

Investment Options and the Business Cycle

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Abstract

I present an RBC model in which investment options raise the volatility of investment compared to the standard adjustment-cost model. When they are embodied only in new capital, new ideas reduce the value of old capital. Thus when the stock of unimplemented ideas rises, the value of stock market falls, i.e., the stock market and Tobin's Q are *negative* indexes of intangibles. In the model, equilibrium is efficient even without markets for knowledge; the stock market suffices.

1 Introduction

An investment option is a profit opportunity that requires an investment to implement. It is postponable if it is a patented invention, or if it is specific to a firm so that others cannot reduce its value by copying it. A firm has investment options that it may use up immediately, or store for future use. A patent, for instance, represents an investment option that only its holder can implement for a certain number of years. In a sense, even a trademark represents an option to produce a product that no one else can produce. Some investment options are protected not by law but by secrecy.

Investment options are a focus of the new Keynesian literature (Shleifer 1986), the strategic delay literature (Chamley and Gale 1994), and the investment literature (MacDonald and Siegel 1986, Dixit and Pindyck 1994). Several papers model business cycles, e.g., Gale (1996), but do not try to fit data.

It is a competitive GE model in which investment options, or “seeds” as I shall call them, are needed for the planting of trees. Seeds are produced by trees that are already planted, which I think of as the result of learning by doing. The number of trees grows over time and, in the absence of the seed constraint on investment, the model would be a standard one-sector Ak model with random TFP shocks. The model

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can be thought of as a GE version of Abel and Eberly (2005) that also endogenizes the supply of what they call “growth options”, but one that focuses on RBC issues and not mainly on the investment-Q relation. It also relates closely to Yorukoglu (2000) who studied the relation between the level of income and the equilibrium variety of goods. And, although the model has no shocks to the investment technology, it behaves a bit like models that do have such shocks, namely Greenwood, Hercowitz and Huffman (1987), and Fisher (2005).

Intangibles reduce Q .—In the model, investment implements new ideas. Because seeds are scarce, the value of planted trees and thus Tobin’s Q , is always above unity. Unimplemented ideas compete for capital, and when there are many around, their price falls, and with it so then does the price of claims to the output of existing ideas. In this sense, the more seeds we have on hand, the larger is the stock of what one would call intangibles, and the lower the value of our planted trees. This result is directly opposite to that of Hall (2000), in whose model are a positive indicator of the stock of intangibles. Technically, the difference arises because Hall assumes variable proportions between tangible and intangible capital in production and no storage of unimplemented intangibles, whereas I assume the opposite: Fixed proportions in production and storage, the latter amounting to variable proportions between consumption goods and intangibles in the investment technology. Abel and Eberly (2005) also predict that their growth options should raise Q , which follows easily in their partial equilibrium setting and which holds in my setting too for any firm that *alone* receives a growth option. When all firms get growth options at the same time, however, interest rates rise immediately and the value of capital in place falls – a standard effect in vintage-capital models. Measures of the aggregate stock of intangibles based on patent applications and trademarks co-move negatively with Tobin’s Q , thus supporting my model. While these relations hold in the aggregate, at the individual-firm level, relation between knowledge stocks and Q is positive, just as Abel and Eberly claim.

More volatile investment.—The standard model with convex adjustment costs induces a smoothness on investment. The seeds model introduces an intertemporal substitutability in investment that raises its volatility just as it raises the volatility of labor supply in the Lucas-Rapping model. The seeds model also has all investment occurring on an extensive margin, just as all the labor supply changes on the extensive margin in the simplest Rogerson-Hansen economy, and this too raises its volatility. The model also relates to that of Khan and Thomas (2005), where firms’ ability to store output raises the volatility of their production and investment.

Decentralization.—These results hold in the planner’s optimum which has two decentralizations. The first is a complete-markets decentralization in which a market for seeds exists. The second decentralization has no seeds market, only a market for shares of firms. The outcome for quantities and prices remains the same. It remains to be seen whether the efficiency of the no-seeds decentralization survives when firms differ. Jovanovic and Braguinsky (2004) develop a related one-period model in which

firms differ in how many seeds they have and in the eventual productivity of trees that they may get to plant; they find that even without a seed market, takeovers (which are still transactions in the market for firms only) achieve efficiency. The results here are consistent with theirs.

Section 2 presents the model, section 3 describes a complete-markets decentralization, section 4 an incomplete-markets one. Section 5 reports simulations, compares the model to the data. Section 6 compares the model to the standard adjustment-cost model. Several proofs and extensions are reported in the Appendix, which also discusses extensions.

2 Model

The model is that of a growing economy with two types of capital – trees, k , and seeds, S . A seed represents an option, storable indefinitely, to plant exactly one tree.

Production of output.—Output of fruit is

$$Y = zk. \tag{1}$$

If X is the number of trees newly planted, k evolves as

$$k' = k + X. \tag{2}$$

Production of seeds.—Let S denote the stock of seeds. New seeds are produced by existing trees. Each period a tree gives rise to λ new seeds, i.e., a total of

$$\text{new seeds} = \lambda k \tag{3}$$

Thus seeds grow via a process like learning by doing that takes up no resources.

The planting of trees.—Planting a tree requires a unit of fruit *and* a seed. Only one tree per seed can be planted, after which the seed is used up. Let S be the stock of seeds and let X be the number of trees planted. Then S evolves as

$$S' = \lambda k + S - X. \tag{4}$$

Since X is subtracted from the stock, a seed can be used to plant exactly one tree. Thus investment is Leontieff in two inputs, seeds and fruit. Their proportions are equal, an assumption that we shall drop when we get to the empirics, along with the assumption that neither k nor S depreciate. Leontieff investment implies that output too is Leontieff in seeds and fruit. Seeds are storable whereas fruit is not.

Timing.—Investment, X , is chosen *after* the trees produce zk units of fruit and after λk new seeds. Since $S' \geq 0$, the constraint on X is

$$X \leq \lambda k + S. \tag{5}$$

Thus investment is Leontieff in two inputs: seeds and fruit. We shall let investment be reversible: ¹

The income identity.—The cost of planting a tree is, as usual, one unit of fruit. Letting C be the consumption of fruit, the income identity is

$$zk = C + X. \quad (6)$$

The shocks.—We assume that the shocks follow the first-order Markov process: $\Pr(z_{t+1} \leq z' \mid z_t = z) = F(z', z)$, and that z' is stochastically increasing in z .

Preferences.—For $\sigma > 0$ and $\beta < 1$, preferences are

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \right\}.$$

The standard one-sector growth model.—It arises when the inequality in (5) never binds. The latter occurs when λ is large enough, e.g., if λ exceeds the largest possible z . It also occurs, *de facto*, when the initial stock of seeds, S_0 , is so large that (5) does not come into play for a very long time.

2.1 The planner's problem

The state is (k, S, z) , and the decision, x , is constrained by (59). The Bellman eq. is

$$v(k, S, z) = \max_{X \leq \lambda k + S} \left\{ \frac{(zk - X)^{1-\sigma}}{1-\sigma} + \beta \int v(k + X, \lambda k + S - X, z') dF \right\}. \quad (7)$$

Lemma 1 *A unique solution v to (7) exists, and is strictly concave in (k, S) . Moreover, X is increasing in S and, if z is i.i.d., in z .*

Proof. (i) Existence, uniqueness: Let T denote the operator on the RHS of (7). The operator is a contraction and maps continuous functions v into continuous and **unbounded** functions (Tv) . Methods of Alvarez and Stokey (2000) show that....(ii) Concavity: We shall show that if \tilde{v} is concave then $T\tilde{v}$ is strictly concave. Let $0 \leq \alpha \leq 1$. The constraint (5) is convex and its boundary is linear in S and k . Therefore if X_1 is feasible and optimal for the state (k_1, S_1) and X_2 is feasible and optimal for (k_2, S_2) , then $X_\alpha \equiv \alpha X_1 + (1 - \alpha) X_2$ is feasible, though not necessarily optimal for $(\alpha k_1 + (1 - \alpha) k_2, \alpha S_1 + (1 - \alpha) S_2)$. Therefore if $0 < \alpha < 1$

$$\begin{aligned} T\tilde{v}(\alpha k_1 + (1 - \alpha) k_2, \alpha S_1 + (1 - \alpha) S_2) &\geq \frac{(zk - X_\alpha)^{1-\sigma}}{1-\sigma} + \beta \int \tilde{v}(k + X_\alpha, \lambda k + S - X_\alpha, z') dF \\ &> \alpha T\tilde{v}(k_1, S_1) + (1 - \alpha) T\tilde{v}(k_2, S_2) \end{aligned}$$

¹Unlike Sargent (1980), we shall not impose the constraint $X \geq 0$. This constraint is never violated in any of the simulations which assume that $\sigma = 2$. With a much lower value of σ and/or with a very persistent and variable z_t process, X would at times be negative.

Therefore the operator transforms weakly into strictly concave functions. Therefore, the operator being a contraction, its unique fixed point v is strictly concave. (iii) Properties of X : (Here I assume the differentiability needed. Later, first derivatives of v will be shown to exist independently of the results of this Proposition). The FOC is

$$\xi(X, S) \equiv -(zk - X)^{-\sigma} + \beta \int \frac{dv}{dX} v(k + X, \lambda k + S - X, z') dF = 0$$

We have dropped k and z from the arguments of ξ as they play no role in the argument to be made. We now argue in 3 steps: (A) If a function of one variable H is twice differentiable with $H'' < 0$, then

$$\frac{\partial}{\partial S} \left(\frac{\partial}{\partial X} H[\lambda k + S - X] \right) = -H''(\cdot) > 0$$

Therefore, concavity of v in S alone implies $\frac{\partial}{\partial S} \frac{dv}{dX} \left(= -\frac{\partial^2 v}{\partial S^2} \right) > 0$; earlier, under (ii) we showed that concavity of v in (k, S) implies concavity of v in X holding (k, S) fixed – i.e., that $\frac{d^2 v}{dX^2} < 0$ and (B) Therefore $\xi_X < 0$ and $\xi_S > 0$. And, when z is i.i.d., $\xi_z =$ (C) Therefore $\frac{\partial X}{\partial S} = -\frac{\xi_S}{\xi_X} > 0$. ■

Reducing the state space.—From (60), $s' = \frac{\lambda + s - x}{1 + x}$. The following result allows us to reduce the state space to just (s, z) :

Lemma 2 For $\sigma \neq 1$, v is of the form

$$v(k, S, z) = w(s, z) k^{1-\sigma},$$

where $w(s, z) = v(1, s, z)$, and where w satisfies

$$w(s, z) = \max_x \left\{ \frac{(z - x)^{1-\sigma}}{1 - \sigma} + (1 + x)^{1-\sigma} \beta \int w\left(\frac{\lambda + s - x}{1 + x}, z'\right) dF \right\} \quad (8)$$

subject to (59). Moreover, v and w are of the same sign as $1 - \sigma$.

The proof (not reported) substitutes the desired functional form for v on the RHS of (7), and verifies that the same functional form emerges on the LHS. The case $\sigma = 1$ is covered separately below. Similar results are in Alvarez and Stokey (2000).

Corollary 1 A unique solution w to (8) exists that is increasing and concave in s .

Proof. *Existence:* Since a unique v exists, $w(s, z) = v(k, S, z) k^{-(1-\sigma)}$ is the unique solution for w . *Increasing:* In (59), a rise in s relaxes the constraint on x . Moreover, if one inserts on the RHS of (8) a function w that increases in s , evidently the property is preserved. *Concave:* The concavity of $v(k, S, z) k^{-(1-\sigma)}$ in S for fixed k implies that w is concave in s . ■

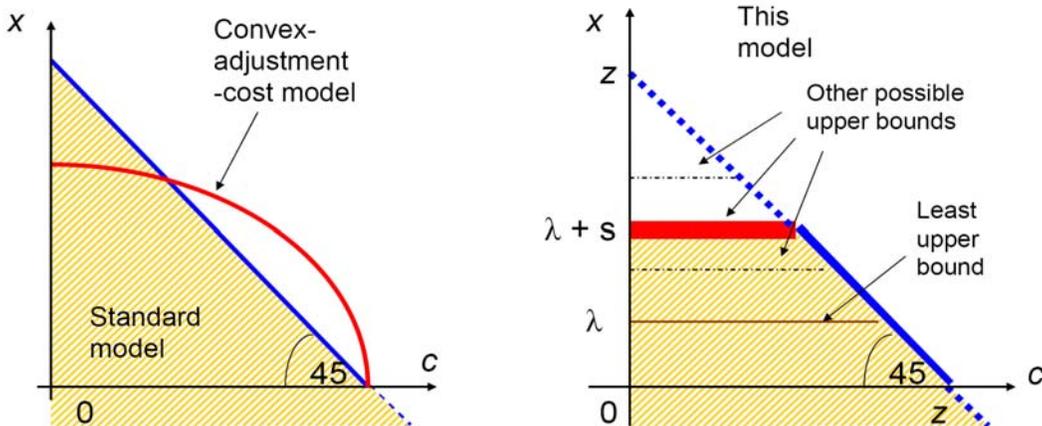


Figure 1: RELATION TO THE CONVEX ADJUSTMENT-COST MODEL

Corollary 2 *The policy $x(s, z)$ is increasing in s and, if z is i.i.d., increasing in z .*

Proof. All changes in $s \equiv S/k$ can be interpreted as changes in S for a given k . By Lemma 2, X is, for all k , increasing in S . For fixed k , a rise in S implies a rise in s and in x . The claim about z follows at once from Lemma 2. ■

The relation to other models is easily seen graphically. In its left panel, Figure 1 shows the consumption-investment trade-off in the standard model and the convex-adjustment-cost model. In its right panel, the Figure shows the constraint imposed by a particular upper bound on x , namely $\lambda + s$. Since $s \geq 0$, investment can never be constrained by any number smaller than λ , and so that's the tightest constraint on x that can possibly arise. The position of the constraint will depend on what has been happening earlier. In particular, an ‘‘seed crunch’’ and with it a high value of Q will turn out to be more likely following a prolonged boom caused by a succession of large realizations of z . Such realizations are likely to draw s to its minimum level of zero, leading the constraint to be at λ .

We can also illustrate in terms of the marginal cost of investment. Let

$$C(x, s) = \frac{\text{investment cost}}{\text{capital stock}} = \begin{cases} x & \text{if } x \leq \lambda + s \\ \infty & \text{otherwise} \end{cases}$$

denote the cost of investment, in units of fruit. The marginal adjustment costs, $\frac{\partial}{\partial x} C(x, s)$, are drawn in Figure 2. Other microfoundations – time to build – is also related, but more complicated. If time to build is T periods, then there are, in principle, T capital stocks, the capital that is productive now, and $T - 1$ capital types, indexed by the number of periods' waiting time until it becomes productive.

In sum there are two differences between this model and the standard one. First, the shape of the feasible set is different, as Figure 1 shows. And, second, there is intertemporal substitution in investment.

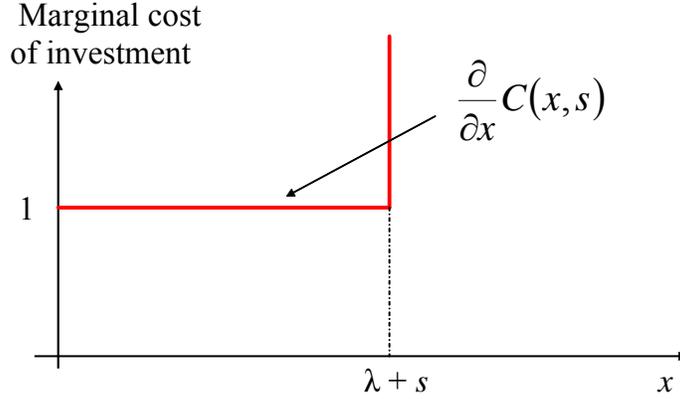


Figure 2: MARGINAL ADJUSTMENT COSTS

Lemma 3 w is strictly increasing in z .

Proof. Since $x \geq -1$ and since z' is stochastically increasing in z , for any function $w(s, z)$ increasing in z' , the second term on the RHS of (8) is increasing in z . Moreover, since $C \geq 0$, the first term on the RHS of (8) is strictly increasing in z .

■

Lemma 4 w is differentiable with respect to s , with derivative

$$w_s = \frac{1}{1 + \lambda + s} ([1 - \sigma] w - (1 + z) [z - x]^{-\sigma}) > 0 \quad (9)$$

for all (s, z) .

The proof is in Appendix 1; it follows the proof of proposition 2 of Lucas (1978) but is complicated by the seed constraint.

Note that the term $(1 - \sigma) w$ is positive for all $\sigma \neq 1$ because for $\sigma > 1$, $w < 0$.

Lemma 5 The optimal policy $x(s, z)$ satisfies

$$1 - \beta \int \left(\frac{(1+x)}{z-x} \right)^{-\sigma} \left[(z' - x')^{-\sigma} (1 + z') + \lambda w'_s \right] dF \begin{cases} = 0 & \text{if } s' > 0 \\ \leq 0 & \text{if } s' = 0 \end{cases} \quad (10)$$

Proof. By Lemma 2, v is differentiable w.r.t. k , and if w is differentiable w.r.t. s , so is v w.r.t. S . Then the FOC is

$$C^{-\sigma} - \beta \int (v_k - v_s) dF \leq 0, \quad (11)$$

with equality if $S' > 0$. We have

$$v(k, S, z) = \max_{S'} \left\{ \frac{(zk + S' - \lambda k - S)^{1-\sigma}}{1-\sigma} + \beta \int v(k + \lambda k + S - S', S', z') dF \right\}.$$

The envelope result (since S does not enter the constraint $S' \geq 0$) is

$$v_S = -C^{-\sigma} + \beta \int v_k dF$$

and

$$v_k = (z - \lambda) C^{-\sigma} + (1 + \lambda) \beta \int v_k dF = (z - \lambda) C^{-\sigma} + (1 + \lambda) (v_S + C^{-\sigma}) = (1 + \lambda) v_S + (1 + z) C^{-\sigma} \quad (12)$$

But by Lemma 2, $v(k, S, z) = w\left(\frac{S}{k}, z\right) k^{1-\sigma}$ so that

$$v_k = (1 - \sigma) w k^{-\sigma} - s w_s k^{-\sigma} \quad \text{and} \quad v_S = w_s k^{-\sigma}$$

Now, from (12), $v_k = (1 + \lambda) v_S + (1 + z) C^{-\sigma}$, so that the FOC becomes

$$C^{-\sigma} - \beta \int (\lambda v'_S + (1 + z) C'^{-\sigma}) dF \leq 0$$

But $v_S = w_s k^{-\sigma}$ and the above equation then reads

$$\begin{aligned} 0 &\geq (z - x)^{-\sigma} k^{-\sigma} - \beta \int \left(\lambda w'_s (k')^{-\sigma} + (1 + z') (z' - x')^{-\sigma} (k')^{-\sigma} \right) dF \\ &= \left(\frac{z - x}{1 + x} \right)^{-\sigma} - \beta \int \left(\lambda w'_s + (1 + z') (z' - x')^{-\sigma} \right) dF, \end{aligned} \quad (13)$$

from which (10) follows. ■

2.1.1 The set on which (5) binds

Consumption is most volatile and investment least volatile when (5) binds. Let $\Delta = \{(s, z) \mid x(s, z) = \lambda + s\}$ be the set of states for which (5) binds. In this region, X cannot respond to z and therefore C moves one-for-one with zk and, hence, is more volatile than in the standard model. True, this statement is conditional on s , but for $(s, z) \in \Delta$, $s' = 0$, and $x' = x(0, z')$. If (s, z) remain in Δ for more than one period, then in period two and beyond,

$$x(0, z) = \lambda \text{ and } c = z - \lambda.$$

The further z is from being a random walk (and it seems to depart substantially from it, see Table 1), the more these rules depart from what the standard model

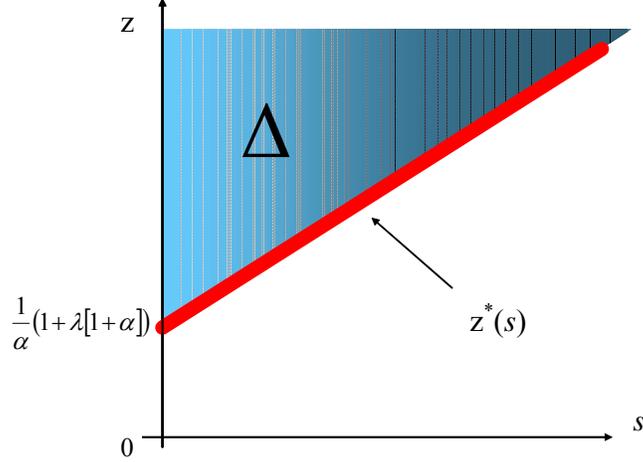


Figure 3: THE SET Δ WHEN z IS I.I.D.

would predict. In contrast, when $s \rightarrow \infty$, we get the standard model, for then the probability that (5) will bind in the foreseeable future goes to zero.

Even when z is i.i.d., x is increasing in z because a higher z today raises wealth and causes a rise in desired future consumption. Because x is increasing in z , Δ contains large z values. For $(s, z) \in \Delta$, $s' = 0$ so that $x' = x(0, z')$. Let $z^*(s) = \inf_{\{z|(z,s) \in \Delta\}} z$ be the boundary of Δ . Then, as Figure 3 illustrates, we can then show the following:

Proposition 1 *If $z \sim F(z)$ is i.i.d., then*

$$z^*(s) = \frac{1 + (1 + \alpha)(\lambda + s)}{\alpha}, \quad (14)$$

where α is the constant:

$$\alpha = \left(\beta \int \frac{\lambda(1 - \sigma) w(0, z') - (1 + z')(z' - x[0, z'])^{-\sigma}}{1 + \lambda} dF(z') \right)^{1/\sigma}. \quad (15)$$

Proof. From (10) and from an updated version of (9) we have

$$\beta \int \left(\frac{1+x}{z-x} \right)^{-\sigma} \frac{\lambda(1 - \sigma) w' - (1 + s')(1 + z')(z' - x')^{-\sigma}}{1 + \lambda + s'} dF \begin{cases} = 1 & \text{if } s' > 0 \\ \geq 1 & \text{if } s' = 0 \end{cases},$$

i.e.,

$$\beta \int \frac{\lambda(1 - \sigma) w' - (1 + s')(1 + z')(z' - x')^{-\sigma}}{1 + \lambda + s'} \begin{cases} = \left(\frac{1+x}{z-x} \right)^\sigma & \text{if } s' > 0 \\ \geq \left(\frac{1+x}{z-x} \right)^\sigma & \text{if } s' = 0 \end{cases}$$

i.e.,

$$\alpha \begin{cases} = \frac{1+x}{z-x} & \text{if } s' > 0 \\ \geq \frac{1+x}{z-x} & \text{if } s' = 0 \end{cases}$$

On the other hand, if x is constrained and held constant at $\lambda + s$ as z varies, the RHS is decreasing in z . Large z 's make the inequality strict. We find the smallest one that will allow strict equality at $x = \lambda + (1 + \lambda) s$. Setting it at equality we have $1 + \lambda + s = \alpha (z - \lambda - s)$, i.e., (14). Moreover, for $z = z^*(s)$ at $s' = 0$ so that $x' = x(0, z')$, and $w' = w(0, z')$, which yields (15). ■

On Δ , only Q responds to changes in z ; x does not, and therefore $s' = \lambda$ is also unchanged. Therefore shocks to output today have no effect on output in any future period. Since Δ contains mainly boom states the model thus implies that the persistence of output shocks is lower in booms. Moreover, in this case where z is *i.i.d.*, changes in Q will not forecast output. This matches the finding of Henry *et al.* (2005) that the stock market is a better predictor of growth in recessions than in booms and, in particular, that in non-recession periods equity returns do not predict growth.

When z is serially correlated, the boundary of Δ is no longer linear but Δ retains a shape similar to that portrayed in Figure 3: $z^*(s)$ still solves (14) in which α is replaced by

$$\alpha(z) = \left(\beta \int \frac{\lambda(1-\sigma)w(0, z') - (1+s')(1+z')(z' - x[0, z'])^{-\sigma}}{1 + \lambda + s'} dF(z', z) \right)^{1/\sigma}.$$

While x is less volatile on Δ , to achieve a given growth rate, x must make up for its low mean on Δ with a higher mean off of Δ , which introduces a force towards bimodality in the distribution of x and a higher volatility of x .

3 Complete markets

Assume that a market for seeds exists. This is not that unrealistic. Serrano (2006) finds that 18 percent of patents granted to small inventors are traded at least once in their lives, and that the citations-weighted percentage is even higher. Large firms also often sell their patents and enter into patent-sharing agreements with one another. Takeovers play a part in achieving transfers of intellectual capital; this is a fairly thick market in which Microsoft and Pfizer, e.g., have been highly active. A firm can be said to sell seeds when it spins off some activity, or when it hires people at wages that include a negative compensating differential for the value that its workers will draw from the experience gained; such a market is modeled, e.g., by Chari and Hopenhayn (1991). An example of employees walking out with seeds is Xerox in the 70's – it had inventions that it was unable or unwilling to implement and that were later marketed by its former employees.

Let $p(s, z)$ be the price of seeds, and $q(s, z)$ the price of a planted tree without a claim on its current-period dividends. A firm pays all its net income in dividends every period. All trade in seeds is between firms.

Firms.—A firm consists of the trees it has planted and of seeds it has stored. The firm maximizes its value. That is, it solves

$$Pk = \max_{X, S'} \{zk - X + (k + X)q + pS'\}$$

subject to (4) but not (5); the firm can support any level of investment X by a seed purchase, so that S' can be negative. Of course, (5) will have to hold in the aggregate. Substituting from (4) for S' , the firm's problem becomes

$$\max_X \{(z + q)k + p(S + \lambda k) + (q - [1 + p])X\}$$

Arbitrage.—If q differed from $1 + p$ the firm could drive dividends to plus infinity by sending X to plus or to minus infinity. A negative X would entail selling off k and the seeds that it embodies at a price of $1 + p$ and paying the net proceeds out as dividends.² These extreme outcomes cannot arise in equilibrium, we must have the “no-arbitrage condition”³

$$q = 1 + p. \quad (16)$$

which, when substituted into the maximand, means that the firm's cum-dividend value is

$$P = (z + q)k + p(S + \lambda k). \quad (17)$$

We shall obtain q from the household's problem, and then (16) gives us p .

Households.—Let $k = \#$ of trees owned by the household. The household's budget constraint therefore is

$$qk' + C = zk + qk. \quad (18)$$

The RHS of (18) gives the household's dividend receipts which are proportional to total resources, the LHS describes how they are spent.

The household's Bellman eq.—The household's personal state is the pair (k, S) , and it takes (s, z) and their laws of motion as given. Its Bellman equation is

$$V(k, s, z) = \max_{k' \geq 0} \left\{ \frac{(zk - q(s, z)[k' - k])^{1-\sigma}}{1 - \sigma} + \beta \int V(k', S', s'(s, z), z') dF \right\} \quad (19)$$

²The most relevant real-life counterpart of this is when a company sells off a division, or when it is acquired.

³This arbitrage condition would hold even if we imposed the constraint that aggregate investment be nonnegative. An individual firm could have $X < 0$ without affecting aggregates. On the other hand, if the salvage value of k were less than unity, (16) would read

$$q \leq 1 + p \quad \text{when } X \leq 0.$$

with $q(s, z)$ and $s'(s, z)$ taken as given.

Since the household gets all the rents, optimality of the equilibrium occurs if and only if $v = V$. For p to equal its marginal social value in consumption units, we should have $p = \frac{v_S}{C^{-\sigma}}$

Proposition 2 *Optimum and equilibrium coincide; for all states,*

$$v = V \quad \text{and} \quad p = \frac{v_S}{C^{-\sigma}}.$$

Proof. The FOC is

$$-C^{-\sigma}q + \beta \int V'_k dF = 0. \quad (20)$$

If $v = V$, (20) reads $-C^{-\sigma} - v_S + \beta \int v'_k dF = 0$, which implies that

$$v_S = -C^{-\sigma} + \beta \int v'_k dF \quad (21)$$

But (7) can be written as

$$v(k, S, z) = \max_{S' \geq 0} \left\{ \frac{(zk - [\lambda k + S - S'])^{1-\sigma}}{1-\sigma} + \beta \int v(k + \lambda k + S - S', S', z') dF \right\}, \quad (22)$$

and differentiating w.r.t. S , (21) follows. This implies that the household's choice of k' should coincide with that of the planner. ■

Finally, let us show that P , the value of the firm, equals the marginal social value, in consumption units, of the capital that it contains.

Corollary 3 *The value of firms equals the marginal social value of the (k, S) bundle that they contain:*

$$P = \frac{v_k + sv_s}{C^{-\sigma}} = \frac{(1-\sigma)w}{c^{-\sigma}}. \quad (23)$$

Proof. The first equality in (23): Since we have established that $p = \frac{v_S}{C^{-\sigma}}$, we need only show that

$$z + q + p\lambda = \frac{v_k}{C^{-\sigma}}$$

But from (22),

$$\frac{v_k}{C^{-\sigma}} = z - \lambda + \frac{(1+\lambda)\beta}{C^{-\sigma}} \int v'_k dF,$$

and therefore we need to show that

$$q + p\lambda = -\lambda + \frac{(1+\lambda)\beta}{C^{-\sigma}} \int v'_k dF.$$

Now, since $q + p\lambda + \lambda = q(1 + \lambda)$, we simply need to show that

$$q = \frac{\beta}{C^{-\sigma}} \int v'_k dF$$

but this follows from (20). **The second equality in (23):** Since $v = w\left(\frac{s}{k}, z\right) k^{1-\sigma}$,

$$v_k = k^{-\sigma} ([1 - \sigma] w - s w_s) \quad \text{and} \quad v_s = w_s k^{-\sigma}$$

and the second equality follows. ■

Calculating q and p .—Optimum and equilibrium are the same, and therefore p must equal the marginal social value of a seed:

$$p(s, z) = \frac{1}{U'(C)} v_s = (z - x)^\sigma w_s(s, z). \quad (24)$$

because $\frac{1}{U'(C)} = \frac{(z-x)^\sigma}{k^{-\sigma}}$ and $v_s = \frac{1}{k} w_s(s, z) k^{1-\sigma}$. Now we can finally prove the result on the relation between seeds and q :

Proposition 3 $p(s, z)$ and, hence, $q(s, z)$ are decreasing in s

Proof. By Corollary 1, w is concave in s which means that w_s is decreasing in s . By Corollary 2, x is increasing in s so that $(z - x)^\sigma$ is decreasing in s . Thus the claim holds for p and, by (16) it also holds for q . ■

3.0.2 The negative dependence of Tobin's Q on s .

Since s is probably not in the firm's book value, by Q or "Tobin's Q " we shall mean the firm's ex-dividend value per unit of k . That is, if D is the firm's dividend, then

$$Q = \frac{P}{k} - \frac{D}{k} \quad (25)$$

We shall now see that if we include capital gains as part of dividends (which is in any case needed if dividend policy is to be neutral in its effect on Q), then $Q = q$.

If the firm were to hold no seeds into the next period but, instead, sell them and pay out the proceeds in dividends along with its net earnings, its dividends per unit of k would equal

$$\frac{\hat{D}}{k} = z - x(1 + p) + p(s + \lambda).$$

In addition to \hat{D} , however, the owners of the firm also enjoy capital gains, the expected value of which is just the current value of the newly-planted trees, i.e., qX . Therefore dividends plus capital gains are

$$\frac{D}{k} = \frac{\hat{D}}{k} + qx = z + p(s + \lambda)$$

Substituting into (25) yields and using (17) yields

$$\begin{aligned} Q &= z + q + p(s + \lambda) - [z - x(1 + p) + p(s + \lambda)] - qx \\ &= q. \end{aligned}$$

By proposition 3, Q is decreasing in s . This is a GE effect, however, that applies to the value of trees when all firms have more seeds. It does not hold in the cross section. A firm that owned an above-average stock of seeds would be more valuable than other firms.

4 Incomplete markets

This section simply assumes that the market for seeds is closed and that, while each firm's state (k, S, z) is public information, separate markets for k and s do not exist. It would of course be better to model the friction that causes the market for seeds to have zero transactions, but this would complicate things. So, let us assume that only (k, S) bundles trade in the form of shares of firms. We use the recursive equilibrium concept of Mehra and Prescott (1980) extended to a growing production economy, as done in Jovanovic (2006, Sec. 4).

Suppose that firms' shares trade but that seeds and trees do not. Seeds then have to be stored by the firms that produced them, and the representative firm holds the tree-seed bundle (k, S) under its roof. The household can own a claim on the dividends paid by such a firm and no other assets exist. Therefore this decentralization has just two markets: A market for output, and a market for firms' shares. Since the number of date- t goods (consumption, capital, and seeds) is three, the number of goods exceeds the number of markets, and we cannot be sure that a recursive competitive equilibrium is optimal.

Assume a continuum of firms of measure one and an equal number of households. Equilibrium then requires that each household hold exactly one share. Firms pay $(z - x[s, z])k$ dividends in state (k, s, z) , and households take firms' policies $x(s, z)$ as given.

4.0.3 The household's decision problem

With n shares, a household's wealth is the current dividend, $(z - x)k$ plus the value of his holdings, $\hat{Q}[s, z]kn$. This wealth is spent on consumption and on future holdings of shares $\hat{Q}(s, z)kn'$. Thus $\hat{Q}kn' + C = ([z - x]k + \hat{Q}k)n$, or after dividing through by k ,

$$\hat{Q}n' + c = (z - x + \hat{Q})n,$$

so that

$$c = (z - x)n + \hat{Q}(n - n')$$

where x is given to the household. The household takes the aggregate law of motion of $k'(s, z)$, $x(s, z)$ and $s'(s, z)$ as given. His state is (k, n, s, z) , and, with some of the arguments (s, z) dropped from the notation, his Bellman equation then is

$$V(k, n, s, z) = \max_{n'} \left\{ \frac{\left((z - x[s, z])n + \hat{Q}(s, z)(n - n') \right)^{1-\sigma} k^{1-\sigma}}{1 - \sigma} + \beta \int V(k'(s, z), n', s', z') dF \right\}.$$

Deriving the pricing equation.—As in the planner's problem, $V(k, n, s, z) = W(n, s, z) k^{1-\sigma}$, where

$$W(n, s, z) = \max_{n'} \left\{ \frac{\left([z - x(s, z)]n + \hat{Q}(s, z)[n - n'] \right)^{1-\sigma}}{1 - \sigma} + \beta (1 + x[s, z])^{1-\sigma} \int W(n', s'[s, z], z') dF \right\} \quad (26)$$

The derivative of W with respect to n , call it W_n , exists for much the same reasons that w_s does. Equilibrium requires that $n'(1, s, z) = 1$. At equilibrium, the first-order condition is

$$(z - x[s, z])^{-\sigma} \hat{Q}(s, z) = \beta (1 + x[s, z])^{1-\sigma} \int W_n(1, s'[s, z], z') dF. \quad (27)$$

The envelope theorem then implies

$$W_n(1, s, z) = (z - x[s, z])^{-\sigma} \left[z - x(s, z) + \hat{Q}(s, z) \right].$$

Updating, substituting into (27), and dividing by $(z - x)^{-\sigma}$ gives our version of the Lucas (1978) pricing formula

$$\hat{Q}(s, z) = \beta (1 + x[s, z]) \int M(s, s', z, z') \left(z' - x[s'(s, z), z'] + \hat{Q}(s'[s, z], z') \right) dF, \quad (28)$$

where

$$M(s, s', z, z') \equiv \left(\frac{[1 + x(s, z)](z' - x[s'(s, z), z'])}{z - x(s, z)} \right)^{-\sigma} \quad (29)$$

is the MRS in consumption between today and tomorrow.

4.0.4 The firm's decision problem

Since markets for s do not exist, the firm's only decision is x . Let us use bold letters to denote aggregate states and decisions $\mathbf{x}(s, z)$ and $\mathbf{s}'(s, z)$. Let P denote the cum-dividend price of $1/k'$ 'th of the representative firm, i.e., the price of the tuple $(1, s)$.

Equilibrium is efficient if $P = v_k + sv_S$, with v defined in (7). The functional equation (in units of the consumption good) for its cum-dividend price per unit of k is

$$P(\mathbf{s}, s, z) = \max_x \left(z - x + \beta(1+x) \int M(\mathbf{s}, \mathbf{s}', z, z') P(\mathbf{s}'[\mathbf{s}, z], s', z) dF \right) \quad (30)$$

Writing P in this way implies that s is public information s even when it differs from \mathbf{s} . I.e., (30) assumes that (s, \mathbf{s}) is a sufficient statistic for the how the market values the firm. If the market did not know a firm's s , it would try to guess s from the firm's choice of x , and incentive constraints would be needed to accompany the problem in (30). Mayers and Majluf (1985) deal with this issue. Thus the seeds market does not exist for reasons other than imperfect information about s .

In equilibrium,

1. All firms must choose the same x , and so we ask that in state $(s, z) = (\mathbf{s}, z)$, the firm will behave like other firms. That is, at the fixed point for P , the RHS of (30) is maximized by $x(s, s, z) = \mathbf{x}(s, z)$. This would imply that $s' = \frac{\lambda + s - x(s, s, z)}{1 + x(s, s, z)} = \mathbf{s}'(s, z) = \frac{\lambda + s - \mathbf{x}(s, z)}{1 + \mathbf{x}(s, z)}$.
2. For all (s, z) , the maximized value of the firm must equal the value that the shareholders hold:

$$P(\mathbf{s}, s, z) = z - x(\mathbf{s}, s, z) + (1 + x(\mathbf{s}, s, z)) \hat{Q}(\mathbf{s}, z). \quad (31)$$

In fact, property 1 implies property 2 as one can deduce by setting $x(s, s, z) = \mathbf{x}(s, z)$ for all (s, z) so that $s' = \mathbf{s}'$, in which case substitution from (31) into (30) makes it identical to (28). Thus it suffices to show that property 1 holds. Recall that $U(C) = \frac{e^{1-\sigma}}{1-\sigma}$ so that $U'(C')/U'(C) = [(1+x)(z' - x')/(z - x)]^{-\sigma}$. Then, evaluated at $x = \mathbf{x}$, the FOC in (30), calculated by solving

$$P(\mathbf{s}, s, z) = \max_{s'} \left\{ z - \hat{x}(s', s) + \beta(1 + \hat{x}[s', s]) \int M(\mathbf{s}, \mathbf{s}'[\mathbf{s}, z], z, z') P(\mathbf{s}'[\mathbf{s}, z], s', z') dF \right\} \quad (32)$$

where

$$\hat{x}(s', s) = \frac{\lambda + s - s'}{1 + s'}, \quad (33)$$

and does not depend on the firm's action.

Differentiability of P .—Similar to the proof of Lemma 4 we can establish that $P_s(\mathbf{s}, s, z) \equiv \frac{\partial}{\partial s} P(\mathbf{s}, s, z)$ exists everywhere. Since

$$\frac{\partial \hat{x}}{\partial s'} = \frac{\partial(1 + \hat{x})}{\partial s'} = \frac{\partial}{\partial s'} \left(\frac{1 + \lambda + s}{1 + s'} \right) = -\frac{1 + x}{1 + s'}$$

the derivative w.r.t. s' is $\frac{1+x}{1+s'} - \beta \frac{1+x}{1+s'} \int M' P' dF + (1+x) \beta \int M' P'_s dF \leq 0$, with an exact equality if $s' > 0$. The term $(1+x)$ cancels, and so the FOC to the problem (32) is

$$1 - \beta \int M' P' dF + (1+s') \beta \int M' P'_s dF \begin{cases} = 0 & \text{if } s' > 0 \\ \leq 0 & \text{if } s' = 0 \end{cases}, \quad (34)$$

Efficiency.—Here P is the cum-dividend price of one- k 'th of the firm in current consumption units. Per unit of its k , a firm is a package of $(1, s)$ units of (k, S) . Therefore, efficiency would appear to require that $P = \frac{1}{v'} (v_k + s v_S)$. In what follows we let $x(s, z)$ denote the planner's optimal policy, and $s'(s, z) = \frac{\lambda + s - x(s, z)}{1 + x(s, z)}$.

The next claim states that if the representative firm used the planner's policy, its market value would equal the marginal social value of the bundle (k, S) :

Lemma 6

$$P(s, s, z) = P, \quad (35)$$

where P is given in (23).

Proof. Updating (35) by a period we have $P(s'[s, z], s', z') = (1 - \sigma) (z' - x[s', z'])^\sigma w(s', z')$. Substituting into the RHS of (30), the latter becomes

$$\begin{aligned} & z - x(s, z) + \beta (1 + x[s, z]) \int M(s, s', z, z') (1 - \sigma) (z - x[s, z])^\sigma w(s'[s, z], z') dF \\ &= z - x + (1 - \sigma) \beta \frac{(1+x)^{1-\sigma}}{(z-x)^{-\sigma}} \int w(s', z') dF \quad \text{in view of (29)} \\ &= (1 - \sigma) (z - x[s, z])^\sigma w(s, z) \\ &= P(s, s, z), \quad \text{as claimed in (35)}. \end{aligned}$$

■

The previous lemma is, however, conditional on the assumption that the representative firm uses the planner's policy, i.e., that

$$x(s, s, z) = x(s, z). \quad (36)$$

Next we shall show that (36) does hold if (35) does.

Lemma 7 *If P satisfies (35), then (36) holds.*

Proof. If (36) holds, the firm's FOC, (34), must coincide with the planner's FOC, (10). In view of (29), LHS of (34) can be written as $1 - \beta \int \left(\frac{(1+x)(z'-x')}{z-x} \right)^{-\sigma} (P' - [1 + s'] P'_s) dF$. This is the same as the LHS of (10) if

$$(z' - x')^{-\sigma} (P' - [1 + s'] P'_s) = \left[(z' - x')^{-\sigma} (1 + z') + \lambda w'_s \right]$$

i.e., if

$$1 + z + \frac{\lambda w_s}{(z - x)^{-\sigma}} = P - (1 + s) P_s \quad (37)$$

Now applying the envelope theorem in (32) and noting that, since $\hat{x}(s', s) = \frac{\lambda + s - s'}{1 + s'}$, $\frac{\partial \hat{x}}{\partial s} = \frac{1}{1 + s'} = \frac{1 + x}{1 + \lambda + s}$, gives

$$\begin{aligned} P_s &= \frac{\partial \hat{x}}{\partial s} \left(-1 + \beta \int M' P' dF \right) = \frac{1 + x}{1 + \lambda + s} \left(-1 + \frac{P - (z - x)}{1 + x} \right), \\ &= \frac{P - 1 - z}{1 + \lambda + s}. \end{aligned}$$

Substituting this into (37) for P_s gives

$$\begin{aligned} 1 + z + \frac{\lambda w_s}{(z - x)^{-\sigma}} &= P - (1 + s) \frac{P - 1 - z}{1 + \lambda + s} \\ &= \frac{\lambda P}{1 + \lambda + s} + (1 + s) \frac{1 + z}{1 + \lambda + s} \end{aligned}$$

Rearranging,

$$\frac{\lambda w_s}{(z - x)^{-\sigma}} = \frac{\lambda P - \lambda(1 + z)}{1 + \lambda + s},$$

i.e.,

$$\begin{aligned} w_s &= (z - x)^{-\sigma} \frac{P - (1 + z)}{1 + \lambda + s} \\ &= (z - x)^{-\sigma} \frac{(1 - \sigma)(z - x[s, z])^\sigma w(s, z) - (1 + z)}{1 + \lambda + s} \quad (35), \\ &= \frac{1}{1 + \lambda + s} \left([1 - \sigma] w - (1 + z) [z - x]^{-\sigma} \right) \end{aligned}$$

But this is the same as (9). ■

Lemmas 6 and 7 then imply the main result of this section:

Proposition 4 *The incomplete-market economy has an efficient equilibrium.*

For general parameter values, we cannot rule out other equilibria that are not efficient. In general, the RHS of (32) is not a contraction operator, and then we cannot tell if more than one solution for P exists. However, for some parameter values, e.g. when z is bounded from above by z_{\max} , then we do have uniqueness.

Proposition 5 *If*

$$\beta z_{\max} < 1 \quad (38)$$

Then the incomplete-markets economy (i) has a unique equilibrium, and (ii) it is efficient.

Proof. (i) When (38) holds, the RHS of (32) is a contraction operator with modulus $\beta z_{\max} < 1$, and then the solution for P is unique. (ii) We apply the previous proposition. ■

4.0.5 The effects of financial-market completion

The results say that if all firms are publicly traded, a stock market exists, the emergence of a seeds market should affect neither prices nor quantities. It is enough that all firms trade on the stock market. Even in a financially developed society like the U.S., however, only about one half of the privately owned capital trades on stock markets, and therefore further enlargement of the stock market would probably raise efficiency. That certainly was the conclusion of Greenwood and Jovanovic (1990) in a model in which different-sized firms gradually join the stock market as they grow.

The efficiency result should extend to a situation in which firms do differ because, e.g., they draw different z 's. Jovanovic and Braguinsky (2004) develop a related one-period model in which firms differ in two dimensions: Project quality which we can interpret as s , and managerial ability, which we can interpret as z . They find that even when s is private information to the firm being acquired, the stock market achieves efficiency.

All this must be qualified by noting that seeds, S , do not share some of the features of inventions that are sometimes thought important. Namely,

1. Seeds are of purely private value, and not costlessly reproducible – as information perhaps is – and cannot raise output in more than one firm;
2. The producer of a seed has a perfect property right to it even when markets for seeds do not exist.

If either assumption did not hold, equilibrium would not be efficient.

5 Numerical solution and fitting the data

Data.—Since $z = Y/k$, we use the output-capital ratio to measure z . We measure k by private non-residential fixed assets, NIPA table 4.1; output and investment are from NIPA table 1.1.5. For Patents we use the total number “utility” (i.e., invention) patents from the U.S. Patent and Trademark Office for 1963-2000, and from the U.S. Bureau of the Census (1975, series W-96, pp. 957-959) for 1946-62. The number of registered trademarks is from the U.S. Bureau of the Census (1975, series W-107, p. 959) for 1946-1969, and from various issues of the Statistical Abstract of the U.S. for later years.

The benchmark model with adjustment costs.—When thinking about empirics, we shall compare the model for the standard model with no seeds constraint and with a quadratic adjustment-cost. Instead of (1), we assume that $Y = zk - h\left(\frac{X}{k}\right)k$, where $h(x) = \frac{b}{2}(x - \delta)^2$ is the adjustment cost. It is described more fully in Appendix 4. We continue to have just three values for z , and maintain the same persistence and volatility.

Process for z .—When de-trended linearly it follows an AR(1) process with autocorrelation coefficient 0.903, and innovation variance 0.026. The Tauchen-Hussey procedure for discretizing the AR yields a first-order Markov chain with 3 evenly spread-out states, $(z_1, z_2, z_3) = (0.092, 0.174, 0.256)$, and the symmetric transition probability matrix

$$\begin{array}{ccc}
 & z_1 & z_2 & z_3 \\
 \hline
 z_1 & 0.787 & 0.210 & 0.004 \\
 z_2 & & 0.667 & 0.167 \\
 z_3 & & & 0.787
 \end{array} \tag{39}$$

Table 1: THE MATRIX OF TRANSITION PROBABILITIES FOR z

which has the stationary distribution $(0.307, 0.387, 0.307)$.

Parameters.—At this point we assume that k depreciates at the rate δ and S at the rate γ so that their laws of motion (2) and (4) become $k' = (1 - \delta)k + X$ and $Y' = (1 - \gamma)Y + \lambda k - X$ respectively. The details are in Appendix 2. The parameter values in Table 2

β	σ	δ	γ	λ	\bar{z}	ρ	std(z)
0.95	2.0	0.08	0.15	0.134	0.174	0.903	0.065

Table 2: PARAMETER VALUES FOR THE SEEDS MODEL

were chosen, among other reasons, so as to match (i) An investment-capital ratio of 0.098, and (ii) An average level of Tobin’s Q of 1.22 and (iii) Some properties of the Q and x series since WW2 which will be shown in Figure 5. Section 6.2 shows that for a constant- z economy the growth rate is bounded by $\lambda - \delta$, and that if $x < \lambda - \delta$, seeds accumulate indefinitely and the seeds constraint becomes irrelevant. The depreciation of S is γ and it was chosen based on estimates of private obsolescence of knowledge by Griliches, Pakes, Schankerman and others.

5.1 Simulated decision rules and Q .

For the parameter values and transition probabilities stated in Tables 1 and 2, Figure 4 plots the equilibrium Q , the decision rules and the value function. In all the plots, the variable on the horizontal axis is s , the beginning-of-period seeds-capital ratio. We may summarize the plots as follows:

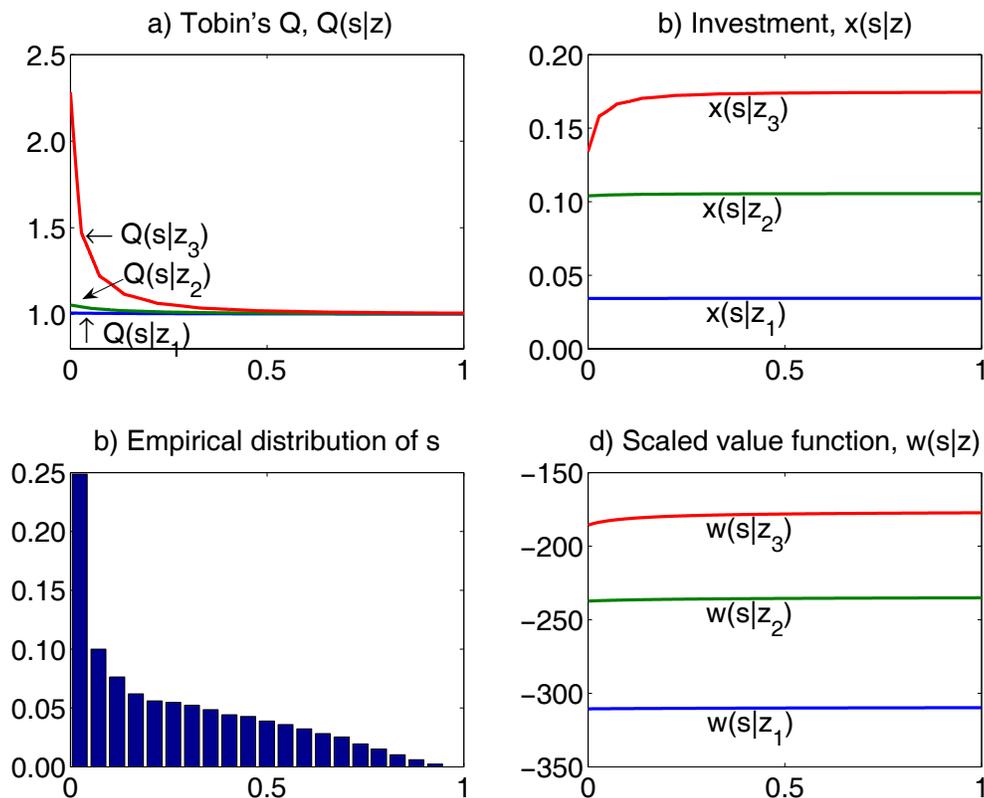


Figure 4: SIMULATED VALUE, DECISION RULES, AND TOBIN'S Q FOR $z_1 < z_2 < z_3$. ON THE HORIZONTAL AXIS IS s .

hence, a negligible correlation between z and s . We now describe how the data were generated and how the model fits. The correlations involving Q will be discussed when we get to intangibles

5.2 Fitting data

The state variables of the model are k, S , and z , and the decision variable is x . In addition, we focused on the price of seeds, p , but the real motivation for it is the role that p plays in the price of the firm, Q . Thus we shall fit the following post-war series: (i) The output-capital ratio, which in the model is z , (ii) The seed-capital ratio, s , (iii) The investment-capital ratio, x , and (iv) Tobin's q as measured by $P - z$, with P given by (23).

Quadratic adjustment costs instead of the seeds constraint.—We shall contrast the seeds model with a standard model with no seeds constraint but with a quadratic

adjustment-cost. That is, instead of (1), the adjustment-cost model assumes that output is $Y = zk - h\left(\frac{x}{k}\right)k$, where $h(x) = \frac{b}{2}(x - \delta)^2$ is the adjustment cost. The model is otherwise the same as the seeds model, except that now λ drops out with the seeds constraint. Table 4 gives the parameter values for the adjustment-cost model:

β	σ	δ	γ	b	\bar{z}	ρ	std(z)
0.95	2.0	0.08	0.15	12.06	0.183	0.902	0.065

Table 4: PARAMETER VALUES FOR THE ADJUSTMENT-COST MODEL

The analysis is in Section 6.2. We continue with the same three values for z , and set $b = 12.06$ which yielded the best fit. The firm's FOC leads to the equation

$$x = \delta + \frac{1}{b}Q = 0.08 + 0.08Q$$

which is numerically quite close to Hayashi's OLS estimate $x = 0.98 + 0.42Q$ for the period 1953-76. In all four Panels of Figure 5, the solid blue lines represent the seeds model, the dashed blue lines represent the adjustment-cost model, and the red lines represent the data. The variables were constructed as follows:

1. The red line in Panel 1 of Figure 5 plots $z = Y/k$ where $Y =$ private non-farm output and $k =$ non-farm stock of capital. The model has just 3 values of z to fit this with: 0.092, 0.174, and 0.256, obtained previously by applying the Tauchen-Hussey procedure to the output-capital ratio.
2. Panel 2 plots the series for s implied by the model as the blue line. Panel 2 also plots several possible proxies for s , each constructed via the formula

$$s' = \frac{n + (1 - \gamma)s - x}{1 - \delta + x}, \quad (40)$$

where n is one of the following: (A) $n =$ patents/ (θk) (red line), (B) $n =$ trademarks/ (θk) (green line), (C) $n = \lambda$ (turquoise line). The constant θ fixes units appropriately; it is explained in Appendix 3.⁵ Red, green, and turquoise lines in the second panel of Figure 5 correspond to cases A, B, and C. In cases A and B the least-squares routine chose $s_0 = 0$ as the initial condition. We noted that the simulated s peaks at 0.32 in the late 80's. The model overpredicts the empirical estimate of s_t , i.e., the estimate of the seeds series that produces the model's best fit to the post-war data. Panel 2 shows all three estimated series for s_t remaining below 0.13. Thus the post-war stock of seeds always was less than thirteen percent of installed capital.

⁵The model is neutral in (λ, θ) . Doubling these two parameters and doubling S_0 doubles S_t for all t but leaves all the other variables unchanged. Therefore θ has been normalized to unity up to this point.

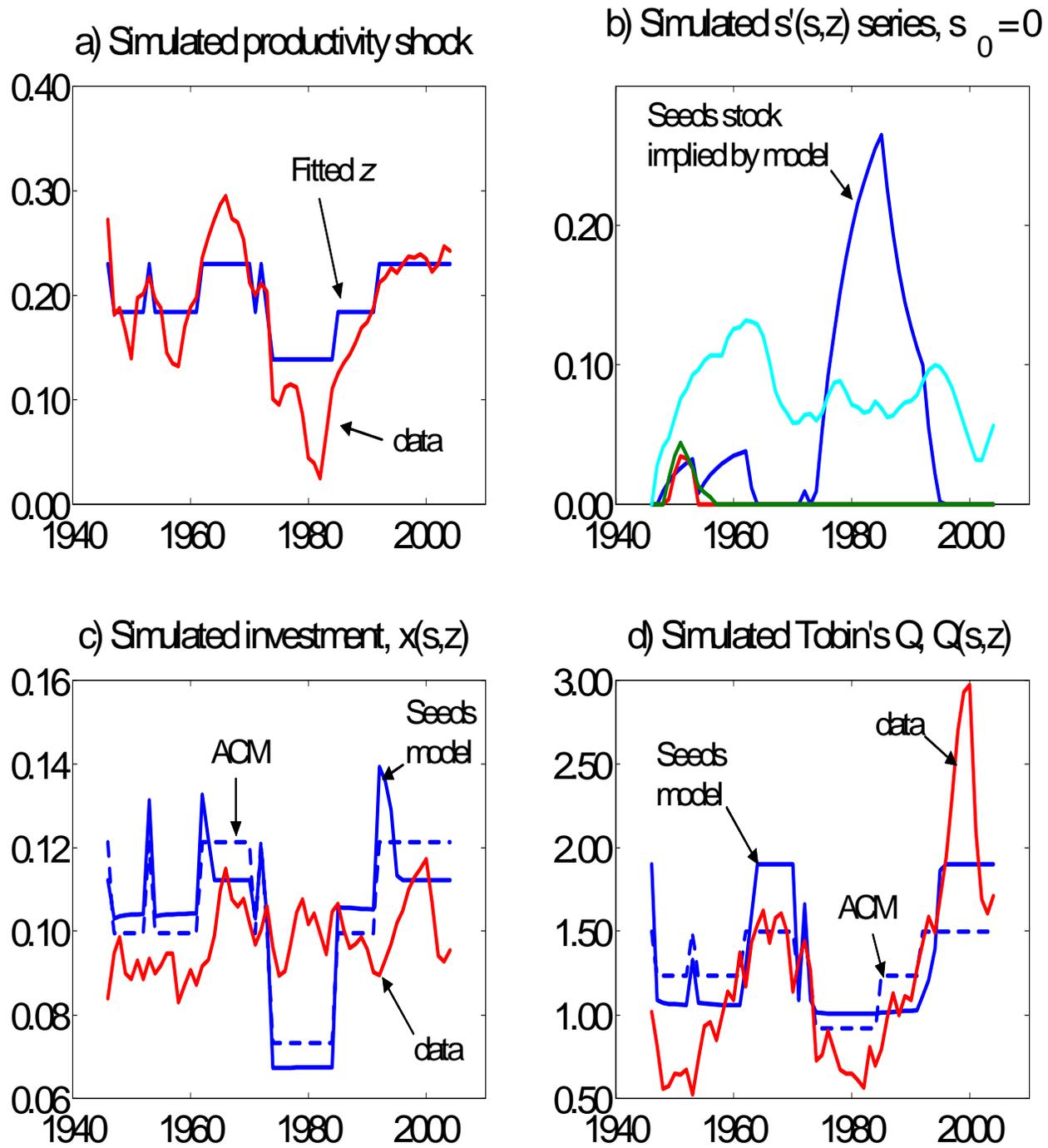


Figure 5: FITTING THE POST-WAR DATA

3. Panel 3 shows that their desire to fit Q and its movements, the seeds and the adjustment-cost models both generate too much investment volatility. This will also be evident in Table 5. Neither model resolves the excess-volatility puzzle. In the seeds model z exerts a more important influence on it than does s . From Panel 2 we see that the simulated s peaks at 0.32
4. In Panel 4 of Figure 5 we plot the actual and fitted Q . For the measured Q , for 1951-1999 we use Hall's series, but since it ends in 1999, for the period 1999-2004 we use Abel's data scaled so that the two Q series match in 1999. This is the red line in Panel 4 of Figure 5. To get a sustained rise in Q we must have a prolonged period during which $z = z_3$. The '90s appear to have been such a period.

The parameters θ and s_0 were chosen to minimize the RSS between the simulated and constructed series. The model has a problem with reconciling the following facts:

- Y/k falls dramatically in the late 70's and early 80's, something that the model interprets as a low- z epoch causing the huge buildup of seeds portrayed in panel 2 and the resulting collapse of Q to its lowest possible level of unity, and
- The rise in Q starting in the early 80s. Even with the accompanying rise in the estimate of z from z_1 to z_2 in the middle 80s and then to z_3 in 1991, it takes time for the model s (the blue line) to be drawn to zero and for Q to rise to its maximal value of 1.75.

5.3 Intangibles and Q

The correlation between s and Q is bolded in the two panels of Table 3. A rise in s represents a rise in the ratio of unimplemented intangible capital to tangible capital. The stock of all intangible capital is $k + S$ with k being the number of seeds already in the ground and being used for production. Therefore the ratio

$$\frac{\text{All intangible capital}}{\text{Tangible capital}} = \frac{k + S}{k} = 1 + s$$

is also monotone in s . This is why the seeds model implies a fall in Q whereas Hall's (2000) implies a rise in Q . In my model, variation in intangibles is caused by variation in the stock of unimplemented seeds. In Hall's model there are variable proportions between intangibles and physical capital in production and there is no storage of intangibles, hence a rise in intangibles gives a rise in the productivity of the firm's measured capital and (barring GE effects) a rise in the firm's Q .

The model matches well the strong positive correlation between z and Q and the negative correlation between s and x . That the latter should be negative in the model may at first seem to contradict Corollary 2 which says that the policy $x(s, z)$ is

increasing in s , a fact that is also borne out by Panel 2 of Figure 4. It turns out that the negative feedback effect of x on s' via (40) is stronger and renders the correlation negative.

We already saw that the model generates too much volatility in investment. To this it is driven by the attempt to also fit Q . But the signs of the $z - x$ correlations are both positive in the two tables, even if their magnitudes differ a lot. The glaring discrepancy is the negative relation between s and z . Like the negative relation between s and x , this one arises because s is constructed using (40), and again reflects the negative effect that x exerts on s' through this accounting relation, and not any negative effect of s on x .

6 Comparison with the adjustment-cost model

We shall now show that our model loosens the relation between investment and Q , so that Abel and Eberly's (2006) argument carries over to the aggregate setting. Figure 3 shows the region Δ on which x cannot respond to z and, hence, to Q . Since Δ contains mainly boom states, we would expect that x should respond more elastically to Q in recessions than in booms.

In the ACM, z is the only state variable and therefore a one-to-one relation emerges between x and Q , and it is shown as the straight black line in Figure 6. When z takes on its lowest value, x is below δ , and Q falls below unity. In the seeds model we also allow negative net investment, but, in contrast to the ACM where the two go hand in hand, in that model this very possibility precludes Q from falling below 1. In the seeds model there are two state variables, s and z .

Next, let us contrast the two models' implications for the second moments of the data. Parameters were chosen for the two models so that they both fit the sample averages $E(z)$, $E(c)$, $E(x)$, and $E(Q)$. Then the two models' abilities to explain the second moments can be contrasted. Column 1 of Table 5 presents summary statistics for the data. As before, the s series was generated via (40). Columns 2 and 3 reports the results of three 100,000-period simulations of two models when the shocks $z \in \{0.092, 0.174, 0.256\}$ are drawn according to the transition matrix in (39).

The first model is the Seeds model; Column 2 presents its implications under the parameter values in Table 2 – the parameters that, together with (39), were used to generate the decision rules in Figures 4 and 5. The initial condition is $s_0 = 0$. One should compare the numbers in Column 2 to the information in Figure 4. For instance, $E(s)$ and $S(s)$ are the mean and standard deviation of the distribution of s plotted in Panel 3 of Figure 4.

Column 3 presents the same set of statistics for the ACM under the parameter values given in Table 4. Both models underpredict the volatility of z by a factor

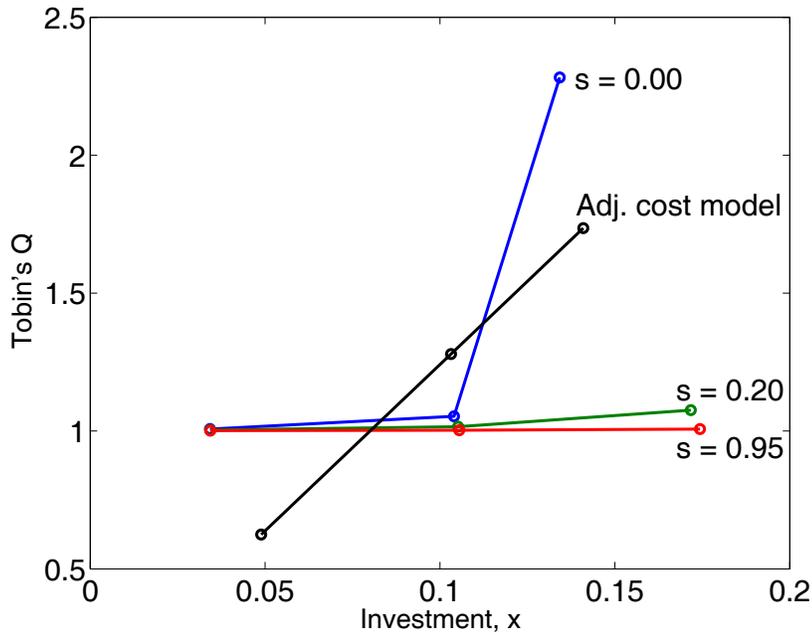


Figure 6: INVESTMENT AND Q IN THE SEEDS AND ADJUSTMENT-COST MODELS

of almost two.⁶ The seeds model overpredicts $S(x)$ by more than the ACM, but it underpredicts $S(Q)$ and $S(c)$ by less. Finally, the seeds model has an additional endogenous variable – seeds – and no new exogenous variables, but as we have seen in Figure 5 it does not explain well any reasonable measure of seeds. Consumption volatility is the same in the two models, but investment is more volatile in the seeds model: Even though the output-capital ratio is equally volatile in the two cases, the seeds constraint induces a negative correlation between c and x on Δ , which allows $S(x)$ to both be higher in the seeds model than in the ACM.

⁶The Tauchen-Hussey procedure is thus pretty far off in this dimension. This seems to be because for a highly persistent process, the approximation with only 3 shocks, the AR coefficient estimated from simulated data underestimates the true coefficient. Since $\text{Var}(z) = \text{Var}(\text{innovation})/(1 - a)$, the ratio is sensitive to a and when a is understated, the ratio can easily be off by a factor of two.

Statistic	Data	Seeds Model $s_0 = 0$	No Seed Constraint Adj-Cost Model
$E(z)$	0.174	0.174	0.188
$E(x)$	0.098	0.098	0.098
$E(s)$	0.195 via (40)	0.262	–
$E(Q)$	1.218	1.218	1.218
$E(c)$	0.076	0.076	0.085
$S(z)$	0.065	0.065	0.065
$S(x)$	0.008	0.048	0.036
$S(s)$	0.047	0.240	–
$S(Q)$	0.571	0.432	0.438
$S(c)$	0.064	0.021	0.022

.Table 5: COMPARISON TO THE ADJUSTMENT-COST MODEL

7 Conclusion

This paper has emphasized the role of new ideas in investment and in the business cycle. It has found that investment options raise the volatility of investment compared to the standard adjustment-cost model. When they facilitate the formation of new capital, new ideas reduce the value of old capital. Thus what we often call intangible capital acts to reduce the value of tangible capital. When intangibles are used up by investment, we found that investment acquires an intertemporal substitution character that is missing in the standard model. Finally, we found that a stock market alone suffices to ensure efficiency of the equilibrium.

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8 Appendix

8.1 Proof of differentiability (Lemma 4)

I use subscripts to denote the state that a policy pertains to. Thus we have the accounting identities

$$s'_s = \frac{\lambda + s - x_s}{1 + x_s} \quad \text{and} \quad s'_{s+h} = \frac{\lambda + s + h - x_{s+h}}{1 + x_{s+h}}.$$

Variations.—We use (33) to figure out the feasible variations.

Variation (i).—If we begin at state $s + h$, and if we want to end up at s'_s , we need an investment of

$$\begin{aligned} \hat{x}(s'_s, s + h) &= \frac{\lambda + s + h - \frac{\lambda + s - x_s}{1 + x_s}}{1 + \frac{\lambda + s - x_s}{1 + x_s}} = \frac{(1 + x_s)(\lambda + s + h) - (\lambda + s - x_s)}{1 + x_s + \lambda + s - x_s} \\ &= \frac{(1 + x_s)h + x_s(\lambda + s) + x_s}{1 + \lambda + s} \\ &= x_s + h \frac{1 + x_s}{1 + \lambda + s}. \end{aligned}$$

Then

$$A_h \equiv \left(\frac{1 + \hat{x}}{1 + x_s} \right)^{1-\sigma} = \left(1 + \frac{h}{1 + \lambda + s} \right)^{1-\sigma},$$

and

$$\hat{x} - x_s = h \frac{1 + x_s}{1 + \lambda + s}.$$

Therefore

$$\begin{aligned} w(s + h, z) &\geq U(z - \hat{x}[s'_s, s + h]) + (1 + \hat{x}[s'_s, s + h])^{1-\sigma} \beta \int w(s'_s, z') dF \\ &= U(z - \hat{x}[s'_s, s + h]) + A_h (1 + x_s)^{1-\sigma} \beta \int w(s'_s, z') dF \\ &= U(z - \hat{x}[s'_s, s + h]) + A_h (w(s, z) - U(z - x_s)) \end{aligned}$$

and

$$\begin{aligned} w(s + h, z) - w(s, z) &\geq U(z - \hat{x}[s'_s, s + h]) - A_h U(c_s) + (A_h - 1)w(s, z) \\ &= U(z - \hat{x}[s'_s, s + h]) - U(c_s) + (A_h - 1)(w(s, z) - U(c_s)). \end{aligned}$$

Dividing both sides by h and taking the limit as $h \searrow 0$ gives

$$\begin{aligned} \frac{d}{ds}w(s, z) &\geq -U'(c_s) \lim_{h \searrow 0} \frac{\hat{x} - x_s}{h} + \lim_{h \searrow 0} \frac{(A_h - 1)}{h} [w(s, z) - U(c_s)] \\ &= -U'(c_s) \frac{1 + x_s}{1 + \lambda + s} + (1 - \sigma) \frac{w(s, z) - U(c_s)}{1 + \lambda + s}. \end{aligned} \quad (41)$$

because, by L'Hôpital's rule,

$$\begin{aligned} \lim_{h \searrow 0} \frac{(A_h - 1)}{h} &= \lim_{h \searrow 0} \frac{dA_h}{dh} = \lim_{h \searrow 0} \frac{d}{dh} \left(1 + \frac{h}{1 + \lambda + s} \right)^{1-\sigma} \\ &= \frac{1 - \sigma}{1 + \lambda + s} \lim_{h \searrow 0} \left(1 + \frac{h}{1 + \lambda + s} \right)^{-\sigma} \\ &= \frac{1 - \sigma}{1 + \lambda + s} \end{aligned}$$

Variation 2: Start from s and end at s'_{s+h} ...

Variation (ii). — If we begin at state s , and if we want to end up at s'_{s+h} , we need an investment of

$$\begin{aligned} \hat{x}(s'_{s+h}, s) &= \frac{\lambda + s - \frac{\lambda + s + h - x_{s+h}}{1 + x_{s+h}}}{1 + \frac{\lambda + s + h - x_{s+h}}{1 + x_{s+h}}} = \frac{(1 + x_{s+h})(\lambda + s) - (\lambda + s + h - x_{s+h})}{1 + x_{s+h} + \lambda + s + h - x_{s+h}} \\ &= \frac{x_{s+h}(\lambda + s) - (h - x_{s+h})}{1 + \lambda + s + h} = \frac{(1 + \lambda + s)x_{s+h} - h}{1 + \lambda + s + h} \\ &= \frac{(1 + \lambda + s + h)x_{s+h} - h(1 + x_{s+h})}{1 + \lambda + s + h} \\ &= x_{s+h} - \frac{h(1 + x_{s+h})}{1 + \lambda + s + h} \end{aligned} \quad (42)$$

$$< x_{s+h} - \frac{h(1 + x_s)}{1 + \lambda + s + h} \quad (43)$$

because by Corollary 2, x is increasing in s . We shall also need the following implication of (42):

$$B_h \equiv \left(\frac{1 + \hat{x}}{1 + x_{s+h}} \right)^{1-\sigma} = \left(1 - \frac{h}{1 + \lambda + s + h} \right)^{1-\sigma}$$

Therefore

$$\begin{aligned} w(s, z) &\geq U(z - \hat{x}) + (1 + \hat{x})^{1-\sigma} \beta \int w(s'_{s+h}, z') dF \\ &= U(z - \hat{x}) + B_h (1 + x_{s+h})^{1-\sigma} \beta \int w(s'_{s+h}, z') dF \\ &= U(z - \hat{x}) - B_h U(z - x_{s+h}) + B_h w(s + h, z). \end{aligned}$$

and therefore

$$w(s, z) - w(s + h, z) \geq U(z - \hat{x}) - B_h U(z - x_{s+h}) + (B_h - 1)w(s + h, z),$$

i.e.,

$$\begin{aligned} w(s + h, z) - w(s, z) &\leq B_h U(z - x_{s+h}) - U(z - \hat{x}) + (1 - B_h)w(s + h, z) \quad (44) \\ &= U(z - x_{s+h}) - U(z - \hat{x}) + (1 - B_h)[w(s + h, z) - U(c_{s+h})] \end{aligned}$$

Now, $[w(s + h, z) - U(z - x_{s+h})]$ is Lipschitz in h for every $z > 0$. This is because it is bounded above by the increment in value when a unit of consumption is added in perpetuity, and the latter is bounded as long as $c > 0$, i.e., as long as $z > 0$). Now, by (43), $x_{s+h} \geq \hat{x} + \frac{h(1+x_s)}{1+\lambda+s+h}$ and therefore

$$U(z - x_{s+h}) - U(z - \hat{x}) \leq U\left(z - \hat{x} + \frac{h(1+x_s)}{1+\lambda+s+h}\right) - U(z - \hat{x})$$

Using the RHS of this expression to replace the first two terms on the RHS of 45) leaves the inequality in (45) undisturbed. Moreover, using L'Hôpital's rule as before,

$$\lim_{h \searrow 0} \frac{1}{h} (1 - B_h) [w(s + h, z) - U(c_{s+h})] = \frac{1 - \sigma}{1 + \lambda + s} [w(s, z) - U(c_s)]$$

Putting this all together,

$$w_s \leq \frac{1}{1 + \lambda + s} (U'(c_s)(1 + x_s) + (1 - \sigma)[w(s, z) - U(c_s)]) \quad (46)$$

Then (41) and (46) imply (9). To see this, (9) says (in this notation) that

$$w_s = \frac{1}{1 + \lambda + s} ([1 - \sigma]w - (1 + z)U'(c)) > 0.$$

For them to be the same we would need that

$$-(1 + x)U' + (1 - \sigma)(w - U) = (1 - \sigma)w - (1 + z)U',$$

i.e.,

$$-(1 + x)U' - (1 - \sigma)U = -(1 + z)U',$$

i.e.

$$(1 - \sigma)U = (z - x)U'$$

which is true because $z - x = c$ so that both sides of the equation equal $c^{1-\sigma}$. Therefore (41) and (46) imply (9).

8.2 Depreciation

Let $\delta =$ depreciation of k and let γ be the depreciation of S . The laws of motion and the value are

$$k' = k(1 - \delta) + X, \quad (47)$$

$$S' = S(1 - \gamma) + \lambda k - X, \quad (48)$$

and

$$v(k, S, z) = \max_{X \leq \lambda k + S} \left\{ \frac{(zk - X)^{1-\sigma}}{1-\sigma} + \beta \int v(k[1 - \delta] + X, \lambda k + S[1 - \gamma] - X, z') dF \right\}. \quad (49)$$

Since

$$\frac{S'}{k'} = \frac{S'}{k} \frac{k}{k'} = \frac{s(1 - \gamma) + \lambda - x}{1 - \delta + x},$$

we have

$$s' = \frac{\lambda + s(1 - \gamma) - x}{1 - \delta + x}, \quad (50)$$

so that $(1 - \delta + x)s' = \lambda + s(1 - \gamma) - x$. Collecting terms, we get

$$xs' + x = \lambda + s(1 - \gamma) - (1 - \delta)s',$$

which leaves us with

$$\hat{x}(s', s) = \frac{\lambda + s(1 - \gamma) - (1 - \delta)s'}{1 + s'}. \quad (51)$$

The auxiliary Bellman equation is

$$w(s, z) = \max_x \left\{ \frac{(z - x)^{1-\sigma}}{1-\sigma} + (1 - \delta + x)^{1-\sigma} \beta \int w\left(\frac{\lambda + s(1 - \gamma) - x}{1 - \delta + x}, z'\right) dF \right\}, \quad (52)$$

and we still have $P = \frac{v_k + sv_s}{C^{-\sigma}}$.

differentiability, i.e., w_s , when there is depreciation I use subscripts to denote the state that a policy pertains to. Thus we have the accounting identities

$$s'_s = \frac{\lambda + s(1 - \gamma) - x_s}{1 - \delta + x_s} \quad \text{and} \quad s'_{s+h} = \frac{\lambda + (s+h)(1 - \gamma) - x_{s+h}}{1 - \delta + x_{s+h}}.$$

If we begin at state $s + h$, and to end up at s'_s we need an investment of

$$\begin{aligned}
\hat{x}(s'_s, s + h) &= \frac{\lambda + s(1 - \gamma) - (1 - \delta)s'_s}{1 + s'_s} \\
&= \frac{\lambda + (s + h)(1 - \gamma) - (1 - \delta)\frac{\lambda + s(1 - \gamma) - x_s}{1 - \delta + x_s}}{1 + \frac{\lambda + s(1 - \gamma) - x_s}{1 - \delta + x_s}} \quad (\text{substituting from [50]}) \\
&= \frac{(1 - \delta + x_s)(\lambda + (s + h)(1 - \gamma)) - (1 - \delta)(\lambda + s(1 - \gamma) - x_s)}{1 - \delta + x_s + \lambda + s(1 - \gamma) - x_s} \\
&= \frac{(1 - \delta + x_s)(\lambda + s(1 - \gamma) + h(1 - \gamma)) - (1 - \delta)(\lambda + s(1 - \gamma) - x_s)}{1 - \delta + \lambda + s(1 - \gamma)} \\
&= \frac{(1 - \delta)(\lambda + s(1 - \gamma) + h(1 - \gamma)) + x_s(\lambda + s(1 - \gamma) + h(1 - \gamma)) - (1 - \delta)(\lambda + s(1 - \gamma) - x_s)}{1 - \delta + \lambda + s(1 - \gamma)} \\
&= \frac{(1 - \delta)h(1 - \gamma) + x_s(\lambda + s(1 - \gamma) + h(1 - \gamma)) + (1 - \delta)x_s}{1 - \delta + \lambda + s(1 - \gamma)} \\
&= \frac{(1 - \delta)h(1 - \gamma) + x_s[1 - \delta + \lambda + s(1 - \gamma) + h(1 - \gamma)]}{1 - \delta + \lambda + s(1 - \gamma)} \\
&= x_s + \frac{(1 - \delta)h(1 - \gamma) + x_sh(1 - \gamma)}{1 - \delta + \lambda + s(1 - \gamma)} \\
&= x_s + h\frac{(1 - \gamma)(1 - \delta + x_s)}{1 - \delta + \lambda + s(1 - \gamma)}.
\end{aligned}$$

Then

$$A_h \equiv \left(\frac{1 - \delta + \hat{x}}{1 - \delta + x_s} \right)^{1 - \sigma} = \left(1 + \frac{h\frac{(1 - \gamma)(1 - \delta + x_s)}{1 - \delta + \lambda + s(1 - \gamma)}}{1 - \delta + x_s} \right)^{1 - \sigma} = \left(1 + h\frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \right)^{1 - \sigma},$$

and

$$\hat{x} - x_s = h\frac{(1 - \gamma)(1 - \delta + x_s)}{1 - \delta + \lambda + s(1 - \gamma)}.$$

Therefore

$$\begin{aligned}
w(s + h, z) &\geq U(z - \hat{x}[s'_s, s + h]) + (1 - \delta + \hat{x}[s'_s, s + h])^{1 - \sigma} \beta \int w(s'_s, z') dF \\
&= U(z - \hat{x}[s'_s, s + h]) + A_h(1 - \delta + x_s)^{1 - \sigma} \beta \int w(s'_s, z') dF \\
&= U(z - \hat{x}[s'_s, s + h]) + A_h(w(s, z) - U(z - x_s))
\end{aligned}$$

and

$$\begin{aligned}
w(s + h, z) - w(s, z) &\geq U(z - \hat{x}[s'_s, s + h]) - A_h U(c_s) + (A_h - 1)w(s, z) \\
&= U(z - \hat{x}[s'_s, s + h]) - U(c_s) + (A_h - 1)(w(s, z) - U(c_s)).
\end{aligned}$$

Dividing both sides by h and taking the limit as $h \searrow 0$ gives

$$\begin{aligned} \frac{d}{ds}w(s, z) &\geq -U'(c_s) \lim_{h \searrow 0} \frac{\hat{x} - x_s}{h} + \lim_{h \searrow 0} \frac{(A_h - 1)}{h} [w(s, z) - U(c_s)] \\ &= -U'(c_s) \frac{(1 - \gamma)(1 - \delta + x_s)}{1 - \delta + \lambda + s(1 - \gamma)} + \frac{(1 - \sigma)(1 - \gamma)}{1 - \delta + \lambda + s(1 - \gamma)} [w(s, z) - U(c_s)] \end{aligned}$$

because, by L'Hôpital's rule,

$$\begin{aligned} \lim_{h \searrow 0} \frac{(A_h - 1)}{h} &= \lim_{h \searrow 0} \frac{dA_h}{dh} = \lim_{h \searrow 0} \frac{d}{dh} \left(1 + h \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \right)^{1 - \sigma} \\ &= \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \lim_{h \searrow 0} \left(1 + h \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \right)^{-\sigma} \\ &= \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)}. \end{aligned}$$

Then

$$\begin{aligned} w_s &= \frac{(1 - \gamma)(1 - \delta + x)}{1 - \delta + \lambda + s(1 - \gamma)} ([1 - \sigma]w - (1 - \delta + x_s)U' - (1 - \sigma)U) \\ &= \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} ([1 - \sigma]w - (1 - \delta + x)(z - x)^{-\sigma} - (z - x)^{1 - \sigma}) \\ &= \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} ([1 - \sigma]w - (z - x)^{-\sigma} [1 - \delta + z]), \end{aligned}$$

which one also could obtain by assuming differentiability in (52) and applying the envelope theorem. The expression collapses to (9) when $\gamma = \delta = 0$.

8.3 Construction of \hat{S}_t

Two practical problems face us when constructing a proxy for S . First, (4) will sometimes lead S to be negative. That is, if we use (??) as a proxy for λk , the resulting S will become negative. To prevent this from happening, we change (4) to

$$S' = \max(0, \lambda k + S - X). \quad (53)$$

Second, we face a units-conversion problem. What we measure, though, is not S but its proxy, \hat{S} , which we shall assume obeys the equation

$$\hat{S} = \theta S = \theta \lambda k \equiv \text{NEW PATENTS \& TRADEMARKS},$$

Since S is measured in consumption units, θ is the number of \hat{S} units per unit of consumption. (Our measures of X and k are already in consumption units). Substituting

for S into (53),

$$\frac{1}{\theta}\hat{S}' = \max\left(0, \lambda k + \frac{\hat{S}}{\theta} - X\right), \text{ i.e., } \hat{S}' = \max\left(0, \theta\lambda k + \hat{S} - \theta X\right),$$

i.e.,

$$\frac{\hat{S}'}{k'}(1 - \delta + x) = \max(0, \theta\lambda + \hat{s} - \theta x),$$

where $\hat{s} = \frac{\hat{S}}{k}$. Therefore the law of motion for \hat{s} is

$$\hat{s}' = \frac{\max(0, \theta\lambda + \hat{s} - \theta x)}{1 - \delta + x},$$

i.e.,

$$\hat{s}' = \frac{\max\left(0, \frac{\text{NEW PATENTS \& TRADEMARKS}}{\text{CAPITAL STOCK}} + \hat{s} - \theta x\right)}{1 - \delta + x},$$

or, dividing both sides by θ ,

$$s' = \frac{\max\left(0, \frac{\text{NEW PATENTS \& TRADEMARKS}}{\theta \cdot \text{CAPITAL STOCK}} + s - x\right)}{1 - \delta + x}.$$

8.4 Details on the standard adjustment-cost model

The adjustment-cost model that simulated and the statistics of which are reported in column 3 of Table 1 goes as follows: Output and dividend is

$$zk - h\left(\frac{X}{k}\right)k$$

where

$$h(x) = \frac{b}{2}(x - \delta)^2.$$

and where k still follows (47) and where the Bellman equation is

$$v(k, z) = \max_X \left\{ \frac{(zk - X - h\left(\frac{X}{k}\right)k)^{1-\sigma}}{1-\sigma} + \beta \int v([1-\delta]k + X, z') dF \right\}. \quad (54)$$

The auxiliary Bellman equation is

$$w(z) = \max_x \left\{ \frac{(z - x - h[x])^{1-\sigma}}{1-\sigma} + (1 - \delta + x)^{1-\sigma} \beta \int w(z') dF \right\}. \quad (55)$$

The FOC is

$$-(z - x - h[x])^{-\sigma} (1 + h'[x]) + (1 - \sigma)(1 - \delta + x)^{-\sigma} \beta \int w(z') dF,$$

i.e.,

$$1 + h'(x) = q,$$

where

$$q = \left(\frac{1 - \delta + x}{z - x - h(x)} \right)^{-\sigma} \beta (1 - \sigma) \int w(z') dF.$$

8.5 Research

Because new seeds are proportional to capital in the model, seeds pile up in recessions, and this depresses Q for a while after the recovery starts. If resources, i.e., research is needed, fewer seeds will be created when p is low. To see how hit might work, let us change (3) to

$$\text{new seeds} = \lambda R^\varepsilon k^{1-\varepsilon}$$

so that (4) becomes

$$S' = \lambda R^\varepsilon k^{1-\varepsilon} + S - X$$

and so that (5) becomes

$$X \leq \lambda R^\varepsilon k^{1-\varepsilon} + S.$$

The planner's Bellman equation becomes

$$v(k, S, z) = \max_{R \geq 0, X \leq \lambda R^\varepsilon k^{1-\varepsilon} + S} \left\{ \frac{(zk - X - R)^{1-\sigma}}{1 - \sigma} + \beta \int v(k + X, \lambda R^\varepsilon k^{1-\varepsilon} + S - X, z') dF \right\}.$$

For $\sigma \neq 1$, v is still of the form

$$v(k, S, z) = w(s, z) k^{1-\sigma},$$

where $w(s, z) = v(1, s, z)$, and where w satisfies

$$w(s, z) = \max_{(r, x) \in \Omega(s)} \left\{ \frac{(z - x - r)^{1-\sigma}}{1 - \sigma} + (1 + x)^{1-\sigma} \beta \int w\left(\frac{\lambda r^\varepsilon + s - x}{1 + x}, z'\right) dF \right\}$$

where

$$r = \frac{R}{k}$$

and

$$\Omega(s) = \{(r, x) \mid x \leq s + \lambda r^\varepsilon\}.$$

We do not worry about the non-negativity of r because the Inada condition given that $\varepsilon < 1$, and we ignore the constraint on the non-negativity of C because when $\sigma > 1$ it is never violated. If z was firm specific and if seeds could not be stored, this version of the model would be close to Klette and Kortum (2004) and Lentz and Mortensen (2005).

The problem with this is that it introduces a Q-elastic supply of seeds, which will limit somewhat how much Q can rise in booms. In sum, it will produce less variation in Q, but maybe a more realistic seeds.

8.6 The deterministic seeds model

Suppose z is a constant. Let

$$x = \frac{X}{k} \quad \text{and} \quad s = \frac{S}{k}.$$

Since k does not depreciate, x then equals the growth rate of k and of C . Let's solve for the constant-growth rate that would obtain in the absence of the constraint (5). We shall call this the "desired" growth rate, x^d . Then $U'(C_{t+1})/U'(C_t) = (1+x)^{-\sigma}$ and the effective discount factor is

$$\hat{\beta} \equiv \beta(1+x)^{-\sigma}. \quad (56)$$

An additional unit of capital produces z units for ever, and so optimal investment leads to a Tobin's Q of unity:

$$Q \equiv \left(\frac{\hat{\beta}}{1-\hat{\beta}} \right) z = 1. \quad (57)$$

Equations (56) and (57) can be solved for x^d :

$$1+x^d = (\beta[1+z])^{1/\sigma}. \quad (58)$$

The model collapses to the standard model if s goes off to infinity. We seek parameter restrictions that will prevent this from happening. From (5),

$$x_t \leq \min(z, \lambda + s_t) \quad (59)$$

This, however, is a short-run constraint, that holds at each t . If k were to grow faster than λ , s_t would eventually become negative. To see this, combine (4) and (3) to get $S' = S - X + \lambda k$ and, hence,

$$s_{t+1} = \frac{\lambda + s_t - x_t}{1 + x_t}. \quad (60)$$

It's easy to show that λ is the maximal feasible long-run growth rate. Let ε be a constant, and suppose that $x = \lambda + \varepsilon$. Then

Lemma 8 For all $s_0 \geq 0$,

- (i) $\varepsilon > 0 \implies s_t \rightarrow -\infty$
- (ii) $\varepsilon < 0 \implies s_t \rightarrow +\infty$

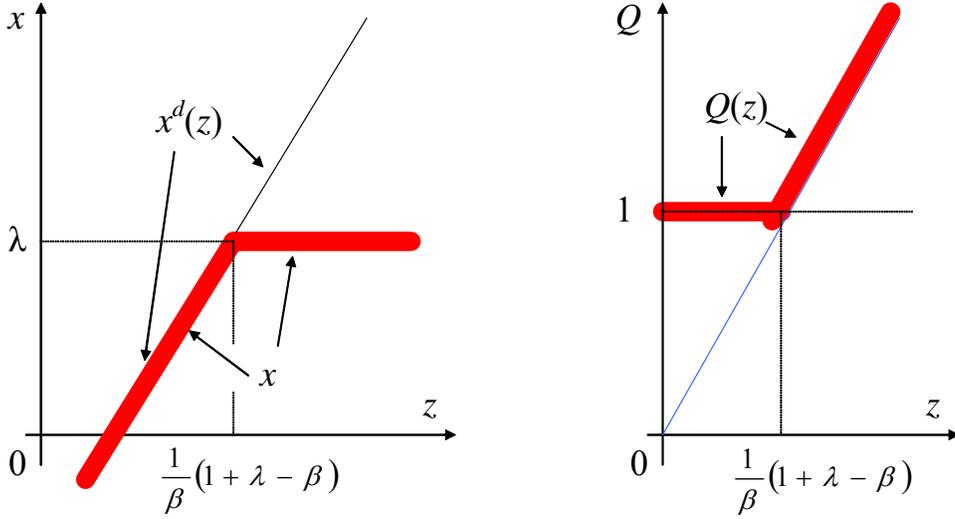


Figure 7: COMPARATIVE STEADY STATES FOR x AND q WHEN $\sigma = 1$.

Proof. (i) Let $\varepsilon > 0$. Then $s_{t+1} = \frac{\lambda + s_t - x}{1+x} = \frac{s_t - \varepsilon}{1+x} < s_t - \frac{\varepsilon}{1+x}$, so that $s_t < s_0 - \left(\frac{\varepsilon}{1+x}\right)t \rightarrow -\infty$. (ii) let $\varepsilon < 0$. Then $s_{t+1} > s_t + \frac{|\varepsilon|}{1+x}$ so that $s_t > s_0 + \frac{|\varepsilon|}{1+x}t \rightarrow +\infty$.

■

Desired growth exceeds λ if

$$[\beta(1+z)]^{1/\sigma} > 1 + \lambda,$$

which is also when the seeds constraint binds in every period. High values of z or β , and low values of σ and λ make it more likely that this inequality will hold. Tobin's Q is just the present value of the marginal products of capital, $\sum_{t=1}^{\infty} \tilde{\beta}^t z$, i.e.,

$$Q = \left(\frac{\tilde{\beta}}{1 - \tilde{\beta}} \right) z, \quad \text{where } \tilde{\beta} = \beta(1 + \lambda)^{-\sigma} > \beta(1 + x^d)^{-\sigma} = \hat{\beta}.$$

Values of Q above unity arise because consumption growth is lower than it would be under x^d ; the rate of interest is thus lower, and this raises the present value of income from capital above its cost.

The case $\sigma = 1$.—From (58), the desired investment and growth rate x is

$$x^d(z) = \beta z - (1 - \beta),$$

and Tobin's Q is

$$Q(z) = \begin{cases} 1 & \text{if } x^d(z) \leq \lambda \\ \frac{\beta}{1 + \lambda - \beta} z & \text{if } x^d(z) > \lambda \end{cases}.$$

The value of z at which $x^d(z) = \lambda$ is $\frac{1}{\beta}(1 + \lambda - \beta)$. Figure 7 plots $x^d(z)$ and $Q(z)$. Of course, $x = \min(\lambda, x^d[z])$.

Transitional dynamics in the deterministic case These are easier to analyze if time is continuous. Let (k_0, S_0) be given, with $S_0 > 0$. Let preferences be

$$\int_0^\infty \frac{1}{1-\sigma} e^{-\rho t} C_t^{1-\sigma} dt.$$

Output is

$$zk = C + X,$$

the seed constraint reads

$$X \leq S$$

and the laws of motion are

$$\dot{S} = \lambda k - X \quad \text{and} \quad \dot{k} = k + X$$

In the absence of the seed constraint we would have $\frac{\dot{C}}{C} = \frac{\dot{k}}{k} = \frac{z-\rho}{\sigma}$, and so we shall assume that

$$\lambda < \frac{z-\rho}{\sigma} \tag{61}$$

so that eventually the seed constraint must bind, and so that eventually we know that $X = \lambda k$. But we want to see how fast this happens from initial conditions. We especially want the time path of Tobin's Q , defined here as the discounted marginal product of k :

$$Q = z \int_t^\infty e^{-\rho(\tau-t)} \frac{U'(C_\tau)}{U'(C_t)} d\tau$$

In the limit, consumption will grow at the rate λ so that $\frac{U'(C_\tau)}{U'(C_t)} = e^{-\sigma\lambda(\tau-t)}$ and Q will converge to

$$Q_\infty = z \int_t^\infty e^{-(\rho+\sigma\lambda)(\tau-t)} d\tau = \frac{z}{\rho + \sigma\lambda}$$

where the rate of interest is

$$\rho + \sigma\lambda$$

which is less than z if (61) holds, so that $Q_\infty > 1$. But if (61) does not hold, then consumption grows at the rate $\frac{z-\rho}{\sigma}$ and $Q_\infty = 1$.

This is a version of the exhaustible-resources problem. The Hamiltonian is

$$\frac{(zk - X)^{1-\sigma}}{1-\sigma} + \mu X + m(\lambda k - X) + n(S - X)$$

The optimality conditions are

$$\begin{aligned} X &: -(zk - X)^{-\sigma} + \mu - m - n = 0 \\ k &: z(zk - X)^{-\sigma} + \lambda m = -\dot{\mu} + \rho\mu \\ S &: n = -\dot{m} + \rho m \end{aligned}$$

and the two constraints must hold.

The region $[0, T)$ where $X < S$.—This is the initial stage, for a finite time, call it $[0, T]$. In this region, $n = 0$ so that the last condition implies

$$m_t = m_0 e^{\rho t} \quad \text{for } t < T$$

The first condition implies, on this region,

$$(zk - X)^{-\sigma} = \mu - m.$$

Substituting all this into the middle condition gives us

$$z(\mu - m_0 e^{\rho t}) + \lambda m_0 e^{\rho t} = -\dot{\mu} + \rho\mu$$

which is the differential equation

$$\dot{\mu} = (\rho - z)\mu + (z - \lambda)m_0 e^{\rho t}$$

Now an equation of the form $\frac{dx}{dt} = Ax + Be^{\rho t}$ has the solution $x = C_1 e^{At} + B \frac{e^{\rho t}}{\rho - A}$. Therefore

$$\mu_t = C_1 e^{(\rho - z)t} + \frac{(z - \lambda)m_0}{1 + z} e^{\rho t}$$

The region $[T, \infty)$.—Here all the multipliers are constant. In particular

$$\mu = Q_\infty = 1 + m.$$