analysis. In particular, dN_t could have jumps, which model a decision by the central bank, or fat-tailed innovations of other kinds (Gabaix et al. 2003, 2006). One does not need to specify the volatility process to obtain the prices of bonds: only the drift part is necessary. This leaves a high margin of flexibility to calibrate volatility, for instance on interest rate derivatives, a topic we do not pursue here.

How can we ensure that the interest rate always remain positive? That is very easy (assuming that the long rate r_* is positive). We could have $dN_t = \sigma (r_t) dz_t$, where z_t is a Brownian process, with $\sigma (r) \sim k' r^{\kappa'}$, $\kappa' > 1/2$ for r in a right neighborhood of 0, and $k' > 0$, so that the local drift at $r_t = 0$ is positive. By the usual Feller conditions on natural boundaries (see Appendix A), the process admits a strong solution, and $r_t \geq 0$ always. And, the bond price (67) is not changed by this assumption about the volatility process. One can indeed change the lower bound for the process (if it is less than r_*) without changing the bond price.

Section 6 will detail the conditions for the existence of the process. The interest rate needs to remains below some upper bound $\overline{r} \in (r_*, r_* + \phi)$, so as to not explode. One way is to assume that $\sigma (r) \sim k (\overline{r} - r)^\kappa$, for r in a left neighborhood of $\overline{r}$, $\kappa > 1/2$ and $k > 0$. Given the drift is negative around $\overline{r}$, that will ensure that $\overline{r}$ is a natural boundary, and $\{\forall t, r_t \leq \overline{r}\}$ almost surely, as detailed in Appendix A.

Example 15 *A model in the spirit of Brennan and Schwartz, where the factors are the short term rate, and the perpetuity rate*

A LG model answers the question that started with Brennan and Schwartz (1979): how to provide an arbitrage-free model interest rates, where the short rate, and the console rate, are factors. To the best of our knowledge, this is the first model that answers this question. Calling $c'_t = V_t - 1/r_*$, the deviation of the perpetuity price from its central value $1/r_*$, consider the following process:

$$\begin{aligned} E_t \left[dr'_t \right] + \left\langle dr'_t, dM_t/M_t \right\rangle &= \left[- (\phi + \psi + r_*) r'_t - (\phi + r_*) (\psi + r_*) c'_t + r'^2_t \right] dt \\ E_t \left[dc'_t \right] + \left\langle dc'_t, dM_t/M_t \right\rangle &= \left[r_t/r_* - r_* c'_t + c_t r'_t \right] dt \end{aligned}$$

and the short-term rate is $r_t = r_* + r'_t$, i.e. $E_t[dM_t/M_t] = -(r_* + r'_t)\,dt$. Again, in the simple case where $M_t = \exp\left(-\int_0^t r_s ds\right)$, then $\langle dr'_t, dM_t/M_t\rangle = \langle dc'_t, dM_t/M_t\rangle = 0$. The price of a zero-coupon bond is:

$$\begin{aligned} Z_t(T) &= e^{-r_*T} + e^{-r_*T}\left(\frac{r_*}{\phi\psi} + \frac{(r_*+\phi)\,e^{-\phi T}}{\phi(\phi-\psi)} + \frac{(r_*+\psi)\,e^{-\psi T}}{\psi(\psi-\phi)}\right) r'_t + \\ &\quad + e^{-r_*T}\frac{r_*(r_*+\phi)(r_*+\psi)}{\phi\psi}\left(1 + \frac{\psi e^{-\phi T} - \phi e^{-\psi T}}{\phi-\psi}\right) c'_t \end{aligned}$$

while the price of the perpetuity is $V_t = 1/r_* + c'_t$. Hence, in this model, the factors are the short term rate $r_t = r_* + r'_t$, and the console price $V_t = 1/r_* + c'_t$.

Example 16 *r_t having a time-varying trend*

In the post-Volcker era, interest rates tended to have predictable trends of increase or increase, which may be captured by the following trend growth rate s_t of the interest rate:

$$\begin{aligned} E_t[dr_t]/dt &= s_t + (r_t - r_*)^2 \\ E_t[ds_t]/dt &= [-\lambda\mu(r_t - r_*) - (\lambda+\mu)s_t] + (r_t - r_*)s_t \end{aligned}$$

with $\lambda, \mu \geq 0$, and $\lambda + \mu > 0$. Economically, s_t is the predicted trend in interest rates, as per the first expression. s_t mean-reverts for two reasons: first, because of the $-\lambda\mu(r_t - r_*)$ term (s_t becomes negative if interest rates are too high); second, because of the $-(\lambda+\mu)s_t$ term.

We apply Theorem 3, with $X'_t = (r_t, s_t)$, $\beta' = (1, 0)$, $\Phi = \begin{pmatrix} 0 & -1 \\ \lambda\mu & \lambda+\mu \end{pmatrix}$. We obtain:

$$Z(T) = e^{-r_*T}\left[1 + \left(e^{-\lambda T} - 1\right)\frac{s_t + \mu(r_t - r_*)}{\lambda(\mu-\lambda)} - \left(e^{-\mu T} - 1\right)\frac{s_t + \lambda(r_t - r_*)}{\mu(\mu-\lambda)}\right].$$

Those examples show it is quite easy to obtain closed forms with processes that are easy to interpret.

4.3 Relation to the affine-yield class

The affine class (Duffie and Kan 1996; Dai and Singleton 2000; Duffie, Pan and Singleton 2000) is a very important class, that contains the processes of Vasicek/Ornstein-Uhlenbeck (1977), Cox, Ingersoll, Ross (1985) and Balduzzi et al. (1996). It is a workhorse of much empirical and theoretical in asset pricing. It comprises processes of the type: $dX_t = (b - \Phi X_t)\,dt + w_t dz_t$, with $w_t w'_t = \sigma^2(H'_1 X_t + H_0)$, with $b, X_t \in \mathbb{R}^n$, $\Phi \in \mathbb{R}^{n\times n}$, $(H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \sigma \in \mathbb{R}$, z_t is a

$n-$dimensional Brownian motion. The interest rate is $r_t = r_* + \beta' (X_t - X_*)$, where $X_* = \Phi^{-1} b$, is assumed to exist.

Under mild technical conditions, bond prices have the expression:

$$Z_t^{\text{Aff}}(T) = \exp\left(-r_* T + \Gamma(T)'(X_t - X_*) + \sigma^2 a(T)\right)$$

where $a(T)$ and $\Gamma(T)$ satisfy coupled ordinary differential equations, that typically need to be solved numerically. This is not a problem for empirical work, but that does hinder theoretical work. The situation is simpler if $H_1 = 0$. In that case, $\Gamma(T) = \gamma(T)$, with $\gamma(T)' = \beta'\left(e^{-\Phi T} - 1\right)/\Phi$. Then: $Z_t^{\text{Aff}}(T) = \exp\left(-r_* T + \gamma(T)'(X_t - X_*) + \sigma^2 a(T)\right)$. This expression can be contrasted with the expression for the LG process (54):

$$Z_t^{\text{LG}}(T) = e^{-r_* T}\left(1 + \gamma(T)'(X_t - X_*)\right). \tag{68}$$

If $\gamma(T)' X_t$ is small, the two expressions are the same, up to terms of second order in $\gamma(T)' X_t$, and second order in σ. Hence, a LG process is a good approximation if the underlying process is in fact affine, and vice-versa. In most cases, the two values are likely to be close, so that existing estimates of parameters in the affine class can be used to calibrate LG processes.[18]

What are the respective merits of the LG and affine classes? First, quantitatively, they will often make close predictions, as the two models yield the same prices to a first order.

The distinctive advantage of the LG class is for stocks. LG yield simple closed forms for stock prices. However, with the affine class, a stock price can be only be expressed

$$P_t^{\text{Aff}}/D_t = \int_0^\infty Z_t^{\text{Aff}}(T)\, dT$$

or, in discrete time, $P_t^{\text{Aff}}/D_t = \sum_{t=0}^{\infty} Z_t^{\text{Aff}}(T)$. Those are infinite sums of exponential expressions, which is a great progress over stochastic sums (see Ang and Liu 2004), but still not very tractable.

Beyond their advantage for stocks, LG processes have two lesser virtues. First, bond prices are also quite simple, and that should prove useful to theorize on bonds (Gabaix 2007). Second, they allow a free functional form for the innovations dN_t, which can include jumps and non-Gaussian behavior, and a free type of heteroskedascity.

[18] That equivalence gives a useful way to calculate easily functionals of LG processes, that can be expressed as a linear combination of bonds. One first works with the affine process, setting volatility to 0, doing a first order Taylor expansion of terms in $(X_t - X_*)$. One gets an expression: $P_t^{\text{Aff}} = a + b(X_t - X_*) + o(X_t - X_*) + o\left(\sigma^2\right)$, for some constant a, b. Then, one knows that for the corresponding LG process, the value of the asset is: $P_t^{\text{LG}} = a + b(X_t - X_*)$, exactly.

On the other hand, affine processes are the central technique to price derivatives, whereas this paper barely begins to study options for LG processes (section 5.3). LG processes look less convenient for options. Some variance needs to go to 0 at the borders, this may make the fit difficult, e.g. when pricing bond options. Also, a potential drawback of pricing bonds with the LG process, is that, in the simplest version at least, bonds have no convexity in the LG framework. However, multifactor LG processes can have a flexible degree of convexity (Example 12).

A nice property of affine process is that if M_t is in the affine class, and γ is a constant, then M_t^γ is always in the affine class too, whereas LG processes do not have that property. Otherwise, advantages of affine models are that they are well-understood, they have been estimated. It would be very desirable to do the same for LG models.

In conclusion, LG processes have a good advantage for stocks, affine processes have a strong advantage for options. For bonds, affine models will continue to be tremendously successful, but LG models may complement them usefully, particularly in theoretical research.

5 Extensions

5.1 Processes with time-dependent coefficients

It is simple to extend the process to time-dependent coefficient. Suppose the process is:

$$\begin{aligned} E_t\left[\frac{d\left(M_t D_t\right)}{M_t D_t}\right] &= -\left(a\left(t\right)+\beta'\left(t\right)X_t\right)dt \\ E_t\left[\frac{d\left(M_t D_t X_t\right)}{M_t D_t}\right] &= -\left(b\left(t\right)+\Phi\left(t\right)X_t\right)dt \end{aligned}$$

With $Y_t = \left(M_t, M_t X_t\right)^\top$, this is $E\left[dY_t\right]/dt = -\omega\left(t\right)Y_t$, where $\omega\left(t\right) = \begin{pmatrix} \alpha\left(t\right) & \beta\left(t\right) \\ b\left(t\right) & \Phi\left(t\right) \end{pmatrix}$. The solution is: $E_0\left[Y_T\right] = \exp\left(-\int_0^T \omega\left(t\right)dt\right)Y_0$. Hence, in the zero-coupon expressions, it is enough to replace ωT by $\int_0^T \omega\left(t\right)dt$. For instance, when $\forall t, b\left(t\right)=0$, the equivalent of (54) is:

$$P_0/D_0 = e^{-\int_0^T a(t)dt} + \left(\int_0^T \beta\left(t\right)dt\right)^{-1} \frac{e^{-\int_0^T \Phi(t)dt} - e^{-\int_0^T a(t)dt}}{\left(\int_0^T \left(\Phi\left(t\right)-a\left(t\right)I_n\right)dt\right)^{-1}} X_t.$$

5.2 Closedness under addition and multiplication

The product of two uncorrelated LG processes is LG. The same reasoning works for the product of two LG process. The product of two uncorrelated LG processes with respective dimensions d_1, d_2 (i.e., with $d_1 - 1$ and $d_2 - 1$ factor respectively) is LG, with dimension $d_1 d_2$ (i.e., with $d_1 d_2 - 1$ factors). The idea is simple, though it requires somewhat heavy notations.

We start in discrete time. Take two LG processes $\left(M_t^i, Y_t^i\right)$, and the product stochastic discount factor $m_t = m_t^1 m_t^2$. Assume that, for any index i, j of the components, $E_t\left[Y_{t+1}^{1(i)} Y_{t+1}^{2(j)}\right] = E_t\left[Y_{t+1}^{1(i)}\right] E_t\left[Y_{t+1}^{2(j)}\right]$, a condition which is for instance verified if the processes are independent. Then, it is easy to verify that for any vector ψ^i, $E_t\left[\left(\psi^1 Y_T^1\right)\left(\psi^2 Y_T^2\right)\right] = E_t\left[\psi^2 Y_T^2\right] E_t\left[\psi^2 Y_T^2\right]$. In particular, $E_t\left[M_T^1 M_T^2\right] = E_t\left[M_T^1\right] E_t\left[M_T^2\right]$

Then, $m_t = m_t^1 m_t^2$ is also the SDF of a LG process.[19] The state vector is $\overline{Y}_t^1 \otimes \overline{Y}_t^2$, i.e. the vector made of the $d_1 d_2$ components $\overline{Y}_t^{1(i)} \overline{Y}_t^{2(j)}$, $i = 0 ... n_X, j = 0 ... n_Y$. The corresponding Ω matrix is $\Omega = \Omega^1 \otimes \Omega^2$. This comes simply from the fact that $E_t\left[m_{t+1}^X m_{t+1}^Y \overline{X}_{t+1} \otimes \overline{Y}_{t+1}\right] = E_t\left[m_{t+1}^X \overline{X}_{t+1}\right] \otimes E_t\left[m_{t+1}^Y \overline{Y}_{t+1}\right]$.

In continuous time, suppose $E\left[d\left(M_t^X \overline{X}_t\right)/dt\right] = -\omega^X M_t^X \overline{X}_t dt$, and $E\left[d\left(M_t^Y \overline{Y}_t\right)/dt\right] = -\omega^Y M_t^Y \overline{Y}_t dt$. Then, $M_t^X M_t^Y$ is also a pricing kernel that comes from a LG process. The state vector is $\overline{X}_t \otimes \overline{Y}_t$ (which has dimension $d_1 d_2$), and the ω matrix is: $\omega^{\overline{X} \otimes \overline{Y}} = I_{n_X} \otimes \omega^Y + \omega^X \otimes I_{n_Y}$.

As an application, consider two LG processes, r_t, and g_t, with:We now merge the two previous examples, to incorporate both a time-varying equity premium and a time-varying dividend growth rate. The stochastic discount factor and dividend are given as follows:

Example 17 *Stock with decoupled LG processes for the growth rate and the risk premium.*

Consider processes with $dM_t/M_t = -rt - \lambda_t dB_t$, $dD_t/D_t = g_t dt + \sigma_t dB_t$, where g_t follows the LG process

$$dg_t = -\phi_g \left(g_t - g_*\right) dt - \left(g_t - g_*\right)^2 dt + dN_t^g.$$

The risk premium, $\pi_t = \lambda_t \sigma_t$, follows the LG process:

$$d\pi_t = -\phi_\pi \left(\pi_t - \pi_*\right) dt + \left(\pi_t - \pi_*\right)^2 dt + dN_t^\pi$$

where N_t^g, N_t^π are martingales. Assume that the processes dN_t^g, dN_t^π and dB_t are uncorrelated. Then, the price of a stock, $P_t = E_0\left[\int_0^\infty M_t D_t dt\right]/M_0$, is, by the reasoning of the previous section, $P_t/D_t = E_t\left[\int_{s=t}^\infty \exp\left(-\int_{u=t}^s \left(r + \pi_u - g_u\right) du\right) ds\right]$. In virtue of the above reasoning,

$$E_t\left[\exp\left(\int_t^s -\pi_u + g_u du\right)\right] = E_t\left[\exp\left(\int_t^s -\pi_u du\right)\right] E_t\left[\exp\left(\int_t^s g_u du\right)\right] \tag{69}$$

[19] This subsection probably contains typos.

For general processes, the above equation would in general require the two processes to be independent – for instance, with stochastic volatility, the respective variance processes should be independent. For LG processes, the property required is the weaker $\langle d\pi_t, dg_t \rangle = 0$ for all t's.

Using the values of the LG processes, and integrating, we obtain, with $R = r + \pi_* - g_*$,[20]

$$P_t/D_t = \frac{1}{R}\left[1 - \frac{\pi_t - \pi_*}{R+\phi_\pi} + \frac{g_t - g_*}{R+\phi_g} - \frac{\left(2R+\phi_\pi+\phi_g\right)\left(\pi_t - \pi_*\right)\left(g_t - g_*\right)}{\left(R+\phi_\pi\right)\left(R+\phi_g\right)\left(R+\phi_\pi+\phi_g\right)}\right]. \tag{70}$$

The central value is again the Gordon formula, $P_t/D_t = 1/R$. It is modified by the current level of the equity premium, and the growth rate of the stock. A stock with a currently high growth rate g_t exhibits a higher price-dividend ratio, and this is amplified when the equity premium is low, as shown by the term $(\pi_t - \pi_*)(g_t - g_*)$.

The difference between formula (70) and formula (16) is the here, the processes for π_t and g_t are decoupled, whereas in (16), they where coupled, i.e. in their drift term there was a term $(g_t - g_*)$. The decoupling forces the presence of a cross term $(\pi_t - \pi_*)(g_t - g_*)$ in the expression of the price. In general, one obtains simpler expressions by having one multifactor LG processes, rather than the product of many different ones.

The sum of two LG processes is LG. This property is quite trivial, and mentioned for completeness. Suppose two LG process $\left(M_t^i, Y_t^i, \nu^i\right)$, with $M_t^i = \nu^i Y_t^i$, for $i = 1, 2$. Call d_i the dimension of Y_t^i, which is the number of factors plus 1. Then, the SDF $M_t = M_t^1 + N_t^2$ comes from a LG process of dimension $d_1 + d_2$. Indeed, define $Y_t = \left(Y_t^1, Y_t^2\right)$, a vector of dimension $d_1 + d_2$ and $\nu = \left(\nu^1, \nu^2\right)$, and $\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$. Then, $E_t\left[Y_{t+1}\right] = \Omega Y_t$, and $M_t = \nu Y_t$.

5.3 A remark on option pricing with LG processes

One can express transforms of options in the LG framework, under some conditions. As in Duffie, Pan and Singleton (2000), this requires Fourier transforms and ordinary differential equations, but not solving partial differential equations.

Consider the case $D_t = 1$, and the price at time 0 of an option giving at time T the right to buy a bond for a price K. Its price is: $P_t = E_t\left[M_T\left(Z_T\left(X_T, S\right) - K\right)^+\right]$. Given $Z_T\left(X_T, S\right)$ is an

[20] Menzly, Santos and Veronesi (2004, Eq. 20) obtain a similar expression. This is natural because their model belong to the LG class, as Example 11 shows.

affine function of X_t, write $Z_T(X_T, S) - K = w' \cdot X_T - K'$, so that the option price at time 0 is:

$$P_0 = E_0\left[M_T\left(Z_T\left(X_T, S\right) - K\right)^+\right] = E_0\left[M_T\left(w' \cdot X_T - K'\right)^+\right] = E_0\left[\left(\psi \cdot Y_T\right)^+\right]$$

with $Y_T = (M_t, M_t X_t)^\top \in \mathbf{R}^{n+1}$, and $\psi = (-K', w)^\top \in \mathbf{R}^{n+1}$.

So the problem is solved if we know how to calculate $E_0\left[(\psi \cdot Y_T)^+\right]$. We can simply transpose the results of Duffie, Pan and Singleton (2000). Assume the following affine process for $Y_t, dY_t = -\omega Y_t dt + dN_t$, where dN_t is a Brownian process with $\langle dN_t, dN_t' \rangle / dt = 2HY_t$, for which[21] $H \in \mathbf{R}^{(n+1)^3}$. Then, for $\lambda \in \mathbf{C}^{n+1}$, when $E_0\left[e^{\lambda Y_Y}\right]$ is well-defined, one has the following "affine-yield" representation:

$$E_0\left[e^{\lambda' Y_T}\right] = e^{B(T)Y_0} \tag{71}$$

where $B(T)$ ensures that, with $V(T, Y) = e^{B(T)Y}$, $\mathcal{A}V - \partial_T V = 0$, which gives:

$$\frac{dB(T)}{dT} = -B(T)\,\omega + B(T)\,HB(T) \tag{72}$$

and $B(0) = \lambda'$. Typically, the ODE (72) needs to be solved numerically.

We are now done. The knowledge of (71) gives the distribution of Y_T by inversion of the Fourier transform, hence the price of the option.

On the other hand, with the above approach, variances of Y_t/M_t are not independent of M_t, whereas it would be better if there were.

Decomposing more complicated functions $g(X)$ on a basis of functions $(w' \cdot X_T - K')^+$, one can (in principle) express any option $E_0[M_T g(X_T)]$ this way. Partial differential equations are avoided, and replaced by comparatively simpler ordinary differential equations and Fourier transforms.

6 Conditions to keep the process well-defined

The results of this paper require that the process be defined for $t \in [0, \infty)$. Appendix A reviews standard sufficient conditions in the one-factor case. The present section present the analogue conditions in the multifactor case. [This section is needs rewriting].

[21] H is a tensor, so that HY_t has dimension $(n+1) \times (n+1)$. More explicitly, $(HY)_{ij} = \sum_k H_{ijk} Y_k$.

6.1 Simple conditions

First, diagonalize the matrix Ω (resp. $-\omega$), i.e. find q and Δ such that $\Omega = q\Delta q^{-1}$, with $\Delta = Diag\left(\psi_1, ..., \psi_{n+1}\right)$, and $\psi_1 \geq ... \geq \psi_{n+1}$. The eigenvector corresponding to eigenvalue ψ_j is $\left(q_{ij}\right)_{i=1...n}$. With $M_t D_t = \nu' Y_t$, call $\xi = q'\nu$, and $K_t = q^{-1} Y_t$. This way: $M_t D_t = \xi' K_t$, and $E_t\left[K_{t+1}\right] = q^{-1}\Omega Y_t = \Delta K_t$. In other terms, the state vector is now K_t, and the process is diagonal, in the sense that $E_t\left[K_{t+1}\right] = \Delta K_t$, where Δ is a diagonal matrix.

We need to find conditions on K_t such that, for all $s \geq t$, $\xi' K_s > 0$.

$$E_t\left[M_{t+T} D_{t+T}\right] = \xi' \Delta^T K_t = \sum_i \xi_i \psi_i^T K_{it}$$

Sufficient conditions are given by the following proposition:

Proposition 3 *(Sufficient conditions for the bond and stock prices to be always positive). Writing the process in diagonal form, $M_t D_t = \xi' K_t$, $E_t K_{t+1} = \Delta K_t$, Δ a diagonal matrix with Δ_{11} the diagonal element with the largest value, a sufficient condition for prices at t to be positive is:*

$$\xi_1 K_{1t} - \sum_{i>1} \left(\xi_i K_{it}\right)^- > 0 \tag{73}$$

where, for a real x, $x^- = \max\left(-x, 0\right)$.

Proof.

$$\begin{aligned} E_t\left[M_{t+T} D_{t+T}\right] &= \xi' \Delta^T K_t = \sum_i \xi_i \psi_i^T K_{it} = \psi_1^T \sum_i \xi_i \left(\frac{\psi_i}{\psi_1}\right)^T K_{it} \\ &\geq \psi_1^T \left(\xi_1 K_{1t} - \sum_i \left(\xi_i K_{it}\right)^-\right) > 0 \end{aligned}$$

as $\psi_i/\psi_1 \in (0,1)$. ■

Applications *Simple stock model.* Take the simplest stock model. The basis is: $K_t = \begin{pmatrix} M_t D_t \left(1 + \gamma_t/\phi\right) \\ -M_t D_t \gamma_t/\phi \end{pmatrix}$, with $\xi = (1,1)$. So, the condition is: $\left(1 + \gamma_t/\phi\right) - \left(-\gamma_t/\phi\right)^- > 0$, i.e. $1 - \gamma_t^-/\phi > 0$, i.e. $\gamma_t > -\phi$, the tightest possible condition.

Bond with n factors. The basis is: $K_{n+1,t} = M_t\left(1 - \sum r_{it}/\phi_i\right)$, $K_{it} = M_t r_{it}/\phi_i$, with $\xi =$

$(1,..,1)$. Then, condition (73) becomes: $1-\sum r_{it}/\phi_i - \sum (r_{it}/\phi_i)^-$, i.e.

$$1-\sum_i r_{it}^+/\phi_i > 0 \tag{74}$$

6.2 Another formulation

[Most of the material will likely go in Cheridito and Gabaix (2007)]

With one factor, the process is well-defined if it stays within $r \leq \overline{r}$, with $\overline{r} < r_* + \phi$. Also, the volatility of the process has to go to 0 near $\overline{r}$. The following is the $n-$factor equivalent. We start with a LG process (47)-(48).

Admissibility of the initial conditions

We start from a process $E_t dY_t = -\omega Y_t dt$.

Step 1 – Diagonalization of the process.

Diagonalize the matrix ω, i.e., find q and Δ such that $\omega = q\Delta q^{-1}$, with $\Delta = Diag\,(\delta_1,...,\delta_{n+1})$, and $\delta_1 \leq ... \leq \delta_{n+1}$. The eigenvector corresponding to eigenvalue δ_j is $(q_{ij})_{i=1...n}$.

Define $Q = Diag\,(q_{1j}) \cdot q^{-1}$. Then, $\omega = Q^{-1}\Delta Q$, with $(1,..,1)\,Q = (1,0,..,0)$. [22]

Define ∇ ("nabla"), a $(n+1)\times(n+1)$ matrix:[23]

$$\nabla_{ij} \quad : \quad = (\delta_{i+1}-\delta_j)\,1_{i\geq j} \text{ for } i = 1...n \tag{75}$$

$$= \quad 1 \text{ for } i = n+1 \tag{76}$$

Step 2 – Admissibility of the initial condition. The initial condition Y_0 should satisfy:

$$\nabla Q Y_0 > 0 \tag{77}$$

where the inequality is meant to hold coordinate by coordinate. Condition (77) is the $n-$dimensional analogue of $r_t - r_* < \phi$ in the one-factor process.

If the initial value of Y_t satisfies (77), and increments are continuous, then all future $Y_{s>t}$ also satisfy (77).

[22] If $q_{1j} = 0$ for some j, one just eliminates the space corresponding to eigenvector j, without changing the economics of the process, in particular $M_t D_t$. (To be fleshed out).

[23] The alternative matrix defined by $\nabla_{ij} = 1_{i\geq j}$ also works. It leads to more stringent conditions.

Making the volatility go to zero near the boundaries We consider the region ϱ^+

$$\varrho^+ = \left\{Y \in \mathbf{R}^{n+1} \mid \nabla Q Y > 0\right\}$$

As $Y_t^1 = M_t D_t = (0_n, 1) \cdot \nabla Q$, $Y \in \varrho^+$ implies $M_t D_t > 0$.

We define a "killing" function $\kappa : \mathbf{R} \to \mathbf{R}_+$, such that (i) $\kappa(x) = 0$ for $x \leq 0$; (ii) for x in a right neighborhood of 0, $\kappa(x) = O(x^\alpha)$, for some $\alpha > 1/2$ and (iii); there is an x_0 (in practice small) such that $\kappa(x) = 1$ for $x > x_0$. Define:

$$K(Y) = \kappa\left(\min_{i=1\dots n} \frac{(\nabla Q Y)_i}{(\nabla Q Y)_{n+1}}\right)$$

That is, $K(Y) = 1$ most of the time, but when Y is close to the boundary of ρ^+, then $K(Y)$ goes to 0.

Transformation of the process to make sure it is defined for $t \in [0, \infty)$.

Start from the "target" process that could be written $d\widetilde{Y}_t = -\omega \widetilde{Y}_t dt + \widetilde{Y}_t dn_t + M_t dN_t$, n_t is a 1-dimensional martingale, N_t a $(n+1)$ dimensional martingale. σ_t captures the log-normal drift in dividend, while dN_t captures innovations to the factors, and $var(dN_t)/dt$ is bounded. The target process $\widetilde{Y}_t$ might explode in finite time, as in the one-factor process. To stabilize it, define the modified process:

$$dY_t = -\omega Y_t dt + Y_t dn_t + K(Y_t) M_t dN_t \tag{78}$$

Then, the modified process is defined for $t \in [0, \infty)$. The modified process is well defined, and has correlations identical to those of the initial process when $K(Y) = 1$, i.e. far enough from the boundary of region ρ^+. The $K(Y) dN_t$ term makes the volatility go to 0 when Y is close to the boundary of (77).[24] Otherwise, it is equal to 1. We note that, in practice, the $K(Y_t)$ term will affect the process very rarely.

6.3 Examples

Take 1-dimensional process, $dM_t/M_t = -r_t dt$, $dr_t = \left(-\phi r_t + r_t^2\right) dt + \sigma(r_t) dz_t$. Take $Y_t = (M_t, M_t r_t)$. Then, $\omega = \begin{pmatrix} 0 & 1 \\ 0 & \phi \end{pmatrix}$, $Q = \begin{pmatrix} 1 & -1/\phi \\ 0 & 1/\phi \end{pmatrix}$, $\nabla = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $\nabla Q Y_t = \begin{pmatrix} M_t(1 - r_t/\phi) \\ M_t \end{pmatrix}$. Condition (77), $\nabla Q Y_t > 0$, is equivalent to $r_t < \phi$ and $M_t > 0$. Also, $K(Y_t) = \kappa(1 - r_t/\phi)$. The conditions above implies that it goes to 0 as r_t is in a left neighborhood of ϕ.

[24] The above procedure works with continuous increments. When there are jumps, the jumps should not transport Y_t outside of ρ^+.

6.4 Justification

The key lemma is the following.

Lemma 1 *Given a matrix $\omega \in \mathbb{R}^{m\times m}$, with a diagonalization $\omega = Q^{-1}\Delta Q$, with $Q_{1j} = 1$ for $j = 1...m$, and $\Delta = Diag\left(\psi_0, ..., \psi_{m-1}\right)$. Define:*

$$\begin{aligned} \nabla_{ij} \quad &: \quad = \frac{\psi_i - \psi_{j-1}}{\psi_i - \psi_0} 1_{i\geq j} \text{ for } i = 1...m-1 & (79) \\ &= \quad 1 \text{ for } i = m & (80) \end{aligned}$$

Define $V := \nabla Q \omega Q^{-1} \nabla^{-1}$. Then, for $i < j$, $V_{ij} = 0$, and for $i > j$, $V_{ji} \leq 0$. Also, $V_{ii} = \delta_i$, and $V(1, ..., 1)' = \delta_0 (1, ..., 1)'$. Finally, $(0, .., 0, 1) \nabla Q = (1, 0, ..., 0)$.

Consider then $Z_t = \nabla Q Y_t$. We have $E_t dZ_t/dt = -VZ_t$, which has non-negative non-diagonal elements. Hence, an element $Z_t^i = 0$, while $Z_t^j > 0$, then $EdZ_t^i/dt \geq 0$. This means that, in the deterministic version of the process, if $Z_0 > 0$, then for all $t > 0, Z_t > 0$.

In discrete time, we suppose that Ω has positive eigenvalues. We start from $Y_{t+1} = \Omega Y_t$, and call $\omega = I_{n+1} - \Omega$. $Z_t = \nabla Q Y_t$. We have $Z_{t+1} = KZ_t$, with $K = I_{n+1} - \nabla Q \omega Q^{-1} \nabla^{-1}$. K which has weakly positive non-diagonal elements, and as diagonal elements, the eigenvalues of Ω, so that finally K has weakly positive coefficients. Hence, if $Z_t \geq 0$, $Z_{t+1} \geq 0$.

7 Conclusion

Linearity-generating processes are very tractable, as they yield closed forms for stocks and bonds, and prices that are linear in factors. They are likely to be useful in several parts of economics, when trend growth rates, or risk premia, are time-varying.

The results of this paper suggest the following questions.

First, it would be desirable to study explicit, non-toy, economic models that take advantage of the tractability offered by the LG structure. Gabaix (2007) presents such a model.

Second, since the LG processes are defined by moment conditions (Eq. 20-21), they lend themselves to estimation and testing by GMM techniques.

Third, LG processes suggest a new way to linearize models. Given a model, one could do a Taylor expansion expressing moments $E_t [m_{t+1}]$ and $E_t [m_{t+1} Y_{t+1}]$ as a linear function of the factors, thereby making equations 20-21 hold to a first order approximation. The projected model is then in the LG class, and its asset prices are approximations of the prices of the initial problem. Hence the LG class offers a way to derive linear approximations of the asset prices of more complicated models. Appendix C studies such an example, where a non-LG process can be approximated by an LG process to an arbitrary degree of precision.

Fourth, the LG class suggests a way to create further discount factor processes. The background state vector Y_t could follow a process richer than an autoregressive process, and the stochastic discount factor, which simply a linear projection of the state vector in LG processes, could be a richer function of it.

We conclude that LG processes might be a useful addition to the economists' toolbox.

Appendix A. Regularity conditions for the one-factor process

This appendix details conditions for the existence and uniqueness of the solutions. We recommend Karatzas and Shreve (1991 Chapter 5.5) and Revuz and Yor (1999, Chapter IX) for systematic treatments, and Ait-Sahalia (1996, Appendix) for a pedagogical overview. We call $\mathcal{D} = (\underline{r}, \overline{r})$ the domain of existence of r, and c an arbitrary point in $\mathcal{D}$. We call $\mu(r)$ the drift of r, and assume $dN_t = \sigma(r_t)\, dz_t$. We make the following assumptions.

(i) The drift and diffusion functions are continuously differentiable in r in $\mathcal{D}$, and $\sigma^2(r) > 0$ in $\mathcal{D}$.

(ii) The integral of $m(r) = \exp\left(\int_c^r 2\mu(u)/\sigma^2(u)\, du\right)/\sigma^2(r)$ converges at both boundaries of $\mathcal{D}$.

(iii) The integral of $s(r) = \exp\left(-\int_c^r 2\mu(u)/\sigma^2(u)\, du\right)$ diverges at both boundaries of $\mathcal{D}$.

(iv) μ is Lipschitz continuous, and there is a function $\rho(x) : \mathbf{R}_+ \to \mathbf{R}_+$, with $\rho(0) = 0$, such that for any $\varepsilon > 0$, $\int_{(0,\varepsilon)} \rho(x)^{-2}\, dx = +\infty$, and $|\sigma(x) - \sigma(y)| \leq \rho(|x-y|)$.

If conditions (i)-(iv) are satisfied, then there is a unique Ito process $\{r_t, t \geq 0\}$ which is a strong solution of the stochastic differential equation (86) with initial condition $r_0 = r$. Moreover, $\{r_t, t \geq 0\}$ is Markov.

The key substantive point is that the process is defined for all $t \geq 0$, and does not explode. This condition is crucial, as if we started with $r_0 > \beta$, the process would explode in finite time with positive probability, so that the process would not be defined for all times.

Conditions (i), (ii) and (iv) guarantee the existence and uniqueness of the solution up to the variable may hit the boundaries. Condition (iii) implies that the boundaries are actually not reached. The intuition is as follows. Consider the correct boundary. Condition (iii) implies $\mu(\overline{r}) < 0$, so that the process tends to return inside $\mathcal{D}$, and also requires that $\sigma^2(r)$ tends to 0 fast enough as $r \uparrow \overline{r}$.

Sufficient conditions to ensure (i)-(iv) Conditions (i) and (ii) guarantee that the stochastic differential equation (86) admits a unique strong solution. Those conditions are verified in the following cases. Condition (iii) guarantees that the end points $\mathcal{D}$ of are natural boundaries.

We assume $\mu(\overline{r}) < 0$ and $\lim_{r \to \underline{r}} \mu(r) > 0$, so that close to the end points of $\mathcal{D}$, the process tends to go back inside $\mathcal{D}$. In the case $\mu(r) = (r-\alpha)(r-\beta)$, with $\alpha < \beta$, this corresponds to $\overline{r} \in (\alpha, \beta)$ and $\overline{r} \in [-\infty, \alpha)$.

Conditions (ii) and (iii) are verified if the following conditions (C-$\mathcal{D}$) hold. For r in a left-neighborhood of $\overline{r}$, $\sigma^2(r) \sim k(\overline{r} - r)^{\kappa}$, with ($\kappa > 1$ and $k > 0$) or ($\kappa = 1$ and $0 < k < -2m(\overline{r})$). If $\underline{r} > -\infty$, for r in a right neighborhood of $\underline{r}$, $\sigma^2(r) \sim k'(r - \underline{r})^{\kappa'}$, with ($\kappa' > 1$ and $k' > 0$) or

($\kappa' = 1$ and $0 < k' < 2m(\underline{r})$). If $\underline{r} = -\infty$, then $\underline{r}$ is a natural boundary if, for r in a neighborhood of $-\infty$ $\sigma^2(r) \sim k|r|^\beta$, with $k > 0$ and $\beta < 3$. Those last conditions imply assumptions (ii), (iii).

For $\underline{r} = -\infty$, the situation is complex for condition (iv), as the standard conditions found in textbooks do not apply. $\mu(r)$ is not Lipschitz continuous, as $\mu'(r)$ is unbounded. We conjecture that a simple weakening of condition (iv) will allow the case $\underline{r} = -\infty$.

If $\underline{r} > -\infty$, the above conditions (C-$\mathcal{D}$) also imply (iv), as one can take $\rho(x) = K \max\left(x^{\kappa/2}, x^{\kappa'/2}, x\right)$, for a large enough constant K.

Appendix B. Matrix Algebra

In some of the proofs, we will use the following Lemmas, which are standard facts.

Lemma 2 *With $a \in \mathbb{R}, b, c \in \mathbb{R}^n$, and $d \in \mathbb{R}^{n^2}$, suppose that d is invertible and $a - b'd^{-1}c \neq 0$. Then the $(n+1)\times(n+1)$ matrix $\begin{pmatrix} a & b' \\ c & d \end{pmatrix}$ is invertible, and its inverse is:*

$$\begin{pmatrix} a & b' \\ c & d \end{pmatrix}^{-1} = \frac{1}{a - b'd^{-1}c}\begin{pmatrix} 1 & -b'd^{-1} \\ -d^{-1}c & ad^{-1} \end{pmatrix} \tag{81}$$

In the above equation, $a - b'd^{-1}c$ is a real number.

Lemma 3 *With $n \in \mathbb{N}_+^*$, $a \in \mathbb{R}, b \in \mathbb{R}^n$, and $d \in \mathbb{R}^{n^2}$. Call $0_{n\times 1}$ is the zero $n \times 1$ matrix made of 0's, and suppose that $(aI_n - d)$ is invertible. Then, for $t \in \mathbb{N}$,*

$$\begin{pmatrix} a & b' \\ 0_{n\times 1} & d \end{pmatrix}^t = \begin{pmatrix} a^t & b'\left(a^t I_n - d^t\right)(aI_n - d)^{-1} \\ 0_{n\times 1} & d^t \end{pmatrix}$$

and, for $t \in \mathbb{R}$,

$$\exp\left[\begin{pmatrix} a & b' \\ 0_{n\times 1} & d \end{pmatrix} t\right] = \begin{pmatrix} e^{at} & b'\left(e^{at} I_n - e^{dt}\right)(aI_n - d)^{-1} \\ 0_{n\times 1} & e^{dt} \end{pmatrix}$$

Appendix C. Approximating non-LG processes with LG processes

LG processes offer a way to approximate the price of stocks and bonds with non-LG processes, often to an arbitrary degree of precision. This Appendix illustrates this in the example of section 2.2, where the stock dividend growth (detrended) follows an Ornstein-Uhlenbeck process : $dg_t =$

$-\phi g_t dt + \sigma dz_t$. The general properties of approximation with LG processes would require a full paper, but the present appendix simply illustrates that a preliminary investigation justifies being optimistic.

First-order approximation We return to the model of section 2.2, with $R = r - g_*$, and here we call $g_t = \gamma_t$. Define $Y_t^1 = e^{-Rt} D_t$, and $Y_t^2 = e^{-Rt} D_t g_t$. We have: $E_t [dY_{1,t}]/dt = (-R + g_t) Y_{1,t} = -RY_{1,t} + Y_{2,t}$ and

$$dY_{2,t}/dt = Y_{1,t} \left(-(\phi + R) g_t + g_t^2 \right)$$

To approximate g_t^2, we replace it by its steady state mean. To find it, we observing that $E_t \left[dg_t^2 \right]/dt = -2\phi g_t^2 + \sigma^2$, so that taking the expectation at time 0, we obtain $\lim_{t\to\infty} E_0 \left[g_t^2 \right] = \sigma^2/(2\phi)$. Hence we approximate $dY_{2,t} \simeq Y_{1,t} \left(-(\phi + R) g_t + \sigma^2/(2\phi) \right)$. Hence we approximate Y_t by Y_t^*, where

$$E_t [dY_t^*]/dt = - \begin{pmatrix} R & -1 \\ -\sigma^2/(2\phi) & R + \phi \end{pmatrix} Y_t^*$$

Applying Theorem 4, we obtain:

$$P_t^*/D_t = \frac{g_t + R + \phi}{R(R + \phi) - \sigma^2/(2\phi)} \tag{82}$$

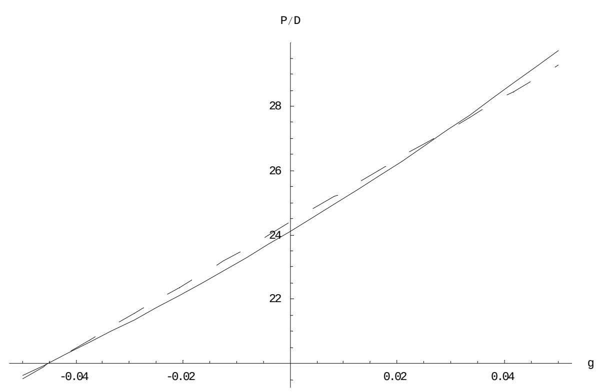

Figure 3: The Figure plots the true value of the P/D ratio of a stock with an Ornstein-Uhlenbeck process (solid line, Eq. 11), and the approximation by a LG process with 1 factor (dashed line, Eq. 82). The annualized values are: $R = 5\%$, $\phi = 15\%$, $\sigma = 4\%$, which corresponds to a stock price volatility of 11% solely caused by changes in g_t. In the range of the Figure, the two curves are within 1.5% of each other.

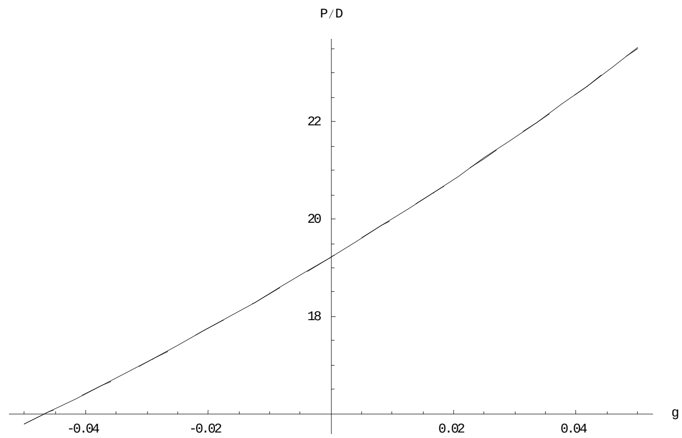

Figure 4: The Figure plots the true value of the P/D ratio of a stock with an Ornstein-Uhlenbeck process (solid line, Eq. 11), and the approximation by a LG process with $n = 5$ factor (solid line), see Eq. 83, truncated at $n = 5$. The annualized values are: $R = 5\%$, $\sigma = 3\%$, $\phi = 15\%$, which corresponds to a stock price volatility of 11% solely caused by changes in g_t. In the range of the Figure, the two curves are within 0.04% of each other.

Figure 3 plots the LG approximation, and the exact expression. We find only a small discrepancy (less than 1.5%) between the two expressions. We conclusion is that the first order approximation of the Ornstein-Uhlenbeck process by a LG process will be rather good, and useful for theoretical purposes.

If the goal is high-level numerical accuracy, we turn to an approximation of arbitrary order.

Approximation of arbitrary order In some examples, and perhaps virtually always (at least, when the processes defining the functions are analytic), it is possible to make LG processes approximate the prices of non-LG process to an arbitrary degree of precision. We provide a simple illustration of this. Define $Y_{it} = e^{-rt} D_t g_t^{i-1}$ for $i = 1, 2, ...$ Hence, the vector of factors is $X_t = \left(g_t, g_t^2, g_t^3, ...\right)$. [25] We have:

$$\begin{aligned} E_t\left[dY_{i,t}\right]/dt &= e^{-rt} D_t \left(g_t^{i+1} + (i-1)(-\phi) g_t^i + (i-1)(i-2)\frac{\sigma^2}{2} g_t^{i-3}\right) - rY_{i,t} \\ &= (i-1)(i-2)\frac{\sigma^2}{2} Y_{i-2,t} - \left[r + (i-1)\phi\right] Y_{i,t} + Y_{i+1,t} \end{aligned}$$

[25] Of course, the same reasoning could be done with another basis $f_i(g_t)$ for the transforms of g_t.

so that $E_t [dY_t] = -\omega Y_t dt$, with $\omega_{i,i-2} = -(i-1)(i-2)\sigma^2/2$,$\omega_{i,i} = r + (i-1)\phi$, $\omega_{i,i+1} = -1$ and $\omega_{ij} = 0$ otherwise. So the price is:

$$P_t/D_t = (1, 0, ..., 0, ..) \, \omega^{-1} \left(1, g_t, g_t^2, ..., g_t^n, ...\right) \tag{83}$$

The sum can be truncated up to step n, i.e. be take to be the restriction of the vector to the first n dimensions. We compare the LG (83) to the exact expression (11). Numerical results, reported in Figure 3, show that the approximation is very good, even for $n = 5$.

It would be good to generalize the above procedure, probably in a future paper. It suggests that LG processes allow to evaluate the price of many non-LG processes (e.g., those with analytic expansions), to an arbitrary degree of precision.

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