SUBSTITUTION AND RISK AVERSION: IS RISK AVERSION IMPORTANT FOR UNDERSTANDING ASSET PRICES?[1]

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The log utility function is widely used to explain asset prices. It assumes that both the elasticity of substitution and relative risk aversion are equal to one. Here I show that much of the same predictions about asset prices can be derived from a time-non-separable expected utility function that assumes an elasticity of substitution close to unity but does not impose restrictions on risk aversion to bets in terms of money.

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1. INTRODUCTION

Since the discovery of the risk premium puzzle by Mehra and Prescott (1985) there has been a lot of debate about the magnitude of risk aversion. In his presidential address Lucas (2003) followed Mehra and Prescott in using the standard power utility function: $\sum_t \beta^t U(C_t)$ where U(C) = $C^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ and U(C) = ln(C) for $\gamma = 1$. He argues for a coefficient of $\gamma = 1$ (the logarithmic function) by using the formula (Equation [6] in Lucas [2003] derived for the power utility function):

$$r = r^s + \gamma g, \tag{1}$$

where r is the interest rate, g is the growth rate of consumption, $r^s = 1/\beta - 1$ is the subjective interest rate and γ is the parameter of the power utility function. Lucas argues that "...this formula makes it clear why fairly low γ values must be used. Per capita consumption growth in the United States is about 0.02 and the after-tax return on capital is around 0.05, so the fact that the subjective interest rate must be positive requires that γ be at most 2.5. Moreover, a value as high as 2.5 would imply much larger interest rate differential than those we see between fast-growing economies like Taiwan and mature economies like the United States. This is the kind of evidence that leads to the use of γ values at or near 1 in applications."[3]

[3] Pages 6 and 7 in Lucas (2003) with some modifications due to difference in notation.

The log utility function (defined for the case $\gamma = 1$) has both a relative risk aversion of 1 and an elasticity of substitution of 1. Here I argue that it is enough to assume that the elasticity of substitution is close to one without worrying too much about the taste towards bets in terms of money. The analysis here is particularly useful for people who are not sure how much the representative consumer is willing to pay to avoid an actuarially fair bet on say 1% of his wealth but are willing to commit to an elasticity of substitution of unity. It is also useful for someone who is not willing to commit to an elasticity of unity but wants to assess the relative importance of the magnitude of risk aversion on asset prices.

I start with a monotonic transformation of the intertemporal log (IL) utility function: The intertemporal Cobb-Douglas (ICD) utility function. This function has the same implications about the return on the market portfolio as the intertemporal log (IL). But the ICD utility function allows us to change the attitude towards bets in terms of money (risk aversion) without changing the elasticity of substitution. We can therefore examine the "net" effect of changes in risk aversion on asset prices and risk premia.

It is shown that changes in risk aversion towards money bets do not affect the expected rate of return on the market portfolio and have only a small effect on risk premium. It is also shown that risk premium does not require risk aversion.

I then consider a closely related function: The intertemporal constant elasticity (ICE) function. It is shown that when the elasticity of substitution is close to unity, the predicted rate of return on the market portfolio is the same as under the ICD and IL functions and the rate of return on the market portfolio does not depend on risk aversion. But unlike the ICD function here also the risk premium does not depend on risk aversion.

The analysis here builds on work of Kihlstrom and Mirman (1974) and on my own work (Eden [1977, 1979]). Kihlstrom and Mirman show that with the Cobb-Douglas utility function risk and the attitude towards risk do not affect savings. This leads to the result that risk does not affect the return on the "market portfolio". Their article still leaves open the questions of the effect of risk and risk aversion on risk premium.

In Eden (1977, 1979) I argue that insurance type phenomena does not require risk aversion and use the Cobb-Douglas utility function to account for the behavior of a gambler who buys insurance. Here I use the same line of reasoning to show that risk premium does not require risk aversion.

To understand why risk premium does not require risk aversion to money bets it is useful to distinguish between aversion to fluctuations and aversion to risk. I now turn to this issue.

2. FLUCTUATIONS AVERSION AND RISK AVERSION

Would you prefer a smooth consumption path to a path that fluctuates around the same mean? In terms of Figure 1 the smooth consumption path *a* promises 3 units of consumption in every period. The fluctuating consumption path *d* starts from 3.5 units and then fluctuates between 3.5 and 2.5. If you prefer the path *a* then a time separable utility function predicts that you will also prefer a smooth consumption path of 2 (*e* in Figure 1) to a bet between a smooth consumption path of 3 and a smooth consumption path of 1 (*a* and *b* in Figure 1). In the time separable utility function aversion to fluctuations implies aversion to risk.

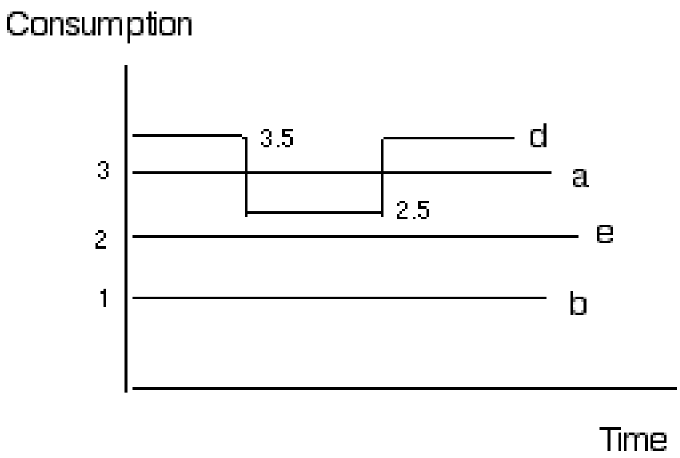

Figure 1

But aversion to fluctuations may have nothing to do with aversion to risk. It is possible that a consumer does not like fluctuations because they require changes in durables. To implement the path *d* one needs to change his house every period or to suffer from a mismatch between his house size and other components of consumption. In addition there are some irreversible choices (like the number of children) that have to be made early on (in most cases). For example, when facing a smooth consumption path one may choose to have 1 child if his permanent consumption is 1, 2 children if his permanent consumption is 2 and 3 children if his permanent consumption is 3. When facing the fluctuating consumption path *d* he may choose to have 3 children but may not enjoy them as much because they will complain whenever his consumption level drops to 2.5 and he has to cut on say the number of movies that they go to.

On the other hand if after a lottery between *a* and *b* he gets to know his permanent consumption early on he will make the optimal choice of the number of children: He will choose one child if his permanent consumption turns out to be 1 and 3 children if it turns out to be 3.

A consumer may thus show aversion to fluctuations but not aversion to risk. This leads to the result that risk premium does not require risk aversion. A consumer who does not like fluctuations does not like uncertainty about his future income and the return on assets. But nevertheless he may be willing to accept bets that are resolved before any irreversible consumption choices are made.[4]

3. BETS IN TERMS OF MONEY AND BETS IN TERMS OF CONSUMPTION

I distinguish between measures of risk aversion to bets in terms of dated consumption and measures of risk aversion to bets in terms of money. Bets in terms of money (wealth) are resolved immediately before any irreversible consumption choice is made. Introspections about money bets require an assumption about borrowing and lending opportunities.

Bets in terms of dated consumption require a different thought experiments. We start from a non-random consumption path and then consider a bet that makes date t consumption a random variable holding

[4] A related argument is in Postlewaite, Samuelson and Silverman (2004). They show that consumption commitments can cause risk neutral agents to care about risk, creating incentives to both insure risks and bunch uninsured risks together.

consumption at all dates other than t constant. The attitude towards this type of bets does not require any assumption about borrowing and lending. But introspection seems more difficult.

The distinction between the two types of bets can be illustrated with the help of Figure 2 that assumes a two-period horizon ($t = 0,1$) and a zero interest rate. The maximum utility that the consumer can get when having the wealth 9, 10 or 11 is a, e and b respectively, where I use these letters to denote numbers (the level of cardinal satisfaction). From observing the indifference map we know that: $a < e < b$. But we do not know by how much. The consumer will prefer a wealth of 10 with certainty to a random wealth {9 or 11 with equal probabilities} if $e > (^1/_2)a + (^1/_2)b$. This will occur for example, if $a = 2$, $e = 9$ and $b = 10$. Otherwise, he will prefer the bet (if for example, $a = 8.5$, $e = 9$ and $b = 10$).

A bet in terms of second period consumption assumes that the level of first period consumption is fixed. For example, in Figure 2 a bet in terms of future consumption (that is of the same size as the money bet just described) has the outcomes {4 or 6}.

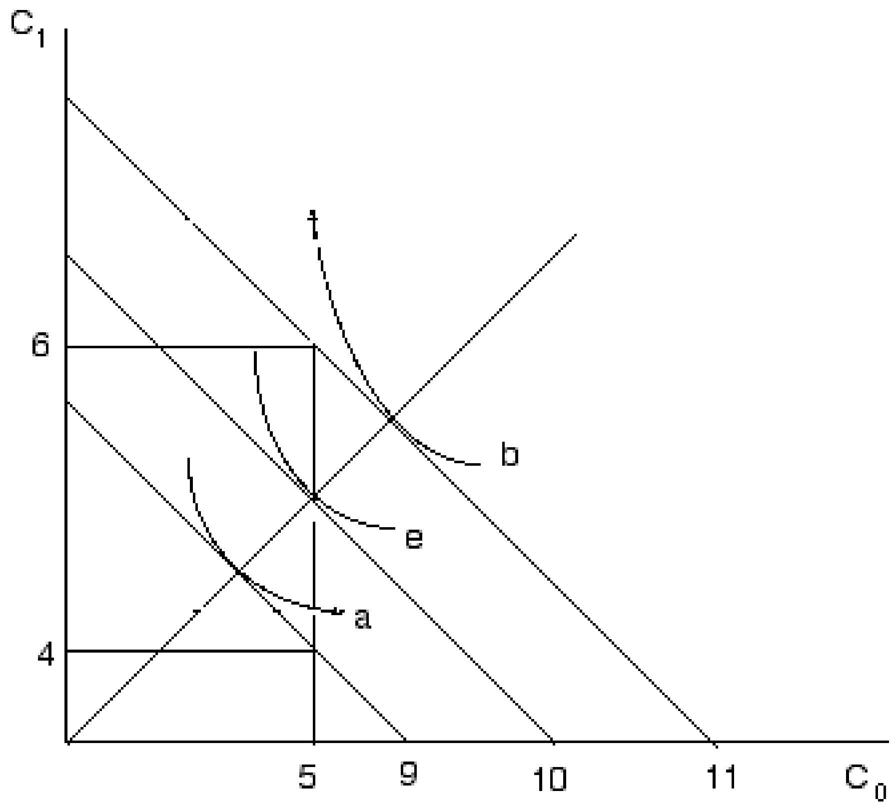

Figure 2

It is clear that the consumer will prefer the money bet {9, 11} to the future consumption bet {4, 6}. But the two bets are of different relative size. The money bet is on 10% of wealth. The consumption bet is on 20% of consumption. The question is whether the consumer will prefer a money bet on x% of wealth to a consumption bet on x% of consumption. To answer this question I compare the relative risk aversion measures to the two kinds of bets. I start by showing that in the time separable case the coefficient of relative risk aversion is the same for the two kinds of bets.

I assume a T+1 periods horizon. The consumer single period strictly concave utility function is U(C) and the discount factor is $0 < \beta \leq 1$. The consumer can lend and borrow at the gross interest rate

$R = 1/\beta$. The consumer's problem when starting with the wealth w is:

$$(2) \qquad V(w) = \max_{C_t} \sum_{t=0}^{T} \beta^t U(C_t) \text{ s.t. } \sum_{t=0}^{T} R^{-t} C_t = w.$$

The attitude towards bets in terms of money is determined by the property of the value function V(w). The solution to (3) is: $C_t = kw$, for all t where $k = 1/\sum_{t=0} R^{-t}$. Therefore:

$$(3) \qquad V(w) = \sum_{t=0}^{T} \beta^t U(kw) = U(kw) \sum_{t=0}^{T} \beta^t = U(kw)/k$$

Taking derivatives leads to:

$$(4) \qquad V''(w)w/V'(w) = U''(kw)(kw)/U'(kw) = U''(c)c/U'(c)$$

Thus under the time separable utility function, the relative risk aversion for bets in terms of money is the same as the relative risk aversion to bets in terms of consumption (at any date). An immediate implication is that relative risk aversion to money bets does not depend on age: When the individual advances with age, the horizon, T+1, gets shorter but consumption per period does not change and therefore relative risk aversion does not change.

I now turn to show that the above result is special to the time-separable case.

4. THE ATTITUDE TOWARDS RISK UNDER THE COBB-DOUGLAS FUNCTION

I now turn to the following utility function:

$$(5) \qquad U(C_1,\ldots,C_T;\ \alpha) = (1/\alpha)\prod_{t=0}^{T}(C_t)^{\alpha\beta^t},\ \alpha \neq 0 \quad (ICD)$$

$$U(C_1,\ldots,C_T;\ \alpha) = \sum_{t=0}^{T}\beta^t \ln(C_t),\ \alpha = 0 \quad (IL)$$

where T+1 is the horizon, $0 < \beta \le 1$ is the discount factor, and $\alpha < 1$ is a parameter that determines the relative risk aversion to money bets (RAM).

To study the attitude towards risk implied by (5) I define the value function:

$$(6) \qquad V(w) = \max U(C_1,\ldots,C_T;\ \alpha) \text{ s.t. } \sum_{t=0}^{T} R^{-t}C_t = w.$$

As before I assume $R = 1/\beta$ and therefore the solution to the maximization problem in (6) is $C_t = kw$ and the value function is:

$$(7) \qquad V(w) = (1/\alpha)(kw)^{\alpha\sum_{t=0}^{T}\beta^t}, \quad (ICD)$$

$$V(w) = \ln(kw)\sum_{t=0}^{T}\beta^t, \quad (IL)$$

The coefficient of relative risk aversion to bets in terms of money (RAM) is:

(8) $$- V''(w)w/V'(w) = 1 - \alpha\sum_{t=0}^{T}\beta^{t} \quad , \text{(ICD)}$$

$$- V''(w)w/V'(w) = 1, \quad \text{(IL)}$$

The coefficient of relative risk aversion to bets in terms of consumption (RAC) is:

(9) $$-U_{tt}C_t/U_t = 1 - \alpha\beta^{t}, \quad \text{(ICD)}$$

$$-U_{tt}C_t/U_t = 1, \quad \text{(IL)}.$$

Comparing (8) to (9) we see that when the utility is not time separable the measure of risk aversion to proportional bets in terms of money is different from the measure of risk aversion to proportional bets in terms of consumption. Note that the assumption $\alpha < 1$ insures RAC > 0.

<u>RAM and age:</u> In the ICD case RAM changes with age. At age τ, RAM = $1 - 1 - \alpha\sum_{t=\tau}^{T}\beta^{t}$. When $\alpha > 0$, RAM increases with age reaching a maximum of $1 - \alpha$ in the last period of one's life. When $\alpha < 0$, RAM decreases with age reaching a minimum of $1 - \alpha$ in the last period of one's life. When α approaches zero RAM approaches 1 (the log utility case). Thus, a prior about the way the RAM coefficient changes with age may help us in choosing the parameter α. My own introspection suggests $\alpha > 0$ and RAM < 1. But as we shall see in the following sections this

parameter is not important for understanding asset prices and therefore there is not much reason to argue about it.

5. A TWO PERIODS SINGLE TREE ECONOMY

I now turn to assess the importance of the RAM coefficient for understanding asset prices - the question in the title. I start with a simple version of Lucas (1978) tree economy. There is a representative consumer who lives for two periods. He is born with an endowment of a tree that yields y units of consumption in the first period of his life and d_s units in the second period state s. After the first period dividends are distributed there is a market for trees. The price of a tree is p and the representative consumer chooses (in the first period of his life) present consumption (C_0) and the amount of trees (A) subject to the budget constraint:

$$C_0 + pA = y + p \tag{10}$$

Consumption in the second period in state s is given by:

$$C_{1s} = Ad_s \tag{11}$$

Substituting (10) into (11) leads to: $C_1 = d(y + p - C_0)/p$. The consumer chooses C_0 to solve:

$$\max_{C_0} \sum_{s=1}^{S} \Pi_s U[C_0,\ d_s(y + p - C_0)/p], \tag{12}$$

where Π_s is the probability of state s. The first order condition to (12) is:

(13) $$\sum_{s=1}^{S} \Pi_s (U_{0s} - U_{1s} d_s / p) = 0$$

where $U_{0s} = \partial U(C_0, C_{1s}) / \partial C_0$ and $U_{1s} = \partial U(C_0, C_{1s}) / \partial C_{1s}$.

The ICD-IL case:

We now assume the Cobb-Douglas case: $U(C_0, C_1) = (1/\alpha)(C_0)^{\alpha}(C_1)^{\delta}$, where $\delta = \alpha\beta$. In this case:

(14) $$U_{0s} - d_s U_{1s}/p = (1/\alpha)\left(\frac{\alpha}{C_0} - \frac{\delta}{y + p - C_0}\right)(C_0)^{\alpha}\left(\frac{(y + p - C_0)d_s}{p}\right)^{\delta}$$

Therefore the first order condition (13) requires $\left(\frac{\alpha}{C_0} - \frac{\delta}{y + p - C_0}\right) = 0$ and $C_0 = \alpha(y+p)/(\alpha+\delta)$.

To solve for p we substitute the market clearing condition $C_0 = y$ in $C_0 = \alpha(y+p)/(\alpha+\delta)$. This leads to:

(15) $$p = (\delta/\alpha)y = \beta y.$$

The asset pricing formula (14) can also be obtained for the IL case. The rate of return on the asset is:

(16) $$D/p = D/\beta y = G/\beta,$$

where D = $\sum_{s=1}^{S} \Pi_s d_s$ is expected dividends and G = 1 + g = D/y is the expected gross rate of growth of consumption. Since (8) implies RAM = 1 - α(1 + β), varying α will change it without affecting the expected returns on the asset. We have thus shown,

Claim 1: When the representative agent's utility function is ICD-IL, the expected rate of return on the asset does not depend on the RAM measure of relative risk aversion and does not depend on the variance of the return. It depends only on the expected rate of growth in consumption (G) and the time preference parameter β.

Claim 1 is generalized in the Appendix to the finite horizon case and to any monotonic transformation of the ICD utility function. Since a monotonic transformation does not change the intertemporal elasticity of substitution (IES) we conclude that IES = 1 leads to (15).

6. A TWO PERIODS MANY ASSETS ECONOMY

I now turn to the many assets economy. I endow the representative agent with n trees. These n trees yield a total of y units of consumption (fruits) in the first period. Tree i yields d_{is} units in the second period in state s. The budget constraint of the representative agent is now:

$$C_0 + \sum_{i=1}^{n} p_i A_i = y + \sum_{i=1}^{n} p_i \tag{16}$$

(17) $$C_{1s} = \sum_{i=1}^{n} d_{is} A_i$$

The agent problem is:

(18) $$\max_{A_i} \ E\{U(C_0, C_1)\} \text{ s.t. (16) and (17).}$$

Substituting the constraints in the objective function we can write (18) as:

(19) $$\max_{A_i} \sum_{s=1}^{S} \Pi_s U(y + \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} p_i A_i, \sum_{i=1}^{n} d_{is} A_i)$$

The first order condition for this problem is:

(20) $$\sum_{s=1}^{S} \Pi_s(-U_{0s} p_i + U_{1s} d_{is}) = 0$$

I use $D_s = \sum_{i=1}^{n} d_{is}$ for the aggregate dividends. I also assume that we can write the dividends of asset i in state s as a linear function of D_s:

(21) $$d_{is} = a_i + b_i D_s + e_{is},$$

where $\sum_{i=1}^{n} e_{is} = 0$ for all s; $\sum_{i=1}^{n} b_i = 1$ and $\sum_{i=1}^{n} a_i = 0$. We assume that the error terms e_{is} is determined by a zero sum purely distributive lottery, has zero mean and is independent of D_s. A riskless asset is an asset with non-random dividends. The market portfolio is an asset for which $d_{is} = D_s$. The assumption about the error terms insures that the expected return on an asset with $b_i = 0$ is the same as the return on a riskless asset and the expected return on an asset with $a_i = 0$ is the same as the expected returns on the market portfolio. I now show this for the ICD case.

The ICD case:

We now turn to the ICD case: $U(C_0, C_1) = (1/\alpha)(C_0)^{\alpha}(C_1)^{\delta}$. Using the first order condition (20) and the market clearing conditions $C_0 = y$ and $C_{1s} = D_s$, we arrive at the equilibrium condition:

$$\sum_{s=1}^{S} \Pi_s \{-p_i (\sum_{i=1}^{n} d_{is})^{\delta} \alpha y^{\alpha-1} + d_{is}\delta(\sum_{i=1}^{n} d_{is})^{\delta-1} y^{\alpha}\} = 0 \tag{22}$$

Substituting (21) into (22), rearranging and using the assumption that e_i does not depend on D, leads to:

$$p_i = \beta y \frac{\sum_{s=1}^{S} \Pi_s d_{is} (D_s)^{\alpha\beta-1}}{\sum_{s=1}^{S} \Pi_s (D_s)^{\alpha\beta}} = \beta y \frac{\sum_{s=1}^{S} \Pi_s (a_i + b_i D_s)(D_s)^{\alpha\beta-1}}{\sum_{s=1}^{S} \Pi_s (D_s)^{\alpha\beta}} \tag{23}$$

When $a_i = 0$, $p_i = \beta b_i y$. For this asset, $d_{is}/p_i = (b_i D_s + e_{is})/\beta b_i y$. Taking expectations leads to the following Claim.

Claim 2: The rate of return on an asset that its dividends are proportional to the aggregate dividends ($a_i = 0$) is G/β.

I now turn to show that risk premium does not require risk aversion.

Claim 3: When $\delta < 1$, the rates of return on all assets with $b_i = 0$ is the same and is less than G/β.

Note that when $\delta = \alpha\beta < 1$ the coefficient of risk aversion $RAM = 1 - \alpha(1 + \beta)$ may be positive or negative. For example if $\beta = 1$ and $\alpha = 0.5$ then $RAM = 0$.

<u>Proof:</u> The rate of return on asset i is:

$$(24) \qquad (a_i + b_i D_s + e_{is})/p_i = (1/\beta y)(a_i + b_i D_s + e_{is})\, G(a_i, b_i),$$

where $G(a_i, b_i) = \dfrac{1}{b_i + a_i \sum_{s=1}^{S} \Pi_s (D_s)^{\delta - 1} / \sum_{s=1}^{S} \Pi_s (D_s)^{\delta}}$ is a non linear term. Since we assume $\delta < 1$, the covariance between D and $D^{\delta-1}$ is negative and

$$(25) \quad G(1, 0) = \frac{\sum_{s=1}^{S} \Pi_s (D_s)^{\delta}}{\sum_{s=1}^{S} \Pi_s (D_s)^{\delta-1}} = \frac{\sum_{s=1}^{S} \Pi_s (D_s)^{\delta-1} D_s}{\sum_{s=1}^{S} \Pi_s (D_s)^{\delta-1}} = \frac{Cov(D^{\delta-1}, D)}{\sum_{s=1}^{S} \Pi_s (D_s)^{\delta-1}} + \sum_{s=1}^{S} \Pi_s D_s$$

$$< \sum_{s=1}^{S} \Pi_s D_s .$$

Substituting this in (24) and taking expectations leads to the conclusion that the expected rate of return on any asset with $b_i = 0$ is less than G/β. □

The intuition is in the observation that when $\delta < 1$, $RAC = 1 - \delta > 0$ and the representative consumer is averse to uncertainty about future consumption. He will therefore hold the market portfolio rather than the risk free asset only if there is a risk premium.

For the log case the asset pricing formula is given by:

$$(26) \qquad p_i = \beta y \sum_{s=1}^{S} \Pi_s d_{is} / D_s$$

and can be obtained as the limit of (23).

I now turn to a numerical example. It is assumed that the rate of growth in aggregate dividends (consumption) is 1 or 1.04 with equal probabilities and β = 1. I consider three assets and use the following notation:

R^b = the return on an asset with a_i = 1 and b_i = 0 (the risk free return);

R^1 = the return on an asset with a_i = 0 and b_i = 1 (the market portfolio);

R^2 = the return on an asset with a_i = - 2 and b_i = 3.

As we can see from Table 1 the rate of return on the market portfolio R^1 does not depend on the RAM coefficient and is equal to G/β = 1.02 in our example. The rate of return on the risk free asset is lower and the difference (the risk premium) increases with RAM.

Table 1: Rates of Returns under the ICD-IL utility function (β = 1)

RAM = $1-2\alpha$	R^1 $a_i = 0;\ b_i = 1$ $d_i=\{1,1.04\}$	R^2 $a_i=-2; b_i=3$ $d_i=\{1,1.12\}$	R^b $a_i=1;\ b_i=0$ $d_i=\{1,1\}$	$R^1 - R^b$	$R^2 - R^b$
0	1.02	1.0204	1.0198	0.0002	0.0006
1	1.02	1.0207	1.0196	0.0004	0.0011
3	1.02	1.0215	1.0192	0.0008	0.0023
10	1.02	1.0241	1.0179	0.0021	0.0062

The risk premium seems small and changes in RAM do not affect asset prices by much. I now turn to examine the robustness of this conclusion for a different utility function that also implies IES = 1 and for a utility function that allows for changes in IES.

"Ordinal certainty equivalent preferences":

Selden (1978) has proposed a non-expected utility function that separates between the elasticity of substitution and risk aversion. Kreps and Porteus (1978) and Epstein and Zin (1989) have extended Selden's analysis to the multi-period case in a time-consistent manner. I now show that when IES = 1, Selden's procedure may be observationally equivalent to the ICD-IL utility function.

Selden evaluates consumption paths in two stages. He first uses a "certainty equivalence function" to substitute a certainty equivalent for the random future consumption and then an "aggregator function" to evaluate current consumption and the certainty equivalence of future consumption.

To illustrate, let C denotes current consumption and x denotes a random future consumption. The consumer first uses the certainty equivalence function μ to convert x to a scalar: $Z = \mu(x)$. He then uses the aggregator function $G(C, Z)$ to evaluate the consumption path. In this formulation IES is determined by the properties of the aggregator function G while RAC is determined by the properties of the certainty equivalence function μ.

I now turn to the special case:

(27) $\quad G(C, Z) = \log(C) + \log(Z); \quad Z = (Ex^{\sigma})^{1/\sigma}$ where $0 \neq \sigma < 1$.

In (27) the aggregator function is logarithmic and as in Epstein and Zin (1991), the certainty equivalence function is of the CES type. For the single asset case, the consumer's problem is:

$$\max_A \log(y + p - pA) + \beta \log\{[\sum_{s=1}^{S} \Pi_s (D_s A)^{\sigma}]^{1/\sigma}\} \tag{28}$$

The first order condition for this problem is (15). Thus as in the log expected utility case the price of the asset depends only on current dividends ($p = \beta y$) and not on the certainty equivalent of future consumption. Therefore risk aversion and aggregate risk do not affect the price of the asset and the expected return.

For the many asset case, the consumer problem under Selden's utility function is:

$$\max_{A_i} \log[y + \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} p_i A_i] + \beta \log\{[\sum_{s=1}^{S} \Pi_s (\sum_{i=1}^{n} d_{is} A_i)^{\sigma}]^{1/\sigma}\} \tag{29}$$

The equilibrium prices (which we obtain after substituting $A_i = 1$ and in the first order conditions) are:

$$p_i = \beta y \frac{\sum_{s=1}^{S} \Pi_s d_{is} (D_s)^{\sigma-1}}{\sum_{s=1}^{S} \Pi_s (D_s)^{\sigma}} \tag{30}$$

This is exactly the formula (23). We have thus shown the following Claim.

Claim 4: The Selden-Epstein-Zin utility function (27) and the ICD utility function are observationally equivalent (in the sense that they both yield the same asset prices) when $\sigma = \alpha\beta$.

Under the ICD utility function the relative risk aversion measure for bets in terms of second period consumption is: RAC = $1 - \alpha\beta$. Thus we may interpret the coefficient σ in (27) as a measure of RAC. We may also use Table 1 to get the predictions of (27) about asset returns.

Note that RAC = $1 - \beta\alpha$ = (RAM + $1/\beta$)/($1/\beta$ + 1). Therefore a unit change in RAC is equivalent to roughly 2 units change in RAM and this will make the RAC measure of risk aversion look more important than our RAM measure. For example, in Table 1 with $\beta = 1$, RAM varies from 0 to 10 while RAC varies from 0.5 to 5.5.

7. THE CONSTANT ELASTICITY FUNCTION

When we change the parameter in the power utility function we get a relatively large effect on asset prices. Is the effect due to the implied change in risk aversion or the implied change in the elasticity of substitution? To discuss this question I introduce now an intertemporal constant elasticity (ICE) utility function that allows for a separation between the RAM and the IES. I now assume:

$$U(C_0, C_1) = (1/\psi)[(C_0)^\rho + \beta(C_1)^\rho]^{\psi/\rho}. \tag{31}$$

where $\rho < 0$ and the elasticity of substitution $1/(1 - \rho)$ is less than unity. There are difficulties in extending the ICE utility function to many periods. These difficulties are discussed in Appendix D where it is argued that the difficulties may not be severe when IES is close to unity.

To interpret the coefficient ψ in (31) I consider the problem under certainty:

(32) $$V(w) = \max\ (1/\psi)[(C_0)^\rho + \beta(C_1)^\rho]^{\psi/\rho} \text{ s.t. } C_0 + C_1/R = w,$$

where $R = 1/\beta$ is the gross interest rate. The solution to (32) is the smooth consumption: $C_0 = C_1 = kw$, where $k = R/(R + 1)$. Substituting the solution in (32) leads to:

(33) $$V(w) = (1/\psi)(1+\beta)^{\psi/\rho}(kw)^\psi$$

It follows that:

(34) $$- V''(w)/V'(w) = 1 - \psi.$$

Thus, ψ is the parameter that governs the RAM coefficient. I now turn to the single asset case. Under the ICE utility function, the first order condition (13) is:

(35) $$p = \beta y^{1-\rho} \frac{\sum_{s=1}^{S} \Pi_s [y^\rho + \beta(d_s)^\rho]^{\psi/\rho - 1}(d_s)^\rho}{\sum_{s=1}^{S} \Pi_s [y^\rho + \beta(d_s)^\rho]^{\psi/\rho - 1}}$$

Thus when the elasticity of substitution is different from unity the price does depend on the RAM parameter ψ.

Note that when ρ is small (35) is close to (15). Thus,

<u>Claim 5:</u> The ICE predicted asset price (35) is approximately equal to the ICD-IL predicted price (15) when ρ is close to zero (and IES is close to 1).

This says that when IES is close to unity the ICE utility function and the ICD-IL utility function have the same predictions about the asset price. I now turn to make the connection with the standard utility function.

The standard power (SP) utility function is:

$$U(C_0, C_1) = (1/\rho)[(C_0)^\rho + \beta(C_1)^\rho] \ , \ \rho < 0. \tag{36}$$

Also here IES = $1/(1 - \rho)$. I restrict $\rho < 0$ to facilitate the comparison with the ICE function. Under SP the first order condition (13) is:

$$p = \sum_{s=1}^{S} \Pi_s U_{1s} d_s / U_0 = \beta y^{1-\rho} \sum_{s=1}^{S} \Pi_s (d_s)^{\rho} \tag{37}$$

Comparing (37) to (35) leads to the following Claim.

<u>Claim 6:</u> The ICE utility function with ψ close to zero (RAM close to unity) yields approximately the same predicted asset price as the standard power utility function with the same ρ (IES) parameter.

This Claim says that if we accept RAM = 1 we may work with standard power utility function to study the effect of variations in IES on the asset's price. It also allows us to interpret a change in the coefficient of SP as a change in IES rather than a change in RAM. Since (1) implies that a change in the parameter of SP has a large effect on the price of the asset (the interest rate) we may say that under (31) this change occurs because of the implied change in IES. I now turn to a numerical example.

Example: I assume that G is a random variable that can take two possible realizations: 1 and 1.04 with equal probabilities. Table 2 calculates the gross rate of return (D/p) for alternative values of the elasticity of substitution parameter and the risk aversion parameter. In this example, changes in the elasticity of substitution have a large effect on the gross expected interest rate while changes in risk aversion have a relatively small effect. We also note that when the elasticity of substitution is less than one, the expected rate of return on the asset decreases with the RAM coefficient. The predictions of the standard power utility function are in the columns with RAM = 1.

Table 2*: The ICE utility function: (D/p) as a function of IES and RAM (β = 1)

IES\RAM	RAM = 0	RAM = 1 (SP)	RAM = 3	RAM = 10
IES=1 (ICD)	1.0200	1.0200	1.0200	1.0200
IES=0.5	1.0401	1.0400	1.0398	1.0391
IES=0.333	1.0603	1.0600	1.0595	1.0577

Under the SP utility function a change in the power parameter that leads to a change in IES from 1 to 0.333 also leads to a change in the RAM from 1 to 3. It is therefore seems natural to compare the change from IES = 1 to IES = 0.333 and from RAM = 1 to RAM = 3 under (31). The change from IES = 1 to IES = 0.333 increases the interest rate by about about 300%. The change from RAM = 1 to RAM = 3 decreases the interest rate by less than 0.05%. We may thus say that for changes that are thought to be identical under the SP function, the changes in IES produces a much larger effect on the real interest rate.

Can we say the same thing about risk premium $R^1 - R^b$. To examine this question I turn to the price formula for the many assets case (20). Under (31) this formula is:

$$p_i = \beta y^{1-\rho} \frac{\sum_{s=1}^{S} \Pi_s [(y)^\rho + \beta (D_s)^\rho]^{\psi/\rho - 1} (D_s)^{\rho-1} d_{is}}{\sum_{s=1}^{S} \Pi_s [(y)^\rho + \beta (D_s)^\rho]^{\psi/\rho - 1}} \tag{38}$$

<u>Claim 7</u>: Under the ICE function with IES close to unity (ρ close to zero) risk premium is strictly positive but does not depend on the RAM coefficient.

<u>Proof</u>: The price of a riskless asset that yields $d_{is} = 1$ for all s is:

$$p_b = \beta y^{1-\rho} \frac{\sum_{s=1}^{S} \Pi_s [(y)^\rho + \beta (D_s)^\rho]^{\alpha/\rho - 1} (D_s)^{\rho-1}}{\sum_{s=1}^{S} \Pi_s [(y)^\rho + \beta (D_s)^\rho]^{\alpha/\rho - 1}} \tag{39}$$

When ρ is close to zero, (39) is close to:

(40) $$p_b = \beta y \sum_{s=1}^{S} \Pi_s (D_s)^{-1}$$

The gross rate of return on the riskless asset is therefore close to:

(41) $$1/p_b = (1/\beta y)\left[\sum_{s=1}^{S} \Pi_s (D_s)^{-1}\right]^{-1}$$

The risk premium is:

(42) $$(D/p) - (1/p_b) = (1/\beta y)\left(D - \left[\sum_{s=1}^{S} \Pi_s (D_s)^{-1}\right]^{-1}\right) \geq 0.$$

This does not depend on the risk aversion parameter ψ. The inequality follows from Jensen's inequality. □

Table 3 uses (38) to compute the price of the riskless asset (by substituting d_{is} = 1 in [38]) and the risk premium. The results support the claim that changes in IES are relatively more important. The risk premia are small and are in the range of 0.04% to 0.2%. When IES goes from 1 to 0.333 the risk premium goes up by 280% to 620%. When RAM goes from 1 to 3 the risk premium goes up by 0% (when IES = 1) to 220% (when IES = 0.333). The example also suggests that a given change in RAM will have a larger effect the lower IES is.

Table 3*: The ICE utility function: Risk premium (R^1 - R^b) as a function of IES and RAM (β = 1)

IES\RAM	RAM = 0	RAM = 1	RAM = 3	RAM = 10
IES=1	0.00039	0.00039	0.00039	0.00039
IES=0.99	0.00039	0.00040	0.00040	0.00041
IES=0.95	0.00040	0.00041	0.00043	0.00050
IES=0.5	0.00070	0.00080	0.00100	0.00169
IES=0.333	0.00109	0.00122	0.00149	0.00241

The discussion up to this point suggests that RAM is not important if we are willing to assume IES close to unity. I now turn to examine this conclusion for the case in which not all assets can be traded.

7. INCOMPLETE MARKETS

I assume N households indexed h. There are n+N types of trees: n types (of physical capital) are traded and N types (of human capital) are not traded. Each household starts with a portfolio of n+1 trees one tree from each of the traded-physical-capital type and human capital. The aggregate per capita amount of fruit (income) in state s is D_s.

The amount of dividends from trees of type 1,...,n is given by (21) and is repeated here for convenience.

$$d_{is} = a_i + b_i D_s + e_{is} \tag{43}$$

where $\sum_{i=1}^{n} e_{is} = 0$ and e_{is} are independent of D_s. The amount of dividends from human capital H_s^h is given by:

(44) $$H_s^h = a_h + b_h D_s + u_s^h$$

where $\sum_{h=1}^{N} u_s^h = 0$ and the u_s^h are independent of D_s. Per capita income is given by:

(45) $$D_s = \sum_{i=1}^{n} d_{is} + (1/N)\sum_{h=1}^{N} H_s^h$$

We may think in terms of three independent lotteries that occur at the beginning of period 1. The first lottery determines the aggregate per capita magnitude D. The second is a zero sum lottery that determines e and the third is a zero sum lottery that determines u. A state of nature s is a description of the outcome of all three lotteries.[5] I assume $a_h = 0$ and $b_h = 0.7$ for all h. It is also assumed that $\sum_{i=1}^{n} b_i = 0.3$ and $\sum_{i=1}^{n} a_i = 0$.

Household h consumption is:

(46) $$C_0^h + \sum_{i=1}^{n} p_i A_i^h = y + \sum_{i=1}^{n} p_i$$

(47) $$C_{1s}^h = \sum_{i=1}^{n} d_{is} A_i^h + H_s^h$$

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[5] Since the lotteries are independent the number of states of nature is: $S = L_1 \times L_2 \times L_3$ where L_i is the number of possible realizations of lottery i.

where A_i^h is household h choice of the quantity of asset i (i=1,...,n).

Since labor share is 0.7, the typical agent problem can now be written as:

$$\max_{A_i} \sum_{s=1}^{S} \Pi_s U(y + \sum_{i=1}^{n} p_i(1 - A_i), \sum_{i=1}^{n} d_{is} A_i + 0.7 D_s + u_s), \tag{48}$$

where the superscript h is suppressed. The first order condition for this problem is still given by (20).

Using symmetry all consumers will make the same first period consumption choice and therefore the clearing of the first period consumption market requires: $C_0 = y$. Symmetry also implies that consumption of household h in the second period is given by $C_s^h = D_s + u_s^h$. The first order condition (20) should hold for all h and therefore I suppress the superscript h and write $C_{1s} = D_s + u_s$ for the representative consumer. Substituting this in the first order condition (20) leads to the following pricing formula:

$$p_i = \beta y \frac{\sum_{s=1}^{S} \Pi_s (a_i + b_i D_s)(D_s + u_s)^{\alpha\beta - 1}}{\sum_{s=1}^{S} \Pi_s (D_s + u_s)^{\alpha\beta}} \tag{49}$$

I now turn to a numerical example in which aggregate consumption may take the realizations 1 and 1.04. For each realization of the aggregate consumption we add a bet in which the typical household can win or lose 0.08 units. This is consistent with the standard deviations of aggregate consumption in the data (0.02) and the Deaton-Paxon estimate of the standard deviation of individual consumption (0.08).[6]

Table 4 presents the results of the numerical example. The risk premia in Table 4 are almost identical to the risk premia in Table 1. The rates of return themselves are not. The rate of return on the market portfolio is now declining in the RAM coefficient. It seems that allowing for incomplete markets will affect our estimate of β but will have little or no effect on our estimate of the risk premia.

Table 4: Predicted Rates of Returns when markets are incomplete (β = 1 and C_1 = $\{1 \pm 0.08 , 1.04 \pm 0.08\}$)

RAM = $1-2\alpha$	R^1 $a_i = 0$; $b_i = 1$ $d_i=\{1,1.04\}$	R^2 $a_i=-2$; $b_i=3$ $d_i=\{1,1.12\}$	R^b $a_i=1$; $b_i=0$ $d_i=\{1,1\}$	$R^1 - R^b$	$R^2 - R^b$
0	1.017	1.017	1.017	0.0002	0.0006
1	1.014	1.014	1.013	0.0004	0.0011
2	1.011	1.012	1.010	0.0006	0.0017
3	1.008	1.009	1.007	0.0008	0.0023
10	0.987	0.991	0.985	0.0022	0.0063

[6] Deaton and Paxson (1994) finds that the variance of log consumption within each age cohort increases by 0.07 every decade in the US (page 446). Their random walk assumption in equations (1) - (3) imply a variance in consumption of 0.007 per year which is roughly equal to a standard deviation of 0.08.

I now turn to the effect of changes in RAM on welfare calculations.

8. WELFARE CALCULATIONS

To examine the effect of RAM on welfare (holding constant IES = 1) I start with a consumption path of $C_0 = 1$ and $C_1 = \{1 \text{ or } 1.04\}$. Following Lucas I calculate the required compensation (λ) for the consumption risk, where λ solves:

$$(50)\quad (\tfrac{1}{2})U(1 + \lambda,\ 1 + \lambda) + (\tfrac{1}{2})U[1 + \lambda,\ 1.04(1 + \lambda)] = U(1,\ 1.02)$$

Thus the consumer is fully compensated for the risk if his consumption in all periods and states of nature is increased by a fraction of λ.

Assuming the ICD utility function with $\beta = 1$ leads to:

$$(51)\qquad \lambda = [2(1.02)^{\alpha}/(1.04^{\alpha} + 1.00^{\alpha})]^{1/2\alpha} - 1\,.$$

The second column of Table 3 reports the required compensation in percentage terms (100λ) for various levels of RAM. Not surprising, risk aversion matters. For example, going from RAM = 0 to RAM = 1 doubles the required compensation. But as in Lucas (2003) all the magnitudes are a small fraction of a percent.

Table 5: Required compensations in percentage terms (100λ); C_1 = {1 or 1.04}

Ram	2 periods, 1 shock	2 periods, 2 shocks	3 periods, 3 shocks	2 periods, 1 shock SD = 0.08
0	0.005	0.005	0.009	0.077
1	0.010	0.029	0.038	0.154
2	0.014	0.053	0.068	0.232
3	0.019	0.077	0.098	0.309
10	0.053	0.245	0.307	0.839

To check for robustness I also considered cases in which consumption follows a random walk. The third column in Table 3 reports the required compensation when both current and future consumption are random: C_0 = {1 or 1.04} and C_1 = {C_0 or 1.04C_0}. In this case the required compensation are substantially higher relative to the single shock case but are still a fraction of a percent. The three periods random walk case, reported in the fourth column assumes: C_0 = {1 or 1.04}, C_1 = {C_0 or 1.04C_0} and C_2 = {C_1 or 1.04C_2}. In this case the welfare cost is larger than in the previous case. This suggests that adding shocks whose effect are being eliminated by "good policy" increases the welfare gain. Note also that the required compensation is almost proportional to the RAM coefficient.

Allowing for incomplete markets may increase the welfare gain. If by "good policy" we eliminate aggregate risk we may also greatly reduce the number of markets required for completeness. This may therefore improve the allocation of diversifiable risk in the economy. In the last column of Table 5 I assume that the "good policy" eliminates all risk in a two periods one shock economy assuming: $C_1 = \{0.94 \text{ or } 1.1\}$ initially and then by "good policy" is converted to $C_1 = 1.02$. The assumed standard deviation of consumption is thus 0.08 and is consistent with the Deaton-Paxson estimate discussed above. Note that if RAM = 10 the welfare gain is 0.8%. This starts to look like real money.[7]

9. RATES OF RETURN FOR HYPOTHETICAL CLAIMS UNDER THE ICD-IL FUNCTION

Under the ICD-IL function the expected return on the market portfolio does not depend on the RAM coefficient. But the RAM coefficient does affect the expected rates of return on claims on parts of GDP that are not proportional to consumption. To get a sense of the importance of the RAM coefficient, I consider now hypothetical claims on (a) GDP, (b) the wage bill, (c) non-wage income (profits) and (d) corporate profits, all in real per-capita terms.

Our first task is to express equation (25) in terms of rates of change. Equation (25) is conditional on all the information available at time t and we may therefore write:

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[7] It has also been argued that a good policy may improve production efficiency. For example, it is possible that the consumer is not averse to fluctuations in consumption but is averse to fluctuations in labor supply. It is also possible that average capacity utilization will improve as a result of policy. For a recent survey of the literature see Barlevy (n.d.).

(52) $$d_{it+1} = a_{it+1} + b_{it+1}D_{t+1} + e_{it+1},$$

where the coefficients are time dependent. I normalize y = 1 and assume: $a_{it+1} = a_i d_{it}$ and $b_{it+1} = b_i d_{it}$. This means that the predicted share in the pie is proportional to the time t share. Dividing (48) by d_{it} yields:

(53) $$G_{it+1} = a_i + b_i G_{t+1} + \varepsilon_{it+1},$$

where $G_{it+1} = d_{it+1}/d_{it}$ is the gross rate of growth in asset i dividends, $G_{t+1} = D_{t+1}$ is the gross rate of growth in consumption and $\varepsilon_{it+1} = e_{it+1}/d_{it}$ is an error term. I also assume that ε_{it+1} has a zero mean and is not correlated with G_{t+1}. The time invariant coefficients a_i and b_i can therefore be estimated from running the regression (52).

Note that multiplying the coefficients a_i and b_i by the same constant does not change the expected rate of return (24). Therefore after estimating the regression coefficients in (53) we can plug the coefficients directly (without multiplying it by d_{ti}) in (24) to compute the predicted gross rate of return on asset i.

Equation (53) requires data on the gross rates of change of flows (fruits) and these data are easier to get than data on prices. For example there is no market for slaves and therefore no data on the price of human capital defined as body plus the knowledge embodied in it. But we can predict the gross rate of return on human capital even without observing its price. Similarly and maybe more relevant, we do not observe the price of unincorporated equity. But nevertheless we can predict the rate of return on it if we observe the flow of profits it yields.

I use NIPA US post war data (from January 1948 to January 2004) taken from the Saint Louis Fed web page to compute the gross rate of growth in real per capita terms of the following variables: consumption (c), wage earnings (w), corporate profits (pr), GDP (y) and non-wage income (y-w). The detail of the calculations of these variables and the description of the data are in Appendix c.

Table 4 provides summary statistics for the annual data. All rates of change are close to 2%. The smallest rate is for the wage bill (1.6%) and the highest is for corporate profits (2.2%). The standard deviation is in the range 0.02 - 0.04 except for corporate profits where it is much higher (0.16).

Table 6: Summary Statistics about Annual Per Capita Gross Rates of Change

	Average	Standard deviation
Consumption (c)	1.019	0.02
GDP (y)	1.018	0.03
Wage earnings (w)	1.016	0.03
Profits (y-w)	1.020	0.04
Corporate Profits (pr)	1.022	0.16

Table 7 provides the regression results from running (50). Most intercepts are small and barely significant. The intercept on corporate profits is an exception.

Table 7*: Regressions of the rate of change of asset i on the rate of change in consumption

Dependent var.	Intercept	Slope	Rsquare
y	-0.19 (0.10)	1.18 (0.10)	0.73
w	0.07 (0.14)	0.93 (0.14)	0.46
y-w	-0.45 (0.14)	1.44 (0.14)	0.67
pr	-2.06 (0.93)	3.03 (0.91)	0.17

* Standard errors in parentheses.

Table 1 may therefore provide a good approximation for the rates of return on the four hypothetical claims when $\beta = 1$. The expected return on the market portfolio (R^1) is a good approximation for the returns on claims on the wage bill, non-wage income and GDP. The expected return on the more risky portfolio (R^2) is an estimate of the return on a claim on corporate profits.

The prediction of the model for various β can be approximated by multiplying Table 1 by $1/\beta$. This is done in Table 8 for $1/\beta = 1.025$.

Table 8: Predicted Rates of Returns (IES = 1; $1/\beta$ = 1.025)

RAM $= 1-2\alpha$	R^1 $a_i = 0; b_i = 1$ $d_i=\{1,1.04\}$	R^2 $a_i=-2; b_i=3$	R^b $a_i=1; b_i=0$	$R^1 - R^b$	$R^2 - R^b$
0	1.0455	1.0459	1.0453	0.0002	0.0006
1	1.0455	1.0463	1.0451	0.0004	0.0012
2	1.0455	1.0466	1.0449	0.0006	0.0017
3	1.0455	1.0470	1.0447	0.0008	0.0023
10	1.0455	1.0497	1.0433	0.0022	0.0064

The expected rates of returns in Table 8 are consistent with the estimates in McGrattan and Prescott (2003) who took an explicit account of taxes and frictions and found average returns in the 4-5 percent range. The expected rate of return on the market portfolio is 1.0455. The expected rate of return on a claim on corporate profits is 1.0459 when RAM = 0 and 1.05 when RAM = 10. The corresponding risk premia on the more risky portfolio (corporate profits) are 0.05% and 0.6% respectively.

10. CONCLUDING REMARKS

It was argued that Lucas' observations about the return on the market portfolio do not require an assumption about the attitude towards money bets. They do require the assumption that the elasticity of substitution is close to unity and that markets are complete.

In the Cobb-Douglas case risk premium depends on the RAM coefficient but is not sensitive to changes in it. In the ICE case with IES close to unity, risk premium does not depend on the RAM coefficient. This says that if we are willing to commit to IES close to unity we should not worry too much about the correct magnitude of the RAM coefficient.

But there is no agreement about the magnitude of IES. Hall (1988), Campbell and Mankiw (1989) and Beaudry and Wincoop (1996) provide estimates of the IES between zero and one. Under the ICE utility function changes in IES are important for asset prices but changes in RAM are not. Even when IES = 0.333 the risk premium is not very sensitive to changes in RAM. It is 0.1% when RAM = 0 and 0.2% when RAM = 10.

Allowing for incomplete markets does not change risk-premia in the ICD example we worked out. But it does affect the rates of returns on the assets and introduces a negative relationship between the rates of return and the RAM coefficient. This may be the result of a precautionary savings type behavior.

Not surprisingly changes in the RAM coefficient affect the calculation of the welfare gains from eliminating business cycle risks. This point is well recognized by Lucas (2003) and other authors on this subject.

Data on flows can be used to compute the rates of returns on various claims. I used post war US NIPA data and found that claims on the wage bill and on total profits are close to a claim on the market portfolio (aggregate consumption). But a claim on corporate profits is more risky than a claim on the market portfolio. The predictions of the

ICD-IL utility function are consistent with the findings in McGrattan and Prescott (2003) but cannot account for the original Mehra and Prescott (1985) puzzle.

APPENDIX A: A FINITE HORIZON SINGLE ASSET ICD ECONOMY

I now consider an economy in which the representative agent lives for T periods. At $t = 0$ he gets endowment of one tree that provides fruits for T periods and then dies (together with the agent).

I allow a general dividend (income) process. It is assumed that the representative agent at $t = 0$ assigns positive probabilities, π_s, to all states $s = 1,...,S$. Over time he updates this probabilities when he learns that some states did not occur. The set of possible states at time t (the information available at time t) is denoted by I_t. The updated probability of state s is denoted by $(\pi_s | I_t)$. Note that $(\pi_s | I_t) = 0$ if $s \notin I_t$. The agent also knows the information that he will have at time $j > t$ if state s occurred. This information is denoted by I_{js}. At time t the choices of $(A_0,...,A_{t-1})$ was already made. Since there is one tree per agent we assume $A_j = 1$ for $j < t$. The agent chooses A_t and makes a contingent plan that specifies the amount of trees he will own at future dates: $(A_{t+1s},...,A_{T-1s})$. The agent has to choose $A_{js} = A_{js'}$ if at time j he cannot distinguish between the two states. Thus, he faces the informational constraint: $A_{js} = A_{js'}$ if $s,s' \in I_{js}$. Assuming an ICD utility function we can state the time t problem as follows.

(A1) $$V_t(k_{t-1}, I_t) = \max_{A_t, A_{t+1s},...,A_{T-1s}}$$

$$k_{t-1}(d_t + p_t - A_t p_t)^{\alpha\beta^t} \sum_{s=1}^{S}(\pi_s \mid I_t)\,[A_t(d_{t+1s} + p_{t+1s}) - A_{t+1s}p_{t+1s}]^{\alpha\beta^t}\, k_{t+1s}$$

s.t.

$$k_{t-1} = \prod_{j=0}^{t-1} (d_j)^{\alpha\beta^j}$$

$$k_{t+1s} = \prod_{j=t+2}^{T} [A_{j-1s}(d_{js} + p_{js}) - A_{js}p_{js}]^{\alpha\beta^j}$$

$$A_{js} = A_{js'} \text{ if } s,s' \in I_{js}$$

I now define equilibrium as follows.

Equilibrium at time t is a vector $(A_t,\ A_{t+11},\ldots,A_{T-11},\ldots,A_{t+1S},\ldots,A_{T-1S};$ $p_t,\ p_{t+11},\ldots,p_{T-11},\ldots,p_{t+1S},\ldots,p_{T-1S})$ such that

(a) given prices $(p_t,\ p_{t+1s},\ldots,p_{T-1s})$, the quantity vector $(A_t\ ,A_{t+1s},\ldots,A_{T-1s})$ solves (A1) and

(b) market clearing: $A_t = 1$ and $A_{js} = 1$ for all $j > t$ and all s.

I now generalize the asset pricing formula (10) to the finite horizon case.

<u>Claim A1:</u> Equilibrium prices at time t are given by:

(A2) $$p_t = (\beta + \beta^2 + \ldots + \beta^{T-t})d_t \text{ and}$$

$$p_{js} = (\beta + \beta^2 + \ldots + \beta^{T-j})d_{js} \text{ for all } t < j < T$$

Note that when $T = \infty$ (A2) implies $p_t = d_t/\rho$, where the subjective interest rate $1 + \rho = 1/\beta$. This formula is in the logarithmic preference example in Ljungqvist and Sargent (2000, page 239).

<u>Proof:</u> When T = 1, there is trade in the asset only in period $t = T - 1 = 0$ and (A2) coincides with (17). We now proceed by induction. We assume that equilibrium prices when the horizon is T-t-1 (at time t+1) satisfy (A2) and show that equilibrium prices when the horizon is T-t (at time t) satisfy (A2).

Given our induction hypothesis we can write the problem (A1) as:

(A3) $$V(k_{t-1};\ I_t) =$$

$$\max_{A_t} k_{t-1}(d_t + p_t - A_t p_t)^{\alpha\beta^t} \textstyle\sum_{s=1}^{S}(\pi_s \mid I_t)\,[A_t(d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1}} k_{t+1s}$$

Now $k_{t+1s} = \prod_{j=t+2}^{T}(d_{js})^{\alpha\beta^j}$ is a constant and $p_{t+1s} = (\beta + \beta^2 + ... + \beta^{T-t-1})d_{t+1s}$. Note that the assumption $A_{t+1s} = 1$ follows from the induction hypothesis. The first order condition for the problem (A3) is:

(A4) $$-\alpha\beta^t p_t(d_t + p_t - A_t p_t)^{\alpha\beta^t - 1} \textstyle\sum_{s=1}^{S}(\pi_s \mid I_t)\,[A_t(d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1}} k_{t+1s}$$
$$+\ (d_t + p_t - A_t p_t)^{\alpha\beta^t} \textstyle\sum_{s=1}^{S}(\pi_s \mid I_t)\,\alpha\beta^{t+1}(d_{t+1s} + p_{t+1s})[A_t(d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1}-1} k_{t+1s} = 0$$

Substituting $A_t = 1$ and $p_{t+1s} = (\beta + \beta^2 + ... + \beta^{T-t-1})d_{t+1s}$ in (A4) leads to:

(A5) $$p_t = (\beta + \beta^2 + ... + \beta^{T-t})d_t$$

This completes the proof. □

We can now use Claim A to compute the rate of return on the asset as follows.

$$\text{(A6)} \qquad (d_{t+1s} + p_{t+1s})/p_t =$$

$$= (1+\beta+\beta^2+\ldots+\beta^{T-t-1})d_{t+1s}/(\beta+\beta^2+\ldots+\beta^{T-t})d_t = d_{t+1s}/\beta d_t$$

Using $G_t = \sum_{s=1}^{S}(\pi_s \mid I_t)(d_{t+1s}/d_t)$ to denote the expected consumption growth we can write the expected rate of return at time t as:

$$\text{(A7)} \qquad G_t/\beta = G_t(1+\rho),$$

where ρ is the subjective rate of interest. This is exactly the formula (15) that we got in the two periods horizon.

APPENDIX B: MONOTONIC TRANSFORMATION OF THE COBB-DOUGLAS UTILITY FUNCTION

In Table 1 we have seen that the prediction of the log utility function about the average return in the economy is the same as the prediction of the Cobb-Douglas functions. We now show that this is also the case for other monotonic transformation of the Cobb-Douglas function.

We assume a utility function F(U), where F' > 0. The problem (14) is now:

$$\text{(B1)} \qquad \max_{C_0} \sum_{s=1}^{S}\Pi_s F\{U[C_0, d_s(y+p-C_0)/p]\}$$

The first order condition for this problem is:

(B2) $$\sum_{s=1}^{S} \Pi_s F_s^{'} (U_{0s} - d_s U_{1s}/p) = 0$$

where $F_s^{'} = F'\{U[C_0, d_s(y + p - C_0)/p]\}$. In general a monotonic transformation will change the price of a tree. In the Cobb-Douglas case $C_0 = \alpha(y+p)/(\alpha+\delta)$ and

(B3) $$U_{0s} - d_s U_{1s}/p = 0 \text{ for all s.}$$

It follows that a monotonic transformation that changes the derivatives $F_s^{'}$ will not change p.

We may now consider the family of utility functions that are monotonic transformation of the log utility function. This is a much larger family than the Cobb-Douglas utility function. It includes for example, $[\ln(C_0) + \ln(C_1)]^{\gamma}$. We can now generalize Claim 1 as follows.

Claim B1: If the utility function of the representative agent is a monotonic transformation of the log utility function, then the expected rate of return in a single asset economy is G/β.

APPENDIX C: DATA

I took the following series from the St. Louis Fed web site. Population (POP): Civilian Labor Force (M, SA), Wage bill (NW): Compensation of Employees: Wages and Salary Accruals (Q, SAAR),

Consumption (NC): Personal Consumption Expenditures (Q, SAAR)

Price level (P): Gross Domestic Product Chain-type Price Index

Corporate Profits (NPR): Corporate Profits After Tax with Inventory Valuation Adjustment (IVA) and Capital Consumption Adjustment (CCADi)

Nominal GDP (NGDP): Gross Domestic Product, 1 Decimal

These data are available from January 1948 until January 2004. The data are available on a quarterly basis (except for population which is given on a monthly basis and was converted to a quarterly series). The data are in billions of current dollars and were divided by the price level and by population to obtain real per capita magnitudes:

W = NW/P(POP) real per capita wage earnings

C = NC/P(POP) real per capita consumption

PR = NPR/P(POP) real per capita Corporate Profits

Y = NGDP/P(POP) real per capita GDP

Y-W = (NGDP-NW)/P(POP) real per capita non wage income

I computed the following gross rates of change: $c_t = C_t/C_{t-1}$, $w_t = W_t/W_{t-1}$, $pr_t = PR_t/PR_{t-1}$, $y_t = GDP_t/GDP_{t-1}$, $(y-w)_t = (Y-W)_t/(Y-W)_{t-1}$.

APPENDIX D: TIME INCONSISTENCY AND EXTENSION TO MANY PERIODS

Will an agent that lives in an Arrow-Debreu world and make consumption plan at t = -1 will want to change it as he learns more about the true history? Assuming an expected utility function is sufficient to guarantee that the agent will not want to revise his plan over time and in this sense his plan is time consistent. This claim is well known. I show it here because many people have argued, erroneously in my opinion, that the ICE utility function suffers from a time inconsistency problem when extended to many periods. I also show that the MRS does not depend on past consumption for the ICD case and may not be very sensitive to past consumption in the ICE case when IES is close to unity.

To show the claim that expected utility guarantees time consistency, I assume a three periods horizon: t = 0, 1, 2. Events at each date may take S possible realizations. The probability that "state of nature" k will occur at t = 0 is denoted by π_k. The probability that "state of nature" i will occur at date 1 given that "state of nature" k has occurred at t = 0 is denoted by π_{ki} and the probability that "state of nature" j will occur at date 2 given that "state of nature" k has occurred at t = 0 and "state of nature" i has occurred at t = 1 is denoted by π_{kij}. Similarly, C_{0k} denotes consumption at t = 0 state k, C_{1ki} denotes consumption at t = 1 state (k,i) and C_{2kij} denotes consumption at t = 2 state (k,i,j). The most general formulation used in Arrow (1964) assumes that the consumer evaluates consumption plans by the utility function:

(D1) $$Z(C_{01},\ldots,C_{0S};C_{111},\ldots,C_{1SS};C_{2111},\ldots,C_{2SSS})\,.$$

At t = -1 he faces the budget constraint:

(D2) $$\sum_{k,i,j} P_{0k}C_{0k} + P_{1ki}C_{1ki} + P_{2kij}C_{2kij} = w,$$

where P are the prices of the contingent commodities. He maximizes (D1) subject to (D2). The first order condition requires:

(D3) $$Z_{2msr} / Z_{1ms} = P_{2msr} / P_{1ms},$$

where $Z_i = \partial Z/\partial C_i$. In this general formulation an agent that learns about the state at t = 0 will in general want to change his consumption plan because the conditional probabilities of the states at t = 1 and t = 2 will change and as a result the function Z will change.

I now turn to the expected utility case assuming that there exists a function U such that:

(D4) $$Z = \sum_k \pi_k \sum_i \pi_{ki} \sum_j \pi_{kij} U(C_{0k}, C_{1ki}, C_{2kij})$$

In this case, the marginal utilities are:

(D5) $$Z_{1ms} = \pi_m \pi_{ms} \sum_j \pi_{msj} U_1(C_{0m}, C_{1ms}, C_{2msj}), \quad Z_{2msr} = \pi_m \pi_{ms} \pi_{msr} U_2(C_{0m}, C_{1ms}, C_{2msr})$$

The marginal rate of substitution (MRS) is:

(D6) $$Z_{2msr}/Z_{1ms} = \pi_{msr} U_2(C_{0m}, C_{1ms}, C_{2msr}) \ / \ \sum_j \pi_{msj} U_1(C_{0m}, C_{1ms}, C_{2msj})$$

The MRS does not change when at t = 0 the consumer learns that state m has occurred. In this sense the expected utility assumption is sufficient for guaranteeing time consistency.

This does not say that having a time non-separable utility function can easily be extended to many periods horizon. The difficulty emerges when MRS depends on all past consumption. In general (D6) depends on C_0. I now turn to show that this is not a problem for the ICD function and that it may also not be a significant problem for the ICE function when IES is close to one.

To simplify, I now assume that the state of nature at t = 0 is known and I therefore omit the index k. I start with the Cobb-Douglas case assuming:

(D7) $$Z = (1/\alpha)\sum_i \pi_i \sum_j \pi_{ij}(C_0)^{\alpha}(C_{1i})^{\beta\alpha}(C_{2ij})^{\beta^2\alpha}$$

The marginal utilities in this case are:

(D8) $$Z_{1s} = \pi_s \beta (C_0)^{\alpha}(C_{1s})^{\beta\alpha-1}\sum_j \pi_{sj}(C_{2sj})^{\beta^2\alpha}\ ,\ Z_{2sr} = \pi_s \pi_{sr}(C_0)^{\alpha}(C_{1s})^{\beta\alpha}\beta^2(C_{2sr})^{\beta^2\alpha-1}$$

The marginal rate of substitution is:

(D8) $$Z_{2sr}/Z_{1s} = \frac{\pi_{sr}\beta(C_{1s})(C_{2sr})^{\beta^2\alpha-1}}{\sum_j \pi_{sj}(C_{2sj})^{\beta^2\alpha}}$$

This does not depend on C_0.

I now turn to the ICE case assuming:

(D9) $$Z = (1/\psi)\sum_i \pi_i \sum_j \pi_{ij}[(C_0)^{\rho} + \beta(C_{1i})^{\rho} + \beta^2(C_{2ij})^{\rho}]^{\psi/\rho}$$

The marginal utilities in this case are:

(D10) $$Z_{1s} = \pi_s \sum_j \pi_{sj}[(C_0)^{\rho} + \beta(C_{1s})^{\rho} + \beta^2(C_{2sj})^{\rho}]^{\frac{\psi}{\rho}-1}\beta(C_{1s})^{\rho-1}\ ,$$

$$Z_{2sr} = \pi_s \pi_{sr}[(C_0)^{\rho} + \beta(C_{1s})^{\rho} + \beta^2(C_{2sr})^{\rho}]^{\frac{\psi}{\rho}-1}\beta^2(C_{2sr})^{\rho-1}$$

The marginal rate of substitution is:

(D11) $$Z_{2sr}/Z_{1s} = \frac{\pi_{sr}[(C_0)^\rho + \beta(C_{1s})^\rho + \beta^2(C_{2sr})^\rho]^{\frac{\psi}{\rho}-1}\beta(C_{2sr})^{\rho-1}}{\sum_j \pi_{sj}[(C_0)^\rho + \beta(C_{1s})^\rho + \beta^2(C_{2sj})^\rho]^{\frac{\psi}{\rho}-1}(C_{1s})^{\rho-1}}$$

This does depend on C_0 and suggests that extension to many periods will prove difficult. However, it seems that when ρ is close to zero and the elasticity is close to 1 this problem is not severe. To show this point I now turn to a numerical example that assumes: $\beta = 1$,

$C_0 = 2$ or 3 with equal probabilities,

$C_1 = 3$

$C_2 = 2$ or 3 with equal probabilities.

I calculate the marginal rate of substitution between $C_1 = 3$ and $C_2 = 3$ as a function of C_0. This is given by:

(D12) $$MRS(C_0) = \frac{((C_0)^\rho + 3^\rho + 3^\rho)^{\frac{\psi}{\rho}-1}}{[(C_0)^\rho + 3^\rho + 3^\rho]^{\frac{\psi}{\rho}-1} + [(C_0)^\rho + 3^\rho + 2^\rho]^{\frac{\psi}{\rho}-1}}$$

Table D1 calculates the Ratio = $MRS(C_0=3)/MRS(C_0=2)$. This ratio may measure the sensitivity of MRS to changes in C_0. As can be seen from the Table the ratio is close to 1 when the elasticity of substitution is close to 1.

Table D1: Ratio of MRS as a function of IES and RAM

IES\RAM	RAM = 0	RAM = 1	RAM = 3
0.99	1.00018665	1.00000187	0.9996324
0.95	1.00101972	1.00005096	0.99811625
0.9	1.00229153	1.00022892	0.99611642
0.8	1.00591692	1.00118059	0.99177473

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