On Measuring the Contribution of Entering and Exiting Firms to Aggregate Productivity Growth

by

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Abstract

The problem of assessing the impact of firm entry and exit on aggregate productivity growth is addressed. The proposed method overcomes some problems with currently proposed methods. The paper also addresses some of the problems involved in aggregating outputs and inputs when firms enter and exit so that the one output and one input aggregate productivity decompositions can be applied. It turns out that multilateral index number theory is useful in performing the aggregation of many outputs (inputs) into a single output (input).

Key words:
Productivity, index numbers, industry dynamics, entry and exit of firms, multilateral index number theory.

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1. Introduction

A recent development in productivity analysis is the increased focus on the impact of firm entry and exit into an industry on aggregate levels of productivity growth. Haltiwanger and Bartelsman and Doms in their survey papers make the following observations:[1]

"There are large and persistent differences in productivity across establishments in the same industry (see Bartelsman and Doms (2000) for an excellent discussion). The differences themselves are large – for total factor productivity the ratio of the productivity level for the plant at the 75th percentile to the plant at the 5th percentile in the same industry is 2.4 (this is the average across industries) – the equivalent ratio for labour productivity is 3.5." John Haltiwanger (2000; 9).

"The ratio of average TFP for plants in the ninth decile of the productivity distribution relative to the average in the second decile was about 2 to 1 in 1972 and about 2.75 to 1 in 1987." Eric J. Bartelsman and Mark Doms (2000; 579).

Thus the recent productivity literature has demonstrated empirically that increases in the productivity of the economy can be obtained by reallocating resources[2] away from low productivity firms in an industry to the higher productivity firms.[3] However, different investigators have chosen different methods for measuring the contributions to industry productivity growth of entering and exiting firms and the issue remains open as to which method is "best". We propose yet another method for accomplishing this decomposition. It differs from existing methods in that it treats time in a symmetric fashion so that the industry productivity difference in levels between two periods reverses sign when the periods are interchanged as do the various contribution terms.[4] Our proposed productivity decomposition is explained in sections 2 and 3 below, assuming that each

[1] See their papers for many additional references to the literature. Some of the more important references are Baldwin and Gorecki (1991), Baily, Hulten and Campbell (1992), Griliches and Regev (1995), Baldwin (1995), Haltiwanger (1997), Ahn (2001), Foster, Haltiwanger and Krizan (2001), Aw, Chen and Roberts (2001), Fox (2002), Baldwin and Gu (2002), Balk (2003), and Bartelsman, Haltiwanger and Scarpetta (2004).

[2] A more precise meaning for the term "reallocating resources" is "changing input shares".

[3] This conclusion has also emerged from the extensive literature on benchmarking and on Data Envelopment Analysis; e.g., see Coelli, Prasada Rao and Battese (1998).

[4] Balk (2003; 29) also emphasized the importance of a symmetric treatment of time. A symmetric decomposition was proposed earlier by Griliches and Regev (1995) and a modification of it was used by Aw, Chen and Roberts (2001).

firm in the industry produces only one homogeneous output and uses only one homogeneous input.

Another problem with the various productivity decompositions that have been suggested in the literature is that they often assume that there is only one output and one input that each production unit in the industry produces and uses. If the list of outputs being produced and inputs being used by each firm is constant across firms, then there is no problem in using normal index number theory to construct output and input aggregates for each continuing firm that is present for the two periods under consideration.[5] However, this method for constructing output and input aggregates does not work for entering and exiting firms, since there is no natural base observation to compare the single period data for these firms. This problem does not seem to have been widely recognized in the literature with some notable exceptions.[6] Hence in the remainder of this paper, we focus our attention on solutions to this problem. Our suggested solution to this problem is to use *multilateral index number theory* so that each firm's data in each time period is treated as if it were the data pertaining to a "country". Unfortunately, there are many possible multilateral methods that could be used. In section 5 below, we construct an artificial data set involving three continuing firms, one entering and one exiting firm and then in the remaining sections of the paper, we use various multilateral aggregation methods in order to construct firm output and input aggregates, which we then use in our suggested productivity growth decomposition formula. The multilateral aggregation methods that we consider are: the star system (section 6); the GEKS system (section 7); the own share system (section 8); the "spatial" linking method due to Robert Hill (section 9) and a simple deflation of value aggregates method (section 10).

Section 11 concludes.

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[5] An economic justification for using a superlative index to accomplish this aggregation can be supplied under some separability assumptions; see Diewert (1976).

[6] Aw, Chen and Roberts (2001) and Aw, Chung and Roberts (2003) recognized the importance of this problem and they used a modification of a multilateral method originally proposed by Caves, Christensen and Diewert (1982). The modification that they used is due to Good (1985) and is explained in Good, Nadiri and Sickles (1997). The original Caves, Christensen and Diewert method was designed for use in a single cross section and is not suitable for use in a panel data context if there is considerable inflation between the periods in the panel.

2. The Measurement of Aggregate Productivity Levels in the One Output One Input Case

We begin by considering a very simple case where firms produce one output with one input so that it is very straightforward to measure the productivity of each firm by dividing its output by its input used.[7] We assume that these firms are all in the same industry, producing the same output and using the same input, so that it is also very straightforward to measure industry productivity in each period by dividing aggregate industry output by aggregate industry input. *Our measurement problem is to account for the contributions to industry productivity growth of entering and exiting firms*.

In what follows, C denotes the set of continuing production units that are present in periods 0 and 1, X denotes the set of exiting firms which are only present in period 0, and N denotes the set of new firms that are present only in period 1.

Let $y_{Ci}^t > 0$ and $x_{Ci}^t > 0$ respectively denote the output produced and input utilized by continuing unit $i \in C$ during period t = 0, 1. Let $y_{Xi}^0 > 0$ and $x_{Xi}^0 > 0$ respectively denote the output produced and input used by exiting firm $i \in X$ during period 0. Finally, let $y_{Ni}^1 > 0$ and $x_{Ni}^1 > 0$ respectively denote the output produced and input used by the new firm $i \in N$ during period 1.

The *productivity level* Π_{Ci}^t of a continuing firm $i \in C$ in each period t can be defined as output y_{Ci}^t divided by input x_{Ci}^t:

$$(1)\ \Pi_{Ci}^t \equiv y_{Ci}^t / x_{Ci}^t ; \qquad i \in C ; t = 0,1.$$

The *productivity levels of the exiting firms* in period 0 and the *entering firms* in period 1 are defined in a similar fashion, as follows:

[7] We will consider the case of many outputs and many inputs in sections 4-10 below.

(2) $\Pi_{Xi}{}^0 \equiv y_{Xi}{}^0 / x_{Xi}{}^0$; $\quad i \in X$;

(3) $\Pi_{Ni}{}^1 \equiv y_{Ni}{}^1 / x_{Ni}{}^1$; $\quad i \in N$.

Since the production units are all producing the same output and are using the same input, a natural definition for *industry productivity* Π^0 in period 0 is aggregate output divided by aggregate input:[8]

$$(4)\ \Pi^0 \equiv [\sum_{i\in C} y_{Ci}{}^0 + \sum_{i\in X} y_{Xi}{}^0] / [\sum_{i\in C} x_{Ci}{}^0 + \sum_{i\in X} x_{Xi}{}^0]$$
$$= S_C{}^0 \sum_{i\in C} s_{Ci}{}^0 \Pi_{Ci}{}^0 + S_X{}^0 \sum_{i\in X} s_{Xi}{}^0 \Pi_{Xi}{}^0$$

where the *aggregate input shares* of the continuing and exiting firms in period 0, $S_C{}^0$ and $S_X{}^0$, are defined as follows:

$$(5)\ S_C{}^0 \equiv \sum_{i\in C} x_{Ci}{}^0 / [\sum_{i\in C} x_{Ci}{}^0 + \sum_{i\in X} x_{Xi}{}^0] ;$$
$$(6)\ S_X{}^0 \equiv \sum_{i\in X} x_{Xi}{}^0 / [\sum_{i\in C} x_{Ci}{}^0 + \sum_{i\in X} x_{Xi}{}^0] .$$

In addition, the period 0 *micro input share*, $s_{Ci}{}^0$, for a *continuing* firm $i\in C$ is defined as follows:

$$(7)\ s_{Ci}{}^0 \equiv x_{Ci}{}^0 / \sum_{k\in C} x_{Ck}{}^0 ; \qquad i\in C.$$

Thus $s_{Ci}{}^0$ is the input of continuing firm i in period 0, $x_{Ci}{}^0$, divided by the total input used by all continuing firms in period 0, $\sum_{k\in C} x_{Ck}{}^0$. Similarly, the period 0 *micro input share* for *exiting* firm $i\in X$, $s_{Xi}{}^0$, is defined the input of exiting firm i in period 0, $x_{Xi}{}^0$, divided by the total input used by all exiting firms in period 0, $\sum_{k\in X} x_{Xk}{}^0$:

$$(8)\ s_{Xi}{}^0 \equiv x_{Xi}{}^0 / \sum_{k\in X} x_{Xk}{}^0 ; \qquad i\in X.$$

[8] It is possible to rework our analysis by reversing the role of inputs and outputs so that output shares replace input shares in the decomposition formulae. Then at the end, we can take the reciprocal of the aggregate inverse productivity measure and obtain an alternative productivity decomposition. We owe this suggestion to Bert Balk.

Of course, the period 0 *aggregate productivities for continuing and exiting firms*, Π_C^0, and Π_X^0, can be defined in a similar manner to the definition of Π^0 in (4), as follows:

$$(9)\quad \Pi_C^0 \equiv \sum_{i\in C} y_{Ci}^0 / \sum_{i\in C} x_{Ci}^0$$
$$= \sum_{i\in C} s_{Ci}^0 \Pi_{Ci}^0 ;$$
$$(10)\quad \Pi_X^0 \equiv \sum_{i\in X} y_{Xi}^0 / \sum_{i\in X} x_{Xi}^0$$
$$= \sum_{i\in X} s_{Xi}^0 \Pi_{Xi}^0 .$$

Substitution of (9) and (10) back into definition (4) for the aggregate period 0 level of productivity leads to the following decomposition of aggregate period 0 productivity into its continuing and exiting components:

$$(11)\quad \Pi^0 = S_C^0 \Pi_C^0 + S_X^0 \Pi_X^0$$
$$(12)\quad\quad = \Pi_C^0 + S_X^0 (\Pi_X^0 - \Pi_C^0)$$

where (12) follows from (11) using $S_C^0 = 1 - S_X^0$.

Expression (12) is a useful decomposition of the period 0 aggregate productivity level Π^0 into two components. The first component, Π_C^0, represents the productivity contribution of continuing production units while the second term, S_X^0 $(\Pi_X^0 - \Pi_C^0)$, represents the contribution of exiting firms relative to continuing firms to the overall period 0 productivity level. Usually the exiting firm will have lower productivity levels than the continuing firms so that Π_X^0 will be less than Π_C^0 and thus under normal conditions, the second term on the right-hand side of (12) will make a *negative contribution* to the overall level of period 0 productivity.[9]

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[9] Olley and Pakes (1996; 1290) have an alternative covariance type decomposition of the overall level of productivity in a given period into firm effects but it is not suitable for our purpose, which is to highlight the differential effects on overall period 0 productivity of the exiting firms compared to the continuing firms.

Substituting (9) and (10) into (12) leads to the following decomposition of the period 0 productivity level Π^0 into its individual firm contributions:

$$(13)\ \Pi^0 = \sum_{i\in C} s_{Ci}^0\, \Pi_{Ci}^0 + S_X^0 \sum_{i\in X} s_{Xi}^0\, (\Pi_{Xi}^0 - \Pi_C^0)$$

where we have also used the fact that $\sum_{i\in X} s_{Xi}^0$ sums to unity.

Obviously, the above material can be repeated with minimal modifications to provide a decomposition of the industry period 1 productivity level Π^1 into its constituent components. Thus, Π^1 is defined as follows:

$$(14)\ \Pi^1 \equiv [\sum_{i\in C} y_{Ci}^1 + \sum_{i\in N} y_{Ni}^1] / [\sum_{i\in C} x_{Ci}^1 + \sum_{i\in N} x_{Ni}^1]$$
$$= S_C^1 \sum_{i\in C} s_{Ci}^1\, \Pi_{Ci}^1 + S_N^1 \sum_{i\in N} s_{Ni}^1\, \Pi_{Ni}^1$$

where the period 1 aggregate input shares of continuing and new firms, S_C^1 and S_N^1, and individual continuing and new firm shares, s_{Ci}^1 and s_{Ni}^1, are defined as follows:

$$(15)\ S_C^1 \equiv \sum_{i\in C} x_{Ci}^1 / [\sum_{i\in C} x_{Ci}^0 + \sum_{i\in X} x_{Xi}^0]\ ;$$
$$(16)\ S_N^1 \equiv \sum_{i\in N} x_{Ni}^1 / [\sum_{i\in C} x_{Ci}^0 + \sum_{i\in X} x_{Xi}^0]\ ;$$
$$(17)\ s_{Ci}^1 \equiv x_{Ci}^1 / \sum_{k\in C} x_{Ck}^1\ ;\ i\in C\ ;$$
$$(18)\ s_{Ni}^1 \equiv x_{Ni}^1 / \sum_{k\in N} x_{Nk}^1\ ;\ i\in N\ .$$

The period 1 counterparts to Π_C^0 and Π_X^0 defined by (9) and (10) are the *aggregate period one productivity levels of continuing firms* Π_C^1 and *entering firms* Π_N^1, defined as follows:

$$(19)\ \Pi_C^1 \equiv \sum_{i\in C} y_{Ci}^1 / \sum_{i\in C} x_{Ci}^1$$
$$= \sum_{i\in C} s_{Ci}^1\, \Pi_{Ci}^1\ ;$$
$$(20)\ \Pi_N^1 \equiv \sum_{i\in N} y_{Ni}^1 / \sum_{i\in N} x_{Ni}^1$$
$$= \sum_{i\in N} s_{Ni}^1\, \Pi_{Ni}^1\ .$$

Substitution of (19) and (20) back into definition (14) for the aggregate period 1 level of productivity leads to the following decomposition of aggregate period 1 productivity into its continuing and new components:

$$(21)\ \Pi^1 = S_C^{\,1}\ \Pi_C^{\,1} + S_N^{\,1}\ \Pi_N^{\,1}$$
$$(22)\qquad = \Pi_C^{\,1} + S_N^{\,1}\ (\Pi_N^{\,1} - \Pi_C^{\,1})$$

where (22) follows from (21) using $S_C^{\,1} = 1 - S_N^{\,1}$. Thus the aggregate period 1 productivity level Π^1 is equal to the aggregate period 1 productivity level of continuing firms, $\Pi_C^{\,1}$, plus a second term, $S_N^{\,1}\ (\Pi_N^{\,1} - \Pi_C^{\,1})$, which represents the contribution of the new entrants' productivity levels, $\Pi_N^{\,1}$, relative to that of the continuing firms, $\Pi_C^{\,1}$.[10]

Substituting (19) and (20) into (22) leads to the following decomposition of the aggregate period 1 productivity level P^1 into its individual firm contributions:

$$(23)\ \Pi^1 = \sum_{i\in C} s_{Ci}^{\,1}\ \Pi_{Ci}^{\,1} + S_N^{\,1} \sum_{i\in N} s_{Ni}^{\,1}\ (\Pi_{Ni}^{\,1} - \Pi_C^{\,1}).$$

This completes our discussion of how the levels of productivity in periods 0 and 1 can be decomposed into individual contribution effects for each firm. In the following section, we study the much more difficult problem of decomposing the aggregate productivity change, Π^1/Π^0, into individual firm growth effects, taking into account that not all firms are present in both periods and hence, there is a problem in calculating growth effects for those firms present in only one period.

[10] Baldwin (1995) in his study of the Canadian manufacturing sector showed that on average, the productivity levels of new entrants was below that of continuing firms but if the new entrant survived, then they reach the average productivity level of continuing firms in about a decade. For additional empirical evidence on the relative productivity levels of entering and exiting firms, see Bartelsman and Doms (2000; 581), Aw, Chen and Roberts (2001) (who also find that the productivity level of new entrants is below that of incumbents) and section 5 of Bartelsman, Haltiwanger and Scarpetta (2004).

3. The Measurement of Productivity Change Between the Two Periods

It is traditional to define the productivity change of a production unit going from period 0 to period 1 as a ratio of the productivity levels in the two periods rather than as a difference between the two levels. This is because the ratio measure will be independent of the units of measurement while the difference measure will depend on the units of measurement (unless some normalization is performed). However, in the present context, as we are attempting to calculate the contribution of new and disappearing production units to overall productivity change, it is more convenient to work with the difference concept, at least initially.

Using formula (13) for the period 0 productivity level Π^0 and formula (23) for the period 1 productivity level Π^1, we obtain the following decomposition of the *productivity difference*:

$$(24)\ \Pi^1 - \Pi^0 = \Sigma_{i\in C}\, s_{Ci}^1 \Pi_{Ci}^1 - \Sigma_{i\in C}\, s_{Ci}^0 \Pi_{Ci}^0 + S_N^1 \Sigma_{i\in N}\, s_{Ni}^1 (\Pi_{Ni}^1 - \Pi_C^1) - S_X^0 \Sigma_{i\in X}\, s_{Xi}^0 (\Pi_{Xi}^0 - \Pi_C^0)$$

$$(25)\ \quad = \Pi_C^1 - \Pi_C^0 + S_N^1 (\Pi_N^1 - \Pi_C^1) - S_X^0 (\Pi_X^0 - \Pi_C^0)$$

where (25) follows from (24) using (12) and (22). Thus the overall industry productivity change, $\Pi^1 - \Pi^0$, is equal to the productivity change of the continuing firms, $\Pi_C^1 - \Pi_C^0$, plus a term that reflects the contribution to overall productivity change of new entrants, $S_N^1 (\Pi_N^1 - \Pi_C^1)$,[11] plus a term that reflects the contribution to overall productivity change of exiting firms, $- S_X^0 (\Pi_X^0 - \Pi_C^0)$.[12] Note that the reference productivity levels that the productivity levels of the entering and exiting firms are compared with, Π_C^1 and Π_C^0 respectively, are *different* in general, so even if the average productivity levels of entering and exiting firms are the *same* (so that Π_N^1 equals Π_X^0), the contributions to

[11] This term is positive if and only if the average level of productivity of the new entrants in period 1, Π_N^1, is *greater* than the average productivity level of continuing firms in period 1, Π_C^1.

[12] This term is positive if and only if the average level of productivity of the firms who exit in period 0, Π_X^0, is *less* than the average productivity level of continuing firms in period 0, Π_C^0.

overall industry productivity *growth* of entering and exiting firms can still be nonzero, provided that $\Pi_N^1 \neq \Pi_C^1$ and $\Pi_X^0 \neq \Pi_C^0$.[13]

The first two terms on the right-hand side of (24) give the aggregate effects of the changes in productivity levels of the continuing firms. It is useful to further decompose this aggregate change in the productivity levels of continuing firms into two sets of components; the first set of terms measures the productivity change of each continuing production unit, $\Pi_{Ci}^1 - \Pi_{Ci}^0$, and the second set of terms reflects the shifts in the share of resources used by each continuing production unit, $s_{Ci}^1 - s_{Ci}^0$. As Balk (2003; 26) noted, there are two natural decompositions for the difference in the productivity levels of the continuing firms, (27) and (29) below, that are the difference counterparts to the decomposition of a value ratio into the product of a Laspeyres (or Paasche) price index times a Paasche (or Laspeyres) quantity index:

(26) $\Pi_C^1 - \Pi_C^0 = \sum_{i\in C} s_{Ci}^1 \Pi_{Ci}^1 - \sum_{i\in C} s_{Ci}^0 \Pi_{Ci}^0$

(27) $\qquad = \sum_{i\in C} s_{Ci}^0 (\Pi_{Ci}^1 - \Pi_{Ci}^0) + \sum_{i\in C} \Pi_{Ci}^1 (s_{Ci}^1 - s_{Ci}^0)$;

(28) $\Pi_C^1 - \Pi_C^0 = \sum_{i\in C} s_{Ci}^1 \Pi_{Ci}^1 - \sum_{i\in C} s_{Ci}^0 \Pi_{Ci}^0$

(29) $\qquad = \sum_{i\in C} s_{Ci}^1 (\Pi_{Ci}^1 - \Pi_{Ci}^0) + \sum_{i\in C} \Pi_{Ci}^0 (s_{Ci}^1 - s_{Ci}^0)$.

We now note a severe disadvantage associated with the use of either (27)[14] or (29): *these decompositions are not invariant with respect to the treatment of time*. Thus if we

[13] Haltiwanger (1997) (2000; 10) argues that if the productivity levels of entering and exiting firms or establishments are exactly the same, then the sum of the contribution terms of entering and exiting firms should be zero. However, our perspective is different: we want to measure the differential effects on productivity *growth* of entering and exiting firms and so what counts in our framework are the productivity levels of entering firms relative to continuing firms in period 1 and the productivity levels of exiting firms relative to continuing firms in period 0. Thus if continuing firms show productivity growth over the two periods, then if the entering and exiting firms have the same productivity levels, the effects of entry and exit will be to decrease productivity growth compared to the continuing firms. Balk (2003; 28) follows the example of Haltiwanger (1997) in choosing a common reference level of productivity to compare the productivity levels of entering and exiting firms but Balk chooses the arithmetic average of the industry productivity levels in periods 0 and 1 (which is at least a symmetric choice) whereas Haltiwanger chooses the industry productivity level of period 0 (which is not a symmetric choice). In any case, our approach seems to be different from other approaches suggested in the literature.

reverse the roles of periods 0 and 1, we would like the decomposition of the aggregate productivity difference for continuing firms, $\prod_C^0 - \prod_C^1 = \sum_{i\in C} s_{Ci}^0 \prod_{Ci}^0 - \sum_{i\in C} s_{Ci}^1 \prod_{Ci}^1$, into terms involving the individual productivity differences $\prod_{Ci}^0 - \prod_{Ci}^1$ and the individual share differences $s_{Ci}^0 - s_{Ci}^1$ that are the *negatives* of the original difference terms.[15] It can be seen that the decompositions defined by (26) and (28) do not have this desirable symmetry or invariance property.

A solution to this lack of symmetry is to simply take an arithmetic average of (26) and (28), leading to the following *Bennet* (1920) *type decomposition of the productivity change of the continuing firms*:

$$(30)\ \prod_C^1 - \prod_C^0 = \sum_{i\in C} (1/2)(s_{Ci}^0 + s_{Ci}^1)(\prod_{Ci}^1 - \prod_{Ci}^0) + \sum_{i\in C} (1/2)(\prod_{Ci}^0 + \prod_{Ci}^1)(s_{Ci}^1 - s_{Ci}^0).$$

The use of this decomposition for continuing firms dates back to Griliches and Regev (1995; 185).[16] Balk (2003; 29) also endorsed the use of this symmetric decomposition.[17] We endorse the use of this decomposition since it is symmetric and can also be given a strong axiomatic justification.[18]

Substitution of (30) into (24) gives our final "best" decomposition of the aggregate productivity difference $\prod^1 - \prod^0$ into micro firm effects:

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[14] The decomposition defined by (26) is the one used by Baily, Hulten and Campbell (1992; 193) for continuing firms except that they used *logs* of the TFP levels $\prod_{Ci}^t$ instead of the levels themselves.

[15] In other words, we want the difference decomposition to satisfy a differences counterpart to the usual time reversal test that occurs in index number theory.

[16] Griliches and Regev (1995; 185) also have a symmetric treatment of the industry difference in TFP levels but firms that exit and enter during the two periods being compared are treated as one firm and they make a direct comparison of the change in productivity of all entering and exiting firms on this basis. It can be seen that there are problems in interpretation if there happen to be no entering (or exiting) firms in the sample or more generally, if there are big differences in the shares of entering and exiting firms. Aw, Chen and Roberts (2001; 73) also use this symmetric methodology, except they work with logs of TFP.

[17] "In view of its symmetry it should be the preferred one." Bert M. Balk (2003; 29).

[18] Diewert (2005) showed that the Bennet decomposition of a difference of the form $\sum_i p_i^1 q_i^1 - \sum_i p_i^0 q_i^0$ into a sum of terms reflecting price change and a sum of terms reflecting quantity change can be given an axiomatic justification that is analogous to the axiomatic justification for the use of the Fisher (1922) ideal index in index number theory. The adaptation of this axiomatic theory to provide a decomposition of $\sum_i p_i^1 s_i^1 - \sum_i p_i^0 s_i^0$ is straightforward.

$$(31)\ \Pi^1 - \Pi^0 = \sum_{i\in C} (1/2)(s_{Ci}{}^0 + s_{Ci}{}^1)(\Pi_{Ci}{}^1 - \Pi_{Ci}{}^0) + \sum_{i\in C} (1/2)(\Pi_{Ci}{}^0 + \Pi_{Ci}{}^1)(s_{Ci}{}^1 - s_{Ci}{}^0)$$
$$+ S_N{}^1 \sum_{i\in N} s_{Ni}{}^1 (\Pi_{Ni}{}^1 - \Pi_C{}^1) - S_X{}^0 \sum_{i\in X} s_{Xi}{}^0 (\Pi_{Xi}{}^0 - \Pi_C{}^0).$$

The first set of terms on the right hand side of (31), $\sum_{i\in C} (1/2)(s_{Ci}{}^0 + s_{Ci}{}^1)(\Pi_{Ci}{}^1 - \Pi_{Ci}{}^0)$, gives the contribution of the productivity growth of each continuing firm to the aggregate productivity difference between periods 0 and 1, $\Pi^1 - \Pi^0$; the second set of terms, $\sum_{i\in C} (1/2)(\Pi_{Ci}{}^0 + \Pi_{Ci}{}^1)(s_{Ci}{}^1 - s_{Ci}{}^0)$, gives the contribution of the effects of the reallocation of resources between continuing firms going from period 0 to 1; the third set of terms, $S_N{}^1\sum_{i\in N} s_{Ni}{}^1 (\Pi_{Ni}{}^1 - \Pi_C{}^1)$, gives the contribution of each new entering firm to productivity growth and the final set of terms, $- S_X{}^0 \sum_{i\in X} s_{Xi}{}^0 (\Pi_{Xi}{}^0 - \Pi_C{}^0)$, gives the contribution of each exiting firm to productivity growth.

Note that the decomposition (31) is symmetric: if we reverse the role of periods 0 and 1, then the new aggregate productivity difference will equal the negative of the original productivity difference and each individual firm contribution term of the new right hand side will equal the negative of the original firm contribution effect. None of the contribution decompositions suggested in the literature have this time reversal property, with the exception of the decomposition (51) due to Balk (2003; 28) but Balk's decomposition compares the productivity levels of entering and exiting firms to the arithmetic average of the industry productivity levels in periods 0 and 1 instead of to the average productivity level of continuing firms in period 1 (in the case of entering firms) and to the average productivity level of continuing firms in period 0 (in the case of exiting firms).

We now make a final adjustment to (31) in order to make it invariant to changes in the units of measurement of output and input: we divide both sides of (31) by the base period productivity level Π^0.[19] With this adjustment, (31) becomes:

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[19] Instead of dividing by Π^0, we could divide by the logarithmic mean of Π^0 and Π^1. The left hand side of the resulting counterpart to (32) reduces to $\ln(\Pi^0/\Pi^1)$, which is completely symmetric in the data whereas the left hand side of (32) is not. We owe this suggestion to Bert Balk.

$$(32)\ [\Pi^1/\Pi^0] - 1 = [\sum_{i\in C} (1/2)(s_{Ci}{}^0 + s_{Ci}{}^1)(\Pi_{Ci}{}^1 - \Pi_{Ci}{}^0) + \sum_{i\in C}(1/2)(\Pi_{Ci}{}^0 + \Pi_{Ci}{}^1)(s_{Ci}{}^1 - s_{Ci}{}^0) + S_N{}^1 \sum_{i\in N} s_{Ni}{}^1 (\Pi_{Ni}{}^1 - \Pi_C{}^1) - S_X{}^0 \sum_{i\in X} s_{Xi}{}^0 (\Pi_{Xi}{}^0 - \Pi_C{}^0)]/\Pi^0.$$

In the following sections, we will illustrate the aggregate productivity decomposition (32) using an artificial data set. *Note that (32) is only valid for an industry that produces a single output and uses a single input*. However, in practice, firms in an industry produce many outputs and use many inputs. Hence, before the decomposition (32) can be implemented, it is necessary to aggregate the many outputs produced and inputs used by each firm into aggregate firm output and input. This aggregation problem is not straightforward because some firms are entering and exiting the industry. In the following section, we address this unconventional aggregation problem.[20]

4. How can the Inputs and Outputs of Entering and Exiting Firms be Aggregated?

The aggregate productivity decomposition defined by (32) above assumes that each firm produces only one output and uses only one input. However, firms in the same industry typically produce many outputs and utilize many inputs. Thus in order to apply (32), we have to somehow aggregate all of the outputs produced by each firm into an aggregate output that is comparable across firms and across time periods (and aggregate all of the inputs utilized by each firm into an aggregate input that is comparable across firms and across time periods). It can be seen that these two aggregation problems are in fact *multilateral aggregation problems*;[21] i.e., the output vector of each firm in each period must be compared with the corresponding output vectors of all other firms in the industry over the two time periods involved in the aggregate productivity comparison.[22] In the following sections of this paper, we will illustrate how these firm output and input

[20] As noted earlier, Aw, Chen and Roberts (2001) and Aw, Chung and Roberts (2003) addressed this aggregation problem using the multilateral method explained in Good, Nadiri and Sickles (1997).

[21] Bilateral index number theory compares the price and quantity vectors pertaining to two situations whereas multilateral index number theory attempts to construct price and quantity aggregates when there are more than two situations to be compared. See Balk (1996) (2001) and Diewert (1999) for recent surveys of multilateral methods.

[22] Fox (2002) seems to have been the first to notice that aggregating firm outputs and inputs into aggregate outputs and inputs should be treated as a multilateral aggregation problem in order to avoid paradoxical results.

aggregates can be formed using several methods that have been suggested in the multilateral aggregation literature.

In order to make the comparison of alternative multilateral methods of aggregation more concrete, we will utilize an artificial data set. In the following section, we table our data set and calculate the aggregate productivity of the industry using normal index number methods.

5. Industry Productivity Aggregates Using an Artificial Data Set

We consider an industry over two periods, 0 and 1, that consists of five firms. Each firm f produces varying amounts of the same two outputs and uses varying amounts of the same two inputs. The output vector of firm f in period t is defined as $y_f^t \equiv [y_{f1}^t,y_{f2}^t]$ and the corresponding input vector is defined as $x_f^t \equiv [x_{f1}^t,x_{f2}^t]$ for t = 0,1 and f = 1,2,…,5. Firms 1,2 and 3 are continuing firms, firm 4 is present in period 0 but not in period 1 (and hence is the exiting firm) and firm 5 is not present in period 0 but is present in period 1 (and hence is the entering firm). Firm 1 is a medium sized firm, firm 2 is a tiny firm and firm 3 is a very large firm. The output price vector of firm f in period t is defined as $p_f^t \equiv [p_{f1}^t,p_{f2}^t]$ and the corresponding input price vector is defined as $w_f^t \equiv [w_{f1}^t,w_{f2}^t]$ for t = 0,1 and f = 1,2,…,5. The industry price and quantity data are listed in Table 1.

Table 1: Firm Price and Quantity Data for Periods 0 and 1

	Firm 1		Firm 2		Firm 3		Firm 4		Firm 5	
Output prices										
	p_{11}^t	p_{12}^t	p_{21}^t	p_{22}^t	p_{31}^t	p_{32}^t	p_{41}^t	p_{42}^t	p_{51}^t	p_{52}^t
t = 0	1.0	1.0	0.8	1.2	0.9	0.8	1.2	1.1	___	___
t = 1	15.0	7.0	13.0	8.0	14.0	7.0	___	___	16.0	8.0
Output quantities										
	y_{11}^t	y_{12}^t	y_{21}^t	y_{22}^t	y_{31}^t	y_{32}^t	y_{41}^t	y_{42}^t	y_{51}^t	y_{52}^t
t = 0	12.00	8.00	1.00	1.00	50.00	50.00	7.00	9.00	___	___
t = 1	15.00	8.00	3.00	2.00	60.00	45.00	___	___	16.00	8.00
Input prices										
	w_{11}^t	w_{12}^t	w_{21}^t	w_{22}^t	w_{31}^t	w_{32}^t	w_{41}^t	w_{42}^t	w_{51}^t	w_{52}^t
t = 0	1.0	1.0	0.7	0.8	0.9	1.1	1.2	1.0	___	___
t = 1	10.0	23.0	13.0	16.0	8.0	26.0	___	___	14.0	20.0

Input quantities

	x_{11}^t	x_{12}^t	x_{21}^t	x_{22}^t	x_{31}^t	x_{32}^t	x_{41}^t	x_{42}^t	x_{51}^t	x_{52}^t
t = 0	10.00	10.00	1.00	1.00	45.00	35.00	13.00	12.00	___	___
t = 1	8.00	6.00	2.00	2.00	35.00	30.00	___	___	7.00	6.00

Thus the period 0 output price vector for firm 1 is $p_1^0 = [1,1]$, the period 1 output price vector for firm 1 is $p_1^1 = [15,7]$ and so on. Note that there has been a great deal of general price level change going from period 0 to 1.[23]

In the following sections, we will look at various methods for forming output and input aggregates for each firm and each period but before we do this, it is useful to compute *total industry supplies* of the two outputs, $y^t \equiv [y_{\bullet 1}^t, y_{\bullet 2}^t]$ for each period t and *total industry demands* for each of the two inputs $x^t \equiv [x_{\bullet 1}^t, x_{\bullet 2}^t]$ for each period t as well as the corresponding *unit value prices*, $p^t \equiv [p_{\bullet 1}^t, p_{\bullet 2}^t]$ and $w^t \equiv [w_{\bullet 1}^t, w_{\bullet 2}^t]$.[24] This information is listed in (32) below.

(33) $p^0 = [0.946, 0.869]$; $p^1 = [14.468, 7.159]$; $w^0 = [0.968, 1.057]$; $w^1 = [9.308, 24.318]$;
$y^0 = [70, 68]$; $y^1 = [94, 63]$; $x^0 = [69, 58]$; $x^1 = [52, 44]$.

Note that industry output 1 has increased from 70 to 94 but industry output 2 decreased slightly from 68 to 63. However, both industry input demands dropped markedly; input 1 decreased from 69 to 52 and input 2 decreased from 58 to 44. Thus overall, industry productivity improved markedly going from period 0 to 1.

In order to benchmark the reasonableness of the various productivity decompositions given by (32) above for different multilateral methods to be discussed in the following four sections, it is useful to use the industry data in (33) in order to construct normal index number estimates of industry Total Factor Productivity Growth (TFPG). Following

[23] In some applications of the literature on the contribution of entry and exit to aggregate productivity growth, the comparison periods are a decade apart and so in high inflation countries, the period 0 and 1 price levels can differ considerably.

[24] The unit value price of output n in period t is defined as $p_{\bullet n}^t \equiv \sum_{f=1}^5 p_{fn}^t y_{fn}^t / \sum_{f=1}^5 y_{fn}^t$ for n = 1,2 and t = 0,1. The unit value price of input n in period t is defined as $w_{\bullet n}^t \equiv \sum_{f=1}^5 w_{fn}^t x_{fn}^t / \sum_{f=1}^5 x_{fn}^t$ for n = 1,2 and t = 0,1.

Jorgenson and Griliches (1967) (1972),[25] TFPG can be defined as a quantity index of output growth, $Q(p^0,p^1,q^0,q^1)$, divided by a quantity index of input growth, $Q^*(w^0,w^1,x^0,x^1)$:

(34) $TFPG \equiv Q(p^0,p^1,q^0,q^1)/Q^*(w^0,w^1,x^0,x^1)$.

In order to implement (34), one needs to choose an index number formula for Q and Q*. From an axiomatic perspective, the "best" choices suggested in the literature seem to be the Fisher (1922) ideal formula[26] or the Törnqvist (1936) Theil (1967) formula.[27] With these two choices of index number formula, the resulting TFP growth rates[28] for the data listed in (33) are as follows:

(35) $TFPG_F = 1.5553$; $TFPG_T = 1.5573$.

If we subtract 1 from the above TFPG rates, we obtain industry aggregate counterparts to the left hand side of (32), $[\Pi^1/\Pi^0] - 1$. Thus using the Fisher formula, industry productivity improved 55.53% and using the Törnqvist Theil formula, industry productivity improved 55.73%. These productivity growth rates should be kept in mind as we look at alternative multilateral methods for constructing output and input aggregates for each firm in each period so that we can implement the decomposition formula (32). In other words, a multilateral method that leads to an aggregate productivity growth rate $[\Pi^1/\Pi^0] - 1$ that is very different from the range .5553 to .5573 is probably not very reliable.

We now turn to our first multilateral method for constructing output and input aggregates for each firm in each period.

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[25] For recent surveys on how to measure TFPG, see Balk (2003) and Diewert and Nakamura (2003).

[26] See Diewert (1992). The Fisher output quantity index is defined as $Q_F(p^0,p^1,q^0,q^1) \equiv [p^0\cdot q^1\ p^1\cdot q^1/p^0\cdot q^0\ p^1\cdot q^0]^{1/2}$ where $p\cdot q$ denotes the inner product of the vectors p and q.

[27] See Diewert (2004). Both of these formulae can be given economic justifications as well; see Diewert (1976).

[28] Actually these rates are 1 plus the total factor productivity growth rates.

6. The Star System for Making Multilateral Comparisons

Recall that in the previous section, we defined the firm f and period t output and input vectors as $y_f^t \equiv [y_{f1}^t,y_{f2}^t]$ and $x_f^t \equiv [x_{f1}^t,x_{f2}^t]$ for t = 0,1 and f = 1,2,…,5. However, for t = 0 and f = 5 and also for t = 1 and f = 4, there are no data, since these two firms are entering and exiting respectively. Thus there are actually a total of 8 output and input quantity vectors instead of 10. It will prove to be more convenient to relabel our data so that there are only 8 distinct output and input quantity vectors. Thus define the output quantity vectors y^1, y^2, y^3 and y^4 as the previously defined vectors y_1^0, y_2^0, y_3^0 and y_4^0 respectively (these are the nonzero period 0 output quantity vectors) and define the vectors y^5, y^6, y^7 and y^8 as the previously defined vectors y_1^1, y_2^1, y_3^1 and y_5^1 respectively (these are the nonzero period 1 nonzero output quantity vectors). Similarly, define the output price vectors p^1, p^2, p^3 and p^4 as the previously defined vectors p_1^0, p_2^0, p_3^0 and p_4^0 respectively and define the vectors p^5, p^6, p^7 and p^8 as the previously defined vectors p_1^1, p_2^1, p_3^1 and p_5^1 respectively. Undertake the same reordering of the data for inputs. Now we are in a position to apply multilateral methods in order to construct output and input aggregates for each firm in each period. In effect, we treat each of the 8 output (or input) price and quantity vectors as if they corresponded to the data that pertained to a country and we choose a multilateral method in order to construct an output (or input) aggregate for each of our 8 "countries".[29]

The first multilateral method that we will consider is the Star System.[30] In order to implement this method, we choose our favorite bilateral index number formula, say the Fisher formula Q_F, and we choose one observation as the base (or star), say observation k, and then we compute the Fisher quantity aggregate of each observation relative to the base k, $Q_F(p^k,p^1,y^k,y^1)$, $Q_F(p^k,p^2,y^k,y^2)$, … , $Q_F(p^k,p^8,y^k,y^8)$. The resulting sequence of 8 numbers can serve as comparable output aggregates for our 8 observations.

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[29] Note that we need to make two multilateral comparisons: one for outputs and one for inputs.
[30] This terminology is due to Kravis (1984; 10).

Of course, the problem with the Star System aggregates is that it is necessary to asymmetrically choose one observation as the "star" and usually, it is not clear which observation should be chosen to be the star.[31] Thus in Tables 2 and 3 below, we list each of the 8 output and input aggregates respectively, choosing each observation as the base in turn. In order to make these output and input aggregates comparable, we divide each set of parities by the parity for the first observation. Thus the output and input parities listed in Tables 2 and 3 are the following normalized parities for outputs and inputs respectively, for k = 1,…,8:[32]

(36) $1,\ Q_F(p^k,p^2,y^k,y^2)/Q_F(p^k,p^1,y^k,y^1),\ \ldots,\ Q_F(p^k,p^8,y^k,y^8)/Q_F(p^k,p^1,y^k,y^1)$;

(37) $1,\ Q_F{}^*(w^k,w^2,x^k,x^2)/Q_F{}^*(w^k,w^1,x^k,x^1),\ \ldots,\ Q_F{}^*(w^k,w^8,x^k,x^8)/Q_F{}^*(w^k,w^1,x^k,x^1)$.

Table 2: Fisher Star Output Aggregates

Outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
Base=1	1.000	0.102	4.971	0.794	1.170	0.250	5.203	1.225
Base=2	1.000	0.102	5.103	0.824	1.199	0.256	5.405	1.247
Base=3	1.000	0.099	4.971	0.790	1.216	0.256	5.365	1.270
Base=4	1.000	0.098	4.997	0.794	1.243	0.260	5.482	1.296
Base=5	1.000	0.100	4.785	0.748	1.170	0.247	5.070	1.232
Base=6	1.000	0.100	4.857	0.764	1.184	0.250	5.169	1.243
Base=7	1.000	0.098	4.821	0.754	1.201	0.252	5.203	1.261
Base=8	1.000	0.100	4.794	0.751	1.163	0.246	5.052	1.225

Table 3: Fisher Star Input Aggregates

Inputs	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
Base=1	1.000	0.100	3.975	1.252	0.680	0.200	3.183	0.646
Base=2	1.000	0.100	3.958	1.251	0.677	0.200	3.175	0.644
Base=3	1.000	0.100	3.975	1.243	0.692	0.201	3.281	0.650
Base=4	1.000	0.100	4.005	1.252	0.690	0.200	3.229	0.650
Base=5	1.000	0.100	3.904	1.234	0.680	0.201	3.260	0.644
Base=6	1.000	0.100	3.949	1.250	0.675	0.200	3.170	0.643
Base=7	1.000	0.100	3.856	1.235	0.664	0.201	3.183	0.637
Base=8	1.000	0.100	3.946	1.244	0.682	0.201	3.228	0.646

[31] In our particular example, a case could be made for choosing either observation 3 or 7; i.e., the observations that correspond to the very large firm. However, there are still two choices and again, it is not clear which of these two should be chosen.

[32] Recall that our final decomposition of the industry productivity change defined by (32) does not depend on our rather arbitrary units of measurement for aggregate firm outputs and inputs.

Note that the input aggregates for observations 1 and 2 using any of the observations as the base are always the same. This is due to the use of the Fisher formula and the fact the input vectors for observations 1 and 2 are proportional[33]; it turns out that if the quantity vectors for the two observations being compared are proportional, then the Fisher quantity index that compares these two observations will reflect this factor of proportionality.[34] However, in general, it can be seen that the choice of the base "country" or observation does affect the output and input parities.

Now go along each row of Table 2 and divide each output aggregate by the input aggregate that corresponds to that observation that is listed in the corresponding row of Table 3. This determines the productivity level of each observation using each of the 8 observations as the base in the index number comparisons in turn. These star productivity levels are listed in Table 4.

Table 4: Fisher Star Productivity Levels

Base=1	1.000	1.021	1.251	0.634	1.721	1.250	1.635	1.897
Base=2	1.000	1.021	1.289	0.659	1.771	1.279	1.703	1.937
Base=3	1.000	0.990	1.251	0.636	1.756	1.271	1.635	1.953
Base=4	1.000	0.982	1.247	0.634	1.802	1.296	1.698	1.993
Base=5	1.000	0.992	1.226	0.606	1.721	1.226	1.555	1.914
Base=6	1.000	0.997	1.230	0.612	1.754	1.250	1.630	1.933
Base=7	1.000	0.980	1.250	0.611	1.809	1.253	1.635	1.981
Base=8	1.000	1.000	1.215	0.604	1.706	1.227	1.565	1.897

There can be a considerable amount of variation in the productivity levels for each observation, depending on which observation is chosen as the base in the star system comparison. Thus if we choose the base to equal 1 (firm 1 in period 0), the productivity level of firm 2 in period 0 is 1.021 whereas if we choose the base to equal 7 (firm 3 in period 1), the productivity level of firm 2 in period 0 is 0.980, which is a 4% variation in

[33] The input vector for firm 1 in period 0 is $x^1 = [10,10]$ and for firm 2 in period 0 is $x^2 = [1,1]$.

[34] Similarly, if the two price vectors are proportional, then the Fisher price index between the two observations will reflect this factor of proportionality. The Fisher formula seems to be the only superlative formula that is consistent with both Hicks' and Leontief's aggregation theorems; see Allen and Diewert (1981).

productivity levels due to the choice of a different base for our bilateral index number comparisons.[35]

Aggregate output prices that correspond to the 8 output aggregates that are listed in Table 2 for each choice of base observation can be obtained by dividing the value of output produced by each firm in each period by the corresponding output listed for that observation in Table 2. Similarly, aggregate input prices that correspond to the 8 input aggregates that are listed in Table 3 for each choice of base observation can be obtained by dividing the value of inputs used by each firm in each period by the corresponding input listed for that observation in Table 3. Once these aggregate output and input prices have been constructed, then we are in a position to apply the decomposition analysis that was discussed in sections 2 and 3 above.

We define the various terms that occur on the right and left hand sides of the aggregate productivity growth decomposition (31) as follows:

(38) $\Gamma \equiv [\Pi^1/\Pi^0] - 1$ (aggregate industry productivity growth);

(39) $\Gamma_{CD} \equiv \sum_{i\in C} (1/2)(s_{Ci}^0 + s_{Ci}^1)(\Pi_{Ci}^1 - \Pi_{Ci}^0)/\Pi^0$

(direct productivity growth contribution of continuing firms);

(40) $\Gamma_{CR} \equiv \sum_{i\in C}(1/2)(\Pi_{Ci}^0 + \Pi_{Ci}^1)(s_{Ci}^1 - s_{Ci}^0)/\Pi^0$

(reallocation contribution of continuing firms);

(41) $\Gamma_N \equiv S_N^1 \sum_{i\in N} s_{Ni}^1 (\Pi_{Ni}^1 - \Pi_C^1)/\Pi^0$ (contribution of entering firms to TFPG);

(42) $\Gamma_X \equiv - S_X^0 \sum_{i\in X} s_{Xi}^0 (\Pi_{Xi}^0 - \Pi_C^0)]/\Pi^0$ (contribution of exiting firms to TFPG).

In our example, there are three continuing firms in each of the summations in (39) and (40) but only one term in each of the summations in (41) and (42) since we have only one exiting and one entering firm.

[35] Ideally, we would like all the entries in each column of Table 4 to be identical so that the productivity levels of each firm observation would not depend on the choice of index number base.

The terms defined by (38)-(42) are listed in Table 5 below for each choice of base; i.e., we use the data listed in Tables 2-4 above (along with the corresponding prices) in order to construct an aggregate industry productivity growth decomposition for each of our 8 bases.

Table 5: Aggregate Productivity Growth Decompositions for Each Choice of Base

	Γ	Γ_{CD}	Γ_{CR}	Γ_N	Γ_X
Base=1	0.5356	0.4054	-0.0061	0.0337	0.1025
Base=2	0.5496	0.4247	-0.0062	0.0300	0.1010
Base=3	0.5471	0.4128	-0.0063	0.0391	0.1015
Base=4	0.6025	0.4704	-0.0071	0.0374	0.1017
Base=5	0.5174	0.3739	-0.0066	0.0440	0.1061
Base=6	0.5684	0.4311	-0.0070	0.0387	0.1056
Base=7	0.5678	0.4249	-0.0075	0.0425	0.1080
Base=8	0.5296	0.3887	-0.0065	0.0418	0.1056

It can be seen that the choice of base matters. Aggregate productivity growth using observation 5 (data of firm 1 in period 1) as the index number formula base leads to industry productivity growth of 51.74% whereas if observation 4 (data of the disappearing firm 4 in period 0) is used as the base, then industry productivity growth is much larger at 60.25%.[36] Looking at the last 4 columns in Table 5, it can be seen that the direct productivity growth of continuing firms accounts for most of the industry productivity growth (somewhere between 37.39% and 47.04%), the contribution of the exiting firm is between 10% and 11%, the contribution of the entering firm is between 3.0% and 4.4% and the reallocation of resources between continuing firms sums to a negligible contribution to overall productivity growth.

It is of some interest to look at the direct productivity growth contribution and the reallocation contribution of each continuing firm. Thus define the three terms on the right hand side of (39) as Γ_{CD1}, Γ_{CD2} and Γ_{CD3}, the direct productivity growth contributions of continuing firms 1, 2 and 3 respectively and define the three terms on the right hand side of (40) as Γ_{CR1}, Γ_{CR2} and Γ_{CR3}, the reallocation contributions of

[36] The choice of observations 2, 3, 6 and 7 as the index number base gives rise to industry TFP growth rates that are closest to our target rates of around 55.53% and 55.73%; recall (35) above. Note that the average of the industry productivity growth rates for the large firm observations (3 and 7) is 55.74%.

continuing firms 1, 2 and 3 respectively. These terms are listed in Table 6 for each of our 8 choices of index number base.

Table 6: Direct and Reallocation Contributions to Aggregate Productivity Growth for Each Continuing Firm and for Each Choice of Base

	Γ_{CD1}	Γ_{CD2}	Γ_{CD3}	Γ_{CR1}	Γ_{CR2}	Γ_{CR3}
Base=1	0.1210	0.0073	0.2771	-0.0372	0.0309	0.0002
Base=2	0.1262	0.0080	0.2905	-0.0381	0.0305	0.0014
Base=3	0.1263	0.0088	0.2777	-0.0396	0.0296	0.0037
Base=4	0.1345	0.0099	0.3261	-0.0367	0.0305	-0.0009
Base=5	0.1234	0.0076	0.2429	-0.0456	0.0298	0.0093
Base=6	0.1289	0.0082	0.2939	-0.0403	0.0312	0.0021
Base=7	0.1372	0.0089	0.2788	-0.0492	0.0304	0.0112
Base=8	0.1216	0.0074	0.2597	-0.0414	0.0305	0.0044

Viewing Table 6, it can be seen that the largest contribution to industry TFP growth is the direct TFP growth of firm 3 (the large firm); it contributes an amount somewhere between 24.29% (the index base 5 estimate) and 32.61% (the index base 4 estimate). The next largest contribution comes from the medium sized firm 1; it contributes an amount between 12.10% (the index base 1 estimate) and 13.72% (the index base 7 estimate). The other contribution terms are all less than 5%.

Obviously, some form of averaging of the different star decompositions is called for. Our next multilateral method simply takes the geometric averages of the output and input aggregates listed in Tables 2 and 3 and then implements the decomposition (31) using these new output and input aggregates.

7. The GEKS Method for Making Multilateral Comparisons

The GEKS method for making multilateral comparisons dates back to Gini (1931), Eltetö and Köves (1964) and Szulc (1964). As was indicated in the previous section, this method simply takes each of the star output and input parities and takes the geometric mean of them.[37] These GEKS relative output and input aggregates are listed in Table 7.

[37] The GEKS aggregates can be defined in a number of equivalent ways but this is one way; see for example, Diewert (1999; 31-37).

Once the output and input aggregates have been constructed, then the GEKS productivity levels can be constructed by dividing the output aggregate by the corresponding input aggregate. The resulting 8 productivity levels are also listed in Table 7.

Table 7: GEKS Output and Input Aggregates and Productivity Levels

Outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
	1.000	0.100	4.911	0.777	1.193	0.252	5.242	1.250
Inputs	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	1.000	0.100	3.946	1.245	0.680	0.201	3.213	0.645
Prod Levels	y_1/x_1	y_2/x_2	y_3/x_3	y_4/x_4	y_5/x_5	y_6/x_6	y_7/x_7	y_8/x_8
	1.000	0.998	1.245	0.624	1.755	1.256	1.631	1.938

Aggregate output prices that correspond to the 8 output aggregates that are listed in Table 7 can be obtained by dividing the value of output produced by each firm in each period by the corresponding output listed for that observation in Table 7. Similarly, aggregate input prices that correspond to the 8 input aggregates that are listed in Table 7 can be obtained by dividing the value of inputs used by each firm in each period by the corresponding input listed for that observation in Table 7. Once these aggregate output and input prices have been constructed, then we can repeat the decomposition analysis that was implemented in the previous section.

The productivity growth decomposition terms defined by (38)-(42) are listed in Table 8 below. We also list the direct and reallocation contribution terms defined by the individual terms in (39) and (40) for each continuing firm in Table 8.

Table 8: The GEKS Aggregate Productivity Growth Decomposition

Γ	Γ_{CD}	Γ_{CR}	Γ_N	Γ_X	
0.5521	0.4162	-0.0066	0.0384	0.1040	
Γ_{CD1}	Γ_{CD2}	Γ_{CD3}	Γ_{CR1}	Γ_{CR2}	Γ_{CR3}
0.1274	0.0083	0.2806	-0.0410	0.0304	0.0039

From Table 8, the GEKS aggregate productivity growth Γ is 55.21%, which is reasonably close to our target rates of around 55.53% to 55.73%; recall (35) above. Thus we conclude that the GEKS method for constructing relative output and input levels for each firm in each period is satisfactory, at least for our particular numerical example.

One problem with the (unweighted) GEKS method is that each firm observation is given an equal weighting when the output and input aggregates are constructed. Since small firms may have different data than large firms and hence their star parities could be quite different from those of large firms, it may not be wise to give these small firms equal weighting in the construction of the output and input aggregates. Thus in the following section, we look at a multilateral method for constructing output and input aggregates that gives large firms more weight than small firms.

8. The Own Share Method for Making Multilateral Comparisons

Recall our discussion in section 6 when we described how the star output aggregates could be constructed using observation k as the base. We noted that the sequence of 8 numbers, $Q_F(p^k,p^1,y^k,y^1)$, $Q_F(p^k,p^2,y^k,y^2)$, … , $Q_F(p^k,p^8,y^k,y^8)$, could serve as comparable output aggregates for our 8 observations. Hence, using observation k as the base, the *share of total output of observation k* is:

$$(43)\quad s_k^* \equiv Q_F(p^k,p^k,y^k,y^k)/[Q_F(p^k,p^1,y^k,y^1) + Q_F(p^k,p^2,y^k,y^2) + \ldots + Q_F(p^k,p^8,y^k,y^8)]$$
$$= 1/[Q_F(p^k,p^1,y^k,y^1) + Q_F(p^k,p^2,y^k,y^2) + \ldots + Q_F(p^k,p^8,y^k,y^8)] ; \qquad k = 1,\ldots,8,$$

where the last equation in (43) follows from the fact that the Fisher ideal quantity index satisfies an identity test and hence $Q_F(p^k,p^k,y^k,y^k)$ equals 1. Thus, using the metric of observation k to make the index number comparisons, the share of observation k in "world" output, s_k^*, is defined by (43) for k = 1,2, …, 8. Thus each observation's *own share* of "world" output is defined by (43). Put another way, if we look at the entries in Table 2 above, the numbers listed in the Base=1 row determine the share of observation 1 in total output over the two periods, s_1^*; the numbers listed in the Base=2 row determine the share of observation 2 in total output over the two periods, s_2^*; …; and the numbers listed in the Base=8 row determine the share of observation 8 in total output over the two periods, s_8^*. Thus each row in Table 2 determines only one share of "world" output and so the rows that correspond to smaller shares of world output get a smaller influence in the overall multilateral comparison; i.e., the own share system does weight the individual star parities according to their economic importance as opposed to the more democratic GEKS method where each star parity has the same importance.

Unfortunately, the own shares s_k* defined by (43) do not sum up to unity and so we renormalize these "shares" to sum up to unity as follows:[38]

(44) $y_k \equiv s_k^*/[\sum_{j=1}^{8} s_j^*]$; $\qquad$ $k = 1,\ldots,8.$

The output aggregates y_k defined by (44) are the *own share aggregates*.[39] The same procedure can be used in order to define own share input aggregates.

These own share relative output and input output and input aggregates are listed in Table 9. Once the output and input aggregates have been constructed, then the own share productivity levels can be constructed by dividing the output aggregate by the corresponding input aggregate. The resulting 8 productivity levels are also listed in Table 9.[40]

Table 9: Own Share Output and Input Aggregates and Productivity Levels

Outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
	0.068	0.007	0.332	0.052	0.082	0.017	0.357	0.085
Inputs	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0.091	0.009	0.357	0.113	0.062	0.018	0.293	0.058
Prod Levels	y_1/x_1	y_2/x_2	y_3/x_3	y_4/x_4	y_5/x_5	y_6/x_6	y_7/x_7	y_8/x_8
	0.750	0.742	0.931	0.465	1.322	0.943	1.218	1.462

Aggregate output prices that correspond to the 8 output aggregates that are listed in Table 9 can be obtained by dividing the value of output produced by each firm in each period by the corresponding output listed for that observation in Table 9. Similarly, aggregate

[38] In our empirical example, the s_k* summed up to 0.99996 so that the differences between the y_k and the s_k* were negligible. The corresponding input shares summed up to 0.99997.
[39] The own share system was proposed by Diewert (1988; 69). For the axiomatic properties of this method, see Diewert (1999; 37-39).
[40] Note that the units of measurement for the output and input aggregates are quite different in Tables 7 and 9. This illustrates the importance of providing a productivity growth decomposition that is independent of the units of measurement.

input prices that correspond to the 8 input aggregates that are listed in Table 9 can be obtained by dividing the value of inputs used by each firm in each period by the corresponding input listed for that observation in Table 9. Once these aggregate output and input prices have been constructed, then we can repeat the decomposition analysis that was implemented in the previous sections.

The productivity growth decomposition terms defined by (38)-(42) are listed in Table 10 below. We also list the direct and reallocation contribution terms defined by the individual terms in (39) and (40) for each continuing firm in Table 10.

Table 10: The Own Share Aggregate Productivity Growth Decomposition

Γ	Γ_{CD}	Γ_{CR}	Γ_N	Γ_X	
0.5545	0.4165	-0.0067	0.0403	0.1044	
Γ_{CD1}	Γ_{CD2}	Γ_{CD3}	Γ_{CR1}	Γ_{CR2}	Γ_{CR3}
0.1290	0.0086	0.2789	-0.0423	0.0302	0.0054

From Table 10, the own share aggregate productivity growth Γ is 55.45%, which is very close to our target rate of around 55.53% to 55.73%; recall (35) above.[41] Thus we conclude that the own share method for constructing relative output and input levels for each firm in each period is very satisfactory, at least for our particular numerical example.

9. Hill's Method for Making Multilateral Comparisons

Another method for finding output and input aggregates can be based on the following idea: observations which are most similar in their price structures (i.e., their output prices are closest to being proportional across items) should be linked using a bilateral index number formula first. Then the observation outside of the first two observations that has the most similar relative prices to the first two observations should be added to the chain

[41] Note that the own share decomposition is very close to the GEKS decomposition listed in Table 8 above. Diewert (1988; 69) (1999; 38) showed that the own share aggregates and the GEKS aggregates will usually approximate each other fairly closely.

of links, etc. This basic idea has been successfully exploited by Robert Hill at higher levels of aggregation,[42] where complete price and expenditure data are available.

In order to apply this idea, it is necessary to choose a measure of how *dissimilar* are the (relative) output prices corresponding to any two observations. There are many measures of relative price dissimilarity that could be chosen[43] but we choose the following one that measures the degree of dissimilarity between the output prices of observations j and k:

$$(45)\ D(p^j,p^k) \equiv [\ln(p_1{}^k/P_F(p^k,p^j,q^k,q^j)p_1{}^j)]^2 + [\ln(p_2{}^k/P_F(p^k,p^j,q^k,q^j)p_2{}^j)]^2\ ;\ j,k = 1,\ldots,8$$

where $P_F(p^k,p^j,q^k,q^j)$ is the Fisher output price index of observation j relative to k.[44] Thus instead of comparing the price of output 1 for observation k, $p_1{}^k$, with the price of output 1 for observation j, $p_1{}^j$, we multiply $p_1{}^j$ by the Fisher price index for observation k relative to j, $P_F(p^k,p^j,q^k,q^j)$, which inflates the base prices j by a general inflation factor that makes the prices of k comparable to the inflated j prices. In particular, if the j prices are equal to λ times the k prices, so that $p^j = \lambda p^k$, then the Fisher index that compares the j prices to the k prices will pick up this factor of proportionality so that $P_F(p^k,p^j,q^k,q^j) = \lambda$ and it can be seen that under these circumstances, the dissimilarity measure defined by (45) will be zero; i.e., we will have $D(p^j,p^k) = 0$. It can also be verified that the dissimilarity measure defined by (45) satisfies the following *symmetry property*:

$$(46)\ D(p^k,p^j) = D(p^j,p^k)\ ; \qquad j,k = 1,\ldots,8.$$

[42] See Robert Hill (1999a) (1999b) (2001) (2005). The basic idea of spatially linking countries that have the most similar price and quantity structures dates back to Fisher (1922; 271-272). Here we apply the same idea to observations, treating each observation as a "country".

[43] See Diewert (2002) for an axiomatic treatment of the topic.

[44] This dissimilarity measure is essentially equal to that used by Allen and Diewert (1981) except that they used the Törnqvist index $P_T(p^k,p^j,q^k,q^j)$ to adjust for general price level change in place of the Fisher index $P_F(p^k,p^j,q^k,q^j)$ in (45). Diewert (2002; 20) defined a weighted counterpart to (45) which he called the weighted log quadratic index of relative price dissimilarity.

Table 11 lists the Fisher output price indexes $P_F(p^k,p^j,q^k,q^j)$ between each pair of observations.[45]

Table 11: Fisher Output Price Indexes Between Each Pair of Observations

Base k=1	1.000	0.980	0.855	1.152	12.007	11.000	11.100	13.064
Base k=2	1.021	1.000	0.850	1.133	11.963	10.969	10.904	13.093
Base k=3	1.170	1.176	1.000	1.354	13.518	12.570	12.591	14.738
Base k=4	0.868	0.883	0.738	1.000	9.811	9.190	9.145	10.720
Base k=5	0.083	0.084	0.074	0.102	1.000	0.927	0.949	1.081
Base k=6	0.091	0.091	0.080	0.109	1.079	1.000	1.016	1.170
Base k=7	0.090	0.092	0.079	0.109	1.054	0.985	1.000	1.143
Base k=8	0.077	0.076	0.068	0.093	0.925	0.855	0.875	1.000

Note that Fisher output price levels for firms present in period 1 are 9.145 to 14.738 times the levels of prices for firms present in period 0 (see the entries in the northeast corner of Table 11).

Table 12 lists the dissimilarity measures $D(p^j,p^k)$ defined by (45). Note that this (j=8, k=8) matrix is symmetric.

Table 12: Log Quadratic Output Price Dissimilarity Measures

0.00000	0.08220	0.00705	0.00380	0.34067	0.12932	0.26641	0.28161
0.08220	0.00000	0.13759	0.12365	0.71777	0.40242	0.61510	0.63505
0.00705	0.13759	0.00000	0.00047	0.23304	0.07164	0.17715	0.18557
0.00380	0.12365	0.00047	0.00000	0.24608	0.08182	0.19079	0.19811
0.34067	0.71777	0.23304	0.24608	0.00000	0.04837	0.00305	0.00324
0.12932	0.40242	0.07164	0.08182	0.04837	0.00000	0.02565	0.02722
0.26641	0.61510	0.17715	0.19079	0.00305	0.02565	0.00000	0.00000
0.28161	0.63505	0.18557	0.19811	0.00324	0.02722	0.00000	0.00000

Note that the dissimilarity measure between observations 7 and 8 is 0; this is due to the fact that the output price vectors for these two observations are proportional.

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[45] The Fisher (1922) output price index is defined as $P_F(p^0,p^1,q^0,q^1) \equiv [p^1{\cdot}q^0\ p^1{\cdot}q^1/p^0{\cdot}q^0\ p^0{\cdot}q^1]^{1/2}$. Row k of Table 11 is equal to $P_F(p^k,p^1,q^k,q^1)$, $P_F(p^k,p^2,q^k,q^2)$, … , $P_F(p^k,p^8,q^k,q^8)$.

Inspection of Table 12 shows that the lowest dissimilarity measures which link the data are: 7-8; 7-5; 7-6; 3-4; 1-4; 1-2 and 3-5. This set of links will enable us to construct output aggregates, $y_1,\ldots,y_8$, which are listed in Table 15 below.

The same strategy that was used to construct Hill output aggregates can be used to construct input aggregates. The input counterparts to Tables 11 and 12 are Tables 13 and 14.

Table 13: Fisher Input Price Indexes Between Each Pair of Observations

Base k=1	1.000	0.750	0.994	1.102	16.029	14.500	16.650	16.884
Base k=2	1.333	1.000	1.331	1.471	21.474	19.333	22.257	22.570
Base k=3	1.006	0.752	1.000	1.117	15.842	14.497	16.254	16.867
Base k=4	0.907	0.680	0.895	1.000	14.338	13.131	14.896	15.214
Base k=5	0.062	0.047	0.063	0.070	1.000	0.898	1.014	1.056
Base k=6	0.069	0.052	0.069	0.076	1.114	1.000	1.153	1.169
Base k=7	0.060	0.045	0.062	0.067	0.986	0.867	1.000	1.028
Base k=8	0.059	0.044	0.059	0.066	0.947	0.855	0.972	1.000

Note that Fisher input price levels for firms present in period 1 are 13.131 to 22.570 times the levels of prices for firms present in period 0 and so input prices grew faster than output prices over the two periods.

Table 14: Log Quadratic Input Price Dissimilarity Measures

0.00000	0.00893	0.02014	0.01669	0.35300	0.02161	0.73589	0.06377
0.00893	0.00000	0.00225	0.04993	0.25127	0.00277	0.58761	0.02507
0.02014	0.00225	0.00000	0.07378	0.20284	0.00002	0.50447	0.01219
0.01669	0.04993	0.07378	0.00000	0.51780	0.07605	0.95664	0.14529
0.35300	0.25127	0.20284	0.51780	0.00000	0.20208	0.06805	0.11723
0.02161	0.00277	0.00002	0.07605	0.20208	0.00000	0.51192	0.01122
0.73589	0.58761	0.50447	0.95664	0.06805	0.51192	0.00000	0.36697
0.06377	0.02507	0.01219	0.14529	0.11723	0.01122	0.36697	0.00000

Inspection of Table 14 shows that the lowest dissimilarity measures which link the data are: 3-6; 2-3; 1-2; 1-4; 6-8; 5-7 and 5-8. This set of links will enable us to construct Hill input aggregates, $x_1,\ldots,x_8$, which are listed in Table 15. The eight Hill productivity levels, $y_1/x_1,\ldots,y_8/x_8$, are also listed in Table 15.

Table 15: Hill Output and Input Aggregates and Productivity Levels

Outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
	1.000	0.102	4.997	0.794	1.222	0.256	5.295	1.284
Inputs	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	1.000	0.100	3.958	1.252	0.681	0.200	3.263	0.644
Prod Levels	y_1/x_1	y_2/x_2	y_3/x_3	y_4/x_4	y_5/x_5	y_6/x_6	y_7/x_7	y_8/x_8
	1.000	1.021	1.262	0.634	1.795	1.277	1.623	1.992

Comparing the entries in Table 15 with the corresponding GEKS entries in Table 7, it can be seen that with the exceptions of observations 1 and 7, the Hill productivity levels tend to be greater than the corresponding GEKS productivity levels.

Aggregate output prices that correspond to the 8 output aggregates that are listed in Table 15 can be obtained by dividing the value of output produced by each firm in each period by the corresponding output listed for that observation in Table 15. Similarly, aggregate input prices that correspond to the 8 input aggregates that are listed in Table 15 can be obtained by dividing the value of inputs used by each firm in each period by the corresponding input listed for that observation in Table 15. Once these aggregate output and input prices have been constructed, then we can repeat the decomposition analysis that was implemented in the previous sections.

The productivity growth decomposition terms defined by (38)-(42) are listed in Table 16 below. We also list the direct and reallocation contribution terms defined by the individual terms in (39) and (40) for each continuing firm in Table 16.

Table 16: The Hill Aggregate Productivity Growth Decomposition

Γ	Γ_{CD}	Γ_{CR}	Γ_N	Γ_X	
0.5401	0.3986	-0.0063	0.0440	0.1038	
Γ_{CD1}	Γ_{CD2}	Γ_{CD3}	Γ_{CR1}	Γ_{CR2}	Γ_{CR3}
0.1318	0.0080	0.2588	-0.0428	0.0301	0.0064

From Table 16, the Hill aggregate productivity growth Γ is 54.01%, which is not as close to our target rates of around 55.53% to 55.73% compared to the GEKS and own share decompositions of productivity growth. Thus for this particular numerical example, we conclude that the Hill method for constructing relative output and input levels for each firm in each period is satisfactory but not as good at the GEKS and own share estimates.

10. An Approximate Method for Constructing Output and Input Aggregates

The multilateral methods for constructing output and input aggregates that have been discussed in the previous 3 sections are theoretically satisfactory methods. However, they suffer from two major disadvantages:

- They may not be practical for very large data sets; i.e., they are somewhat computation intensive.
- Detailed price and quantity information on outputs and inputs may not be available for each production unit; i.e., only information on output revenues and input costs by unit may be available.

Thus in the present section, we assume that we have only information on firm revenues and costs by period and that we also have aggregate intertemporal price indexes for both outputs and inputs available. In particular, we assume that we have the aggregate Fisher output and input price indexes at our disposal. Using the aggregate period 0 and 1 information on the industry's two outputs and inputs listed in section 5 above,[46] the Fisher and Törnqvist output price index numbers for period 1 are 12.283 and 12.239 respectively[47] while the Fisher and Törnqvist input price index numbers for period 1 are 16.035 and 15.998 respectively. We will use the Fisher industry price index values for outputs and inputs for period 1, $P_F(p^0,p^1,q^0,q^1)$ and $P_F{}^*(w^0,w^1,x^0,x^1)$ respectively, to deflate all of the period 1 firm revenues and costs in order to make them at least approximately comparable to the period 0 firm revenues and costs. Recalling the notation that was introduced at the beginning of section 6, for observations 1-4, we define firm aggregate outputs and inputs as firm revenues and costs respectively; i.e., define the *output and input aggregates*, y_1,y_2,y_3,y_4 and x_1,x_2,x_3,x_4 respectively, as follows:

[46] See the listing of the industry data in (32) above.
[47] The corresponding index values are 1 in period 0.

(47) $y_k \equiv p^k \cdot y^k$; $k = 1,2,3,4$; $x_k \equiv w^k \cdot x^k$; $k = 1,2,3,4$.

For observations 5-8 (the period 1 observations), we define firm aggregate outputs and inputs as Fisher index deflated firm revenues and costs respectively; i.e., define the *output and input aggregates*, y_5,y_6,y_7,y_8 and x_5,x_6,x_7,x_8 respectively, as follows:

(48) $y_k \equiv p^k \cdot y^k/P_F(p^0,p^1,q^0,q^1)$; $k = 5,6,7,8$; $x_k \equiv w^k \cdot x^k/P_F{}^*(w^0,w^1,x^0,x^1)$; $k = 5,6,7,8$.

Obviously, the output and input aggregates defined by (47) and (48) are not going to be as accurate as the output and input aggregates defined in the previous 3 sections. However, it is still of some interest to see how close these approximate aggregates are to the previously defined multilateral aggregates. The approximate output and input aggregates are listed in Table 17 along with the corresponding plant productivity levels.

Table 17: Approximate Output and Input Aggregates and Productivity Levels

Outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
	20.000	2.000	85.000	18.300	22.877	4.478	94.031	26.052
Inputs	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	20.000	1.500	79.000	27.600	13.595	3.617	66.105	13.595
Prod Levels	y_1/x_1	y_2/x_2	y_3/x_3	y_4/x_4	y_5/x_5	y_6/x_6	y_7/x_7	y_8/x_8
	1.000	1.333	1.076	0.663	1.683	1.238	1.422	1.916
GEKS	1.000	0.998	1.245	0.624	1.755	1.256	1.631	1.938

In order to make the units of measurement for outputs and inputs listed in Table 17 comparable to the units listed in the corresponding GEKS Table 7, it is necessary to divide the outputs row by 20 and the inputs row by 20. The productivity levels row in Table 17 is comparable to the corresponding row in Table 7. For easy reference, the GEKS productivity levels are listed as the last row in Table 17. It can be seen that there are some rather substantial differences in the GEKS productivity levels compared to the corresponding approximate ones.

As usual, aggregate output prices that correspond to the 8 output aggregates that are listed in Table 17 can be obtained by dividing the value of output produced by each firm in each period by the corresponding output listed for that observation in Table 17. Similarly, aggregate input prices that correspond to the 8 input aggregates that are listed in Table 17 can be obtained by dividing the value of inputs used by each firm in each period by the corresponding input listed for that observation in Table 17. Once these aggregate output and input prices have been constructed, then we can repeat the decomposition analysis that was implemented in the previous sections.

The productivity growth decomposition terms defined by (38)-(42) are listed in Table 18 below. We also list the direct and reallocation contribution terms defined by the individual terms in (39) and (40) for each continuing firm in Table 18. For ease of comparison, we list the decompositions for the GEKS, own share and Hill methods in Table 18 as well.

Table 18: The Approximate Method Aggregate Productivity Growth Decomposition

	Γ	Γ_{CD}	Γ_{CR}	Γ_N	Γ_X	
Approx Method	0.5553	0.4033	-0.0023	0.0659	0.0885	
GEKS	0.5521	0.4162	-0.0066	0.0384	0.1040	
Own Share	0.5545	0.4165	-0.0067	0.0403	0.1044	
Hill	0.5401	0.3986	-0.0063	0.0440	0.1038	
	Γ_{CD1}	Γ_{CD2}	Γ_{CD3}	Γ_{CR1}	Γ_{CR2}	Γ_{CR3}
Approx Method	0.1264	-0.0028	0.2798	-0.0491	0.0374	0.0094
GEKS	0.1274	0.0083	0.2806	-0.0410	0.0304	0.0039
Own Share	0.1290	0.0086	0.2789	-0.0423	0.0302	0.0054
Hill	0.1318	0.0080	0.2588	-0.0428	0.0301	0.0064

From Table 18, the approximate method aggregate productivity growth Γ is 55.53%, which is exactly equal to our target Fisher rate of 55.53%. This exact equality is not a statistical fluke but is a consequence of the fact that we have used the industry Fisher price indexes to deflate the period 1 value data. Thus our approximate method works extremely well in terms of replicating the industry's aggregate productivity growth. However, the other terms on the right hand side of (32) are not always well predicted by the approximate method. In particular, the approximate method leads to a contribution of entry term Γ_N equal to 6.59% whereas the other methods lead to contribution terms in the 3.84 to 4.40% range. Also, the approximate method leads to a contribution of exit term Γ_X equal to 8.85% whereas the other methods lead to contribution terms in the 10.38 to

10.44% range. However, considering the simplicity of the approximate method, we have to conclude that at least for this example, the approximate method for constructing output and input aggregates for the purposes of implementing the productivity growth decomposition (32) was satisfactory but of course, it was not as good at the GEKS and own share methods.

11. Conclusion

The paper suggested a new formula (32) for decomposing industry productivity growth into terms that reflect the productivity growth of individual production units that operate in both the base and comparison periods, the reallocation of resources among continuing firms from lower productivity to higher productivity units and to entry and exit contribution terms. Unfortunately, this formula (and the other formulae derived in the literature) is derived under the assumption that each production unit produces a single homogeneous output and uses a single homogeneous input. Most of the paper (sections 4-10) is concerned with the problems involved in aggregating many outputs and many inputs into output and input aggregates. In order to accomplish this aggregation, we suggested the use of multilateral methods and we implemented four multilateral methods on a test data set that is described in section 5 above. For our test data set, we found that the own share method worked best but the GEKS method was very close. The Hill methods and an approximate method that used value aggregates in the base period and deflated value aggregates in the comparison period also worked reasonably well for our data set. The fact that the approximate method worked so well is very encouraging for empirical work in this area, since variants of it are what have been used in empirical applications of productivity decompositions that involve entry and exit.[48]

[48] In our test example, we used the actual "industry" Fisher output and input price indexes as the deflators. In empirical work, the deflators that are available are unlikely to be the exact industry deflators and so there will be some extra error due to this unavailability. Also, in real life, it is unlikely that all of the production units in a given industry are producing positive amounts of a common list of outputs and using positive amounts of a common list of inputs, as was the case in our example. Hence, there will be additional errors due to the heterogeneity of establishment production.

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