Identification and Price Determination with Taylor Rules: A Critical Review

John H. Cochrane*

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Abstract

The parameters of the Taylor rule relating interest rates to inflation and other variables are not identified in new-Keynesian models. Thus, Taylor rule regressions cannot be used to argue that the Fed conquered inflation by moving from a 'passive' to an 'active' policy in the early 1980s. Since there is nothing in economics to rule out explosive hyperinflations, price level determinacy requires ingredients beyond the Taylor principle, such as a non-Ricaridan fiscal regime.

1 Introduction

How is the price level determined, in modern fiat-money economies in which the central bank follows an interest rate target? The "new-Keynesian" or "Taylor Rule" approach to monetary economics provides the current "standard answer" to this question. There are two core propositions to this view: 1) The price level (or inflation) is determinate because the Fed systematically raises nominal interest rates more than one-for-one with inflation. This 'active' interest rate-target eliminates the indeterminacy of the price level that holds under standard fixed interest rate targets. 2) U. S. inflation *was* stabilized in the early 1980s by a change from a 'passive' policy in which interest rates did not respond sufficiently to inflation to an 'active' monetary policy in which they did do so. Most famously, Clarida, Gali and Gertler (2000) fit Taylor rules to the U.S. Federal reserve, running regressions of interest rates on inflation. They find coefficients below one up to 1980, and above one since. Any good theory needs a stylized reading of history. Monetarism has Friedman and Schwartz' *Monetary theory.* This is it for Taylor rules.

I argue against both propositions. To see the key point, we need to understand how new-Keynesian models work. They *do not* say that higher inflation causes the Fed to raise real interest rates, which in turn lowers "demand" and hence stabilizes future inflation. That's "old-Keynesian" logic. That logic produces stable dynamics, in which we solve backward for endogenous variables as a function of past shocks. Instead, new-Keynesian models say that higher inflation would lead the Fed to raise future inflation explosively. For only one value of

*Graduate School of Business, University of Chicago and NBER. 5807 S. Woodlawn Ave. Chicago IL 60637. 773 702 3059. john.cochrane@gsb.uchicago.edu. I gratefully acknowledge research support from the CRSP and the NSF via a grant administered by the NBER. I am grateful to Eric Leeper, Lars Hansen, Chris Sims and participants at the Spring 2005 Inidiana University monetary economics conference for useful suggestions.

inflation today will we fail to see such an explosion. Ruling out explosive paths, new-Keynesian modelers conclude that inflation today is determined as that value.

Here, then, is the empirical problem: In this logic, explosive inflation represents an off-equilibrium threat. *In equilibrium* the threat of explosive inflation is never seen or measured. The dynamics of inflation (and real variables) *in equilibrium* can tell us *nothing* about the nature of off-equilibrium threats. The crucial coefficients, i.e. the parameters of the "Taylor rule" and the "forward looking roots" are not *identified.* The change in regression coefficients pre- and post- 1980 tells us *nothing* about determinacy. If you run a Taylor-rule regression in artificial data generated by a new-Keynesian model in which the structural coefficient of interest rates on inflation is greater than one, what you measure can be both greater or less than one, and it has no information about the true Taylor-rule coefficient.

Back to theory, nothing in economics rules out explosive *nominal* paths in the first place. (Transversality conditions rule out *real* explosions, but not nominal ones.) Therefore, nothing in economics allows us to insist on the unique "locally bounded" equilibrium. The Taylor principle, in the context of a new-Keynesian model, does not determine the price level.

So what *does* determine the price level? A completely free commodity (gold) standard or exchange rate peg can determine the price level, at least relative to the chosen standard. But while logically transparent, this possibility cannot apply to modern fiat-money economies.

The quantity theory MV=PY can determine the price level. This mechanism requires two crucial ingredients. First, there must be a "special" medium of exchange, rendered unique in that capacity by law or by custom. Second, that medium must be held in artificially low supply. If people can make transactions with bonds, foreign currency, freely created "inside money" (banknotes, iou's) or if the central bank leaves the money supply passive, MV=PY no longer determines the price level. The first requirement is tenuous in modern economies. The second requirement is blatantly violated. Central banks target interest rates. Even if the quantity theory *could* determine the price level in a modern economy– if "money demand" were "stable" enough – interest rate targeting means that it does not do so.

There is one economic theory remaining: the fiscal theory of the price level. New-Keynesian models adopt "Ricardian" regimes, in which the government stands ready to raise taxes or lower spending to accommodate any inflation-induced changes in the value of government debt. By doing so, they throw away an equilibrium condition with the potential to determine the price level. The price level *can* be determined by changing this assumption, and adopting (or recognizing) a fiscal regime that is at least partially non-Ricardian. Since none of commodity standards, the quantity theory, or new-Keynesian / Taylor rule models can determine the price level in a fiat money economy with interest rate targets, I conclude that the "non-Ricardian" fiscal regime is the only economic model that can do so.

"Economic" is an important qualifier. Most of the case for Taylor rules in popular writing, and sometimes in academic contexts (Taylor 1999 for example) switches roots and emphasizes a stabilizing, rather than destabilizing, story: higher interest rates reduce demand which reduces future inflation. This is a pleasant and intuitively pleasing story to many, but it throws out the edifice of theoretical coherence – explicit underpinnings of optimizing agents, clearing markets etc. – built up by the new-Keynesian effort. If this is, in fact, how inflation is stabilized in modern economies, economists really have no idea *why* this is so.

The theoretical point is not as dramatic as it appears. Though I argue against two cornerstones of the new-Keynesian world-view, removing them does not necessarily bring down the edifice. I do not argue that we must start and finish all price level analysis with nominal debt and prospective deficits. The determinacy issue is one of pruning multiple equilibria. If we replace the rule "insist on explosive dynamics and keep only the unique equilibrium that does not explode" with "keep only equilibria consistent with the government valuation equation," (and if that can be done) much of the nature and dynamics of the equilibria can remain unaffected. In particular, if we can specify a set of fiscal constraints that rules out explosive inflation paths, then the equilibrium dynamics of the new-Keynesian model can remain completely unchanged.

However, it is crucial to spell out the ultimately *fiscal* considerations that rule out such equilibria in order to say that the price level is determined, and if so at what level. In thinking about the ultimate determinants of price stability, those fiscal considerations are at a minimum a crucial part of the story.

Readers may greet my description of new-Keynesian models with some scepticism. How could we possibly think that the price level could be determined in a frictionless model, even a cashless model? How can we possibly think that the price level is determined by a commitment by the Fed to explosively inflate or deflate in response to the tiniest off-equilibrium realization? Yet that *is* how the models work, and if one is sceptical, then that scepticism should motivate a search for different models.

2 A simple model

2.1 Identification

We can see the points in a very simple model consisting only of a Fisher equation and a Taylor rule describing Fed policy:

$$i_t = r + E_t \pi_{t+1} \tag{1}$$

$$i_t = r + \phi \pi_t + x_t \tag{2}$$

where i_t = nominal interest rate, π_t = inflation, r constant real rate, and x_t = random component to monetary policy. The coefficient ϕ measures how sensitive the central bank is to inflation. The "random component" x_t is not necessarily a "shock", so I allow it to be serially correlated,

$$x_t = \rho x_{t-1} + \varepsilon_t. \tag{3}$$

The random component can represent a response to GNP or other variables, but those variables (in this simple example) do not enter the "IS" or Fisher equation.

We can solve this model by substituting out the nominal interest rate, leaving only inflation,

$$E_t \pi_{t+1} = \phi \pi_t + x_t. \tag{4}$$

The basic points do not require the Phillips - IS curve features of new-Keynesian models, and thus any frictions. This claim needs to be shown, and do this by expanding the analysis to include fully-specified new-Keynesian models below. It is routine in the new-Keynesian literature to study determinacy in such a stripped down and frictionless model (King 2000 p. 76, Woodford 2001 for example).

Solutions and determinacy with $\phi > 1$

Following standard procedure in the new-Keynesian tradition, when $\phi > 1$ we solve this difference equation forward, and we restrict attention to the unique locally bounded (nonexplosive) solution, giving us

$$\pi_t = -\sum_{j=0}^{\infty} \frac{1}{\phi^{j+1}} E_t \left(x_{t+j} \right) = -\sum_{j=0}^{\infty} \frac{\rho^j}{\phi^{j+1}} x_t = -\frac{x_t}{\phi - \rho}. \tag{5}$$

Since π_t is proportional to x_t, the dynamics of equilibrium inflation are simply those of the shock x_t,

$$\pi_t = \rho \pi_{t-1} + w_t \tag{6}$$

$(w_t = -\varepsilon_t/(\phi - \rho))$. Since this is the unique locally bounded solution, the new-Keynesian literature concludes that the inflation rate is determinate in this model.

Identification with $\phi > 1$

If we accept this solution, however, the contrast between equation (4) and equation (6) makes the central identification point: *the dynamics of equilibrium inflation identify the serial correlation ρ of the monetary policy disturbance x_t not the Taylor-rule coefficient ϕ.*

Perhaps if we ran explicit Taylor-rule regressions we would get a different answer? Using (1) and the solution (6), we know that in equilibrium interest rates follow

$$i_t = r + E_t \pi_{t+1} = r + \rho \pi_t \tag{7}$$

There is no error term, so an OLS regression will recover ρ exactly. Thus, *a Taylor rule regression of i_t on π_t will estimate the shock serial correlation parameter ρ rather than the Taylor rule parameter ϕ.*

What happened to the Fed policy rule, Equation (2)? The solution (5) shows that the right hand variable π and the error term x are correlated – perfectly correlated in fact.

Since the issue is correlation of right hand variables and errors, perhaps we can be clever (as, for example, Clarida, Gali and Gertler 2000 are) and run the Taylor rule regression by instrumental variables? Alas, the only instruments at hand are lags of π_t and i_t, themselves endogenous and thus invalid instruments. For example, if we use all available variables lagged once as instruments, we have

$$\begin{aligned} E(\pi_t | \pi_{t-1}, i_{t-1}, \pi_{t-2}, i_{t-2} ....) &= \rho \pi_{t-1} \\ E(i_t | \pi_{t-1}, i_{t-1}, \pi_{t-2}, i_{t-2} ....) &= r + \rho^2 \pi_{t-1} \end{aligned}$$

Thus the instrumental variables regression gives exactly the same estimate

$$E(i_t | \Omega_{t-1}) = r + \rho E(\pi_t | \Omega_{t-1})$$

Is there nothing clever we can do? No. The equilibrium dynamics of the observable variables are given by (6) and (7),

$$\begin{aligned} \pi_t &= \rho \pi_{t-1} + w_t \\ i_t &= r + \rho \pi_t \end{aligned}$$

The equilibrium dynamics *do not involve ϕ*. They are *the same for every value of $\phi > 1$. ϕ is not identified from data on $\{i_t, \pi_t\}$ in the equilibrium of this model.* More formally, the likelihood function for $\{\pi_t, i_t\}$ does not involve ϕ. The only point of ϕ is to threaten hyperinflation in order to rule out equilibria. It does not matter at all *how fast* that hyperinflation comes.

2.2 Lubik and Schorfheide; testing regions

Lubik and Schorfheide (2004) claim to test for determinacy vs. indeterminacy. Obviously, we need to understand this claim. The short answer is that the appearance of identification in Lubik and Schorfheide's analysis comes only from arbitrary restrictions on the dynamics of the shocks[1].

Lubik and Schorfheide explain their ideas in the same single-equation setup as I use above, simplifying even further by assuming a white noise monetary policy disturbance

$$x_t = \varepsilon_t$$

(i.e., $\rho = 0$). The equilibrium is characterized again by (4) which becomes

$$E_t \pi_{t+1} = \phi \pi_t + \varepsilon_t.$$

The solutions are, generically,

$$\pi_{t+1} = \phi \pi_t + \varepsilon_t + \delta_{t+1}$$

where δ_{t+1} represents the inflation forecast error. If $\phi > 1$, the unique locally bounded solution is

$$\pi_t = -\frac{\varepsilon_t}{\phi}.$$

If $\phi < 1$, then any δ_{t+1} with $E_t \delta_{t+1} = 0$ is possible – there are multiple equilibria.

Lubik and Schorfheide show that ϕ is not identified when $\phi > 1$. For example, the likelihoods in their Figure 1 are flat functions of ϕ for the region $\phi > 1$. However, they still claim to be able to test for determinacy. The essence of their test is a claim that the model with indeterminacy $\phi < 1$ can produce time-series patterns that the model with determinacy cannot produce.

They explain the result with this simple example. Since δ_{t+1} is arbitrary, it does no harm to restrict $\delta_{t+1} = M\varepsilon_{t+1}$ with M an arbitrary parameter. In this example, then, the (local or bounded) solutions are

$$\begin{aligned} \phi &> 1\colon \pi_t = -\frac{\varepsilon_t}{\phi} \\ \phi &< 1\colon \pi_t = \phi \pi_{t-1} + \varepsilon_{t-1} + M\varepsilon_t \end{aligned} \tag{8}$$

Thus, if $\phi > 1$, the model can only produce white noise inflation π_t. If $\phi < 1$, the model produces an ARMA (1,1) in which ϕ is identified as the AR root. Thus, if you saw an ARMA(1,1), you would know you're in the *region* of indeterminacy. They go on to construct a likelihood ratio test for determinacy vs. indeterminacy.

Alas, even this kind of identification is achieved only by restricting the nature of the shock process x_t. If the shock process x_t is *not* white noise, than the $\phi > 1$ solution can display complex dynamics in general, and an ARMA(1,1) in particular. Since the shock process is unobserved, we cannot in fact tell even the region $\phi > 1$ from the region $\phi < 1$. I can sum up this point in a proposition:

Proposition: *For any time series process of $\{i_t, \pi_t\}$, and for any $\tilde{\phi}$, one can construct an x_t process that generates given process for the observables $\{i_t, \pi_t\}$. If $\tilde{\phi} > 1$, the observables are generated as the unique bounded forward-looking solution. In either case, given $\pi_t = a(L)\varepsilon_t$ we construct $x_t = b(L)\varepsilon_t$ with $b_j = a_{j+1} - \tilde{\phi} a_j$.*

[1] Beyer and Farmer (2006), reviewed below, make this point with a series of examples.

In particular, any observed time series process for $\{i_t, \pi_t\}$ that is consistent with a $\phi < 1$ model is also consistent with a different $\tilde{\phi} > 1$ model. Thus, absent restrictions on the unobserved forcing process $\{x_t\}$, there is no way to tell the regime with determinacy from the regime with indeterminacy. (Equivalently, the *joint* set of parameters including ϕ and the parameters of the x_t process are unidentified; one can only identify some of these parameters, e.g. $\phi < 1$ vs. $\phi > 1$, by fixing others, e.g., the parameters of x_t.)

Proof. Start with *any* process for inflation $\pi_t = a(L)\varepsilon_t$. Choose an arbitrary $\tilde{\phi} > 1$.Then, we *construct* a disturbance process $x_t = b(L)\varepsilon_t$ so that the forward-looking equilibrium with arbitrary $\tilde{\phi} > 1$ generates the the desired time-series process for inflation, i.e. we construct $b(L)$ so that Equation (5) holds,

$$a(L)\varepsilon_t = -E_t \sum_{j=0}^{\infty} \frac{1}{\tilde{\phi}^{j+1}} x_{t+j} = -E_t \sum_{j=0}^{\infty} \frac{1}{\tilde{\phi}^{j+1}} b(L)\varepsilon_{t+j}.$$

Finding $b(L)$ is a simple bit of time-series algebra. We can write the answer explicitly as

$$b_j = a_{j+1} - \tilde{\phi} a_j. \tag{9}$$

It's easy enough to check that this answer is correct:

$$-E_t \sum_{j=0}^{\infty} \frac{1}{\tilde{\phi}^{j+1}} b(L)\varepsilon_{t+j} = -E_t \sum_{j=0}^{\infty} \frac{1}{\tilde{\phi}^{j+1}} \sum_{k=0}^{\infty} \left(a_{k+1} - \tilde{\phi} a_k\right) \varepsilon_{t+j-k}$$

$$\begin{aligned} = \quad & -\frac{1}{\tilde{\phi}} \left[\left(a_1 - \tilde{\phi} a_0\right) \varepsilon_t + \left(a_2 - \tilde{\phi} a_1\right) \varepsilon_{t-1} + \left(a_3 - \tilde{\phi} a_2\right) \varepsilon_{t-2} + ... \right] \\ & -\frac{1}{\tilde{\phi}^2} \left[\left(a_2 - \tilde{\phi} a_1\right) \varepsilon_t + \left(a_3 - \tilde{\phi} a_2\right) \varepsilon_{t-1} + \left(a_4 - \tilde{\phi} a_5\right) \varepsilon_{t-2} + ... \right] \\ & -\frac{1}{\tilde{\phi}^3} \left[\left(a_3 - \tilde{\phi} a_2\right) \varepsilon_t + \left(a_4 - \tilde{\phi} a_3\right) \varepsilon_{t-1} + \left(a_5 - \tilde{\phi} a_4\right) \varepsilon_{t-2} + ... \right] + ... \end{aligned}$$

$$= a_0\varepsilon_t + a_1\varepsilon_{t-1} + a_2\varepsilon_{t-2} + ...$$

Deriving the answer takes a little more work, and is therefore presented in the Appendix (see Equations (49)-(50)).

If we choose a $\tilde{\phi} < 1$, then the construction is even easier. The solutions to (4) are

$$\pi_{t+1} = \tilde{\phi}\pi_t + x_t + \delta_{t+1}$$

where δ_t is an arbitrary unforecastable shock. To construct an x_t we need therefore

$$(1 - \tilde{\phi} L)\pi_{t+1} = x_t + \delta_{t+1}.$$

Obviously, forecast errors must be equated, so we must have $\delta_{t+1} = a_0\varepsilon_{t+1}$. Then,

$$\begin{aligned} (1 - \tilde{\phi} L)a(L)\varepsilon_{t+1} &= b(L)\varepsilon_t + a_0\varepsilon_{t+1} \\ (1 - \tilde{\phi} L)a(L) &= a_0 + Lb(L) \\ (1 - \tilde{\phi} L)a(L) - a_0 &= Lb(L) \end{aligned}$$

or, explicitly,

$$\left(a_1-\tilde{\phi}a_0\right)L+\left(a_2-\tilde{\phi}a_1\right)L^2+..=b_0L+b_1L^2+...$$

or, once again

$$b_j=a_{j+1}-\tilde{\phi}a_j.$$

i_t is just given by $i_t=r+E_t\left(\pi_{t+1}\right)$, and so adds nothing once we match π dynamics.

Example: Suppose we generate data from the Lubik-Schorfheide example with $\phi<1$, i.e. $x_t=\varepsilon_t$ is i.i.d., and therefore π_t follows the ARMA(1,1) process (8). We can generate *exactly* the same solution from a model with *arbitrary* $\tilde{\phi}>1$ if we let the policy disturbance x_t be an ARMA(1,1) rather than restrict it to be white noise. From (8), we have

$$\begin{aligned}\pi_t &= \phi\pi_{t-1}+M\varepsilon_t+\varepsilon_{t-1}\\ &= M\varepsilon_t+(1+\phi M)\varepsilon_{t-1}+\phi(1+\phi M)\varepsilon_{t-2}+\phi^2(1+\phi M)\varepsilon_{t-3}+...\end{aligned}$$

Equation (9) then says we choose

$$\begin{aligned}x_t &= \left((1+\phi M)-\tilde{\phi}M\right)\varepsilon_t+\left[\phi(1+\phi M)-\tilde{\phi}(1+\phi M)\right]\varepsilon_{t-1}+\left[\phi^2(1+\phi M)-\tilde{\phi}\phi(1+\phi M)\right]\varepsilon_{t-2}+...\\ x_t &= \left(1+\left(\phi-\tilde{\phi}\right)M\right)\varepsilon_t+\phi\left(\phi-\tilde{\phi}\right)\left(\phi^{-1}+M\right)\varepsilon_{t-1}+\phi^2\left(\phi-\tilde{\phi}\right)\left(\phi^{-1}+M\right)\varepsilon_{t-2}+..\end{aligned}$$

$$\begin{aligned}x_t-\phi x_{t-1} &= \left[1+\left(\phi-\tilde{\phi}\right)M\right]\varepsilon_t+\left[\phi\left(\phi-\tilde{\phi}\right)\left(\phi^{-1}+M\right)-\phi\left(1+\left(\phi-\tilde{\phi}\right)M\right)\right]\varepsilon_{t-1}\\ x_t-\phi x_{t-1} &= \left[1+\left(\phi-\tilde{\phi}\right)M\right]\varepsilon_t-\tilde{\phi}\varepsilon_{t-1}\end{aligned}$$

2.3 Determinacy and identification in the other equilibria

Lubik and Schorfheide's analysis raises a related question: Granted that we cannot identify the Taylor parameter ϕ_π in the unique "determinate" equilibrium with $\phi_\pi>1$, can we identify ϕ_π if the economy is in one of the (many) "inderterminate" equilibria with $\phi_\pi<1$, or in one of the "explosive" equilibria with $\phi_\pi>1$? Are at least the estimates with $\phi_\pi<1$ from the 1970s meaningful? Would estimates from a hyperinflating regime be meaningful?

Again, the answer appears initially to be "yes", but that "yes" hinges on lag-length restrictions of the forcing process x_t. Without such restrictions, we cannot identify ϕ in *any* equilibria, not just the forward-looking $\phi>1$ bounded equilibrium.

The appearance of identification.

Return to the example (1)-(2)-(3) with an AR(1) monetary policy disturbance. The difference equation (4) is still all we have to characterize solutions. Thus, we solve the model by writing

$$\pi_{t+1}=\phi\pi_t+x_t+\delta_{t+1}. \tag{10}$$

where δ_{t+1} is an arbitrary random variable with $E_t\left(\delta_{t+1}\right)=0$.

If $\phi<1$, following the standard procedure, we solve this equation backward,

$$\pi_t=\sum_{j=0}^{\infty}\phi^j\delta_{t-j}+\sum_{j=0}^{\infty}\phi^j x_{t-j-1}.$$

Now there are multiple locally bounded (nonexplosive) solutions. We can use any additional shock series $\{\delta_t\}$ we like. These are called "sunspots" or "multiple equilibria" in the new-Keynesian tradition. With $\phi < 1$, new-Keynesian authors conclude, inflation is not determinate.

In the $\phi > 1$ case, the requirement that $E_t\pi_{t+j}$ not explode gives a unique choice of δ_t at each time period; a unique locally-bounded equilibrium. The corresponding forecast error δ_t is typically not zero. In our example, the unique non-explosive solution (5) requires the unique equilibrium indexed by[2] the choice of forecast error

$$\delta_t = \frac{x_t - \rho x_{t-1}}{\rho - \phi} = \frac{w_t}{\rho - \phi} \tag{11}$$

Other forecast errors are possible, but lead to explosive solutions.

In our example with an AR(1) x process, ϕ *is* identified in almost all of the $\phi < 1$ equilibria, and in all the explosive equilibria with $\phi > 1$. The "unique forward looking" equilibrium – the choice (11) is also the unique equilibrium in which ϕ is *not* identified.

This point is easiest to see if we remove the shock to the interest rate rule, setting $\sigma_\varepsilon = 0$. Then, with $\phi > 1$ the unique bounded forward-looking equilibrium is

$$\pi_t = 0$$

Obviously, you can't measure any dynamics out of that. Almost all of the backward-looking equilibria (and explosive forward looking equilibria) are

$$\pi_t = \phi^t \pi_0$$

Obviously, ϕ *is* identified and easily measurable in these equilibria. The one exception is if $\pi_0 = 0$ – the same special case.

The point is almost as easy to see in the special case of an i.i.d disturbance, with $\rho = 0$. Writing the Fisher equation (1) in terms of the inflation forecast error δ_t, we have

$$\pi_t = i_{t-1} - r + \delta_t$$

Now we can solve for the interest rate dynamics

$$i_t = \phi i_{t-1} - \phi r + \phi\delta_t + \varepsilon_t \tag{12}$$

Since the errors ε_t and δ_t are both orthogonal to i_{t-1}, ϕ is identified by a regression of i_t on i_{t-1}. But there is a special case – if $\delta_t = -\varepsilon_t/\phi$ and $i_0 = 0$ then $i_t = 0$, and ϕ is not identified. This is exactly the special choice of δ_t in (11) that we choose in the forward-looking solution! $\delta_t = -\varepsilon_t/\phi$ is also a possible equilibrium of the backward-looking solution, so the statement that "ϕ is identified in the backward-looking solutions" must exclude this equilibrium as well.

Identification disappears without restrictions on the x process

Alas this happy state of affairs disappears if we do not restrict the x process, e.g. to white noise or an AR(1). If we allow an arbitrary disturbance process, then we can no longer identify ϕ, even if $\phi < 1$. This conclusion is already proved in the above proposition. Given *any* equilibrium $\pi_t = a(L)\varepsilon_t$, and given any $\tilde{\phi} > 1$, the proposition shows us how to construct a $x_t = b(L)\varepsilon_t$ that generates the desired $\pi_t = a(L)\varepsilon_t$.

[2] Algebra: Plug

$$\pi_t = -\frac{v_t}{\phi - \rho}$$

into

$$\pi_t = \phi\pi_{t-1} + v_{t-1} + \delta_t.$$

and sove for δ_t.

3 Local equilibria?

Perhaps it is wiser to reconsider the proposition that an 'active' monetary policy, i.e. raising an interest rate target more than one-for-one with inflation, and a Fisher equation relating nominal interest and inflation are *alone* sufficient to determine the price level.

Nothing in *economics* rules out the explosive solutions[3]. Thus, we cannot say that the Taylor rule in this class of models *determines* the price level. We need some other criterion to rule out multiple solutions if the price level or inflation are to be determined. Price level determinacy must come from somewhere else.

To make this sort of claim, one really must write down the model in a more explicit fashion. To keep the discussion compact, I just simplify the standard source, Woodford 2000, and refer the reader to that source for technical details and elaboration.

Consumers maximize a standard utility function

$$\max E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$$

Consumers receive a constant nonstorable endowment $Y_t = Y$. They trade in complete financial markets described by real contingent claims prices $m_{t,t+1}$ and hence nominal contingent claims prices $Q_{t,t+1} = \frac{P_t}{P_{t+1}} m_{t+1}$. The interest rate is determined from contingent claim prices by

$$\frac{1}{1+i_t} = E_t \left[Q_{t,t+1} \right]$$

I follow Woodford and many others in describing a frictionless economy. One may be a bit disturbed by the presence of prices and no money, but this specification does make sense. At the simplest level, we can think about a monetary model in which the government pays interest on money, equal to the interest it pays on one-period nominal debt. At this point, money is exactly equivalent to nominal debt, so there is no real point in carrying around two letters for the same thing. (Woodford 2001 trades a few more symbols for a bit of comfort and presents the model with interest-paying money in this way.) Alternatively, remember that money M in such models represents money held overnight, usually subject to an interest cost. Thus, a "cashless economy" can operate quite well if agents exchange some maturing government bonds for cash in the morning, use the cash for transactions during the day, and then pay taxes in cash and buy new government debt with cash (repurchase agreements) at the end of the day, holding no money overnight $M = 0$ when interest is charged. The "price level" still refers to the tradeoff between cash and goods. Why the price level is *determinate* in such an economy is the question we are after, but if it is, there is no harm in talking about nominal prices even though no money is held overnight. The economy can also be truly cashless, using electronic claims to maturing government debt as medium of exchange. A "dollar" is then defined by the right to exchange a "dollar" of maturing debt to extinguish a "dollar" of tax liability.

The government issues one-period nominal debt; the face value issued at time $t-1$ and coming due at date t is $B_{t-1}(t)$. The household then faces a present-value budget constraint

[3] I stress "economics" for a reason. A number of authors advocate additional principles on which to rule out nominal explosions. Most prominently in a series of papers culminating in McCallum (2003), McCallum argues for a "minimum-state variable" criterion. See Woodford's (2003) discussion for the other side of that debate.

(equivalent to a flow constraint and a no-Ponzi condition)

$$E_t \sum_{j=0}^{\infty} Q_{t,t+j} \left(P_{t+j} C_{t+j}\right) = B_{t-1}(t) + E_t \sum_{j=0}^{\infty} Q_{t,t+j} \left(P_{t+j} Y_{t+j} - P_{t+j} T_{t+j}\right) \tag{13}$$

where T_t denotes real net lump-sum taxes.

The consumer's first order conditions for optimal choice are, first that marginal rates of substitution equal the contingent claims price ratio

$$\beta \frac{u_c(C_{t+1})}{u_c(C_t)} = \frac{P_{t+1}}{P_t} Q_{t,t+1},$$

and second the "transversality condition", the limit of the condition that the consumer neither leaves unused wealth nor unpaid debts at the end of life,

$$\lim_{T\to\infty} E_t \left[Q_{t,T} B_{t-1}(T)\right] = 0.$$

Equilibrium $C_t = Y$ thus requires that the contingent claims prices are given by

$$\beta \frac{u_c(Y_{t+1})}{u_c(Y_t)} = \beta \equiv \frac{1}{1+r} = \frac{P_{t+1}}{P_t} Q_{t,t+1} \tag{14}$$

$$Q_{t,t+1} = \frac{1}{1+r} \frac{P_t}{P_{t+1}}. \tag{15}$$

In particular, the Fisher relation results,

$$\frac{1}{1+i_t} = E_t(Q_{t,t+1}) = \beta E_t \frac{P_t}{P_{t+1}} = \frac{1}{1+r} E_t \left(\frac{1}{\Pi_{t+1}}\right) \tag{16}$$

Loglinearizing, we obtain the usual Fisher relation

$$i_t = r + E_t \pi_{t+1}.$$

From 13, equilibrium $C_t = Y$ also requires

$$B_{t-1}(t) = \sum_{j=0}^{\infty} E_t Q_{t,t+j} \left(P_{t+j} T_{t+j}\right) \tag{17}$$

and using contingent claim prices from 14,

$$\frac{B_{t-1}(t)}{P_t} = \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} T_{t+j}. \tag{18}$$

This is an *equilibrium condition* that derives from the *consumer's* present value budget constraint or, equivalently, the transversality condition for the consumer's choice to be an optimum. It is not a "government budget constraint." (Cochrane 2005 gives an extended discussion of this point.) We do not know yet whether it describes what set of net taxes $\{T_{t+j}\}$ must be satisfied for a given P_t, or whether it describes the initial price level P_t given a set of taxes T_{t+j}, or some combination of the two directions of causality. I call it the "government debt valuation equation" to keep in mind that it is an equilibrium condition and not a constraint.

The Fisher equation (16) and the government debt valuation equation (18) are the *only* two conditions that need to be satisfied to give an equilibrium. So long as a path of variables $\{i_t, P_t, B_{t-1}(t), T_t\}$ satisfy these two equations, that path is an equilibrium of this economy. Consumers have optimized and markets (goods, government bonds, contingent claims) have cleared. Obviously and hardly surprisingly, the equilibrium is not yet *unique*, in that many different price levels and inflation paths will work. We need *some* specification of monetary policy to determine the price level.

The new-Keynesian/Taylor rule analysis adds a rule of the form

$$i_t = r + \phi \pi_t$$

to this analysis, and maintains a "Ricardian" fiscal regime; net taxes T_{t+j} are assumed to adjust so that the government debt valuation equation (18) holds. Our only equilibrium condition is then the Fisher equation (16).

Nothing in this system rules out explosive inflation or deflation. Nothing requires "local" equilibria. *Nominal hyperinflations* are perfectly valid equilibria.

The only solution is to strengthen the government valuation equation – to specify at least some measure of a non-Ricardian regime in which that equation can help to determine the price level or the rate of inflation.

3.1 Woodford agrees (if you read closely)

The central theoretical question is, again, why should we restrict attention to local, nonexplosive equilibria? It is this criterion in new-Keynesian models that forces us to look for $\phi > 0$ and then lets us select one equilibrium, and claim that the model determines the price level. The only way to answer this question is, as above, to consider equilibria *globally*, and then search for some *economic* reason to rule out the explosive paths.

Woodford (2001) treats this issue in Ch 2.4, starting in p. 123. In turn, Woodford follows Schmitt-Grohé and Uribe (2000) and Benhabib Schmitt-Grohé and Uribe (2001). We specialize to perfect foresight. We consider a Fed policy rule

$$i_t = \phi(\Pi_t);\ \Pi_t = P_t/P_{t-1} \tag{19}$$

$\phi(\cdot)$ is a function; we consider the possibility of nonlinear relations (Woodford 4.1). Consumers have subjective discount rate β and with constant endowments; with a frictionless economy or in the cashless limit, the Fisher relation, equivalently the consumer's first order condition is

$$\Pi_{t+1} = \beta(1 + i_t). \tag{20}$$

As in my linear example, then, we are looking for solutions to the pair (19) and (20). In the same way, we can substitute out the interest rate and study directly (Woodford 4.6)

$$\Pi_{t+1} = \beta \left[1 + \phi(\Pi_t)\right].$$

Schmitt-Grohé and Uribe's main point is that a Taylor rule with slope greater than one cannot apply *globally*, because such a rule would violate the lower bound that nominal interest rates cannot be less than zero[4]. Therefore, the Taylor principle that interest rates react more than one-for-one to inflation cannot apply globally.

Figure 1 illustrates the situation. The equilibrium at Π^* satisfies the "Taylor principle" $\phi_\pi > 1$. It is therefore a "unique *local* equilibrium" of the model. Any value of Π_0 other than Π^* leads away from the neighborhood of Π^* as shown. The point is to try establish that Π^* is the unique equilibrium of the model.

With a lower bound on nominal interest rates, the function $\phi(\Pi)$ must also have another stationary point. In addition, this stationary point must violate the Taylor principle; it must cut the 45° line from above and thus have $\phi_\pi < 1$ as shown. This is a "price-indeterminate" equilibrium. As shown, many paths lead to the $\phi_\pi < 1$ point, and we have no way of telling which one is the right one. There are "multiple local equilibria" near this point.

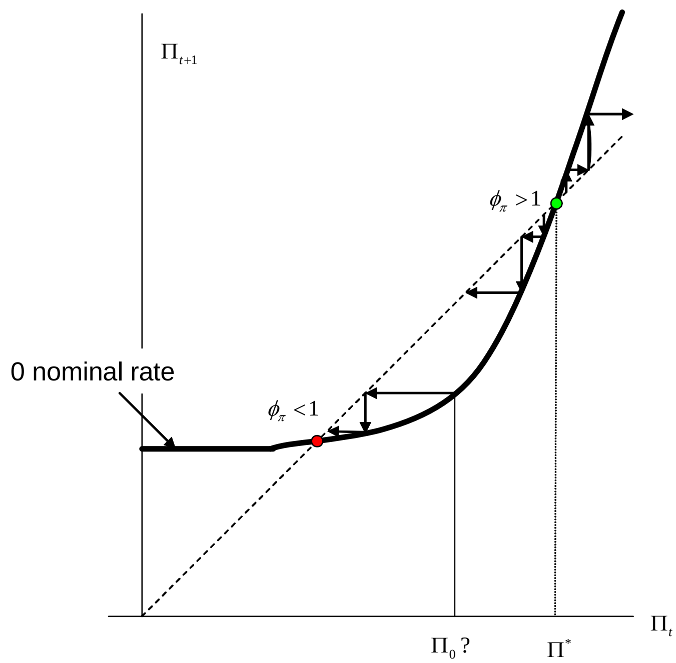

Figure 1: Global equilibria in a perfect foresight model.

There were always multiple equilibria, as any of the paths in Figure 1 is an equilibrium. Schmitt-Grohé and Uribe's point is that merely restricting attention to "locally bounded" or "nonexplosive" equilibria is not enough to ensure global determinacy. The function $\phi(\Pi)$ must also have a stationary point such as $\phi_\pi < 1$ which is a "locally bounded" but *not* "determinate" equilibrium. In addition, ruling out explosions is not enough to rule out the equilibria that start slightly below Π^* and lead down to the lower equilibrium.

My point is larger – what rules out the *explosive* equilibria on the right, as well as the nonexplosive, but still multiple, equilibria on the left? And what does Woodford have to say about that question?

[4]Actually, nominal interest rates can be less than zero in an economy that only has government bonds and consumers may not hold cash overnight. But real economies all do have cash, so there is no point in spending time on this theoretical possiblity.

First, (p.128) Woodford notes that

> "The equilibrium ..[Π^*].. is nonetheless *locally* unique, which may be enough to allow expectations to coordinate upon that equilibrium rather than on one of the others."

Similarly, King (2000, p. 58-59) writes

> "By specifying $\tau > 0$ [$\phi_\pi > 0$ in our notation] then, the monetary authority would be saying, 'if inflation deviates from the neutral level, then the nominal interest rate will be increased relative to the level which it would be at under a neutral monetary policy.' If this statement is believed, then it may be enough to convince the private sector that the inflation and output will actually take on its neutral level.

This strikes me as a rather weak argument for pruning equilibria.

Woodford argues that we should not think of an economy making an ε mistake and slipping from Π^* into a more explosive equilibrium; instead we should think of expectations of future inflation driving inflation today, and he argues that explosive inflation is an unrealistic expectation. But I think this argument really means that people don't believe the Fed is really so pig-headed as to keep raising interest rates 2 for 1 once inflation reaches 1000%; they don't believe the function ϕ. If so, this violates the rules of the game; we are asking for equilibria given that people really do believe the $\phi(\Pi)$ function with complete certainty. If the fact, correctly anticipated, is a different $\phi(\Pi)$ function, then let that be analyzed. I think the expectation of an explosion is perfectly reasonable – *If* the Fed were committed to raising interest rates more than 1-1 with inflation, *if* we lived in a world of constant real rates, so this translates into a commitment to raise future inflation more than 1-1 with past inflation, then my expectation is that we'll see hyperinflation.

Woodford's real answer though lies in section 4.2 "Policies to prevent a deflationary trap" (i.e. to cut off equilibria to the left of Π^*) and 4.3 "Policies to prevent an inflationary panic" (i.e. to cut off equilibria to the right of Π^*). (If the preceding arguments were convincing, we wouldn't need these sections of course). And in both of these sections, Woodford fundamentally argues for price-level determinacy by moving to a non-Ricardian regime, and having *fiscal* policy prune equilibria. (Woodford notes on p. 124 that the model we have discussed so far is completed by an explicitly Ricardian fiscal regime.)

Thus, he eliminates the left hand equilibria as follows (p. 132): "let total nominal government liabilities D_t be specified to grow at a constant rate $\bar{\mu} > 1$ while monetary policy is described by the Taylor rule (4.1). [My (19)]." A zero nominal interest rate requires steady deflation at the subjective discount rate, so the real value of a constant stock of nominal debt explodes. Such a real explosion violates consumers' transversality conditions, and so cannot be an equilibrium. (Woodford specifies $\bar{\mu} > 1$ to take care of the case that the $\phi_\pi < 1$ equilibrium has a small positive nominal interest rate.) "Thus," (p. 133) "in the case of an appropriate *fiscal* (my emphasis) policy rule, a deflationary trap is not a possible rational expectations equilibrium."

"Let total nominal Government liabilities D_t be specified.." is an *additional* assumption, and *different* from the explicitly Ricardian assumptions of the model described so far. Growing nominal debt with deflation and means real debt that grows explosively, while net real taxes remain unchanged. This is a "Non-Ricardian" threat to violate the government debt valuation equation. We are pruning equilibria and determining the price level by adding a *fiscal* regime.

Woodford takes an apparently different approach in section 4.3, p. 135 "Policies to prevent an inflationary panic," i.e. to rule out equilibria to the right of Π^*. First, he suggests a strengthening of the Taylor principle (p.136). He suggests that the Fed commit to a policy in which the graph in Figure 1 becomes *vertical* at some finite inflation $\bar{\pi}$. Then, there is no rational-expectations equilibrium with exploding inflation; at least if "equilibrium" requires a finite price level. At one level, this proposal is not as extreme as it sounds. After all, the Taylor principle as understood in the new-Keynesian modeling tradition amounts to the Fed making unpleasant (but somehow credible) threats about off-equilibrium behavior. The more unpleasant the threat (if believed) the more effective. On the other hand, the real economy would presumably substitute to other moneys before the value of money reached zero, making hyperinflation possible.

Recognizing, I think, that this is a bit far-fetched[5], Woodford comes to a different suggestion (p. 138):

> ...self-fulfilling inflations may be excluded through the addition of policy provisions that apply only in the case of hyperinflation. For example, Obstfeld and Rogoff (1986) propose that the central bank commit itself to peg the value of the monetary unit in terms of some real commodity by standing ready to exchange the commodity for money in event that the real value of the total money supply ever shrinks to a certain very low level. If it is assumed that this level of real balances is one that would never be reached except in the case of a self-fulfilling inflation, the commitment has no effect except to exclude such paths as possible equilibria. ...[This proposal could] well be added as a hyperinflation provision in a regime that otherwise follows a Taylor rule.

This proposal is inherently *fiscal* as well. In order for the government to exchange the money stock (nominal liabilities) for some real commodity, it has to have sufficient stocks of that commodity on hand, or a commitment to raise enough tax revenue to obtain the commodity. A purely Ricardian fiscal regime cannot defend a commodity standard. Governments facing hyperinflations, in fact, notoriously do not have the resources to redeem their money stocks. (Even if one does not want to call this a "fiscal" proposal, it is certainly a reversion to a commodity standard. It remains true then, that the Taylor principle alone does not determine the price level.)

In sum, then, I read Woodford's analysis as an agreement on the central points. Nothing in economics rules out non-local equilibria, since nothing in economics rules out nominal hyperinflation or deflation. Hence, Woodford agrees that we cannot jump from "unique *local* equilibrium" to "unique equilibrium" without further economic analysis. And finally, the central ingredient Woodford adds to rule out the undesired equilibrium is a non-Ricardian fiscal policy (at least in some states), in which the price level is determined by the valuation equation for government debt.

Since they are not fully worked out, even these proposals to trim equilibria are open to some criticism and at least a variety of interpretations. Any equilibrium requires coordination

[5] Woodford also considers (p. 137) a related proposal involving extreme but finite inflation, plus limits on money demand elasticity. This proposal won't work in the frictionless model on which I have focused however.

between fiscal and monetary policy, as monetary economists since Friedman (1946) have stressed. Woodford's suggestions amount to a commitment that in certain states the government will adopt an uncoordinated policy that rules out any equilibrium. I do not think this is a credible threat, or a credible set of expectations.

For example, the proposal to rule out deflation with nominal debt amounts to the time-honored prescription to escape deflation by printing unbacked money. Clearly, this policy cannot coexist with the low interest rate target. In Woodford's suggestion, both money-printing and the low interest rate target pig-headedly continue, resulting in no equilibrium, period. In most analysis of such "uncoordinated" policy, however, and surely in people's expectations of what would happen should we actually reach this state, one or the other policy (printing unbacked money, low interest rate target) soon must give way so the policy is "coordinated" after all. For example, one might hope that the interest rate policy rule jumps to a higher level. But this is then just a different policy rule, a different function $\phi(\Pi)$; perhaps one with a jump. And, most importantly, now the paths that lead first to deflation and then to something else *are* valid equilibria and cannot be ruled out.

Similarly, we might ask exactly how does a contingent commodity standard rule out a hyperinflation? Why can't we just hyperinflate, hit the bound, redeem the currency, and then continue on our merry way with a new currency? Again, Woodford keeps the Taylor rule interest rate policy alive along with the redemption to a new currency. This of course is impossible.

In sum, then, Woodford cuts off the undesirable equilibria by having the *government* fully commit (this is a perfect foresight model) to *impossible* actions (uncoordinated policy), and have the private sector *completely believe* this commitment, even though the states of the world in which the uncoordinated policies are to take effect have never been observed nor even approached. It would be no different, and a lot simpler, if the government were simply to say "if inflation gets to x or y, the government commits to blow everything up."

This criticism is a bit unfair of course. Woodford is merely setting out verbally a rough sketch of a path one might follow to use fiscal commitments to prune multiple equilibria. My point only is that some more painting needs to be done to fill out that sketch in a satisfactory way, a point I suspect Woodford would agree with.

Woodford on identification

Woodford also notices the identification problem. on p.93, he discusses Taylor's (1999) and Clarida, Gali and Gertler's (2000) regression evidence that the Fed responded less than 1-1 to inflation after 1980 and more than 1-1 afterwards.

> Of course, such an interpretation depends on an assumption that the interest-rate regressions of these authors correctly identify the character of systematic monetary policy during the period. In fact, an estimated reaction function of this kind could easily be misspecified.

(An example in which the measured ϕ coefficient is 1/2 of the true value follows.) However, though Woodford sees the possibility of a bias in the estimated coefficients, he does not say that the structural parameter ϕ is unidentified.

4 Identification in new-Keynesian models

One may well object at the whole idea of studying identification and determinacy in such a stripped down model, with no monetary friction and no means by which the central bank can affect real rates. It turns out that the simple models *do* in fact capture the relevant issues, but one can only show that by examining "real" new-Keynesian models in detail and seeing, at the cost of some algebra, that the same points emerge.

4.1 The standard three-equation model

Throughout, I will base the analysis on standard New-Kenesian IS-LM models, for example as in the excellent expositions in King (2000) and Woodford (2001). The basic model is

$$\begin{aligned} y_t &= E_t y_{t+1} - \sigma\left[r_t - r\right] + x_{dt} & (21)\\ i_t &= r_t + E_t \pi_{t+1} & (22)\\ \pi_t &= \beta E_t \pi_{t+1} + \gamma\left[y_t - \bar{y}_t\right] + x_{\pi t} & (23) \end{aligned}$$

where y = output, r = real interest rate, i =nominal interest rate, π = inflation, and x are disturbances.

While seemingly ad-hoc, the point of the entire literature is that this structure has exquisite micro-foundations. The first equation is labeled "IS." Now, that should probably stand for "Intertemporal Substitution," as it is typically derived from first order conditions for consumption vs. interest rate and the equation of output to consumption (in this simple model that ignores capital). The second equation is simply the Fisher relation between interest rates and inflation.

The last equation is the "new-Keynesian Phillips curve." What makes it "new" is the timing of inflation on the right hand side. Phillips might have had a constant: output is higher when inflation is higher. An "accelerationist" might put π_{t-1} on the right hand side: output is higher when inflation is *increasing*. Friedman (1968) and Lucas (1972) might put $E_{t-1}(\pi_t)$ on the right hand side: output is higher if inflation is higher than expected. "new-Keynesian" put $E_t(\pi_{t+1})$ on the right hand side. Optimizing firms setting prices subject to adjustment costs set prices today based on expectations of prices tomorrow. This change in timing has dramatic effects on the properties of the model. Many writers interpret this equation causally, larger output gaps *cause* inflation to change. However, like the other equations, it is simply a first order condition that must hold in equilibrium. The disturbances x_{dt} and $x_{\pi t}$ are not necessarily unforecastable. I use a roman letter (x not ε) to remind us of that fact.

There is an active debate on the right specification of (23). Some authors including Furher and Moore (1995) and Mankiw and Reis (2002) point out that (23) specifies high output when inflation is high relative to *future* inflation, i.e. when inflation is *declining*, and that this prediction is contrary to fact. (An instance of the point that the slight change in timing has dramatic implications). Mankiw and Reis argue for a return to mechanical or adaptive expectations, i.e. π_{t-1} on the right hand side. Others such as Gali (2003) and Sbordone (2002, 2005) respond that the right specification puts marginal cost in place of output on the right hand side of (23). This can save the estimate of the sign of ϕ, but the cost is that the "gap" series and the whole new-Keynesian setup now has nothing to do with recessions as conventionally understood. Gali's Figure[6] 2 is particularly dramatic on this point – the "gap" has essentially no correlation with

[6] Page 46 of http://www.econ.upf.edu/crei/people/gali/pdf_files/wcpaper.pdf

detrended GDP. For my purposes, it doesn't really matter whether the "y" series represents output or cost. However, in deference to this debate it does seem worth considering the possibility that $\gamma < 0$, and I'll consider this parameter in what follows.

4.2 Deviations from equilibrium

For determinacy and identification questions, we can simplify the system somewhat by studying *deviations* from an equilibrium rather than the equilibrium itself. Start with an equilibrium process for output $\{y_t^*\}$. The Fed's control of interest rates is enough in this model to attain any such path as an equilibrium. From (23) we can find the required path for equilibrium inflation π_t^*; from (21) we can find the required path for the equilibrium real rate r_t^*, and then from (22) we can find the required equilibrium nominal interest rate i_t^*:

$$\begin{aligned}
\pi_t^* &= E_t \sum \beta^j \left[\gamma(y_{t+j}^* - \bar{y}_{t+j}) + x_{\pi t+j}\right] \\
r_t^* &= r + \frac{1}{\sigma}\left(E_t y_{t+1}^* - y_t^*\right) + \frac{1}{\sigma} x_{dt} \\
i_t^* &= r_t^* + E_t \pi_{t+1}^*
\end{aligned}$$

The "neutral" (King 2000) or "no-gap" equilibrium $y_t^* = \bar{y}_t$ is a particularly interesting baseline. Using overbars to denote this case, we have

$$\begin{aligned}
\bar{\pi}_t &= E_t \sum \beta^j x_{\pi t+j} \\
\bar{r}_t &= r + \frac{1}{\sigma}\left(E_t \bar{y}_{t+1} - \bar{y}_t\right) + \frac{1}{\sigma} x_{dt}
\end{aligned}$$

so the equilibrium interest rate must be

$$\bar{\imath}_t = \bar{r}_t + E_t \bar{\pi}_{t+1} = r + \frac{1}{\sigma}\left(E_t \bar{y}_{t+1} - \bar{y}_t + \varepsilon_{dt}\right) + E_t \bar{\pi}_{t+1}. \tag{24}$$

I use the general case y_t^* to emphasize that the lack of identification holds for *any* equilibrium.

Defining $\tilde{y}_t = y_t - y_t^*$ as the deviation of output from the $*$ equilibrium, we can subtract the values of (21)-(23) from those of the $*$ equilibrium to describe *deviations* from equilibrium as

$$\tilde{\imath}_t = \tilde{r}_t + E_t \tilde{\pi}_{t+1} \tag{25}$$
$$\tilde{y}_t = E_t \tilde{y}_{t+1} - \sigma \tilde{r}_t \tag{26}$$
$$\tilde{\pi}_t = \beta E_t \tilde{\pi}_{t+1} + \gamma \tilde{y}_t. \tag{27}$$

This is the same model, but without constants or shocks.

4.3 Identification

Now it is clearly true (by construction) that if the Fed sets $i_t = i_t^*$, i.e. $\tilde{\imath}_t = 0$, then $\pi_t = \pi_t^*$ and $y_t = y_t^*$, i.e. $\tilde{\pi}_t = 0$, $\tilde{y}_t = 0$ are *an* equilibrium. But it is also now clearly true that setting $i_t = i_t^*$ is not enough to determine that this is the *only* equilibrium; it is not enough to determine the price level.

To see this point, I find it useful to write (25)-(27) with $\tilde{\imath}_t = 0$ as[7]

$$\left[\begin{array}{c} E_t\tilde{y}_{t+1} \\ E_t\tilde{\pi}_{t+1} \end{array}\right] = \frac{1}{\beta}\left[\begin{array}{cc} \beta+\sigma\gamma & -\sigma \\ -\gamma & 1 \end{array}\right]\left[\begin{array}{c} \tilde{y}_t \\ \tilde{\pi}_t \end{array}\right] \tag{28}$$

Since the model only restricts the dynamics of *expected* future output and inflation, we have multiple equilibria, i.e. any

$$\left[\begin{array}{c} \tilde{y}_{t+1} \\ \tilde{\pi}_{t+1} \end{array}\right] = \frac{1}{\beta}\left[\begin{array}{cc} \beta+\sigma\gamma & -\sigma \\ -\gamma & 1 \end{array}\right]\left[\begin{array}{c} \tilde{y}_t \\ \tilde{\pi}_t \end{array}\right] + \left[\begin{array}{c} \delta_{y,t+1} \\ \delta_{x,t+1} \end{array}\right]$$

is valid, not just $\delta_{yt} = \delta_{xt} = 0$ and hence $\tilde{y}_t = \tilde{\pi}_t = 0$.

Perhaps however the dynamics of (28) are explosive, so at least $\tilde{y} = \tilde{\pi} = 0$ is the only *local* or *nonexplosive* equilibrium. Alas, this hope is dashed as well: the eigenvalues of the transition matrix in (28) are less than one. Since the algebra can be handled in a slightly more general case, I defer the proof for a moment.

The fact that setting $\tilde{\imath} = 0$ does not determine the price level should not surprise us. This is a pure interest rate peg; The $\bar{y}$ and $\bar{\pi}$ that appear for example on the right hand side of equation (24) are not Taylor-type policies. They are exogenous constants at each date, not the endogenous output and inflation series y_t, π_t, etc. We have just shown again the familiar result that an interest rate peg – even one that varies exogenously over time – does not determine the price level (or output, in this model).

To determine output and the price level, it is not enough to say what nominal interest rates will be *in* equilibrium, the Fed must say something about how policy would respond *out* of equilibrium. King thus considers Taylor-type rules of the form

$$i_t = i_t^* + \phi_0\left(\pi_t - \pi_t^*\right) + \phi_1\left(E_t\pi_{t+1} - E_t\pi_{t+1}^*\right) \tag{29}$$

For example, with $\phi_1 = 0$ the dynamics of deviations from the $*$ equilibrium are generalized to

$$\left[\begin{array}{c} E_t\tilde{y}_{t+1} \\ E_t\tilde{\pi}_{t+1} \end{array}\right] = \left[\begin{array}{cc} 1+\frac{\sigma\gamma}{\beta} & -\frac{\sigma}{\beta}+\sigma\phi_0 \\ -\frac{\gamma}{\beta} & \frac{1}{\beta} \end{array}\right]\left[\begin{array}{c} \tilde{y}_t \\ \tilde{\pi}_t \end{array}\right].$$

[7] Algebra:

$$\begin{aligned} 0 &= \tilde{r}_t + E_t\tilde{\pi}_{t+1} \\ E_t\tilde{y}_{t+1} &= \tilde{y}_t + \sigma\tilde{r}_t \\ E_t\tilde{\pi}_{t+1} &= \frac{1}{\beta}\tilde{\pi}_t - \frac{\gamma}{\beta}\tilde{y}_t. \end{aligned}$$

Hence

$$\begin{aligned} E_t\tilde{y}_{t+1} &= \tilde{y}_t - \sigma E_t\tilde{\pi}_{t+1} \\ E_t\tilde{\pi}_{t+1} &= \frac{1}{\beta}\tilde{\pi}_t - \frac{\gamma}{\beta}\tilde{y}_t \end{aligned}$$

or

$$\begin{aligned} E_t\tilde{y}_{t+1} &= \left(1+\frac{\sigma\gamma}{\beta}\right)\tilde{y}_t - \frac{\sigma}{\beta}\tilde{\pi}_t \\ E_t\tilde{\pi}_{t+1} &= -\frac{\gamma}{\beta}\tilde{y}_t + \frac{1}{\beta}\tilde{\pi}_t. \end{aligned}$$

As King shows, and I verify in a moment, with suitably "active" ϕ_0 such as $\phi_0 > 1$, both eigenvalues of this system are greater than one, so output and inflation are again "determinate," in the sense that there is a unique non-explosive equilibrium, $\tilde{y} = 0$, $\tilde{\pi} = 0$.

King's form of a Taylor rule is particularly useful for my point here. *In equilibrium*, we will always see $\pi_t - \pi_t^* = 0$. Thus, a regression estimate of (29) cannot possibly estimate ϕ_0, ϕ_1. *There is no movement in the right hand variables in equilibrium.* Taylor determinacy depends entirely on what the Fed would do *out* of equilibrium, which we can never see from data *in* that equilibrium.

King recognizes the problem, writing in footnote 41,

> "The specification of this rule leads to a subtle shift in the interpretation of the policy parameters τ_i; these involve specifying how the monetary authority will respond to deviations of inflation from target. But if these parameters are chosen so that there is a unique equilibrium, then no deviations of inflation will ever occur."

King does not address the implications of this non-identification for empirical work.

4.4 Out of equilibrium response

Again, the "Old-Keynesian" intuition is that the Fed will react to inflation by raising real rates; this action will lower output and via the Phillips curve, lower future inflation. To emphasize that the "new-Keynesian" model works in a fundamentally different way, I graph in Figure 2 the path of output, inflation, and interest rates in response to an "off-equlibrium" one percent innovation to inflation at period one, together with no unexpected change in output. This is the response of the system (25)-(27) to $\tilde{\pi}_1 = 1$, $\tilde{y}_1 = 0$, together with the policy rule $\tilde{\imath}_t = 1.3 \times \tilde{\pi}_t$, using parameters $\beta = 0.95, \sigma = 1, \gamma = 1, \phi = 1.3$

At period 1, inflation $\tilde{\pi}_1 = 1$, $\tilde{y}_1 = 0$. The Fed responds by setting interest rates $\tilde{\imath}_1 = 1.3 \times \tilde{\pi}_1 = 1.3$. I find the following paths by simulating forward the system (60). Real interest rates rise throughout the simulation, as one might have hoped. However, output *increases* uniformly and eventually explodes in the *positive* direction, while inflation explodes in the *negative* direction, precisely the opposite of what we might have expected.

In the context of the model, however, this behavior makes sense. In the new-Keynesian IS curve (21), a high real rate lowers current output *relative to future output*, not on its own. Given no change in current output, a higher real rate must correspond to higher future output. Reading the causality of the IS curve (21) from right to left, in order for current output not to have changed, there must have been an increase in expected future output.

Meanwhile, inflation is exploding off in the *negative* direction. If output is getting large, then from the new-Keynesian Phillips curve (23), current inflation must be large *relative to future inflation.* Given current inflation, then, we must have declining future inflation. Or, again reading causality from right to left, current inflation can only be 1% with large output if people expect future deflation.

In both ways, then, the surprising dynamics of Figure 2 emphasize that the expected future terms in new-Keynesian models essentially change the sign of all the familiar dynamic relationships.

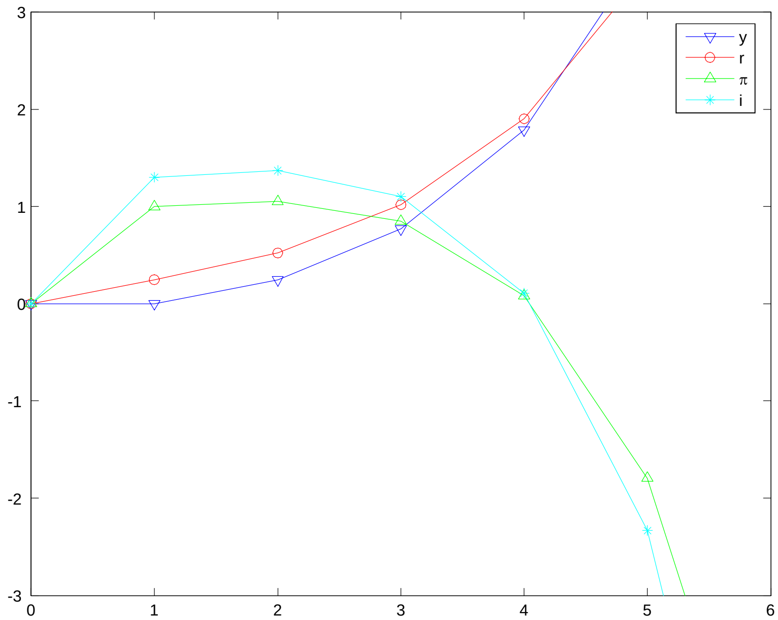

Figure 2: Response of the three-equation new-Keynesian model to a one-percent off-equilibrium inflation innovation, with no change in output. Parameters are $\beta = 0.95, \sigma = 1, \gamma = 1, \phi = 1.3$

4.5 Taylor on Taylor rules

A natural question is, "how does Taylor think Taylor rules work?" The best instance I can find to answer this question is Taylor (1999), and this paper highlights the deep tensions between how many economists view price determinacy and what the models actually do. Taylor adopts a simple model (p. 662),

$$y_t = -\beta(i_t - \pi_t - r) + u_t \tag{30}$$

$$\pi_t = \pi_{t-1} + \alpha y_{t-1} + e_t \tag{31}$$

$$i_t = g_\pi \pi_t + g_y y_t + g_0 \tag{32}$$

Given our previous discussion, we see a striking difference – all the forward-looking terms are absent. The "IS" curve (30) is missing $E_t y_{t+1}$; the "Phillips" curve (31) has *past* rather than current or expected future inflation in it. This is a standard, not "new" Keynesian model. Of course Taylor is perfectly aware of this fact. He writes (p. 662)

> In general α, β and r are reduced form parameters that will depend on the policy parameters g_π, g_y, and g_0. For example, Eq. (30) [my number] could be the reduced form of an optimizing 'IS' curve with future values of the real interest rates... Eq. (31) ... could be the reduced form of a rational expectations model with staggered wage and price setting, in which expectations of future wages and prices have been solved out. If the parameters do not change very much when the policy parameters change, then treating Eqs. (30) and (31) as policy invariant... will be a good approximation. But if the parameters do change by a large amount in response to policy, then the changes must be taken into account in the policy evaluation. Nevertheless, when

viewed as a reduced form, these equations summarize more complex forward-looking models and are useful for illustrating key points.

However, I do not think that the issue is limited to "policy invariance." We want to analyze dynamics for *given* "policy parameters" g_π, g_y, and g_0, so even if α, β and r change with different g, they are constant over time. As we'll see next, the *dynamics* of this system are fundamentally different from those of the "forward looking" models such as I investigate above. (King 2000 p. 72 also details a number of fundamental differences between "new" and "old" Keynesian models of this sort.)

Taylor states (p. 663) that "it is crucial to have the interest rate response coefficient on the inflation rate (or a suitable inflation forecast or smoothed inflation rate) above a critical 'stability threshold' of one," i.e. $g_\pi > 1$ (p. 664)

> The case on the left [$g_\pi > 1$] is the stable case...The case on the right [$g_\pi < 1$] is unstable... This relationship between the stability of inflation and the size of the interest rate coefficient in the policy rule is a basic prediction of monetary models used for policy evaluation research. In fact, because many models are dynamically unstable when g_π is less than one... the simulations of the models usually assume that g_π is greater than one.

This is exactly the *opposite* philosophy from the new-Keynesian models. In new-Keynesian models, $g_\pi > 1$ is the condition for *unstable* dynamics. These models want precisely unstable dynamics to force forward-looking solutions. In this model, $g_\pi > 1$ is the condition for *stable* dynamics, in which we solve for endogenous variables (including inflation) by *backward-looking* solutions.

If inflation π_t goes up – suppose there is a shock e_t – then the interest rate in (32) rises more than the inflation rate. A real interest rate rise in the static IS curve (30) drives output y_t down, and lower output y_t (a larger "gap") in (31) causes inflation next period π_{t+1} to decline. In this way, Taylor's model captures precisely the kind of "old-Keynesian" thinking in which a Taylor rule seems so attractive.

A bit more formally, (and as an alternative to Taylor's graphical analysis), use (30) to eliminate output and (32) to eliminate the nominal rate. As in Taylor (footnote 6), this is simpler with $g_y = 0$, and that simplification does not change the basic point. With $g_y = 0$, inflation dynamics in Taylor's model are

$$\pi_t = [1 + \alpha\beta(1 - g_\pi)]\,\pi_{t-1} - \alpha\beta(g_0 - r) - \alpha u_{t-1} + e_t$$

If $g_\pi > 1$, $[1 + \alpha\beta(1 - g_\pi)] < 1$ and inflation has stable, backward looking dynamics. If $g_\pi < 1$, $[1 + \alpha\beta(1 - g_\pi)] > 1$ and inflation has unstable, "forward-looking" (if one wants locally bounded solutions) dynamics.

Why do the two models disagree so much on the desired kind of dynamics? Because the Taylor model has no expected future terms on the right hand side. Hence, there are no expectational errors. All the shocks driving the system are exogenous shocks to the equations. Thus, if we solve "backward" in terms of these shocks $\{u_t, e_t\}$, then there is no indeterminacy.

Needless to say, if we adopt *Taylor's* model, then the Taylor rule parameter can be identified. (The disturbances are not necessarily uncorrelated over time or cross-sectionally, so measurement is not so simple as an OLS regression, but not impossible).

Alas, our quest is for *economic* models that deliver price determinacy from an interest rate rule. This model fails on the crucial qualification. If in fact inflation has nothing do to with expected future inflation (the heart of the new-Keynesian optimizing Phillips curve), so inflation is mechanistically *caused* by output gaps which are directly under the Fed's control, and if in fact the output gap driving inflation has nothing to do with expected future output, then, yes, the Taylor rule does lead to price determinacy. But despite a half-century of looking for it, *economic* models do not deliver the "if" part of these statements.

5 Rules with leads and lags

So far, I have used only very simple Taylor rules that relate interest rates to the current inflation rate. The literature contains a wide variety of specifications, however, and in particular specifications in which the central bank reacts to expected *future* inflation. It is often claimed that the principle "raise interest rates more than one for one with inflation" is quite robust to details of model and rule specification (See Taylor 1999 for example).

To address this question, start with our simple model (1)-(2), but allow the Fed to respond to expected future inflation rather than current inflation,

$$i_t = r + E_t\pi_{t+1} \tag{33}$$
$$i_t = r + \phi E_t\pi_{t+j} \; j > 0 \tag{34}$$

For $j = 0$ (contemporaneous inflation), the equilibrium condition is

$$E_t\pi_{t+1} = \phi\pi_t$$

as we have seen, the condition for a unique local equilibrium is $\|\phi\| > 1$. As King (2000) emphasizes, however, this condition implies that $\phi < -1$ works just as well as $\phi > 1$ to ensure determinacy. If the Fed threatens oscillating hyperinflation and hyperdeflation, that threat is just as effective in ruling out solutions other than $\pi_t = x_t/(\rho - \phi)$. I think this example is useful to remind us that the economics here are "threats of off-equilibrium explosion" rather than "higher real interest rates will cool off later inflation."

For $j = 1$, a reaction to expected future inflation,we have

$$E_t\pi_{t+1} = \phi E_t\pi_{t+1}.$$

If $\phi = 1$, anything is a solution. For *any* $\phi \neq 1$ (both $\phi > 1$ and $\phi < 1$), we have

$$E_t\pi_{t+1} = 0;\; \pi_t = \delta_{t+1}$$

Inflation must be white noise. No value of ϕ gives even local determinacy.

For $j = 2$, we have

$$E_t\pi_{t+1} = \phi E_t\pi_{t+2}$$

Now a necessary condition for "unstable" or "forward-looking" equilibrium is reversed, $\|\phi\| < 1$. Since interest rates react to inflation *two* periods ahead, and interest rates control expected inflation *one* period ahead, the interest rate and *one*-period ahead inflation must move less than *two* period ahead inflation if we want an explosive root. And even this specification is now not enough to give us a unique local equilibrium, since there is an E_t on both sides of the equation. $\pi_{t+1} = \delta_{t+1}$, $E_t(\delta_{t+1}) = 0$ is a solution for *any* value of ϕ.

In sum, in this simple model, Taylor determinacy disappears as soon as the Fed reacts to expected future rather than current inflation. Since the responses to expected future inflation do not produce equilibria in this model, there is no reason to study identification.

The familiar three-equation model displays similar behavior. The regions of ϕ_π required to produce a forward-looking solution vary considerably whether the central bank reacts to current or expected future inflation, and whether the central bank reacts to output. For example, King (2000) shows that when the central bank responds to expected future inflation, there is a second region of very large $\phi_\pi >> 1$ that again leads to indeterminacy. Allowing $\gamma < 0$ gives further and quite complex regions.

6 General case

One might suspect that these results depend on the details of the three equation model. What if one specifies a slightly different Taylor rule, or slightly different IS or Phillips curves?

The bottom line is that when you estimate dynamics from stationary variables, you must find "stable" dynamics. You cannot measure eigenvalues greater than one. In the forward-looking bounded solution, shocks corresponding to eigenvalues greater than one are set to zero; consequently the system is always observationally indistinguishable from one with eigenvalues less than one, but in which the shocks also happen to be zero. In some specifications it is possible to correctly identify the Taylor coefficient of Fed policy – but then you will misidentify some other parameter (σ, γ), so that the overall estimate of the *system* is unidentified, and other structural parameters which can lead to "determinacy" or "indeterminacy" are unidentified.

The most general form of the model can be written

$$x_{t+1} = Ax_t + C\varepsilon_{t+1} \tag{35}$$

where x_t is a vector of variables, e. g. $x_t = \left[\begin{array}{ccccc} y_t & \pi_t & i_t & x_{\pi t} & x_{dt} \end{array} \right]'$. As in this example, not all elements of x are directly observable. The point of the model may be to link endogenous observables (y, π, i) to disturbances (x_π, x_d), hence the solution may end up with fewer shocks than variables in x.

In most economic models, the shocks ε_t are shocks to taste and technology. Hence, we want stable dynamics and we solve the model by iterating backwards,

$$x_t = \sum_{j=0}^{\infty} A^j \varepsilon_{t-j}.$$

However, in new-Keynesian models at least some of the shocks are forecast errors. The model stops at $E_t x_{it+i} =$ something else. In this case the backwards solution leads to indeterminacy since forecast errors can be anything. Hence, new-Keynesian models want at least some of the roots to be explosive (forward-looking) so that the forecast errors are uniquely determined and there is a unique local solution.

Thus, we solve the model by breaking it in to the part corresponding to roots greater and less than zero. By an eigenvalue decomposition of the matrix A, write

$$x_{t+1} = Q\Lambda Q^{-1} x_t + C\varepsilon_{t+1}$$

where Λ is a diagonal matrix of eigenvalues,

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix},$$

and Q is a corresponding matrix of eigenvectors. Hence, we can write the model as

$$z_{t+1} = \Lambda z_t + \xi_{t+1}$$

where

$$z_t = Q^{-1} x_t ; \xi_{t+1} = Q^{-1} C \varepsilon_{t+1}$$

Since Λ is diagonal, we can solve for each z variable separately. We solve the unstable roots forwards and the stable roots backwards. (Again, there is no economics in this, it is simply the rules of the game in new-Keynesian models to find the unique locally bounded equilibrium.)

$$\|\lambda_i\| > 1 : \; z_{it} = \sum_{j=1}^{\infty} \frac{1}{\lambda_i^j} E_t \xi_{t+j}^i = 0 \tag{36}$$

$$\begin{aligned} \|\lambda_i\| &< 1 : \; z_{it} = \sum_{j=0}^{\infty} \lambda_i^j \xi_{t-j}^i \qquad (37) \\ z_{it} &= \lambda_i z_{it-1} + \xi_{it} \end{aligned}$$

The forward-looking solution, then, simply corresponds to setting to zero the shocks corresponding to eigenvalues greater than one.

Call the vector of the z variables in corresponding to eigenvalues less than one in (37) z_t^* and corresponding shocks ξ_t^*; call the diagonal matrix of eigenvalues less than one Λ^*, and call the matrix consisting of rows of Q corresponding to eigenvalues less than one Q^*. Then, we can characterize the dynamics of the original x as

$$z_t^* = \Lambda^* z_{t-1}^* + \xi_t^* \tag{38}$$
$$x_t = Q_t^* z_t^* \tag{39}$$

The roots $\|\lambda\|$ that are greater than one do not show up anywhere in the solution. Thus, *we cannot measure roots greater than one from a sample of data taken from the equilibrium of this model.* Equation (36) shows why: *there is no variation in the linear combinations of variables you need to measure* $\|\lambda\| > 1$. Equations (38) and (39) are obviously indistinguishable from a full solution of the model in which the z variables in (36) have eigenvalues $\|\lambda_i\| < 1$, but happen to have exactly zero shocks ξ^i, so the z happen to be zero, just as the shocks and z must all happen to be zero in the forward-looking solution with $\|\lambda\| > 1$. Thus, *the dynamics are indistinguishable from those of a model with the eigenvalues greater than one changed to eigenvalues less than one.*

6.1 System non-identification: an example

An example is useful to digest the general case. This example also investigates a related issue: There *are* specifications in which the *Taylor rule* coefficient can be identified. But the above general case states that in such specifications we will not be able to identify or measure *other* parameters. There is no way to measure whether the *system as a whole* is determinate.

To see this point, suppose the Fed follows

$$i_t = \phi\pi_t \tag{40}$$

There is no time-varying constant (i_t^*) in this rule, no error term, and the Fed follows the same reaction to *in*-equilibrium inflation as it does to *off*-equilibrium inflation, unlike the general case of (29) (in that equation, i_t^* can depend on inflation π^*, giving the response of interest rates to inflation *in* equilibrium. That response need not equal the response ϕ_0 of interest rates to *off*-equilibrium inflation.) If the Fed follows this rule, we *can* identify ϕ by a simple regression of i on π. Of course, this specification cannot revive the empirical literature, as the implication of a perfect fit can be quickly rejected. Still, it is worth examining the theoretical possibility it raises of measuring the Taylor coefficient.

But if the new-Keynesian determinacy rules are followed, *other* structural parameters will be unidentified. The system *as a whole* must display stable dynamics in equilibrium; the eigenvalues we measure from an equilibrium must be less than one. We can never measure the eigenvalues that are greater than one. The only question is *which* structural parameters get mis-measured along the way.

To see this point, suppose the Fed follows (40). To make this theoretical point, I examine a simple version of the three-equation model,

$$\begin{aligned} i_t &= r_t + E_t\pi_{t+1} && (41)\\ y_t &= E_t y_{t+1} - \sigma r_t + x_{dt} && (42)\\ \pi_t &= \beta E_t\pi_{t+1} + \gamma y_t && (43)\\ x_{dt} &= \rho x_{dt-1} + \varepsilon_t && (44) \end{aligned}$$

For simplicity I have set $r = 0$, $\bar{y} = 0, x_\pi = 0$ and assumed an AR(1) process for the remaining shock x_d. Recursively substituting, we can express the model in the standard form[8]

$$\begin{bmatrix} E_t y_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\sigma\gamma}{\beta} & -\frac{\sigma}{\beta} + \sigma\phi \\ -\frac{\gamma}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \tilde{\pi}_t \end{bmatrix}$$

[8] Using the Phillips curve (43) we have

$$\beta E_t \pi_{t+1} = \pi_t - \gamma y_t - x_{\pi t}.$$

Using the Fisher relation (41) and the policy (40), we have

$$r_t = \phi\pi_t - E_t\pi_{t+1}$$

Substituting for r_t and $\beta E_t \pi_{t+1}$ in (42) and rearranging,

$$\beta E_t y_{t+1} = \beta y_t + \beta\sigma\phi\pi_t - \sigma\pi_t + \sigma\gamma y_t + \sigma x_{\pi t} - \beta x_{dt}$$

In sum, we have

$$\begin{aligned} E_t y_{t+1} &= \left[1 + \frac{\sigma\gamma}{\beta}\right] y_t + \sigma\left[\phi - \frac{1}{\beta}\right]\pi_t + \frac{\sigma}{\beta}x_{\pi t} - x_{dt} \\ E_t\pi_{t+1} &= \frac{1}{\beta}\pi_t - \frac{\gamma}{\beta}y_t - \frac{1}{\beta}x_{\pi t}. \end{aligned}$$

plus Equation (44). ($i_t = \phi\pi_t$ means we do not have to carry a separate i equation around.) Adding forecast errors, we have the standard form (35)

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ x_{dt+1} \end{bmatrix} = \begin{bmatrix} 1+\frac{\sigma\gamma}{\beta} & -\frac{\sigma}{\beta}+\sigma\phi & -1 \\ -\frac{\gamma}{\beta} & \frac{1}{\beta} & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ x_{dt} \end{bmatrix} + \begin{bmatrix} \delta_{yt+1} \\ \delta_{\pi t+1} \\ \varepsilon_{dt+1} \end{bmatrix} \tag{45}$$

I use δ to emphasize that the first two shocks are arbitrary forecast errors.

Eigenvalue decomposing the transition matrix, and assuming parameter values that give two eigenvalues greater than one and one eigenvalue (ρ) less than one, we can write the solution for the observables (y, i, π) as above in the form[9]

$$\begin{aligned} z_{t+1} &= \rho z_t + w_{dt+1} \\ \begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix} &= \begin{bmatrix} 1-\rho\beta \\ \gamma \\ \phi\gamma \end{bmatrix} \begin{bmatrix} z_t \end{bmatrix} \end{aligned}$$

With one shock and three endogenous variables, we should not be surprised to find that they move in lockstep.

Now, we can see which structural parameters are identified. ρ is identified by the regression of any variable on its own lag. ϕ is identified by the ratio $\phi = i_t/\pi_t$. The ratio y_t/π_t identifies the quantity $(1-\rho\beta)/\gamma$, but we cannot separately identify β and γ. Worst of all, σ is completely unidentified, as it appears nowhere in relations between observables.

Since σ is not identified in this model it can be either sign. There is a region of the parameter space with $\phi > 1$ and $\sigma < 0$ in which an eigenvalue is less than one, and hence the model does not have unique bounded solutions. Thus, though we know $\phi > 1$, alas we do not know whether the *other* parameters of the system place us in the zone of "determinacy" or not.

6.2 Taylor regressions in new-Keynesian model output

What happens if you run Taylor-rule regressions in artificial data from a new-Keynesian model? We know the answer for the simple model given above, and we have general theorems that the result will not measure ϕ_π or other crucial parameters needed to establish roots greater than one. Still, it would be comforting and interesting to know the answer in the standard three-equation new-Keynesian model.

In general, the answer is a) not ϕ_π and b) a huge mountain of algebra. While easy enough to evaluate numerically, such answers don't give much intuition. For some special cases, though, we can find intelligible and interesting algebraic formulas. I present the algebra in the Appendix.

Suppose the central bank follows, and we estimate, a rule of the form

$$\begin{aligned} i_t &= \phi_\pi \pi_t + x_{it} \\ x_{it} &= \rho_i x_{it-1}, \end{aligned}$$

and the economy follows the standard three-equation model (21)-(23). When this is the only disturbance to the system (no errors x_{dt} in the IS equation, or $x_{\pi t}$ in the Phillips curve equations,

[9]To find this representation, I take the eigenvalue decomposition of the transition matrix in (45) and keep the eigenvector corresponding to the eigenvalue ρ. Taking analytic eigenvalues and eigenvectors is easy with a symbolic math program; I use Maple included in Scientific Workplace.

no variation in the natural rate $\bar{y}_t$), we can evaluate the *estimated* coefficient in a regression of i_t on π_t as

$$\hat{\phi}_\pi = \rho_i + \frac{(1-\rho_i)(1-\rho_i\beta)}{\sigma\gamma} \tag{46}$$

First, note that ϕ_π appears nowhere in the right hand side. Second, we start with the autocorrelation of the monetary policy shock ρ_i, as in the simple case studied above. Third, there are now extra terms, involving the parameters of the other equations of the model, in particular the intertemporal substitution elasticity σ and the Phillips coefficient γ. Most of all, σ and γ can take on small values, so $\hat{\phi}_\pi$ *can be greater than one.* This observation solves a puzzle of the simple example: How can Clarida, Gali and Gertler (2000) and others find coefficients greater than one? In the simple example, in which $\hat{\phi}_\pi$ estimated the autocorrelation of the policy shock, this was not possible. One might suspect that they had measured *something* by finding a coefficient greater than one. Here we see that estimated coefficients greater than one are perfectly possible – and perfectly uninformative about the true ϕ_π

Suppose instead that the central bank follows, and we estimate, a rule of the form

$$i_t = \phi_\pi E_t\pi_{t+1} + x_{it},$$

i.e. reacting to expected future inflation. In the same case of the three-equation model, the *estimated* coefficient is

$$\hat{\phi}_\pi = 1 + \frac{(1-\rho_i)(1-\rho_i\beta)}{\sigma\gamma\rho_i}. \tag{47}$$

Once again, ϕ_π is absent from the right hand side. In this case, we *generically* find a coefficient greater than one, so long as parameters obey their usual signs, though this finding has *nothing* to do with the actual Taylor rule.

7 Related Literature

Minford, Perugini and Srinivasan (2001, 2002) address a related but different point: does a Taylor-rule regression of interest rates on output and inflation establish that the Fed is in fact following a Taylor rule? The answer is no: Even if the Fed targets the money stock there will be variation of nominal interest rates, output and inflation in equilibrium, so we will see a "Taylor rule" type relation. As a very simple example, just consider a constant money supply equal to money demand,

$$\begin{aligned} m^d - p_t &= \alpha y_t - \beta i_t \\ m_t^d &= m^s \end{aligned}$$

in equilibrium, we see a Taylor-like relation between nominal interest rates, output and the price level

$$i_t = -\frac{1}{\beta}m^s + \frac{\alpha}{\beta}y_t + \frac{1}{\beta}p_t$$

Beyer and Farmer (2006) compare an "indeterminate" AR(1) model

$$p_t = aE_t(p_{t+1})$$

with $\|a\| < 1$ to a "determinate" AR(2),

$$p_t = aE_t(p_{t+1}) + bp_{t-1} + v_t$$

where they choose a and b so that one root is stable and the other unstable. Both models have $AR(1)$ representations, so there is no way to tell them apart. They conjecture based on this result that Lubik and Schorfheide (2004) attain identification by lag length restrictions.

Beyer and Farmer (2004) compute solutions to the three equation new-Keynesian model. They note (p 24) that the equilibrium dynamics are the same for any value of the Fed's Taylor Rule coefficient on inflation, as long as that coefficient is greater than one. Thus, they see that the Taylor Rule coefficient is not identified by the equilibrium dynamics. They examine the model

$$\begin{aligned} u_t &= E_t u_{t+1} + 0.005\left(i_t - E_t\pi_{t+1}\right) - 0.0015 + v_{1t} \\ \pi_t &= 0.97E_t\pi_{t+1} - 0.5u_t + 0.0256 + v_{2t} \\ i_t &= 1.1E_t\pi_{t+1} + 0.028 + v_{3t} \end{aligned}$$

where v_{it} are i.i.d. shocks. They compute the equilibrium dynamics ("reduced form") as

$$\begin{bmatrix} u_t \\ \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.02 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0.05 \\ -0.5 & 1 & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \end{bmatrix}. \tag{48}$$

They state that "all policies of the form

$$i_t = -f_{32}E_t\left[\pi_{t+1}\right] + c_3 + v_{3t},$$

for which

$$|f_{32}| > 1$$

lead to exactly the same reduced form..as long as c_3 and f_{32} are chosen to preserve the same steady state interest rate." They don't state whether this is an analytical result or simply the result of trying a lot of values; since the computation of (48) is numerical, one suspects the latter.

Davig and Leeper (2005) calculate an economy in which the Taylor rule stochastically shifts between "active" $\phi > 1$ and "passive" $\phi < 1$ states. They show that the system can display a unique locally-bounded solution even though one of the regimes is "passive." Intuitively, it is enough that at some date in the future the current course will lead to an explosion to rule out all but one equilibrium. Even if one *could* identify and measure the parameters of the Taylor rule, this model argues against the stylized history that the US moved from "passive" and hence "indeterminate" monetary policy in the 70s to an "active" and hence "determinate" policy in the 1980s. So long as agents understood some chance of moving to an "active" policy, inflation was already "determinate" in the 1970s.

8 Concluding comments

The Taylor rule coefficient, or more generally the parameters needed to measure "local determinacy" vs. "indeterminacy" of equilibrium, are not identified. If you simulate data from a new-Keynesian economy, in which the Taylor principle $\phi > 1$ applies, you will estimate a combination of other parameters, including the persistence of monetary shocks.

Thus evidence that $\phi < 1$ in 1970s and $\phi > 1$ in the 1980s, with stabilizing inflation *does not* argue for new-Keynesian model of price determination. This evidence is the central founding story of the new-Keynesian model of price determination.

On a theoretical basis, there is no reason to throw out the nominal explosions that occur with $\phi > 1$ as equilibria. If the Fed threatens hyperinflation, it might just get hyperinflation, and hyperinflations can and have occurred.

The fiscal theory of the price level is the only coherent *economic* model that can determine the model in a fiat-money economy following an interest rate target. It must at least be added to interest rate rules to obtain a coherent theory of price-level determination. The only logical alternatives are to go back to ad-hoc backward-looking ISLM models, or to discover some new theory of price level determination.

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10 Appendix

10.1 Expected values

We often generate one series (price, consumption, inflation) as an expected discounted sum of another (dividends, income, policy disturbances)

$$\begin{aligned} y_t &= E_t \sum_{j=0}^{\infty} \theta^j x_{t+j} \\ x_t &= a(L)\varepsilon_t \end{aligned}$$

Task 1 Find a representation for $y_t = b(L)\varepsilon_t$. The answer is (Hansen and Sargent 1980)

$$y_t = \left(\frac{La(L) - \theta a(\theta)}{L - \theta} \right) \varepsilon_t.$$

Here's why. Start by writing out

$$y_t^* = \sum_{j=0}^{\infty} \theta^j x_{t+j} = \frac{1}{1 - \theta L^{-1}} x_t = \frac{1}{1 - \theta L^{-1}} a(L)\varepsilon_t.$$

$$y_t^* = \begin{array}{ccccccc} & & & a_0\varepsilon_t & +a_1\varepsilon_{t-1} & +a_2\varepsilon_{t-2} & +... \\ & & +(\theta a_0\varepsilon_{t+1}) & +\theta a_1\varepsilon_t & +\theta a_2\varepsilon_{t-1} & +\theta a_3\varepsilon_{t-2} & +... \\ & +(\theta^2 a_0\varepsilon_{t+2}) & +(\theta^2 a_1\varepsilon_{t+1}) & +\theta^2 a_2\varepsilon_t & +\theta^2 a_3\varepsilon_{t-1} & +\theta^2 a_4\varepsilon_{t-2} & +... \\ +(\theta^3 a_0\varepsilon_{t+3}) & +(\theta^3 a_1\varepsilon_{t+2}) & +(\theta^3 a_2\varepsilon_{t+1}) & +\theta^3 a_3\varepsilon_t & +\theta^3 a_4\varepsilon_{t-1} & +\theta^3 a_5\varepsilon_{t-2} & +... \end{array}$$

Now, y_t is formed by simply getting rid of all the terms involving future ε_{t+j}, which I put in parentheses. Next sum the columns. For example, the ε_{t+1} term is

$$\theta a_0 + \theta^2 a_1 + \theta^3 a_2 + ... = \theta a(\theta)$$

Thus, we can write

$$\begin{aligned} y_t &= \left\{ \frac{a(L)}{1 - \theta L^{-1}} - \left[\theta a(\theta) L^{-1} + \theta^2 a(\theta) L^{-2} + \theta^3 a(\theta) L^{-3} + ... \right] \right\} \varepsilon_t \\ &= \left\{ \frac{a(L)}{1 - \theta L^{-1}} - a(\theta) \left[\theta L^{-1} + \theta^2 L^{-2} + \theta^3 L^{-3} + ... \right] \right\} \varepsilon_t \\ &= \left\{ \frac{a(L)}{1 - \theta L^{-1}} - \frac{a(\theta)\theta L^{-1}}{1 - \theta L^{-1}} \right\} \varepsilon_t \\ &= \left\{ \frac{La(L) - a(\theta)\theta}{L - \theta} \right\} \varepsilon_t \end{aligned}$$

Example. Suppose

$$x_t = \rho x_{t-1} + \varepsilon_t.$$

It's easy to work out by hand that

$$E_t \sum_{j=0} \theta^j x_{t+j} = \sum \theta^j \rho^j x_t = \frac{1}{1 - \rho\theta} x_t = \frac{1}{1 - \rho\theta} \frac{1}{1 - \rho L} \varepsilon_t.$$

Our formula gives

$$\begin{aligned}
E_t \sum_{j=0} \theta^j x_{t+j} &= \left\{ \frac{\frac{L}{1-\rho L} - \frac{\theta}{1-\rho\theta}}{L-\theta} \right\} \varepsilon_t \\
&= \left\{ \frac{\frac{L(1-\rho\theta)-\theta(1-\rho L)}{(1-\rho L)(1-\rho\theta)}}{L-\theta} \right\} \varepsilon_t \\
&= \left\{ \frac{\frac{L-\theta}{(1-\rho L)(1-\rho\theta)}}{L-\theta} \right\} \varepsilon_t \\
&= \frac{1}{(1-\rho L)(1-\rho\theta)} \varepsilon_t
\end{aligned}$$

just as it should.

Task 2, reverse engineering Suppose you have a representation for $y_t = b(L)\varepsilon_t$. Construct an $x_t = a(L)\varepsilon_t$ that justifies it by $y_t = E_t \sum_{j=0}^{\infty} \theta^j x_{t+j}$. We want

$$b(L) = \frac{La(L) - \theta a(\theta)}{L-\theta}.$$

Solving,

$$b(L)(L-\theta) = La(L) - \theta a(\theta).$$

Evaluate at $L=0$ to find $a(\theta)$

$$\begin{aligned}
b(0)(-\theta) &= -a(\theta)\theta \\
b(0) &= a(\theta)
\end{aligned}$$

Then substitute

$$\begin{aligned}
b(L)(L-\theta) &= La(L) - b(0)\theta \\
a(L) &= \frac{b(L)(L-\theta) + b(0)\theta}{L} \\
a(L) &= b(L)(1-\theta L^{-1}) + b(0)\theta L^{-1} \\
a(L) &= b(L) - \theta L^{-1}\left(b(L) - b(0)\right) \qquad (49)
\end{aligned}$$

That's the answer.

We can also write the answer out explicitly:

$$\begin{aligned}
a(L) &= b_0 + b_1 L + b_2 L^2 + b_3 L^3 + ... - \theta L^{-1}\left(b_1 L + b_2 L^2 + ...\right) \\
&= (b_0 - \theta b_1) + (b_1 - \theta b_2) L + (b_2 - \theta b_3) L^2 + ...
\end{aligned}$$

i.e.

$$a_j = b_j - \theta b_{j+1} \qquad (50)$$

We can check,

$$\begin{aligned}
E_t \sum_{j=0}^{\infty} \theta^j x_{t+j} &= E_t \sum_{j=0}^{\infty} \theta^j a(L) \varepsilon_{t+j} \\
&= E_t \sum_{j=0}^{\infty} \theta^j \sum_{k=0}^{\infty} (b_k - \theta b_{k+1}) \varepsilon_{t+j-k}
\end{aligned}$$

$$
\begin{aligned}
&= (b_0 - \theta b_1)\,\varepsilon_t + (b_1 - \theta b_2)\varepsilon_{t-1} + (b_2 - \theta b_3)\,\varepsilon_{t-2} + ... \\
&\quad +\theta\left[(b_1 - \theta b_2)\varepsilon_t + (b_2 - \theta b_3)\,\varepsilon_{t-1} + (b_3 - \theta b_4)\,\varepsilon_{t-2} + ...\right] \\
&\quad +\theta^2\left[(b_2 - \theta b_3)\varepsilon_t + (b_3 - \theta b_4)\,\varepsilon_{t-1} + (b_4 - \theta b_5)\,\varepsilon_{t-2} + ...\right]
\end{aligned}
$$

$$
= b_0\varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2} + ...
$$

10.2 Estimated coefficients from the new Keynesian model.

This section derives Equations (46) and (47)

The system is

$$
\begin{aligned}
y_t &= E_t y_{t+1} - \sigma r_t + x_{dt} \\
i_t &= r_t + E_t \pi_{t+1} \\
\pi_t &= \beta E_t \pi_{t+1} + \gamma y_t + x_{\pi t} \\
i_t &= \phi_\pi \pi_t + x_{it}
\end{aligned}
$$

Eliminate i, r to express the model in standard form,

$$
\begin{aligned}
E_t y_{t+1} &= y_t + \sigma r_t - x_{dt} \\
E_t \pi_{t+1} &= \phi_\pi \pi_t + x_{it} - r_t \\
\beta E_t \pi_{t+1} &= \pi_t - \gamma y_t - x_{\pi t}
\end{aligned}
$$

$$
\begin{aligned}
E_t y_{t+1} &= y_t + \sigma\left(E_t \pi_{t+1} - \phi_\pi \pi_t - x_{it}\right) - x_{dt} \\
E_t y_{t+1} &= y_t + \frac{\sigma}{\beta}\pi_t - \frac{\sigma}{\beta}\gamma y_t - \frac{\sigma}{\beta} x_{\pi t} - \sigma\phi_\pi \pi_t - \sigma x_{it} - x_{dt} \\
E_t y_{t+1} &= \left(1 - \frac{\sigma\gamma}{\beta}\right) y_t + \sigma\left(\frac{1}{\beta} - \phi_\pi\right)\pi_t - \frac{\sigma}{\beta} x_{\pi t} - \sigma x_{it} - x_{dt}
\end{aligned}
$$

Thus, we solve

$$
\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ x_{dt+1} \\ x_{\pi t+1} \\ x_{it+1} \end{bmatrix}
=
\begin{bmatrix}
1 - \frac{\sigma\gamma}{\beta} & \frac{\sigma}{\beta} - \sigma\phi_\pi & -1 & -\frac{\sigma}{\beta} & -\sigma \\
-\frac{\gamma}{\beta} & \frac{1}{\beta} & 0 & -1 & 0 \\
0 & 0 & \rho_d & 0 & 0 \\
0 & 0 & 0 & \rho_\pi & 0 \\
0 & 0 & 0 & 0 & \rho_i
\end{bmatrix}
\begin{bmatrix} y_t \\ \pi_t \\ x_{dt} \\ x_{\pi t} \\ x_{it} \end{bmatrix}
+
\begin{bmatrix} \delta_{yt+1} \\ \delta_{\pi t+1} \\ \varepsilon_{dt+1} \\ \varepsilon_{\pi t+1} \\ \varepsilon_{it+1} \end{bmatrix}
$$

Taking eigenvalues and eigenvectors of the transition matrix, we can express the solution as

$$
\begin{aligned}
z_{dt} &= \rho_d z_{dt-1} + \varepsilon_{dt} \\
z_{\pi t} &= \rho_\pi z_{\pi t-1} + \varepsilon_{\pi t} \\
z_{it} &= \rho_i z_{it-1} + \varepsilon_{it}
\end{aligned}
$$

$$
\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}
=
\begin{bmatrix}
1 - \rho_d\beta & \sigma\left(1 - (1+\rho_\pi)\,\beta + \phi_\pi\beta^2\right) & 1 - \rho_i\beta \\
\gamma & \beta^2\,(1 - \rho_\pi) + \sigma\gamma\,(1 - \beta) & \gamma
\end{bmatrix}
\begin{bmatrix} z_{dt} \\ z_{\pi t} \\ z_{it} \end{bmatrix}
$$

$$
\begin{aligned}
x_{dt} &= \left(\left(1-\rho_d\right)\left(1-\rho_d\beta\right)+\sigma\gamma\left(\rho_d-\phi_\pi\right)\right) z_{dt} \\
x_{\pi t} &= \left(\left(1-\rho_\pi\right)\left(1-\rho_\pi\beta\right)+\sigma\gamma\left(\rho_\pi-\phi_\pi\right)\right)\beta z_{\pi t} \\
x_{it} &= \left(\left(1-\rho_i\right)\left(1-\rho_i\beta\right)+\sigma\gamma\left(\rho_i-\phi_\pi\right)\right)/\sigma z_{it}
\end{aligned}
$$

$$
i_t = \phi_\pi \pi_t + x_{it}
$$

What do you get if you regress i_t on π_t?

$$
\hat{\phi}_\pi = \phi_\pi + cov(\pi_t, x_{it})/var(\pi_t)
$$

Since π_t loads on the shock x_{it}, the covariance is not zero.

$$
\begin{aligned}
var(\pi_t) &= \gamma^2\sigma_{zd}^2 + \left[\beta^2\left(1-\rho_\pi\right)+\sigma\gamma\left(1-\beta\right)\right]^2\sigma_{z\pi}^2 + \gamma^2\sigma_{zi}^2 \\
cov(\pi_t, x_{it}) &= \left(\left(1-\rho_i\right)\left(1-\rho_i\beta\right)+\sigma\gamma\left(\rho_i-\phi_\pi\right)\right)\left(\gamma/\sigma\right)\sigma_{zi}^2
\end{aligned}
$$

$$
\frac{cov(\pi_t, x_{it})}{var(\pi_t)} = \frac{\left[\left(1-\rho_i\right)\left(1-\rho_i\beta\right)+\sigma\gamma\left(\rho_i-\phi_\pi\right)\right]\left(\gamma/\sigma\right)\sigma_{zi}^2}{\gamma^2\sigma_{zd}^2+\left[\beta^2\left(1-\rho_\pi\right)+\sigma\gamma\left(1-\beta\right)\right]^2\sigma_{z\pi}^2+\gamma^2\sigma_{zi}^2}
$$

In the special case that the π and d shocks are zero, we have

$$
\frac{cov(\pi_t, x_{it})}{var(\pi_t)} = \frac{\left(1-\rho_i\right)\left(1-\rho_i\beta\right)}{\sigma\gamma} + \left(\rho_i - \phi_\pi\right)
$$

The ϕ_π cancel, so the answer is

$$
\hat{\phi}_\pi = \frac{\left(1-\rho_i\right)\left(1-\rho_i\beta\right)}{\sigma\gamma} + \rho_i
$$

To evaluate the expected-inflation rule, the system is now

$$
\begin{aligned}
y_t &= E_t y_{t+1} - \sigma r_t + x_{dt} \\
i_t &= r_t + E_t \pi_{t+1} \\
\pi_t &= \beta E_t \pi_{t+1} + \gamma y_t + x_{\pi t} \\
i_t &= \phi_\pi E_t \pi_{t+1} + x_{it}
\end{aligned}
$$

Eliminate i, r to express the model in standard form,

$$
\begin{aligned}
E_t y_{t+1} &= y_t + \sigma r_t - x_{dt} \\
\left(1-\phi_\pi\right) E_t \pi_{t+1} &= x_{it} - r_t \\
\beta E_t \pi_{t+1} &= \pi_t - \gamma y_t - x_{\pi t}
\end{aligned}
$$

$$
\begin{aligned}
E_t y_{t+1} &= y_t + \sigma(1-\phi_\pi)E_t\pi_{t+1} - \sigma x_{it} - x_{dt} \\
E_t y_{t+1} &= y_t + \frac{\sigma}{\beta}(1-\phi_\pi)\left(\pi_t - \gamma y_t - x_{\pi t}\right) - \sigma x_{it} - x_{dt} \\
E_t y_{t+1} &= \left[1 - \frac{\sigma\gamma}{\beta}(1-\phi_\pi)\right] y_t + \frac{\sigma}{\beta}(1-\phi_\pi)\pi_t - \frac{\sigma}{\beta}(1-\phi_\pi)x_{\pi t} - \sigma x_{it} - x_{dt}
\end{aligned}
$$

Thus, we solve

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ x_{dt+1} \\ x_{\pi t+1} \\ x_{it+1} \end{bmatrix} = \begin{bmatrix} 1-\frac{\sigma\gamma}{\beta}(1-\phi_\pi) & \frac{\sigma}{\beta}(1-\phi_\pi) & -1 & -\frac{\sigma}{\beta}(1-\phi_\pi) & -\sigma \\ -\frac{\gamma}{\beta} & \frac{1}{\beta} & 0 & -1 & 0 \\ 0 & 0 & \rho_d & 0 & 0 \\ 0 & 0 & 0 & \rho_\pi & 0 \\ 0 & 0 & 0 & 0 & \rho_i \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ x_{dt} \\ x_{\pi t} \\ x_{it} \end{bmatrix} + \begin{bmatrix} \delta_{yt+1} \\ \delta_{\pi t+1} \\ \varepsilon_{dt+1} \\ \varepsilon_{\pi t+1} \\ \varepsilon_{it+1} \end{bmatrix}.$$

Taking eigenvectors, the solution is

$$\begin{aligned} z_{dt} &= \rho_d z_{dt-1} + \varepsilon_{dt} \\ z_{\pi t} &= \rho_\pi z_{\pi t-1} + \varepsilon_{\pi t} \\ z_{it} &= \rho_i z_{it-1} + \varepsilon_{it} \end{aligned}$$

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1-\rho_d\beta & \sigma(1-\phi_\pi)[1-\beta(1+\rho_\pi)] & 1-\rho_i\beta \\ \gamma & \beta^2(1-\rho_\pi)+\sigma\gamma(1-\beta)(1-\phi_\pi) & \gamma \end{bmatrix} \begin{bmatrix} z_{dt} \\ z_{\pi t} \\ z_{it} \end{bmatrix}$$

$$\begin{aligned} x_{dt} &= [(1-\rho_d)(1-\rho_d\beta)+\sigma\gamma\rho_d(1-\phi_\pi)]\, z_{dt} \\ x_{\pi t} &= \beta\, [(1-\rho_\pi)(1-\rho_\pi\beta)+\sigma\gamma\rho_\pi(1-\phi_\pi)]\, z_{\pi t} \\ x_{it} &= \frac{1}{\sigma}[(1-\rho_i)(1-\rho_i\beta)+\sigma\gamma\rho_i(1-\phi_\pi)]\, z_{it} \end{aligned}$$

$$i_t = \phi_\pi E_t\pi_{t+1} + x_{it} \tag{51}$$

Now, we want to run a regression of i_t on $E_t\pi_{t+1}$. Again, I specialize to $z_d = z_\pi = 0$. Then,

$$\begin{aligned} \pi_t &= \gamma z_{\pi_t} \\ E_t\pi_{t+1} &= \gamma\rho_i z_{it} \end{aligned}$$

With two or fewer shocks, we can recover the shocks from the observable variables, so there is no issue that E_t formed by observable instruments gives less information than E_t formed on the full information set, i.e. seeing the z. Thus, when we run regression (51), the result is

$$\begin{aligned} \hat{\phi}_\pi &= \phi_\pi + \frac{cov(x_{it}, \gamma\rho_i z_{it})}{var(\gamma\rho_i z_{it})} \\ &= \phi_\pi + \frac{1}{\sigma}\frac{[(1-\rho_i)(1-\rho_i\beta)+\sigma\gamma\rho_i(1-\phi_\pi)]\,\gamma\rho_i}{\gamma^2\rho_i^2} \\ \hat{\phi}_\pi &= 1 + \frac{(1-\rho_i)(1-\rho_i\beta)}{\sigma\gamma\rho_i} \end{aligned}$$