Rational Housing Bubble

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Abstract

This paper study an endowment economy inhabited by overlapping generations of homeowners and investors, with the only difference being that homeowners derive utility from housing services while investors do not. Tight collateral constraint limits the borrowing capacity of homeowners and drives down the equilibrium interest rate level to the housing price growth rate, which makes housing attractive as a store of value for investors. As long as the rental market friction is large enough, the investors will hold positive amount of vacant houses in the equilibrium. Housing bubble arises in an equilibrium in which investors hold houses for resale purposes only and not with the expectation of receiving a dividend either in terms of utility or rent. I apply the model to China, in which the caveat of housing bubble can be attributed to the rapid decline in the replacement rate of the pension system.

1 Introduction

Suddenly impoverished consumers have rediscovered the virtues of thrift and the worldwide property boom, which provided an outlet for all those excess savings, has turned into a worldwide bust. Paul Krugman

Housing asset plays a dual role. It is not only an investment good but also a consumption good. With the first property alone, housing asset, like fiat money,

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can have a positive value in the overlapping generation model as shown by Samuelson (1958). We call housing asset has a bubble because its intrinsic value is zero. People are willing to buy it because it serves as a store of value. However, with the second property alone, housing asset, like a Lucas tree, cannot have a rational bubble in the Samuelson's model. Positive dividend (either in terms of rent or in terms of utility) rules out equilibrium with zero or negative interest rate. In this case, housing price is equal to its fundamental value. If there were a bubble asset, its prices would grow at the speed of positive interest rate, which is unsustainable for this endowment economy without growth.

My research question is the following: can housing asset have a rational bubble with both properties? This paper departs from the two-period consumption-loan model by Samuelson (1958) with only one twist. The economy consists of two types of households, homeowners and investors, with only difference being that homeowners derive utility from housing services while investors do not. With two types of households coexisting in the model, the equilibrium can have two possible outcomes, which depend on the degree of financial friction. If the financial friction is low, the model economy ends up in the bubbleless equilibrium, in which investors lend to workers at a positive interest rate. Because the equilibrium interest rate is higher than the return to housing asset (which is zero in the endowment economy without growth), investors have no incentives to hold housing asset.

Tight collateral constraint limits the borrowing capacity of homeowners and drives down the equilibrium interest rate level to the housing price growth rate, which makes housing attractive as a store of value for investors. There are excess supply of funds from the investors and asset shortage due to that homeowners are borrowing constrained at equilibrium interest rate. In the equilibrium, investors use the excess supply of funds to purchase houses which are useless to them and expect the future young investors will purchase the asset from them.

As long as the rental market friction is large enough, rental market cannot absorb all the housing assets bought by investors and the investors will hold some empty houses in the equilibrium. This is because high rental market friction implies a higher rental-price-to-selling-price ratio, which has homeowners substitute rental housing for owner-occupied housing. Therefore, housing bubble arises in an equilibrium in which investors hold houses for resale purposes only and not with the expectation of receiving a dividend either in terms of utility or rent.

There is enormous literature on asset bubbles. My paper is related to the rational bubbles under symmetric information. See Brunnermeier (2009) for other forms of bubbles. Most literature introduces market imperfection into the overlapping generation model.

Early studies focus on exogenous dynamic inefficiency in the overlapping generation model based on Diamond (1965). Tirole (1985) argue the presence of bubble absorb the excess savings in the production sector and achieve dynamic efficiency. He shows that bubble will arise under the presence of Lucas tree as long as the dynamic inefficiency condition holds initially.

Recent studies on bubbles focus on the financial frictions and credit constraint. Kocherlakota (2009) and Martin and Ventura (2010) introduce credit constraint and investor heterogeneity. Bubbles serve as a collateral asset that helps to alleviate the financial constraint of productive firms. Caballero (2006) and Caballero et. al. (2008) argue that speculative bubble alleviate the asset scarcity problem in the emerging market and explain the global imbalance. In terms of theoretical model, Arce and Lopez-Salido (2011) is mostly close to this paper. It introduces housing asset in a three-period OLG model and multiple stationary equilibria exist depending on the financial constraint. This paper has two types of agents in the two-period OLG model and has unique stationary equilibrium.

In all those studies, bubble is Pareto-improving and efficient. In my paper, bubble is good for investors because it is a good substitute for consumption loans. However, bubbles reduce the welfare of homeowners. It raises the borrowing rate and reduces the amount of housing services consumed. In the end, I apply the model to China, in which the caveat of housing bubble can be attributed to the rapid decline in the replacement rate of the pension system.

The structure of paper is organized as follows: Section 2 builds up the benchmark model. Section 3 looks at the housing market data in China and a policy experiment of pension reform. Section 4 considers a model of rental market. Section 5 concludes.

2 Benchmark Model

The benchmark model is a two-period overlapping generation model with exogenous endowment. It is based on the consumption-loan model by Samuelson (1958).

2.1 Preference and Endowment

The economy is inhabited by two types of households: investor and homeowner. Both types live for two periods. The investor has the Cobb-Douglas utility function

$$u^I\left(c_t^t, c_{t+1}^t\right) = \ln c_t^t + \beta \ln c_{t+1}^t \tag{1}$$

where $\beta > 0$. Let c_t^t and c_{t+1}^t denote the non-durable consumption at time t and $t+1$ of households born at t, respectively. The homeowner derives utility not only from non-durable consumption but also from housing services.

$$u^H\left(c_t^t, c_{t+1}^t, h_{t+1}^t\right) = \ln c_t^t + \beta\left(1-\zeta\right)\ln c_{t+1}^t + \beta\zeta \ln h_{t+1}^t \tag{2}$$

where $1 > \zeta > 0$. Because of the homothetic preference, both types of households spend $1/(1+\beta)$ of their total wealth in the first-period consumption in absence of borrowing constraint.

Both investors and homeowners receive y_t^t when young and 0 when old.[1] Denote the growth rate of output by g. Hence,

$$\frac{y_{t+1}^{t+1}}{y_t^t} = 1 + g \tag{3}$$

In each period, there are $N_t \alpha$ young homeowners and $N_t(1-\alpha)$ young investors, $1 > \alpha > 0$. The population growth rate is

$$\frac{N_{t+1}}{N_t} = 1 + n \tag{4}$$

2.2 Social Security

The government is running a pay-as-you-go (PAYG) social security plan. It collects τy_t^t from each young individual at period t and pays $\tau(1+n) y_t^t$ to each old generation, where $\tau \geq 0$. The gross return on PAYG system is given by $(1+g)(1+n)$. Hence, the PAYG is a good substitute for savings if $n+g$ is larger than the interest rate r. There is no government consumption. The government budget constraint is balanced each period.

2.3 Asset Market

The price of owner-occupied houses is given by p_t. Housing asset is completely divisible. For simplicity, I assume away rental market in the benchmark model. It is the extreme case where rental market friction is too high. I will consider the rental market in the model extension part, although it does not affect the main results in the benchmark model.

Both homeowners and investors are subject to the borrowing constraint

$$a_{t+1}^t \geq -(1-\theta) p_t h_{t+1}^t \tag{5}$$

where housing is the only collateral in this economy. The downpayment ratio satisfying $1 > \theta > 0$.

[1] Since I introduce pay-as-you-go social security in the model, the old will receive positive pension benefit. Hence, I can normalize the labor income of the elderly to zero without loss of generality.

2.4 Investor's Problem

The investors' problem is

$$\max_{c_t^t, c_{t+1}^t, h_{t+1}^t, a_{t+1}^t} \ln c_t^t + \beta \ln c_{t+1}^t \tag{6}$$

subject to the following constraint

$$\begin{aligned} c_t^t + a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau) y_t^t \\ c_{t+1}^t &= \tau (1+n) y_{t+1}^{t+1} + R_t a_{t+1}^t + p_{t+1} h_{t+1}^t \\ h_{t+1}^t &\geq 0 \\ a_{t+1}^t &\geq -(1-\theta) p_t h_{t+1}^t \end{aligned}$$

The solution to the investor's problem is given in the appendix. We have the following sufficient conditions for investor's optimal allocations.

Proposition 1 *Given $\tau, g, n, \{R_t, p_t, y_t^t\}_{t=1}^{\infty}$ the optimal decisions of investors are the followings:*

1. *If $R_t = \frac{p_{t+1}}{p_t}$, then*

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta} \left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_t} \right] y_t^t \\ c_{t+1}^t &= \frac{\beta R_t}{1+\beta} \left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_t} \right] y_t^t \\ a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau) y_t^t - c_t^t \\ a_{t+1}^t &> -(1-\theta) p_t h_{t+1}^t \\ h_{t+1}^t &\geq 0 \end{aligned}$$

2. *If $R_t > \frac{p_{t+1}}{p_t}$, then*

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta} \left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_t} \right] y_t^t \\ c_{t+1}^t &= \frac{\beta R_t}{1+\beta} \left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_t} \right] y_t^t \\ a_{t+1}^t &= (1-\tau) y_t^t - c_t^t \\ a_{t+1}^t &> 0 \\ h_{t+1}^t &= 0 \end{aligned}$$

3. *If* $R_t < \frac{p_{t+1}}{p_t}$, *then*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
a_{t+1}^t+(1-\theta)p_t h_{t+1}^t &= 0 \\
p_t h_{t+1}^t &= \frac{\beta\gamma_t(1-\tau)-\tau(1+n)(1+g)}{\theta\gamma_t(1+\beta)}y_t^t \\
h_{t+1}^t &> 0
\end{aligned}$$

where $\gamma_t \equiv \frac{p_{t+1}-(1-\theta)R_t p_t}{\theta p_t}$

2.5 Homeowner's Problem

The homeowner's problem is written as

$$\max_{c_t^t, c_{t+1}^t, h_{t+1}^t, a_{t+1}^t} \ln c_t^t + \beta(1-\zeta)\ln c_{t+1}^t + \beta\zeta\ln h_{t+1}^t \tag{7}$$

subject to the following constraint

$$\begin{aligned}
c_t^t + a_{t+1}^t &= (1-\tau)y_t^t - p_t h_{t+1}^t \\
c_{t+1}^t &= \tau(1+n)y_{t+1}^{t+1} + R_t a_{t+1}^t + p_{t+1}h_{t+1}^t \\
h_{t+1}^t &\geq 0 \\
a_{t+1}^t &\geq -(1-\theta)p_t h_{t+1}^t
\end{aligned}$$

Worker's problem is solved in the appendix. The optimal decision rules are given by the following proposition.

Proposition 2 *Given* $\tau, g, n, \{R_t, p_t, y_t^t\}_{t=1}^{\infty}$ *the optimal decisions of homeowners are the followings*

1. *If homeowner is not borrowing constrained, the optimal allocations are*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta(1-\zeta)R_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\
p_t h_{t+1}^t &= \frac{1}{1-\frac{p_{t+1}}{p_t R_t}}\frac{\beta\zeta}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\
a_{t+1}^t &= (1-\tau)y_t^t - p_t h_{t+1}^t - c_t^t
\end{aligned}$$

2. *If homeowner is borrowing constrained, the optimal allocations are*[2]

$$
\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta(1-\zeta)\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
p_t h_{t+1}^t &= \frac{\Psi_t+\Phi_t}{2\theta\varphi(1+\beta)} \\
a_{t+1}^t &= -(1-\theta)p_t h_{t+1}^t
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_t &= \frac{\lambda_1}{\lambda_2} = \frac{b+\frac{\Psi_t+\Phi_t}{2\theta(1+\beta)}}{\beta(1-\zeta)\left(a-\frac{\Psi_t+\Phi_t}{2\varphi(1+\beta)}\right)} \\
\Psi_t &= a\varphi\beta - b\theta(1+\beta\zeta) \\
\Phi_t &= \sqrt{\Psi_t^2+4ab\theta\beta\zeta\varphi(1+\beta)} \\
\varphi &= \frac{p_{t+1}}{p_t}-(1-\theta)R_t \\
a &= (1-\tau)y_t^t \\
b &= \tau(1+n)(1+g)y_t^t
\end{aligned}
$$

2.6 Competitive Equilibrium

Definition 3 *Given the financial asset $a_1^{1,i}$ and housing stocks $h_1^{1,i}$ for the initial old, the initial interest rate R_0, pension system τ, housing stock $\{H_t\}_{t=1}^{\infty}$, the competitive equilibrium is the sequence of endowment $\left\{y_t^{t,i}\right\}$, prices $\{p_t, R_t\}$, allocations $\left\{c_t^{t,i}, c_{t+1}^{t,i}, h_{t+1}^{t,i}\right\}$ and the initial consumption $c_1^{0,i}$,$i = I, H$ such that*

1. *The allocations solve the problem of investors (6) and homeowners (7)*

2. *The housing market, financial market, and goods market clear*

$$
\begin{aligned}
\int h_{t+1}^{t,i}d\mu^i &= H_{t+1} \\
\int a_{t+1}^{t,i}d\mu^i &= 0 \\
\int c_t^{t,i}d\mu^i+\int c_t^{t-1,i}d\mu^i+p_t\int h_{t+1}^{t,i}d\mu^i &= \int y_t^{t,i}d\mu^i+p_t\int h_t^{t,i}d\mu^i
\end{aligned}
$$

[2] $p_t h_{t+1}^t$ can also be expressed by $\frac{\beta\gamma_t(1-\tau)-\tau(1+n)(1+g)}{\theta\gamma_t(1+\beta)}y_t^t$

In order to characterize the existence and uniqueness of the competitive equilibrium, we study the properties of optimal decision rules.

Lemma 4 *The loan demand (loan supply) of homeowners (investors) is always a strictly decreasing (increasing) function of interest rate.*

Proof. See appendix. ■

We can do the following normalization to detrend the benchmark model into an economy without endowment and population growth. Define

$$
\begin{aligned}
\tilde{y}_t^t &= \frac{y_t^t}{(1+g)^t} \\
\tilde{c}_t^t &= \frac{c_t^t}{(1+g)^t} \\
\tilde{c}_t^{t-1} &= \frac{c_t^{t-1}}{(1+n)(1+g)^t} \\
\tilde{a}_{t+1}^t &= \frac{a_{t+1}^t}{(1+g)^t} \\
\tilde{p}_t &= \frac{p_t}{(1+n)^t (1+g)^t} \\
\tilde{R}_t &= \frac{R_t}{(1+n)(1+g)} \\
\tilde{h}_{t+1}^t &= h_{t+1}^t (1+n)^t
\end{aligned}
$$

Without loss of generality, I assume $g = n = 0$ from now on. Keep in mind that all the variables are detrended.

Lemma 5 *If* $0 < \alpha, \theta < 1$, *there can not be any stationary equilibrium with interest rate* $R^* < 1$

Proof. Suppose that there exists a stationary equilibrium with gross interest $R^* < 1$. First of all, the (detrended) housing price will be constant in the stationary equilibrium. Denote this price level by p^*. Obviously we have $p^* > 0$. Otherwise, workers would purchase infinite amount of houses. The gross return of housing for the investors is 1 ($p_{t+1}/p_t = p^*/p^* = 1$), which is higher than the gross return of savings R^*. From the previous decision rules, the borrowing constraint for both types of households would be binding. The total borrowing of workers is positive and the total borrowing of investors is non-negative. Therefore, the market for loans can not clear. Equilibrium interest rate has to be higher to clear the consumption-loan market and $R^* < 1$ cannot be a equilibrium interest rate. Note that if $\theta = 1$, both investors and households can not borrow in the equilibrium. Any $R^* < 1$ can be the equilibrium interest rate. ■

Proposition 6 *There exists a unique stationary equilibrium. In the stationary equilibrium,*

1. *If* $\theta < \theta_L$*, there are unconstrained homeowners and unconstrained investors holding zero housing asset*
2. *If* $\theta_L < \theta < \theta_H$*, there are borrowing-constrained homeowners and unconstrained investors holding zero housing asset*
3. *If* $\theta > \theta_H$*, then there are constrained homeowners and unconstrained investors holding housing asset*

   where

$$\theta_L = \alpha$$

   and θ_H *is determined by*

$$(1-\alpha)\left[1-\tau-\frac{1}{1+\beta}\right]y-\alpha\left(\frac{1-\theta_H}{\theta_H}\right)\frac{\Psi+\Phi}{2\theta_H(\beta+1)}=0$$

Proof. The optimal demand and supply of loans are continuous. Lemma 4 proves the demand of loans is monotonically decreasing in the interest rate and the supply of loans from investors is also a monotonically increasing function of interest rate. From Lemma 5, there exists a unique stationary equilibrium with $R^* \geq 1$.

Investors will not be borrowing constrained when $R^* \geq 1$. They supply loans in the market. θ will only affect the optimal decision of homeowners, who are the demand side of loan. High θ reduces the borrowing limit of constrained homeowners. If the θ is high enough, the total borrowing from homeowners may become less than the total loan supply from investors. Net interest will converge to zero in order to clear the consumption loan market. Investors would invest extra cash in the housing market. Therefore, there are two threshold θ_L, θ_H and three different cases which we analyze one by one.

1. Unconstrained homeowners and unconstrained investors without housing. In the stationary equilibrium, $y_t^t = y$. $H_t = H$. The equilibrium prices (p_1^*, R_1^*) are determined by

$$\begin{aligned} H &= \alpha\frac{1}{p_1}\frac{R_1}{R_1-1}\frac{\beta\zeta}{1+\beta}\left(1-\tau+\frac{\tau}{R_1}\right)y \\ 0 &= 1-\tau-\frac{1}{1+\beta}\left(1-\tau+\frac{\tau}{R_1}\right)\left(1+\alpha\frac{\beta\zeta R_1}{R_1-1}\right) \end{aligned}$$

The second equation determines a unique $R_1^* > 1$.[3] Hence, housing price can be determined by

$$p_1^* = \alpha \frac{y}{H} \frac{R_1^*}{R_1^* - 1} \frac{\beta\zeta}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1^*}\right)$$

Note that θ can not affect either p_1^* or R_1^*. Now we can solve for the first threshold θ_L when homeowners is borrowing constrained

$$(1-\tau) - \frac{1}{1+\beta}\left(1 - \tau + \frac{\tau}{R_1^*}\right) = \theta_L \frac{R_1^*}{R_1^* - 1} \frac{\beta\zeta}{1+\beta}\left(1 - \tau + \frac{\tau}{R_1^*}\right)$$

Using the market clearing condition for loans, we have $\theta_L = \alpha$. Therefore $\frac{\partial \theta_L}{\partial \alpha} = 1$. The intuition is that more homeowners will increase the equilibrium interest rate. When the interest rate becomes higher, homeowners will reduce the consumption and housing expenditure. They will be borrowing constrained under a stricter borrowing constraint.

2. Constrained homeowners and unconstrained investors without housing. The equilibrium prices (p_2^*, R_2^*) are determined by

$$\begin{aligned} \alpha \frac{1}{p_2} \frac{\Psi + \Phi}{2\theta\varphi(\beta+1)} &= H \\ (1-\alpha)\left[1 - \tau - \frac{1}{1+\beta}\left(1 - \tau + \frac{\tau}{R_2}\right)\right] y - \alpha(1-\theta)\frac{\Psi + \Phi}{2\theta\varphi(\beta+1)} &= 0 \end{aligned}$$

The two equations imply two implicit functions $p_2^*(R_2^*, \theta)$ and $R_2^*(\theta)$. The effect of θ on equilibrium housing price is given by

$$\frac{dp_2^*(R_2^*, \theta)}{d\theta} = \frac{\partial p_2^*(R_2^*, \theta)}{\partial R_2^*} \frac{dR_2^*}{d\theta} + \frac{\partial p_2^*(R_2^*, \theta)}{\partial \theta}$$

On one hand, tighter credit constraint reduces the housing demand, which tends to reduce the price. However, tighter credit constraint also reduces interest rate, which in turns encourages housing consumption. Hence, the total effect is indeterminate.

3. Constrained homeowners and unconstrained investors with empty housing. When $R_3^* = \frac{p_{t+1}}{p_t} = 1$, The market clearing conditions become

$$\begin{aligned} \alpha \frac{1}{p_3} \frac{\Psi + \Phi}{2\theta\varphi(\beta+1)} + (1-\alpha)\frac{I}{p_3} &= H \\ (1-\alpha)\left[(1-\tau)y - \frac{1}{1+\beta} y - I\right] - \alpha(1-\theta)\frac{\Psi + \Phi}{2\theta\varphi(\beta+1)} &= 0 \end{aligned}$$

[3] The other solution $R < 1$ cannot be an equilibrium interest rate.

where I denotes the investor's purchase of housing asset. Combine the two conditions and note that $\varphi = \theta$ when $R = 1$.

$$(1-\alpha)\left[1-\tau-\frac{1}{1+\beta}\right]y+\alpha\frac{\Psi+\Phi}{2\theta(\beta+1)}=p_3H$$

which suggests that p_3^* is independent of θ since $(\Psi+\Phi)/\theta$ does not depend on θ. The total amount of savings is invested in housing asset. The threshold θ_H for investors to hold housing assets is determined by

$$(1-\alpha)\left[1-\tau-\frac{1}{1+\beta}\right]y-\alpha\left(\frac{1-\theta_H}{\theta_H}\right)\frac{\Psi+\Phi}{2\theta_H(\beta+1)}=0$$

It is also true that $\frac{\partial\theta_H}{\partial\alpha}>0$. This is because high α implies fewer loan supply from investors. The collateral constraint has to be higher to clear the loan market.

■

Figure (1) shows the stationary equilibria in three cases. The dotted line is the loan supply of investors. The minimum equilibrium gross interest rate is 1. The solid line is the loan demand from homeowners. As proved in the Lemma 4, it is a decreasing function of interest rate. It is kinked because it consists of two parts. The flatter part is the loan demand of unconstrained homeowners. The steeper part is the loan demand of borrowing-constrained homeowners. The intersection point pins down the equilibrium interest rate.

Proposition 7 *The third case of stationary equilibrium, i.e., constrained homeowners and unconstrained investors with empty housing, is a bubble equilibrium for investors, but not for homeowners.*

Proof. Suppose there is a useless asset called paper. However, its price can be positive in case 3. This is because investor has excess supply of loan in the market. They can invest them in the paper. Since the interest rate is 1 in the equilibrium. The price of paper must grow at the speed of interest rate; otherwise, investors will not hold them. Therefore, the price of paper remains constant. The size of the paper bubble is given by

$$B=(1-\alpha)\left[1-\tau-\frac{1}{1+\beta}\right]y-\alpha\left(\frac{1-\theta}{\theta}\right)\frac{\Psi+\Phi}{2\theta(\beta+1)}>0 \text{ for } \theta>\theta_H$$

This is called pure bubble. However, the bubble can also takes the form housing asset. If investors purchase the housing asset I, then

$$B=(1-\alpha)I$$

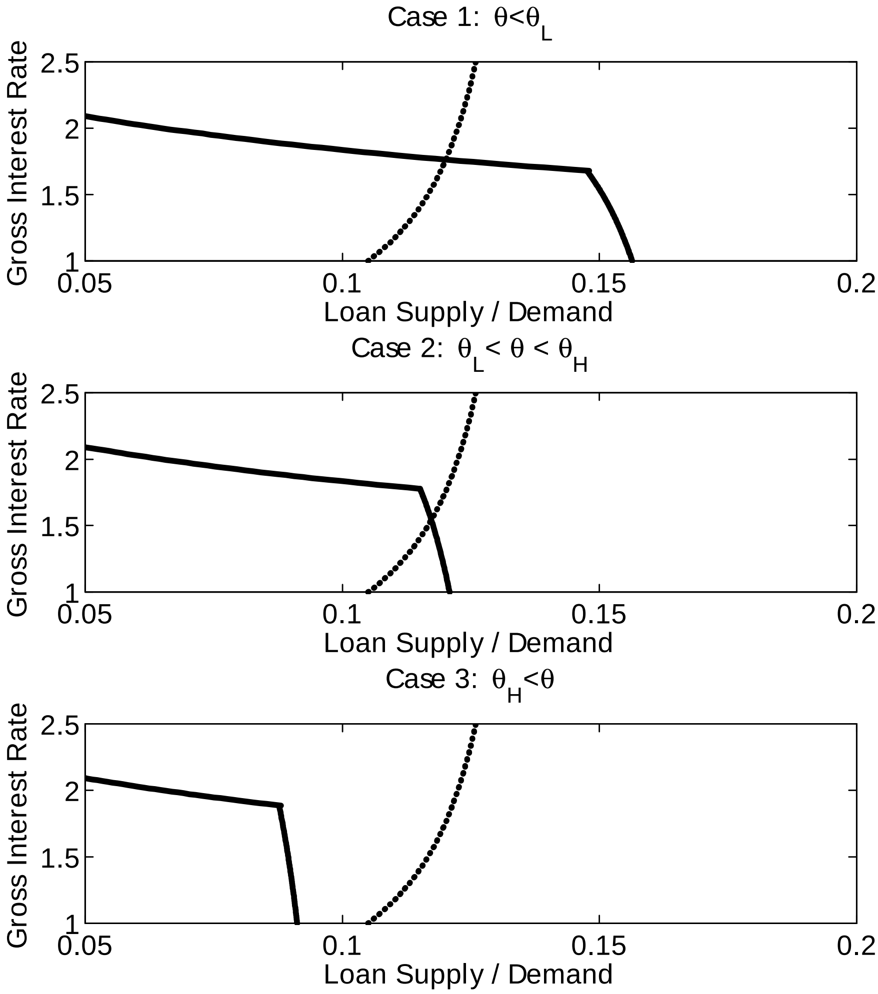

Figure 1: **Three Cases of Stationary Equilibrium.** The fraction of homeowners $\alpha = 0.65$, payroll tax $\tau = 0.2$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$

which means bubble shift from paper market to the housing market. If we define the bubble as the case in which investors hold houses for resale purposes only and not with the expectation of receiving a dividend either in terms of utility or rent, then the case 3 satisfies this definition although here we rule out the rental market. The next question is whether there is bubble for homeowners? The answer is no. First of all, we define the fundamental value of housing asset to homeowners, and then we show that under properly selected interest rate, the housing price is equal to its fundamental value for homeowners in all three cases.

1. Unconstrained homeowners and unconstrained investors without housing. The fundamental value of housing is defined as

$$\begin{aligned} p_t^F &= \frac{p_{t+1} + \frac{\zeta}{1-\zeta}\frac{c_{t+1}^t}{h_{t+1}^t}}{R_t} \\ &= \sum_{\tau=0}^{\infty} \frac{1}{R_t..R_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{t+\tau+1}^{t+\tau}}{h_{t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{R_t..R_{t+T-1}} \end{aligned}$$

Using the first order condition of homeowners

$$p_t^F = \sum_{\tau=0}^{\infty} \frac{1}{R_t..R_{t+\tau}} \left(p_{t+\tau}R_{t+\tau} - p_{t+\tau+1}\right) + \lim_{T\to\infty} p_{t+T} \frac{1}{R_t..R_{t+T-1}}$$

In the stationary equilibrium, $R_1^* > 1, \lim_{T\to\infty} p_1^* \frac{1}{\left(R_1^*\right)^T} = 0$

$$p^F = \sum_{\tau=0}^{\infty} \frac{1}{\left(R_1^*\right)^{\tau+1}} \left(p_1^* R_1^* - p_1^*\right) = p_1^* \sum_{\tau=0}^{\infty} \frac{R_1^* - 1}{\left(R_1^*\right)^{\tau+1}} = p_1^*$$

2. Constrained homeowners and unconstrained investors without housing. The fundamental value of housing can be defined as

$$\begin{aligned} p_t^F &= \frac{p_{t+1} + \frac{\zeta}{1-\zeta}\frac{c_{t+1}^t}{h_{t+1}^t}}{\hat{R}_t} \\ &= \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t..\hat{R}_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{t+\tau+1}^{t+\tau}}{h_{t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t..\hat{R}_{t+T-1}} \end{aligned}$$

where $\hat{R}_t = \theta\frac{\lambda_1}{\lambda_2} + (1-\theta) R_t$. Using the first order condition of constrained homeowners

$$p_t^F = \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t..\hat{R}_{t+\tau}} \frac{\lambda_1 p_t - \lambda_2 p_{t+1} - \mu_1 (1-\theta) p_t}{\lambda_2} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t..\hat{R}_{t+T-1}}$$

In the stationary equilibrium, $\hat{R}_2^* = \theta\frac{\lambda_1}{\lambda_2} + (1-\theta) R_2^* > 1, \lim_{T\to\infty} p_2^* \frac{1}{\left(\hat{R}_2^*\right)^T} = 0$

$$\begin{aligned} p^F &= \sum_{\tau=0}^{\infty} \frac{1}{\left(\hat{R}_2^*\right)^{\tau+1}} \frac{\lambda_1 p_2^* - \lambda_2 p_2^* - (\lambda_1 - \lambda_2 R_2^*)(1-\theta) p_2^*}{\lambda_2} \\ &= p_2^* \sum_{\tau=0}^{\infty} \frac{1}{\left(\hat{R}_2^*\right)^{\tau+1}} \left(\frac{\lambda_1}{\lambda_2}\theta + R_2^*(1-\theta) - 1\right) \\ &= p_2^* \sum_{\tau=0}^{\infty} \frac{\hat{R}_2^* - 1}{\left(\hat{R}_2^*\right)^{\tau+1}} = p_2^* \end{aligned}$$

3. Constrained homeowners and unconstrained investors with empty housing. The fundamental value of housing can be defined as

$$\begin{aligned} p_t^F &= \frac{p_{t+1} + \frac{\zeta}{1-\zeta}\frac{c_{t+1}^t}{h_{t+1}^t}}{\hat{R}_t} \\ &= \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t..\hat{R}_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{t+\tau+1}^{t+\tau}}{h_{t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t..\hat{R}_{t+T-1}} \end{aligned}$$

where $\hat{R}_3 = \theta\frac{\lambda_1}{\lambda_2} + 1 - \theta$. Using the first order condition of homeowners,

$$p_t^F = \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t..\hat{R}_{t+\tau}} \frac{\lambda_1 p_t - \lambda_2 p_{t+1} - (\lambda_1 - \lambda_2 R_t)(1-\theta) p_t}{\lambda_2} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t..\hat{R}_{t+T-1}}$$

In the stationary equilibrium, $p_t = p_3^*, \hat{R}_3^* > 1, \lim_{T\to\infty} p_3^* \frac{1}{\left(\hat{R}_3^*\right)^T} = 0$

$$p^F = p_3^* \sum_{\tau=0}^{\infty} \frac{\hat{R}_3^* - 1}{\left(\hat{R}_3^*\right)^{\tau}} = p_3^*$$

■

The proposition describes the special feature of the equilibrium with bubble, i.e., it is a bubble from investor's point of view only. It may seem strange. However, in order to understand the intuition, let me quote a paragraph from Tirole (1985). Tirole (1985) described two views of money: the fundamentalist view and the bubbly view of money. The fundamentalist view argues that "money is held to finance transactions (or to pay taxes or to satisfy a reserve requirement). To this purpose, money must be a store of value. However, it is not held for speculative purposes as there is no bubble on money." The bubbly view argues that "money is a pure store value à la Samuelson. It does not serve any transaction purpose at least in the long run. This view implies that price of money (bubble) grows at the real rate of interest, and that money is held entirely for speculation."

This paper combines the two views together in one model through different preferences on housing assets. Homeowners derive utility from housing assets. This is similar to the fundamentalist view. Investors treat housing asset as investment tools and store of value. This is same as the bubbly view.

3 Policy Experiment and Data

3.1 Pension Reform

We now consider a policy experiment. Suppose the government remove the PAYG system, i.e., $\tau' = 0$. The removal of PAYG will always increase the supply of loan in the economy. It will reduce the borrowing of unconstrained homeowners. However, for the constrained homeowners, it will increase their loan demand. This is because the borrowing limit is increased by purchasing more housing asset using extra money from tax reduction.

Figure (2) is an illustration of pension reform. The dotted line denotes the demand and supply of loans before the pension reform. The solid line denotes the loan demand and supply after the pension reform. Whether the new equilibrium interest rate will be pushed down towards zero depends on the tightness of collateral constraint. If the borrowing constraint is tight enough, the increase in the loan supply will surpasses the increasing loan demand from constrained homeowners. Therefore, bubble is possible.

Proposition 8 *Suppose the government remove the PAYG system. Bubble will arise if and only if $\theta > \alpha$. If $\tau > \frac{\theta-\alpha}{1-\alpha}$, then housing wealth/GDP ratio must be higher than the pre-reform era.*

Proof. When $\tau = 0$, the total supply of loan by investors becomes $(1-\alpha)\frac{\beta}{1+\beta}y$. The total loan demand from constrained homeowners becomes $\alpha\frac{1-\theta}{\theta}\frac{\beta}{\beta+1}y$. Note that both the supply and demand does not depend on interest rate. Therefore, bubble will arise iff

$$(1-\alpha)\frac{\beta}{1+\beta}y > \alpha\frac{1-\theta}{\theta}\frac{\beta}{\beta+1}y$$

which is equivalent to $\theta > \theta_L = \alpha$. Therefore, if the economy stays at the case 1 of stationary equilibrium in Proposition 6, where both investors and homeowners are unconstrained, then the removal of pension system will not trigger a bubble

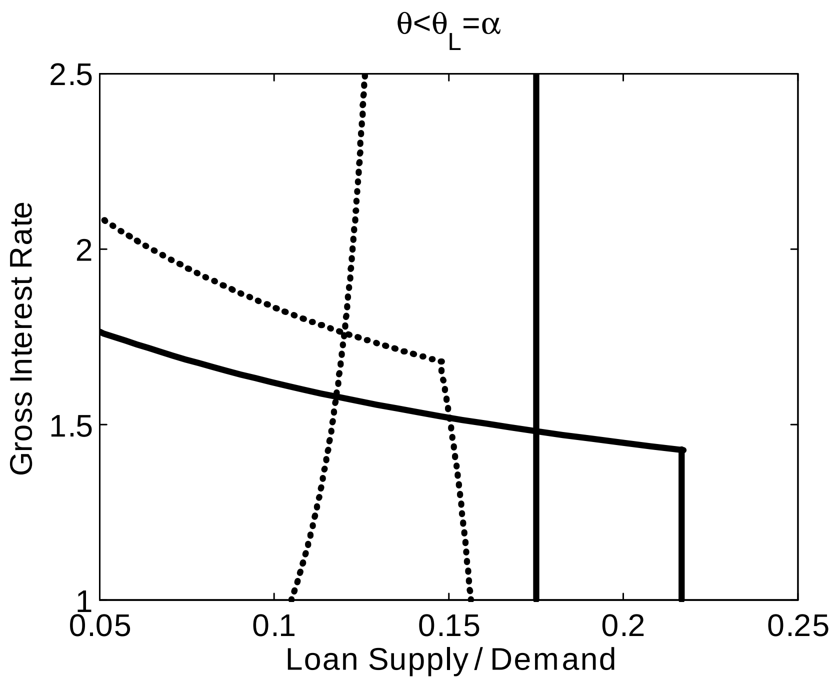

Figure 2: **An Illustration of Pension Reform.** The fraction of homeowners $\alpha = 0.65$, payroll tax $\tau = 0.2$, downpayment ratio $\theta = 0.60$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$. The dotted line denotes the loan demand and supply before the pension reform. The solid line denotes the loan demand and supply after the pension reform.

equilibrium. If the economy stays at case 2 of stationary equilibrium in Proposition 6, we have

$$\frac{p_2 H}{y} = \frac{1-\alpha}{1-\theta}\left[1-\tau-\frac{1}{1+\beta}\left(1-\tau+\frac{\tau}{R_2}\right)\right]$$

In the bubble equilibrium, the housing wealth/GDP ratio is $\frac{\beta}{1+\beta}$. If $\tau > 1-\frac{1-\theta}{1-\alpha} = \frac{\theta-\alpha}{1-\alpha}$, then

$$\frac{p_2 H}{y} < \frac{(1-\alpha)(1-\tau)}{1-\theta}\frac{\beta}{1+\beta} < \frac{\beta}{1+\beta}$$

■

Figure (3) exhibits the policy experiments in all three cases, i.e., $\theta < \theta_L$; $\theta_L < \theta < \theta_H$; $\theta > \theta_H$. According to the Proposition 8, only pension reform in case 2 and case 3 can trigger housing bubble.

3.2 Data

Housing price in China has been increasing strongly over the past decade. The connected solid line in Figure (4) shows that the real land-selling price for the whole country increases at an annual rate 15.7 percent from 2000 to 2009. There is no constant quality official housing price index for China. I also draw the official average commodity building selling price for 35 large cities in China. It shows a slower annual growth rate, 7%, from year 2000 to 2009. Wu et al. (2010) also construct constant quality price index for newly-built private housing in 35 major Chinese cities. According to their estimate, the annual price growth is nearly 10% from year 2000 to 2009. In the meantime, the US has already experienced a housing bubble in 2008.

The unprecedented housing boom in China encourage large increase real estate investment and the boom in the home ownerships. As shown by Figure (5), the share of real estate investment in total fixed investment increases from 13% at 1999 to 20% at 2010. The urban households homeownerships rate estimated from Urban Households Survey shows that China's homeownership rate is nearly 90% in 2010, among the highest in the world.[4] These two facts implies a lot of households

[4]The urban home ownership rate increases from less than 30 percent to 70 percent during 1994-1999, a period when the housing reform takes place. Before the housing reform, it is the state-owned enterprises (SOE) that are responsible for providing employee housing to workers, with a little or no charge for rents. The government liberalizes the housing market in 1994 by selling the public housing to the current employee in state-owned enterprises at heavily subsidized price. Newly employed workers in SOE and workers in the private sectors have to purchase houses that are provided by private real estate developers. The transition into the new housing system ends around 1999, after which no SOE are allowed to provide employee housing to their workers. At the end of year 2010, the home ownership rate of urban households in China is 89.3 percent, which is among the highest in the world. 40.1% of them own privatized houses which previously

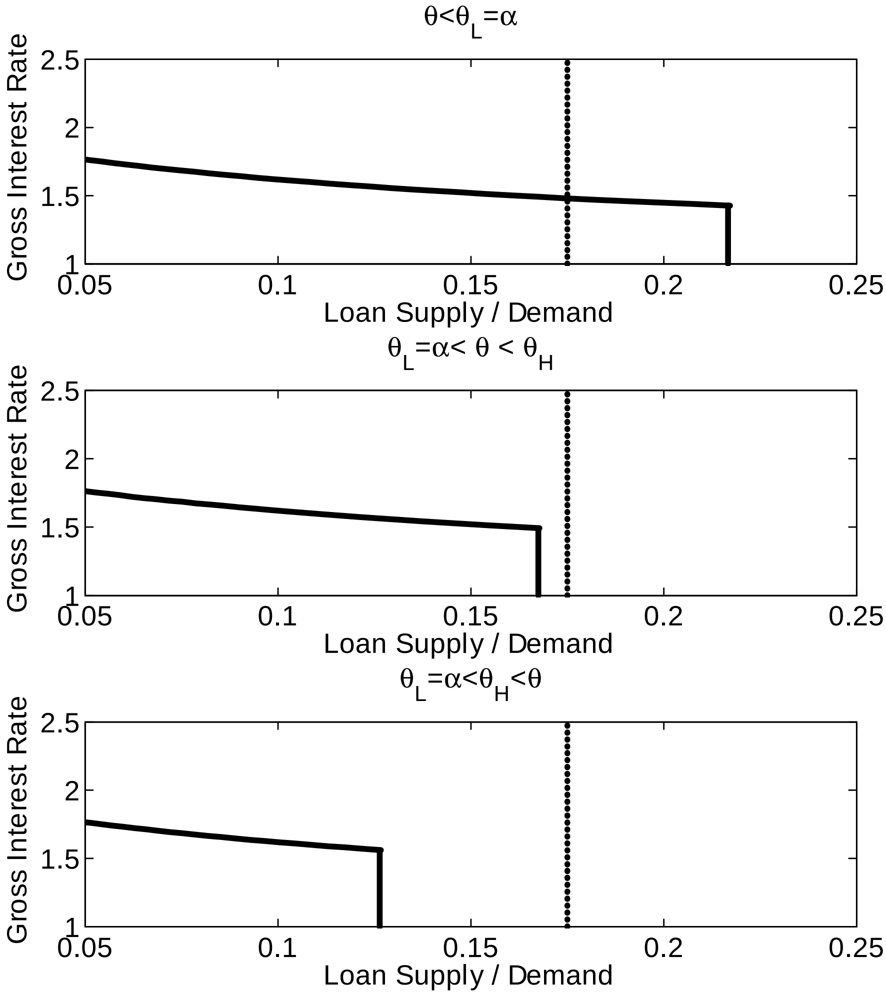

Figure 3: **Stationary Equilibrium After the Pension Reform in Three Cases.** The fraction of homeowners $\alpha = 0.65$, payroll tax $\tau = 0$, downpayment ratio $\theta = 0.60, 0.66, 0.72$, income per capita $y = 1$,discount factor $\beta = 1$, and $\zeta = 0.5$

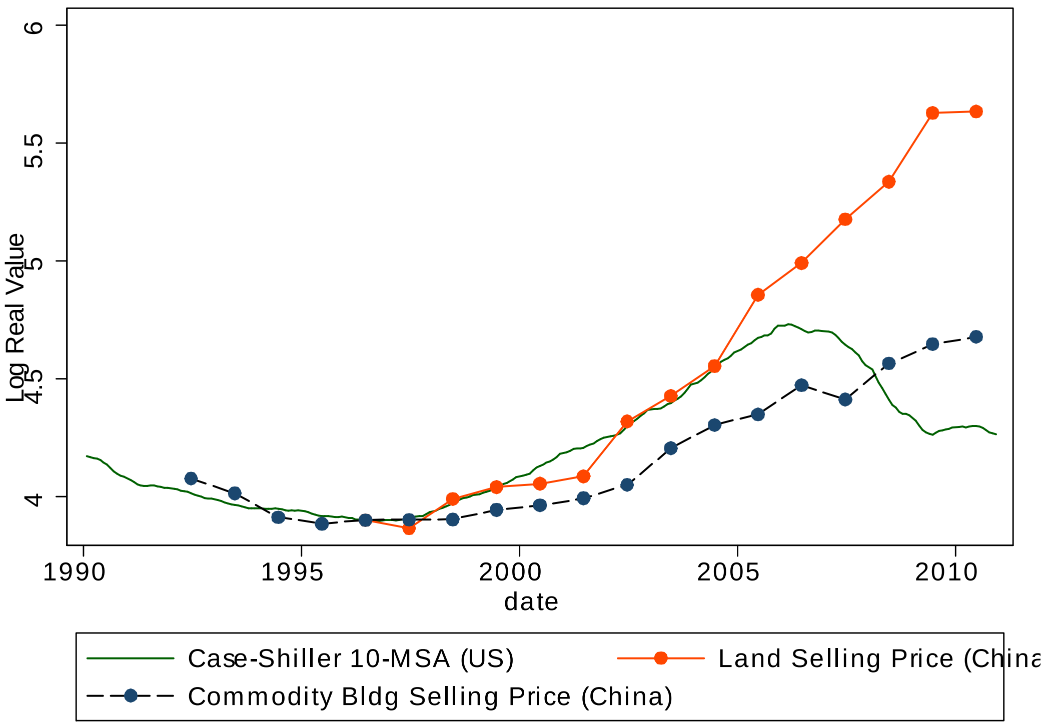

Figure 4: **Housing Price and Land Price: China and US.** The US Housing price index is from S&P/Case-Shiller 10-MSA Index. The land selling price is computed by author using data from China Satistics Year Book. The land price is defined as total value of land purchased divided by total land space purchased. The commodity building sell prices is based on the 35-city average selling price series from National Bureau of Statistics. All series are in log real value deflated by CPI (Urban CPI for Chinese data) and normalized to the same level at year 1996.

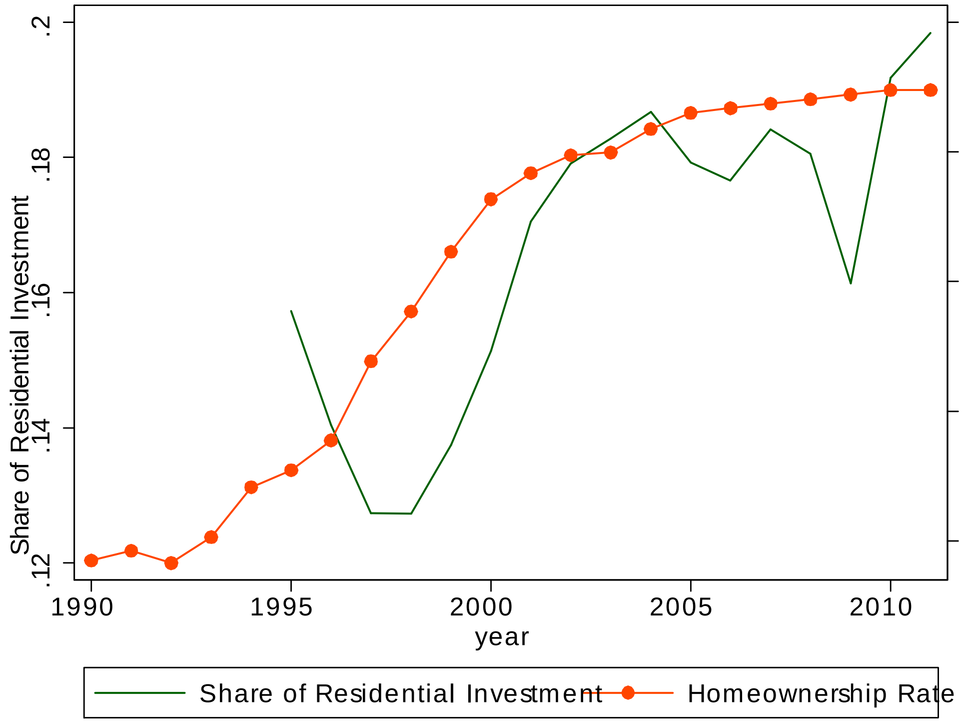

Figure 5: **Urban Residential Investment and Homeownership Rate.** The share of urban residential investment is defined as the real estate development (including land purchase) divided by the total investment in fixed assets in the whole country. Homeownership rate is from China urban households survey.

own more than one apartment. Popular wisdoms claim that there is a housing bubble in China. One feature of the housing bubble is the high vacancy rate in China. A vacant house/apartment is a unit that has been built but is not occupied by anybody. The vacancy rate is defined as all vacant units/all housing units (occupied + vacant). In the US, the gross vacancy rate is The gross vacancy rates are 12.7, 13.0, 13.8, 14.4, 14.5, 14.3 during 2005-2010. In China, according to the China Family Panel Studies 2011, 22% of urban households own more than one apartment. Only 25% of these rich households rent their apartments out. The vacancy rate in year 2010 is 11% according to author's estimate.

According to this paper, the insufficient social security for causing the skyrocketing housing prices because the elderly choose to own empty houses as a store of

are owned by the government or state-owned enterprises. 38% of households have bought houses that are provided at a market price.

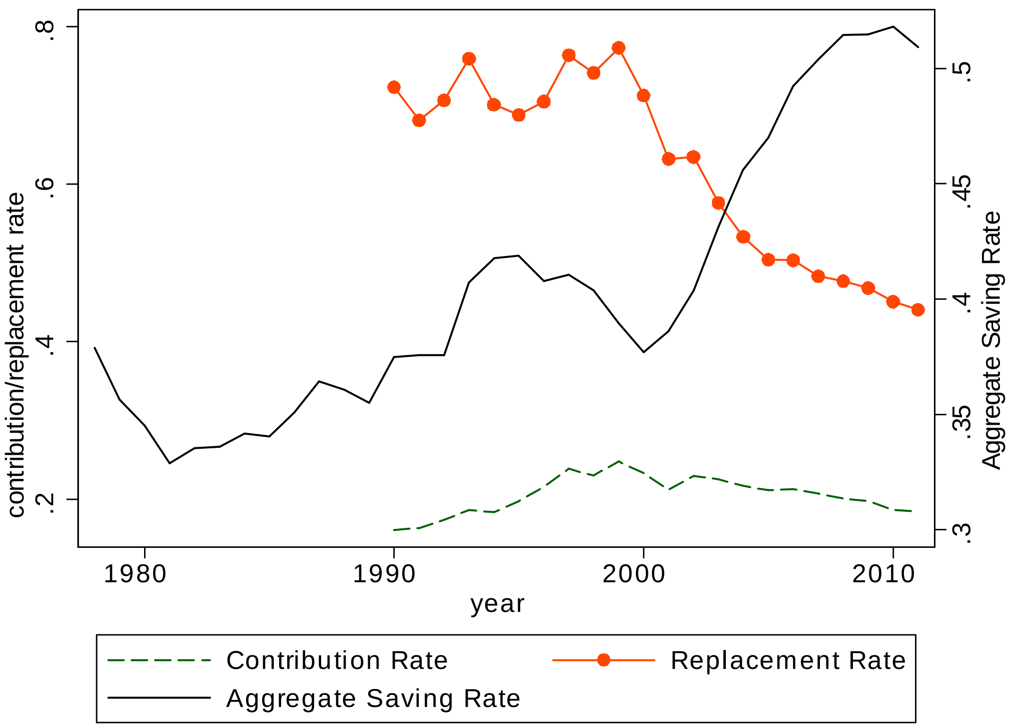

Figure 6: **Social Security Replacement Rate and Contribution Rate.** Data are from China Statistics Year Books 1990-2010. Replacement Rate is defined as the total pension benefit payment per urban retiree covered in the pension system divided by the average urban wage rate. The contribution rate is the total contribution per urban worker covered in the pension system divided by the average urban wage rate.

value to finance their later-life consumption. Figure (6) plots the pension replacement rate and contribution rate in China. The pension reform starts in China from 1999, which changes the traditional pay-as-you-go (PAYG) system into a mixture of PAYG system and fully-funded system. From then on, the replacement rate of pension system decreases from around 75 percent to only 45 percent in 2009. During the same period, the saving rate in China increases by 15 percent, which suggest that Chinese households increase savings partly to compensate the huge decline in the pension payment.

What if those households just invest their pension in terms of stocks and other investment tools? Because the poor development in the financial market, the average return on the stock market over the past twenty years in very low (the average real return on shanghai stock market index is only 2% from year 2000 to 2009) and median households can only access to risk-free bond which delivers almost zero interest actually. Therefore, the missing social security is accompanied by the dynamic inefficiency in China. Figure (7) shows that the real interest rate is China is much lower than the real growth rate, which makes risk-free bond unattractive relative to housing investment for households.

Although there is studies documenting that the capital return in China is very high, however, those projects are not accessible to normal households in China. In fact, Chinese government itself has accumulated great amount of foreign assets and implicitly issue collateralized bonds to Chinese citizens. The low return of government bonds reflects the huge demand for assets/investment tools in China. There are many reasons for causing the dynamic inefficiency problem, e.g., the poor financial development, the absence of social security system, etc.. If the capital account were fully open, Chinese households would have purchased huge amount of assets abroad directly. This dynamic inefficiency creates excess supply of liquidity which allows for speculative bubble.

4 Model Extension

4.1 Rental Market

In this section, I construct a two-period model with rental market. The investor's problem can be written as

$$\max_{c_t^t, c_{t+1}^t, h_{t+1}^t, h_{t+1}^R, a_{t+1}^t} \ln c_t^t + \beta \ln c_{t+1}^t \tag{8}$$

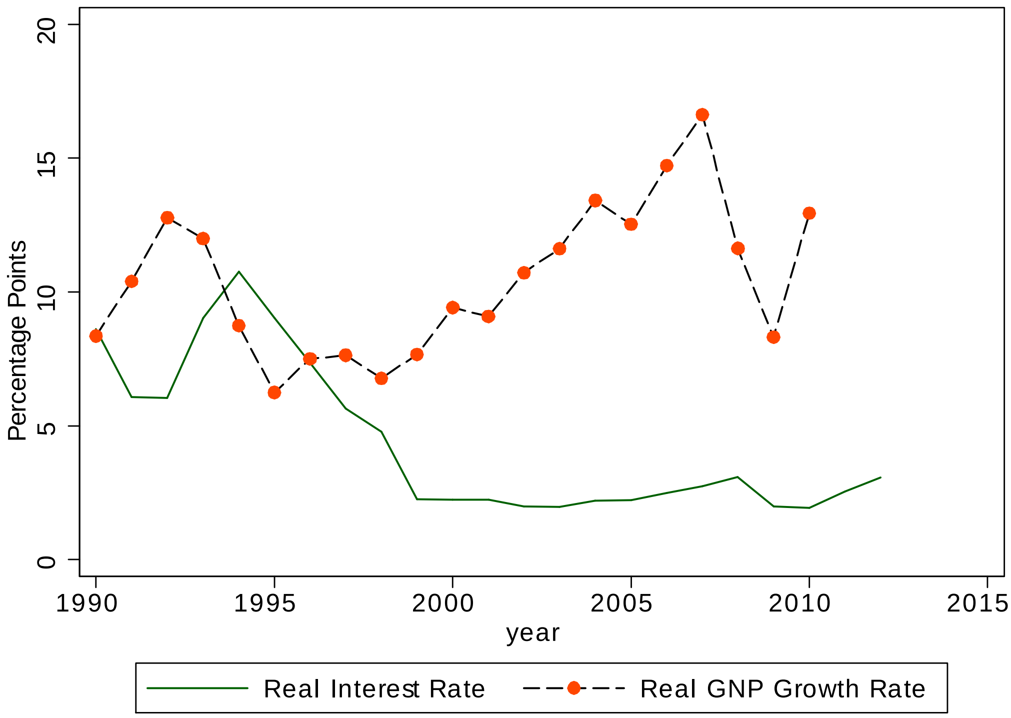

Figure 7: **Dynamic Inefficiency.** The real interest rate is the benchmark interest rate set by the central bank for one-year fixed-term deposit deflated by CPI. The Real GNP annual growth rate is also deflated by CPI.

subject to the following constraint

$$\begin{aligned} c_t^t + a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau) y_t^t + p_t^r h_{t+1}^R \\ c_{t+1}^t &= \tau (1+n) y_{t+1}^{t+1} + R_t a_{t+1}^t + p_{t+1} h_{t+1}^t - \delta_r p_{t+1} h_{t+1}^R \\ h_{t+1}^R &\geq 0 \\ h_{t+1}^t &\geq 0 \\ h_{t+1}^t &\geq h_{t+1}^R \\ a_{t+1}^t &\geq -(1-\theta) p_t h_{t+1}^t \end{aligned}$$

where h_{t+1}^R denotes the amount of houses that are rent out. $\delta_r > 0$ denotes the depreciation rate of rental housing. I will assume frictional rental market in this paper, in the sense that owner-occupied housing will have a smaller depreciation rate than rental housing. This can be interpreted as the moral hazard problem of tenant. I normalize the depreciation rate of owner-occupied housing to zero.

Because of the assumption that investors can not derive utility flow directly from rental housing, the investors will not rent houses in the model. Since all the homeowners are homogenous, they will not provide positive rental housing in the equilibrium. Hence, the homeowners are the only demand side of rental market. The optimization problem

$$\max_{c_t^t, c_{t+1}^t, h_{t+1}^r, h_{t+1}^t, a_{t+1}^t} \ln c_t^t + \beta (1-\zeta) \ln c_{t+1}^t + \beta \zeta \ln \left(h_{t+1}^r + h_{t+1}^t \right) \tag{9}$$

subject to the following constraint

$$\begin{aligned} c_t^t + a_{t+1}^t &= (1-\tau) y_t^t - p_t h_{t+1}^t - p_t^r h_{t+1}^r \\ c_{t+1}^t &= \tau (1+n) y_{t+1}^{t+1} + R_t a_{t+1}^t + p_{t+1} h_{t+1}^t \\ h_{t+1}^t &\geq 0 \\ h_{t+1}^r &\geq 0 \\ a_{t+1}^t &\geq -(1-\theta) p_t h_{t+1}^t \end{aligned}$$

where h_{t+1}^r is the rental housing hold by homeowners.

Definition 9 *Given the financial asset $a_1^{1,i}$ and housing stocks $h_1^{1,i}$ for the initial old, the initial interest rate R_0, pension system τ, housing stock $\{H_t\}_{t=1}^{\infty}$, the competitive equilibrium is the sequence of endowment $\{y_t^{t,i}\}$, prices $\{p_t, R_t, p_t^r\}$, allocations $\left\{c_t^{t,i}, c_{t+1}^{t,i}, h_{t+1}^{t,i}, h_{t+1}^{R,i}, h_{t+1}^{r,i}\right\}$ and the initial consumption $c_1^{0,i}$,$i = I, H$ such that*

1. *The allocations solve the problem of investors (8) and homeowners (9)*

2. *The housing market, financial market, rental market, and goods market clear*

$$\begin{aligned}
\int h_{t+1}^{t,i} d\mu^i &= H_{t+1} \\
\int a_{t+1}^{t,i} d\mu^i &= 0 \\
\int h_{t+1}^{R,i} d\mu^i &= \int h_{t+1}^{r,i} d\mu^i \\
\int c_t^{t,i} d\mu^i + \int c_t^{t-1,i} d\mu^i + p_t \int h_{t+1}^{t,i} d\mu^i &= \int y_t^{t,i} d\mu^i + p_t \int h_t^{t,i} d\mu^i
\end{aligned}$$

Lemma 10 *Unconstrained homeowners will not rent houses in the stationary equilibrium.*

Proof. Suppose homeowners is not borrowing constrained. The Focs of homeowners become

$$\begin{aligned}
-\lambda_1 + \lambda_2 R_t &= 0 \\
\frac{\beta\zeta}{h_{t+1}^r + h_{t+1}^t} - \lambda_1 p_t + \lambda_2 p_{t+1} + v_1 &= 0 \\
\frac{\beta\zeta}{h_{t+1}^r + h_{t+1}^t} - \lambda_1 p_t^r + \nu_2 &= 0
\end{aligned}$$

Suppose $h_{t+1}^r > 0$, then $v_2 = 0$,

$$\lambda_1 p_t^r - \lambda_1 p_t + \lambda_2 p_{t+1} + v_1 = 0$$

Therefore

$$R_t = \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} + \frac{v_1}{\lambda_1}}{p_t - p_t^r} \geq \frac{p_{t+1}}{p_t - p_t^r} > \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r}$$

However, from investor's problem, we know that investor will not hold housing asset. Hence, $h_{t+1}^r = 0$ if homeowners are borrowing constrained. ■

Proposition 11 *If $\theta > \alpha$ and the rental market friction δ_r is large enough, there exists a bubble equilibrium after pension reform. More precisely,*

1. *If $\delta_r > \theta\zeta$, then homeowners will not rent houses and investors will hold empty houses. There exists a housing bubble for investors.*

2. *If $\theta\zeta \geq \delta_r \geq \alpha\zeta$, then homeowners will rent some houses and investors will still hold some empty houses. There exists a housing bubble for investors.*

3. *If $\delta_r < \alpha\zeta$, investors will rent all the houses to homeowners and there is no housing bubble.*

Proof. Since our point of interest is to see whether frictional rental market can resolve the problem of vacant houses and prevent the rise of bubbles, I assume $\theta > \theta_L = \alpha$, such that there exists a bubble after the pension reform when $\delta_r = 0$. From Lemma 10, we know that investors will hold housing asset only if homeowners are borrowing constrained. Therefore, I only consider the equilibrium where homeowners are borrowing constrained and investors lend to homeowners.

When there is a housing bubble, $R = 1$. For the investors to be indifferent between holding empty houses and renting them out, it must be $p^r = \delta_r p$. For the homeowners to rent positive amount of housing, the necessary condition is

$$R < \frac{p}{p - p^r} < \frac{\lambda_1}{\lambda_2} = \gamma = \frac{\theta}{\theta - \delta_r}$$

The demand function for rental housing is given by

$$\begin{aligned} p^r h^r &= y - c - \theta p h \\ &= \frac{\beta}{1+\beta} y - \frac{\theta}{\theta - \delta_r} \frac{\beta (1 - \zeta)}{1 + \beta} y \end{aligned}$$

If $\delta_r > \theta\zeta$, then $p^r h^r < 0$. Homeowners demand zero rental housing if the rental market friction $\delta_r > \theta\zeta$.

Housing bubble can still exist even with active rental market. The financial market clearing condition

$$\int a^I d\mu^i = (1 - \alpha) \left(1 - \frac{1}{1+\beta}\right) y + p^r \int h^R d\mu^i - p \int h^I d\mu^i$$

where

$$h^I \geq h^R$$

Let's suppose $h^I = h^R + h^B$, where h^B is the amount of vacant houses.

$$\int a^I d\mu^i = (1 - \alpha) \frac{\beta}{1+\beta} y + (p^r - p) \int h^R d\mu^i - p \int h^B d\mu^i$$

The loan demand function can be written as

$$\int a^H d\mu^i = -\alpha \frac{1 - \theta}{\theta - \delta_r} \frac{\beta (1 - \zeta)}{1 + \beta} y$$

The loan market clearing condition requires that

$$\int a^I d\mu^i + \int a^H d\mu^i = 0$$

Hence

$$
\begin{aligned}
& p\int h^{B}d\mu^{i} \\
= & \ (1-\alpha)\frac{\beta}{1+\beta}y-(p-p^{r})\int h_{t+1}^{R}-\alpha\frac{1-\theta}{\theta-\delta_{r}}\frac{\beta(1-\zeta)}{1+\beta}y \\
= & \ (1-\alpha)\frac{\beta}{1+\beta}y-\alpha\frac{p-p^{r}}{p^{r}}\left(\frac{\beta}{1+\beta}y-\frac{\theta}{\theta-\delta_{r}}\frac{\beta(1-\zeta)}{1+\beta}y\right)-\alpha\frac{1-\theta}{\theta-\delta_{r}}\frac{\beta(1-\zeta)}{1+\beta}y \\
= & \ \frac{\beta}{1+\beta}y\left(1-\frac{\alpha\zeta}{\delta_{r}}\right)
\end{aligned}
$$

If $\delta_r > \alpha\zeta$, then $p\int h^B d\mu^i > 0$, i.e., there are empty housing held by investors even through the rental market is active. ■

5 Conclusion

This paper study an endowment economy inhabited by overlapping generations of homeowners and investors, with the only difference being that homeowners derive utility from housing services while investors do not. Tight collateral constraint limits the borrowing capacity of homeowners and drives down the equilibrium interest rate level to the housing price growth rate, which makes housing attractive as a store of value for investors. As long as the rental market friction is large enough, the investors will hold positive amount of vacant houses in the equilibrium. Housing bubble arises in an equilibrium in which investors hold houses for resale purposes only and not with the expectation of receiving a dividend either in terms of utility or rent. I apply the model to China, in which the caveat of housing bubble can be attributed to the rapid decline in the replacement rate of the pension system.

This paper also shed some lights on the issue of government debt. If the government lend to much if the borrowing constraint is high, it will only drive the interest too low and investors will start to accumulate too much bubble asset. The Chinese government has issued a rescue package after the financial crisis in the US 2008, which triggered a further wave of housing price boom in China.

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6 Appendix

6.1 Benchmark Model

6.1.1 Investor's problem

The Lagrangian function is

$$\begin{aligned}
L &= \ln c_t^t + \beta \ln c_{t+1}^t \\
&+\lambda_1 \left[(1-\tau)\, y_t^t - c_t^t - a_{t+1}^t - p_t h_{t+1}^t \right] \\
&+\lambda_2 \left[\tau (1+n)\, y_{t+1}^{t+1} + R_t a_{t+1}^t + p_{t+1} h_{t+1}^t - c_{t+1}^t \right] \\
&+\mu_1 \left[a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t \right] \\
&+\nu_1 h_{t+1}^t
\end{aligned}$$

The FOCs become

$$\begin{aligned}
c_t^t &: \frac{1}{c_t^t} - \lambda_1 = 0 \\
c_{t+1}^t &: \frac{\beta}{c_{t+1}^t} - \lambda_2 = 0 \\
a_{t+1}^t &: -\lambda_1 + \lambda_2 R_t + \mu_1 = 0 \\
h_{t+1}^t &: -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t + \nu_1 = 0
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &\geq 0, \text{ if } a_{t+1}^t + (1-\theta) p_t h_{t+1}^t > 0, \text{ then } \mu_1 = 0 \\
\nu_1 &\geq 0, \text{ if } h_{t+1}^t > 0, \text{ then } \nu_1 = 0
\end{aligned}$$

The life-time budget constraint for the investors is

$$c_t^t + \frac{c_{t+1}^t}{R_t} = (1-\tau) y_t^t + \frac{\tau (1+n) y_{t+1}^{t+1}}{R_t} + \left(\frac{p_{t+1}}{R_t} - p_t \right) h_{t+1}^t$$

1. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t > 0$, i.e., the borrowing constraint of the investors is not binding; $h_{t+1}^t > 0$,i.e., the unconstrained investors hold positive amount of housing. Therefore $\mu_1 = \nu_1 = 0$. Plug them into the FOCs

$$\begin{aligned}
-\lambda_1 + \lambda_2 R_t &= 0 \\
-\lambda_1 p_t + \lambda_2 p_{t+1} &= 0
\end{aligned}$$

The following equality holds

$$R_t = \frac{p_{t+1}}{p_t}$$

and the optimal consumption rules are

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta} \left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_t} \right] y_t^t \\
c_{t+1}^t &= \frac{\beta R_t}{1+\beta} \left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_t} \right] y_t^t
\end{aligned}$$

The allocation between the private IOUs and housing asset are indeterminate. The total saving is determined by

$$a_{t+1}^t + p_t h_{t+1}^t = (1-\tau_t) y_t^t - c_t^t$$

2. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t > 0$, i.e., the borrowing constraint of investor is not binding; $h_{t+1}^t = 0$, i.e., the investor holds zero amount of housing. Therefore, $\mu_1 = 0, \nu_1 \geq 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \nu_1 &= 0 \end{aligned}$$

Hence,

$$R_t \geq \frac{p_{t+1}}{p_t}$$

(a) If $\nu_1 = 0$, then we go back to case 1

(b) If $\nu_1 > 0$, then $R_t > \frac{p_{t+1}}{p_t}$. The purchase of housing are less attractive than the lending to the others.

$$\begin{aligned} a_{t+1}^t &= (1-\tau) y_t^t - c_t^t \\ h_{t+1}^t &= 0 \end{aligned}$$

3. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t = 0$, i.e., the borrowing constraint of the investors is binding; $h_{t+1}^t > 0$, i.e., the constrained investors hold positive amount of housing. Therefore, $\mu_1 \geq 0, \nu_1 = 0$.

(a) If $\mu_1 = v_1 = 0$, we go back to case 1. If $\mu_1 > 0, \nu_1 = 0$, then

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &> R_t \\ \frac{\lambda_1}{\lambda_2} &> \frac{p_{t+1}}{p_t} \\ \frac{\lambda_1}{\lambda_2} &= \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t} \end{aligned}$$

Suppose $\frac{p_{t+1}}{p_t} < R_t < \frac{\lambda_1}{\lambda_2}$, then $R_t < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_t p_t}{\theta p_t} < \frac{p_{t+1}-(1-\theta)p_{t+1}}{\theta p_t} = \frac{p_{t+1}}{p_t}$,a contradiction! Therefore,

$$R_t < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t}$$

Let $\gamma_t \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_t p_t}{\theta p_t}$. Rewrite the budget constraints as

$$\begin{aligned} c_t^t &= (1-\tau) y_t^t - \theta p_t h_{t+1}^t \\ c_{t+1}^t &= \tau (1+n) (1+g) y_t^t + \theta \gamma_t p_t h_{t+1}^t \end{aligned}$$

Solve for $p_t h_{t+1}^t$

$$p_t h_{t+1}^t = \frac{\beta\gamma_t (1-\tau) - \tau (1+n)(1+g)}{\theta\gamma_t (1+\beta)} y_t^t$$

Therefore

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{(1+n)(1+g)}{\gamma_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[1-\tau+\frac{(1+n)(1+g)}{\gamma_t}\right] y_t^t \\ a_{t+1}^t &= -(1-\theta) p_t h_{t+1}^t \\ p_t h_{t+1}^t &= \frac{\beta\gamma_t (1-\tau) - \tau (1+n)(1+g)}{\theta\gamma_t (1+\beta)} y_t^t \end{aligned}$$

4. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t = 0$,i.e., the borrowing constraint of the investors is binding; $h_{t+1}^t = 0$,i.e., the investors hold positive amount of housing

$$\begin{aligned} c_t^t &= (1-\tau) y_t^t \\ c_{t+1}^t &= \tau (1+n)(1+g) y_t^t \end{aligned}$$

Then $\mu_1, v_1 \geq 0$.

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t + \nu_1 &= 0 \end{aligned}$$

(a) If $\mu_1, \nu_1 > 0$, either investors have too little endowment when they are young and do not want to save

$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1} - R_t (1-\theta) p_t}{\theta p_t} > \frac{p_{t+1}}{p_t} > R_t$$

or investors' borrowing cost is too large

$$\frac{\lambda_1}{\lambda_2} > R_t > \frac{p_{t+1}}{p_t} > \frac{p_{t+1} - R_t (1-\theta) p_t}{\theta p_t}$$

In this article, I assume the young has enough endowment and wants to save. Therefore, I rule out the case $\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1} - R_t(1-\theta)p_t}{\theta p_t} > \frac{p_{t+1}}{p_t} > R_t$.

(b) If $\mu_1 > 0, v_1 = 0$, We go back to Case 3

(c) If $\mu_1 = 0, \nu_1 > 0$, We go back to Case 2

(d) If $\mu_1 = 0, v_1 = 0$, We go back to Case 1

6.1.2 Homeowner's Problem

The Lagrangian function is

$$\begin{aligned}
&L = \ln c_t^t + \beta\zeta \ln\left(h_{t+1}^t\right) + \beta(1-\zeta)\ln c_{t+1}^t \\
&+\lambda_1\left[(1-\tau)y_t - p_t h_{t+1}^t - c_t^t - a_{t+1}^t\right] \\
&+\lambda_2\left[\tau(1+n)(1+g)y_t + R_t a_{t+1}^t + p_{t+1}h_{t+1}^t - c_{t+1}^t\right] \\
&+\mu_1\left[a_{t+1}^t + (1-\theta)p_t h_{t+1}^t\right]
\end{aligned}$$

The FOCs become

$$\begin{aligned}
c_t^t &: \frac{1}{c_t^t} - \lambda_1 = 0 \\
c_{t+1}^t &: \frac{\beta(1-\zeta)}{c_{t+1}^t} - \lambda_2 = 0 \\
a_{t+1}^t &: -\lambda_1 + \lambda_2 R_t + \mu_1 = 0 \\
h_{t+1}^t &: \frac{\beta\zeta}{h_{t+1}^t} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1(1-\theta)p_t = 0
\end{aligned}$$

where

$$\mu_1 \geq 0, \text{ if } a_{t+1}^t + (1-\theta)p_t h_{t+1}^t > 0, \text{ then } \mu_1 = 0$$

and the life-time budget constraint is given by

$$c_t^t + \frac{c_{t+1}^t}{R_t} + \left(p_t - \frac{p_{t+1}}{R_t}\right)h_{t+1}^t = (1-\tau)y_t^t + \frac{\tau(1+n)y_{t+1}^{t+1}}{R_t} \tag{10}$$

1. $a_{t+1}^t + (1-\theta)p_t h_{t+1}^t > 0$, i.e., the borrowing constraint of the homeowners is not binding. Therefore, $\mu_1 = 0$. Hence,

$$\frac{\lambda_1}{\lambda_2} = R_t = \frac{p_{t+1} + \frac{\zeta}{1-\zeta}\frac{c_{t+1}^t}{h_{t+1}^t}}{p_t} \tag{11}$$

The optimal decision rules are

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta(1-\zeta)R_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\
p_t h_{t+1}^t &= \frac{1}{1-\frac{p_{t+1}}{p_t R_t}}\frac{\beta\zeta}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\
a_{t+1}^t &= (1-\tau)y_t^t - p_t h_{t+1}^t - c_t^t
\end{aligned}$$

2. $a_{t+1}^{t}+(1-\theta)\,p_t h_{t+1}^{t}=0$, i.e., the borrowing constraint of the homeowners is binding. Therefore, $\mu_1 \geq 0$

   (a) If $\mu_1=0$, then we go back to Case 1.

   (b) If $\mu_1>0$

$$\begin{aligned} -\lambda_1+\lambda_2 R_t+\mu_1 &= 0 \\ \frac{\beta\zeta}{h_{t+1}^{t}}-\lambda_1 p_t+\lambda_2 p_{t+1}+\mu_1(1-\theta)\,p_t &= 0 \end{aligned}$$

Hence, the condition for R_t is given by

$$R_t<\frac{\lambda_1}{\lambda_2}$$

Let $\frac{\lambda_1}{\lambda_2}\equiv\gamma_t$, then from the budget constraint

$$c_t^t=(1-\tau)\,y_t^t-\theta p_t h_{t+1}^{t}$$

and

$$c_{t+1}^{t}=\tau(1+n)(1+g)\,y_t^t+(p_{t+1}-R_t(1-\theta)\,p_t)\,h_{t+1}^{t}$$

From the FOC w.r.t. h_{t+1}^{t}, we have

$$\frac{\beta\zeta}{h_{t+1}^{t}}-\lambda_1\theta p_t+\lambda_2(p_{t+1}-R_t(1-\theta)\,p_t)=0$$

Use the expression for λ_1,λ_2, we have

$$\begin{aligned} 1 &= \lambda_1(1-\tau)\,y_t^t-\lambda_1\theta p_t h_{t+1}^{t} \\ \beta(1-\zeta) &= \lambda_2\tau(1+n)(1+g)\,y_t^t+\lambda_2(p_{t+1}-R_t(1-\theta)\,p_t)\,h_{t+1}^{t} \\ \beta\zeta &= \lambda_1\theta p_t h_{t+1}^{t}-\lambda_2(p_{t+1}-R_t(1-\theta)\,p_t)\,h_{t+1}^{t} \end{aligned}$$

Therefore

$$1+\beta=\lambda_1(1-\tau)\,y_t^t+\lambda_2\tau(1+n)(1+g)\,y_t^t$$

Note that

$$\begin{aligned} 1+\beta=&\frac{(1-\tau)\,y_t^t}{(1-\tau)\,y_t^t-\theta p_t h_{t+1}^{t}} \\ &+\beta(1-\zeta)\frac{\tau(1+n)(1+g)\,y_t^t}{\tau(1+n)(1+g)\,y_t^t+(p_{t+1}-R_t(1-\theta)\,p_t)\,h_{t+1}^{t}} \end{aligned}$$

This is a quadratic equation for $p_t h_{t+1}^t$. Let

$$\begin{aligned} x &= p_t h_{t+1}^t \\ \varphi &= \frac{p_{t+1}}{p_t} - (1-\theta) R_t \\ a &= (1-\tau) y_t^t \\ b &= \tau (1+n)(1+g) y_t^t \end{aligned}$$

Then

$$1+\beta = \frac{a}{a-\theta x} + \frac{\beta(1-\zeta) b}{b+\varphi x}$$

It has a unique positive solution

$$p_t h_{t+1}^t = x = \frac{\Psi_t + \Phi_t}{2\theta\varphi(1+\beta)}$$

where $\Psi_t = a\varphi\beta - b\theta(1+\beta\zeta)$, $\Phi_t = \sqrt{\Psi_t^2 + 4ab\theta\beta\zeta\varphi(\beta+1)}$. We can define γ_t

$$\gamma_t = \frac{\lambda_1}{\lambda_2} = \frac{c_{t+1}^t}{\beta(1-\zeta) c_t^t} = \frac{b+\varphi x}{\beta(1-\zeta)(a-\theta x)}$$

and

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta(1-\zeta)\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right] y_t^t \\ p_t h_{t+1}^t &= \frac{\beta\gamma_t(1-\tau) - \tau(1+n)(1+g)}{\theta\gamma_t(1+\beta)} y_t^t \end{aligned}$$

6.1.3 Proof of Lemma 4

We start first by looking the saving function of the unconstrained homeowner/investor. It is obvious to see the saving function of the unconstrained homeowner/investor is a decreasing function of interest rate. When the investor is borrowing constrained, higher interest rate reduces γ_t and implies fewer housing bought. Hence, the amount investor can borrowing is a decreasing function of interest rate. When the homeowner is borrowing constrained, the loan demand function becomes complicated. Differentiate $p_t h_{t+1}^t$ directly w.r.t. φ

$$\begin{aligned} p_t h_{t+1}^t &= \frac{\Psi_t + \sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)}}{2\theta\varphi\left(\beta+1\right)} \\ &= \frac{4ab\beta\zeta\theta\varphi\left(\beta+1\right)}{2\theta\varphi\left(\beta+1\right)\left(\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)} - \Psi_t\right)} \\ &= 2ab\beta\zeta\frac{1}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)} - \Psi_t} \end{aligned}$$

Then

$$\frac{\partial p_t h_{t+1}^t}{\partial\varphi} = -2ab\beta\zeta\left(\frac{1}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)} - \Psi_t}\right)^2$$
$$\times\left(\frac{d}{d\varphi}\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)} - \frac{d}{d\varphi}\Psi_t\right)$$

Note that $\Psi_t = a\varphi\beta - b\theta\left(1+\beta\zeta\right)$

$$\frac{d}{d\varphi}\Psi_t = a\beta$$

and also

$$\begin{aligned} &\frac{d}{d\varphi}\sqrt{\Psi_t^2 + 4ab\theta\beta\zeta\varphi\left(\beta+1\right)} \\ &= \frac{a\beta\Psi_t + 2ab\beta\zeta\theta\left(\beta+1\right)}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)}} \\ &= a\beta\frac{\left(a\varphi\beta - b\theta\left(1+\beta\zeta\right)\right) + 2b\zeta\theta\left(\beta+1\right)}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)}} \\ &< a\beta \end{aligned}$$

because

$$\begin{aligned} &\left(\left(a\varphi\beta - b\theta\left(1+\beta\zeta\right)\right) + 2b\zeta\theta\left(\beta+1\right)\right)^2 - \left(\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)\right) \\ = &-4b^2\zeta\theta^2\left(\beta+1\right)\left(1-\zeta\right) < 0 \end{aligned}$$

Therefore,

$$\frac{\partial p_t h_{t+1}^t}{\partial\varphi} > 0, \frac{\partial p_t h_{t+1}^t}{\partial R} < 0$$

The loan demand of constrained homeowner is an increasing function of interest rate.

6.2 Model Extension

6.2.1 Investor's Problem

The Lagrangian function is

$$\begin{aligned}
&L = \ln c_t^t + \beta \ln c_{t+1}^t \\
&+\lambda_1 \left[(1-\tau) y_t^t + p_t^r h_{t+1}^R - c_t^t - a_{t+1}^t - p_t h_{t+1}^t\right] \\
&+\lambda_2 \left[\tau (1+n) y_{t+1}^{t+1} + R_t a_{t+1}^t + p_{t+1} h_{t+1}^t - \delta_r p_{t+1} h_{t+1}^R - c_{t+1}^t\right] \\
&+\mu_1 \left[a_{t+1}^t + (1-\theta) p_t h_{t+1}^t\right] \\
&+\mu_2 \left[h_{t+1}^t - h_{t+1}^R\right] \\
&+\nu_1 h_{t+1}^t \\
&+\nu_2 h_{t+1}^R
\end{aligned}$$

The FOCs become

$$\begin{aligned}
c_t^t &: \frac{1}{c_t^t} - \lambda_1 = 0 \\
c_{t+1}^t &: \frac{\beta}{c_{t+1}^t} - \lambda_2 = 0 \\
a_{t+1}^t &: -\lambda_1 + \lambda_2 R_t + \mu_1 = 0 \\
h_{t+1}^t &: -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t + \mu_2 + \nu_1 = 0 \\
h_{t+1}^R &: \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} - \mu_2 + \nu_2 = 0
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &\geq 0, \text{ if } a_{t+1}^t + (1-\theta) p_t h_{t+1}^t > 0, \text{ then } \mu_1 = 0 \\
\mu_2 &\geq 0, \text{ if } h_{t+1}^t - h_{t+1}^R > 0, \text{ then } \mu_2 = 0 \\
\nu_1 &\geq 0, \text{ if } h_{t+1}^t > 0, \text{ then } \nu_1 = 0 \\
\nu_2 &\geq 0, \text{ if } h_{t+1}^R > 0, \text{ then } \nu_2 = 0
\end{aligned}$$

The life-time budget constraint for the investors is

$$c_t^t + \frac{c_{t+1}^t}{R_t} = (1-\tau) y_t^t + \frac{\tau (1+n) y_{t+1}^{t+1}}{R_t} + \left(\frac{p_{t+1}}{R_t} - p_t\right) h_{t+1}^t + \left(p_t^r - \frac{\delta_r p_{t+1}}{R_t}\right) h_{t+1}^R$$

1. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t > 0$, $h_{t+1}^t - h_{t+1}^R > 0$, $h_{t+1}^t > 0, h_{t+1}^R > 0$, Then $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. Plug them into the FOCs

$$\begin{aligned}
-\lambda_1 + \lambda_2 R_t &= 0 \\
-\lambda_1 p_t + \lambda_2 p_{t+1} &= 0 \\
\lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} &= 0
\end{aligned}$$

The following equality holds

$$R_t = \frac{p_{t+1}}{p_t} = \frac{\delta_r p_{t+1}}{p_t^r} = \frac{(1-\delta_r)\, p_{t+1}}{p_t - p_t^r}$$

and the optimal consumption rules are

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta R_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right] y_t^t \end{aligned}$$

and the private IOUs, housing asset, and rental housing are jointly determined by

$$a_{t+1}^t + p_t h_{t+1}^t - p_t^r h_{t+1}^R = (1-\tau)\, y_t^t - c_t^t$$

Note that

$$\frac{\delta_r p_{t+1}}{p_t^r} = R_t = \frac{p_{t+1}}{p_t} = \frac{(1-\delta_r)\, p_{t+1}}{p_t - p_t^r}$$

Then

$$R_t = \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t} = \frac{(1-\delta_r)\, p_{t+1} - R_t (1-\theta)\, p_t}{\theta p_t - p_t^r}$$

2. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t > 0, h_{t+1}^t - h_{t+1}^R > 0, h_{t+1}^t > 0, h_{t+1}^R = 0$, then $\mu_1 = \mu_2 = \nu_1 = 0$, $\nu_2 \geq 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} + \nu_2 &= 0 \end{aligned}$$

Hence,

$$R_t = \frac{p_{t+1}}{p_t} \leq \frac{\delta_r p_{t+1}}{p_t^r}$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to the Case 1.

(b) If $\mu_1 = \mu_2 = \nu_1 = 0, \nu_2 > 0$, then

$$\frac{\delta_r p_{t+1}}{p_t^r} > R_t = \frac{p_{t+1}}{p_t} > \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r}$$

and

$$a_{t+1}^t + p_t h_{t+1}^t = (1-\tau)\, y_t^t - c_t^t$$

Under this case, it is also true that

$$R_t = \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t} > \frac{(1-\delta_r)\, p_{t+1} - R_t (1-\theta)\, p_t}{\theta p_t - p_t^r}$$

3. $a^t_{t+1} + (1-\theta)\, p_t h^t_{t+1} > 0, h^t_{t+1} - h^R_{t+1} = 0,\ h^t_{t+1} > 0, h^R_{t+1} > 0$, then $\mu_1 = \nu_1 = \nu_2 = 0, \mu_2 \geq 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_2 &= 0 \\ \lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} - \mu_2 &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} R_t &\geq \frac{p_{t+1}}{p_t} \\ R_t &\geq \frac{\delta_r p_{t+1}}{p^r_t} \\ R_t &= \frac{p_{t+1}(1-\delta_r)}{p_t - p^r_t} \end{aligned}$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to the Case 1.

(b) If $\mu_1 = \nu_1 = \nu_2 = 0, \mu_2 > 0$, then

$$R_t = \frac{p_{t+1}(1-\delta_r)}{p_t - p^r_t} > \frac{p_{t+1}}{p_t} > \frac{\delta_r p_{t+1}}{p^r_t}$$

and

$$\begin{aligned} a^t_{t+1} + (p_t - p^r_t)\, h^t_{t+1} &= (1-\tau)\, y^t_t - c^t_t \\ h^R_{t+1} &= h^t_{t+1} \end{aligned}$$

In this case, it is also true that

$$R_t = \frac{(1-\delta_r)\, p_{t+1} - R_t (1-\theta)\, p_t}{\theta p_t - p^r_t} > \frac{p_{t+1} - (1-\theta)\, R_t p_t}{\theta p_t}$$

4. $a^t_{t+1} + (1-\theta)\, p_t h^t_{t+1} > 0, h^t_{t+1} = h^R_{t+1} = 0$, then $\mu_1 = 0, \mu_2 \geq 0, \nu_1 \geq 0, \nu_2 \geq 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_2 + \nu_1 &= 0 \\ \lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} - \mu_2 + \nu_2 &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} R_t &\geq \frac{p_{t+1}}{p_t} \\ R_t &\geq \frac{(1-\delta_r)\, p_{t+1}}{p_t - p^r_t} \end{aligned}$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to Case 1

(b) If $\mu_1 = \mu_2 = \nu_1 = 0, \nu_2 > 0$, then we go back to Case 2

(c) If $\mu_1 = \nu_1 = \nu_2 = 0, \mu_2 > 0$, then we go back to Case 3

(d) If $\mu_1 = 0, \mu_2 + \nu_1 > 0, \nu_1 + \nu_2 > 0$, then $R_t > \frac{p_{t+1}}{p_t}$ and $R_t > \frac{(1-\delta_r)p_{t+1}}{p_t - p_t^r}$.

$$\begin{aligned} a_{t+1}^t &= (1-\tau)\, y_t^t - c_t^t \\ h_{t+1}^R &= h_{t+1}^t = 0 \end{aligned}$$

It is also true that

$$\begin{aligned} R_t &> \frac{(1-\delta_r)\, p_{t+1} - R_t (1-\theta)\, p_t}{\theta p_t - p_t^r} \\ R_t &> \frac{p_{t+1} - (1-\theta)\, R_t p_t}{\theta p_t} \end{aligned}$$

5. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^S = 0, h_{t+1}^t - h_{t+1}^R > 0, h_{t+1}^t > 0, h_{t+1}^R > 0$, then $\mu_1 \geq 0, \mu_2 = \nu_1 = \nu_2 = 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &\geq R_t \\ \frac{\lambda_1}{\lambda_2} &\geq \frac{p_{t+1}}{p_t} \\ \frac{\lambda_1}{\lambda_2} &= \frac{\delta_r p_{t+1}}{p_t^r} \end{aligned}$$

Discussion:

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to Case 1.

(b) If $\mu_1 > 0, \mu_2 = \nu_1 = \nu_2 = 0$, then

$$\frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta)\, R_t p_t}{\theta p_t}$$

Use the equation $\frac{\lambda_1}{\lambda_2} = \frac{\delta_r p_{t+1}}{p_t^r}$ then we have an expression for R_t

$$R_t = \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta} < \frac{p_{t+1}}{p_t}$$

It follows that

$$R_t, \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r} < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t} = \frac{\delta_r p_{t+1}}{p_t^r}$$

First of all, this suggests that the borrowing cost is smaller than the intertemporal rate of substitution Therefore, the investors must be borrowing constrained. Secondly, the investors are indifferent between constrained-borrow-to-empty and constrained-borrow-to-rent, i.e.,

$$\frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t} = \frac{(1-\delta_r) p_{t+1} - (1-\theta) R_t p_t}{\theta p_t - p_t^r}$$

Let $x \equiv \left(p_t h_{t+1}^t - \frac{p_t^r}{\theta} h_{t+1}^R\right)$ and $\gamma_t \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t}$. Rewrite the budget constraints as

$$\begin{aligned} c_t^t + \theta p_t h_{t+1}^t &= (1-\tau) y_t + p_t^r h_{t+1}^R \\ c_{t+1}^t &= \tau (1+n)(1+g) y_t^t + \left(p_t h_{t+1}^t - \frac{p_t^r}{\theta} h_{t+1}^R\right) \theta \gamma_t \end{aligned}$$

Then

$$\begin{aligned} c_t^t &= (1-\tau) y_t^t - \theta x \\ c_{t+1}^t &= \tau (1+n)(1+g) y_t^t + \theta \gamma_t x \end{aligned}$$

Solve for x

$$x = \frac{\beta \gamma_t (1-\tau) y_t^t - \tau (1+n)(1+g) y_t^t}{\theta \gamma_t (\beta+1)}$$

Therefore

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{\gamma_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta \gamma_t}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{\gamma_t}\right] y_t^t \\ a_{t+1}^t &= -(1-\theta) p_t h_{t+1}^t \\ p_t h_{t+1}^t - \frac{p_t^r h_{t+1}^R}{\theta} &= \frac{\beta \gamma_t (1-\tau) - \tau (1+n)(1+g)}{\theta \gamma_t (\beta+1)} y_t^t \end{aligned}$$

6. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t = 0$, $h_{t+1}^t - h_{t+1}^R > 0, h_{t+1}^t > 0, h_{t+1}^R = 0$, then $\mu_1, \nu_2 \geq 0, \mu_2 = \nu_1 = 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} + \nu_2 &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
\frac{\lambda_1}{\lambda_2} &\geq R_t \\
\frac{\lambda_1}{\lambda_2} &\geq \frac{p_{t+1}}{p_t} \\
\frac{\lambda_1}{\lambda_2} &\leq \frac{\delta_r p_{t+1}}{p_t^r}
\end{aligned}
$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to Case 1

(b) If $\mu_1 > 0, \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to Case 5

(c) If $\mu_1 = \mu_2 = \nu_1 = 0, \nu_2 > 0$, then we go back to Case 2

(d) If $\mu_1 > 0, \nu_2 > 0, \mu_2 = \nu_1 = 0$, then

$$\frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t}$$

Use the condition that $\frac{\lambda_1}{\lambda_2} < \frac{\delta_r p_{t+1}}{p_t^r}$, and the following inequality for R_t holds

$$R_t > \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta}$$

It turns out that $\frac{p_{t+1}-(1-\theta)R_t p_t}{\theta p_t} > \frac{p_{t+1}}{p_t}$ implies $\frac{p_{t+1}}{p_t} > R_t$. Therefore, It follows that

$$R_t, \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r} < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t} < \frac{\delta_r p_{t+1}}{p_t^r}$$

First of all, this suggests that the borrowing cost is smaller than the intertemporal rate of substitution Therefore, the investors must be borrowing constrained. Secondly, the investors prefer the constrained-borrow-to-empty to the constrained-borrow-to-rent, i.e.,

$$\frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t} > \frac{(1-\delta_r) p_{t+1} - (1-\theta) R_t p_t}{\theta p_t - p_t^r}$$

Let $x \equiv p_t h_{t+1}^t$ and $\gamma_t \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_t p_t}{\theta p_t}$. Use the fact that

$$
\begin{aligned}
c_t^t &= (1-\tau) y_t^t - \theta p_t h_{t+1}^t \\
c_{t+1}^t &= \tau (1+n)(1+g) y_t^t - R_t (1-\theta) p_t h_{t+1}^t + p_{t+1} h_{t+1}^t
\end{aligned}
$$

Then

$$\begin{aligned} c_t^t &= (1-\tau)\, y_t^t - \theta x \\ c_{t+1}^t &= \tau\,(1+n)\,(1+g)\, y_t^t + \theta\gamma_t x \end{aligned}$$

Solve for x

$$x = \frac{\beta\gamma_t\,(1-\tau) - \tau\,(1+n)\,(1+g)}{\theta\gamma_t\,(\beta+1)} y_t^t$$

Therefore

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[(1-\tau)+\frac{\tau\,(1+n)\,(1+g)}{\gamma_t}\right] y_t^t \\ a_{t+1}^t &= -\,(1-\theta)\, p_t h_{t+1}^t \\ p_t h_{t+1}^t &= \frac{\beta\gamma_t\,(1-\tau) - \tau\,(1+n)\,(1+g)}{\theta\gamma_t\,(\beta+1)} y_t^t \\ h_{t+1}^R &= 0 \end{aligned}$$

7. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t = 0$, i.e., the borrowing constraint of the investors is binding

   $h_{t+1}^t - h_{t+1}^R = 0$, i.e., the investors rent all the houses out

   $h_{t+1}^t > 0, h_{t+1}^R > 0$, i.e., the investors hold positive amount of housing

   Therefore, $\mu_1, \mu_2 \geq 0$, $\nu_1 = \nu_2 = 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1\,(1-\theta)\, p_t + \mu_2 &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} - \mu_2 &= 0 \end{aligned}$$

   Hence,

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &\geq R_t \\ \frac{\lambda_1}{\lambda_2} &\geq \frac{p_{t+1}}{p_t} \\ \frac{\lambda_1}{\lambda_2} &\geq \frac{\delta_r p_{t+1}}{p_t^r} \end{aligned}$$

   Use the fact that

$$\begin{aligned} -\lambda_1 p_t + \lambda_2 p_{t+1} + (\lambda_1 - \lambda_2 R_t)\,(1-\theta)\, p_t + \mu_2 &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} - \mu_2 &= 0 \end{aligned}$$

Solve for $\frac{\lambda_1}{\lambda_2}$

$$\frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r)\,p_{t+1} - R_t\,(1-\theta)\,p_t}{\theta p_t - p_t^r}$$

(a) If $\mu_1 = 0, \mu_2 = 0, \nu_1 = \nu_2 = 0$, then we go back to Case 1.

(b) If $\mu_1 > 0, \mu_2 = 0, \nu_1 = \nu_2 = 0$, then we go back to Case 5.

(c) If $\mu_1 = 0, \mu_2 > 0, \nu_1 = \nu_2 = 0$, then we go back to Case 3.

(d) If $\mu_1 > 0, \mu_2 > 0, \nu_1 = \nu_2 = 0$, then we have

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &> R_t \\ \frac{\lambda_1}{\lambda_2} &> \frac{p_{t+1}}{p_t} \\ \frac{\lambda_1}{\lambda_2} &> \frac{\delta_r p_{t+1}}{p_t^r} \end{aligned}$$

Use the expression $\frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r)p_{t+1} - R_t(1-\theta)p_t}{\theta p_t - p_t^r}$, the above three inequalities implies

$$\begin{aligned} R_t &< \frac{(1-\delta_r)\,p_{t+1}}{p_t - p_t^r} \\ R_t &< \frac{\frac{p_{t+1}}{p_t} - \theta\frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta} \end{aligned}$$

where I use the assumption $\theta p_t - p_t^r > 0$. Therefore

$$\frac{(1-\delta_r)\,p_{t+1} - R_t\,(1-\theta)\,p_t}{\theta p_t - p_t^r} = \frac{\lambda_1}{\lambda_2} > \frac{\delta_r p_{t+1}}{p_t^r}, \frac{p_{t+1}}{p_t}, R_t$$

It is also true that

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &> \frac{p_{t+1} - R_t\,(1-\theta)\,p_t}{\theta p_t} \\ \frac{\lambda_1}{\lambda_2} &> \frac{p_{t+1}\,(1-\delta_r)}{p_t - p_t^r} \end{aligned}$$

Recall that

$$\begin{aligned} c_t^t &= (1-\tau)\,y_t^t + p_t^r h_{t+1}^R - \theta p_t h_{t+1}^t \\ c_{t+1}^t &= (1+n)\,(1+g)\,y_t^t + R_t a_{t+1}^t + p_{t+1} h_{t+1}^t - \delta_r p_{t+1} h_{t+1}^R \end{aligned}$$

Let $x \equiv \left(p_t - \frac{p_t^r}{\theta}\right) h_{t+1}^t, \gamma_t \equiv \frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r)\frac{p_{t+1}}{p_t} - R_t(1-\theta)}{\theta - \frac{p_t^r}{p_t}}$. Then the above budget constraint becomes

$$\begin{aligned} c_t^t &= (1-\tau) y_t^t - \theta x \\ c_{t+1}^t &= (1+n)(1+g) y_t^t + \theta \gamma_t x \end{aligned}$$

Solve for x

$$x = \frac{\beta \gamma_t (1-\tau) y_t^t - \tau_{t+1} y_{t+1}^t}{\theta \gamma_t (\beta + 1)}$$

Therefore

$$\begin{aligned} \left(p_t - \frac{p_t^r}{\theta}\right) h_{t+1}^t &= \frac{\beta \gamma_t (1-\tau) y_t^t - (1+n)(1+g) y_t^t}{\theta \gamma_t (\beta+1)} \\ h_{t+1}^t &= h_{t+1}^R \\ c_t^t &= \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{\gamma_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta \gamma_t}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{\gamma_t}\right] y_t^t \end{aligned}$$

8. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t = 0, h_{t+1}^t - h_{t+1}^R = 0, h_{t+1}^t = h_{t+1}^R = 0$, then $\mu_1, \mu_2, v_1, v_2 \geq 0$.

$$\begin{aligned} c_t^t &= (1-\tau) y_t^t \\ c_{t+1}^t &= \tau (1+n)(1+g) y_t^t \end{aligned}$$

6.2.2 Sufficient Conditions

Proposition 12 *Given y_t^t, τ, g, n, R_t, the optimal decisions of investors are the followings:*

1. *If $R_t = \frac{p_{t+1}}{p_t}, R_t = \frac{(1-\delta_r) p_{t+1}}{p_t - p_t^r}$, then*

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta R_t}{1+\beta}\left[1 - \tau + \frac{\tau (1+n)(1+g)}{R_t}\right] y_t^t \\ a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau_t) y_t^t - c_t^t + p_t^r h_{t+1}^R \\ a_{t+1}^t &> -(1-\theta) p_t h_{t+1}^t \end{aligned}$$

2. *If* $R_t = \frac{p_{t+1}}{p_t}, R_t > \frac{(1-\delta_r)p_{t+1}}{p_t - p_t^r}$, *then*

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta R_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right] y_t^t \\ a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau_t)\, y_t^t - c_t^t \\ h_{t+1}^R &= 0 \\ a_{t+1}^t &> -(1-\theta)\, p_t h_{t+1}^t \end{aligned}$$

3. *If* $R_t = \frac{p_{t+1}}{p_t}, R_t < \frac{(1-\delta_r)p_{t+1}}{p_t - p_t^r}$, *then*

$$\begin{aligned} \left(p_t - \frac{p_t^r}{\theta}\right) h_{t+1}^t &= \frac{\beta\gamma_t (1-\tau)\, y_t^t - (1+n)(1+g)\, y_t^t}{\theta\gamma_t (\beta+1)} \\ h_{t+1}^t &= h_{t+1}^R \\ c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right] y_t^t \\ a_{t+1}^t &= -(1-\theta)\, p_t h_{t+1}^t \end{aligned}$$

where $\gamma_t \equiv \frac{(1-\delta_r)\frac{p_{t+1}}{p_t} - R_t(1-\theta)}{\theta - \frac{p_t^r}{p_t}}$

4. *If* $R_t > \frac{p_{t+1}}{p_t}, R_t = \frac{(1-\delta_r)p_{t+1}}{p_t - p_t^r}$, *then*

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta R_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right] y_t^t \\ a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau)\, y_t - c_t^t + p_t^r h_{t+1}^R \\ h_{t+1}^R &= h_{t+1}^t \\ a_{t+1}^t &> -(1-\theta)\, p_t h_{t+1}^t \end{aligned}$$

5. *If* $R_t > \frac{p_{t+1}}{p_t}, R_t > \frac{(1-\delta_r)p_{t+1}}{p_t - p_t^r}$, *then*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta R_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\
a_{t+1}^t &= (1-\tau)\,y_t^t - c_t^t \\
h_{t+1}^R &= h_{t+1}^t = 0 \\
a_{t+1}^t &> 0
\end{aligned}$$

6. *If* $R_t > \frac{p_{t+1}}{p_t}, R_t < \frac{(1-\delta_r)p_{t+1}}{p_t - p_t^r}$, *then*

$$\begin{aligned}
\left(p_t - \frac{p_t^r}{\theta}\right)h_{t+1}^t &= \frac{\beta\gamma_t(1-\tau)\,y_t^t - (1+n)(1+g)\,y_t^t}{\theta\gamma_t(\beta+1)} \\
h_{t+1}^t &= h_{t+1}^R \\
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
a_{t+1}^t + (1-\theta)\,p_t h_{t+1}^t &= 0
\end{aligned}$$

where $\gamma_t \equiv \frac{(1-\delta_r)\frac{p_{t+1}}{p_t} - R_t(1-\theta)}{\theta - \frac{p_t^r}{p_t}}$

7. *If* $R_t < \frac{p_{t+1}}{p_t}, R_t = \frac{\frac{p_{t+1}}{p_t} - \theta\frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta}$, *then*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
a_{t+1}^t + (1-\theta)\,p_t h_{t+1}^t &= 0 \\
p_t h_{t+1}^t - \frac{p_t^r h_{t+1}^R}{\theta} &= \frac{\beta\gamma_t(1-\tau) - \tau(1+n)(1+g)}{\theta\gamma_t(\beta+1)}y_t^t
\end{aligned}$$

where $\gamma_t \equiv \frac{p_{t+1} - (1-\theta)R_t p_t}{\theta p_t}$

8. *If* $R_t < \frac{p_{t+1}}{p_t}, R_t > \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta}$*,then*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
a_{t+1}^t + (1-\theta)p_t h_{t+1}^t &= 0 \\
p_t h_{t+1}^t &= \frac{\beta\gamma_t(1-\tau)-\tau(1+n)(1+g)}{\theta\gamma_t(\beta+1)}y_t^t \\
h_{t+1}^R &= 0
\end{aligned}$$

where $\gamma_t \equiv \frac{p_{t+1}-(1-\theta)R_t p_t}{\theta p_t}$

9. *If* $R_t < \frac{p_{t+1}}{p_t}, R_t < \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta}$*, then*

$$\begin{aligned}
\left(p_t - \frac{p_t^r}{\theta}\right)h_{t+1}^t &= \frac{\beta\gamma_t(1-\tau)y_t^t-\tau(1+n)(1+g)y_t^t}{\theta\gamma_t(\beta+1)} \\
h_{t+1}^t &= h_{t+1}^R \\
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
a_{t+1}^t + (1-\theta)p_t h_{t+1}^t &= 0
\end{aligned}$$

where $\gamma_t \equiv \frac{(1-\delta_r)\frac{p_{t+1}}{p_t} - R_t(1-\theta)}{\theta - \frac{p_t^r}{p_t}}$

6.2.3 Homeowner's Problem

The Lagrangian function is

$$\begin{aligned}
&L = \ln c_t^t + \beta(1-\zeta)\ln c_{t+1}^t + \beta\zeta\ln\left(h_{t+1}^r + h_{t+1}^t\right) \\
&+\lambda_1\left[(1-\tau)y_t^t - p_t^r h_{t+1}^r - p_t h_{t+1}^t - c_t^t - a_{t+1}^t\right] \\
&+\lambda_2\left[\tau(1+n)(1+g)y_t + R_t a_{t+1}^t + p_{t+1}h_{t+1}^t - c_{t+1}^t\right] \\
&+\mu_1\left[a_{t+1}^t + (1-\theta)p_t h_{t+1}^t\right] \\
&+\nu_1 h_{t+1}^t \\
&+\nu_2 h_{t+1}^r
\end{aligned}$$

The FOCs become

$$\begin{aligned} c_t^t &: \frac{1}{c_t^t} - \lambda_1 = 0 \\ c_{t+1}^t &: \frac{\beta(1-\zeta)}{c_{t+1}^t} - \lambda_2 = 0 \\ a_{t+1}^t &: -\lambda_1 + \lambda_2 R_t + \mu_1 = 0 \\ h_{t+1}^t &: \frac{\beta\zeta}{h_{t+1}^r + h_{t+1}^t} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t + \nu_1 = 0 \\ h_{t+1}^r &: \frac{\beta\zeta}{h_{t+1}^r + h_{t+1}^t} - \lambda_1 p_t^r + \nu_2 = 0 \end{aligned}$$

where

$$\begin{aligned} \mu_1 &\geq 0, \text{ if } a_{t+1}^t + (1-\theta) p_t h_{t+1}^t > 0, \text{ then } \mu_1 = 0 \\ \nu_1 &\geq 0, \text{ if } h_{t+1}^t > 0, \text{ then } \nu_1 = 0 \\ \nu_2 &\geq 0, \text{ if } h_{t+1}^r > 0, \text{ then } \nu_2 = 0 \end{aligned}$$

and the life-time budget constraint is given by

$$c_t^t + \frac{c_{t+1}^t}{R_t} + p_t^r h_{t+1}^r + \left(p_t - \frac{p_{t+1}}{R_t}\right) h_{t+1}^t = (1-\tau) y_t^t + \frac{\tau(1+n)(1+g) y_t^t}{R_t}$$

1. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t > 0, h_{t+1}^t > 0, h_{t+1}^r > 0$, then $\mu_1 = \nu_1 = \nu_2 = 0$

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t &= 0 \\ \frac{\beta\zeta}{h_{t+1}^r + h_{t+1}^t} - \lambda_1 p_t + \lambda_2 p_{t+1} &= 0 \\ \frac{\beta\zeta}{h_{t+1}^r + h_{t+1}^t} - \lambda_1 p_t^r &= 0 \end{aligned}$$

Hence,

$$\frac{\lambda_1}{\lambda_2} = R_t = \frac{p_{t+1}}{p_t - p_t^r} = \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t - p_t^r}$$

The optimal decision rules are

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{R_t}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta(1-\zeta) R_t}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{R_t}\right] y_t^t \\ h_{t+1}^r + h_{t+1}^t &= \frac{\beta\zeta}{p_t^r} c_t^t \\ (p_t - p_t^r) h_{t+1}^t + a_{t+1}^t &= (1-\tau) y_t^t - (1+\beta\zeta) c_t^t \end{aligned}$$

2. $a^t_{t+1} + (1-\theta) p_t h^t_{t+1} > 0, h^t_{t+1} > 0, h^r_{t+1} = 0$, then $\mu_1 = \nu_1 = 0, \nu_2 \geq 0$.If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to Case 1. If $\mu_1 = \nu_1 = 0, \nu_2 > 0$,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t &= 0 \\ \frac{\beta\zeta}{h^t_{t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} &= 0 \\ \frac{\beta\zeta}{h^t_{t+1}} - \lambda_1 p^r_t + \nu_2 &= 0 \end{aligned}$$

Hence

$$\frac{\lambda_1}{\lambda_2} = R_t < \frac{p_{t+1}}{p_t - p^r_t} < \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t - p^r_t}$$

This suggests that if the rental price is high enough, i.e., $p^r_t > p_t - \frac{p_{t+1}}{R_t}$, unconstrained workers will choose to own houses. The optimal policy rules are

$$\begin{aligned} c^t_t &= \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{R_t}\right] y^t_t \\ c^t_{t+1} &= \frac{\beta(1-\zeta) R_t}{1+\beta}\left[1 - \tau + \frac{\tau(1+n)(1+g)}{R_t}\right] y^t_t \\ h^t_{t+1} &= \frac{\beta\zeta}{p_t - \frac{p_{t+1}}{R_t}} c^t_t \\ a^t_{t+1} &= (1-\tau) y^t_t - \frac{(1+\beta\zeta) p_t - \frac{p_{t+1}}{R_t}}{p_t - \frac{p_{t+1}}{R_t}} c^t_t \end{aligned}$$

3. $a^t_{t+1} + (1-\theta) p_t h^t_{t+1} > 0$, $h^t_{t+1} = 0, h^r_{t+1} > 0$, then $\mu_1 = 0, \nu_1 \geq 0, \nu_2 = 0$.If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to Case 1. If $\mu_1 = \nu_2 = 0, \nu_1 > 0$

$$\begin{aligned} -\lambda_1 + \lambda_2 R_t &= 0 \\ \frac{\beta\zeta}{h^r_{t+1} + h^t_{t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \nu_1 &= 0 \\ \frac{\beta\zeta}{h^r_{t+1} + h^t_{t+1}} - \lambda_1 p^r_t &= 0 \end{aligned}$$

Hence

$$\frac{\lambda_1}{\lambda_2} = R_t > \frac{p_{t+1}}{p_t - p^r_t} > \frac{p_{t+1} - (1-\theta) R_t p_t}{\theta p_t - p^r_t}$$

The optimal policy rules are

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\ c_{t+1}^t &= \frac{\beta}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \\ p_t^r h_{t+1}^r &= \frac{\beta\zeta}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_t}\right]y_t^t \end{aligned}$$

4. $a_{t+1}^t+(1-\theta)\,p_t h_{t+1}^t=0,\ h_{t+1}^t>0, h_{t+1}^r>0$, then $\mu_1\geq 0, \nu_1=\nu_2=0$.If $\mu_1=\nu_1=\nu_2=0$, then we go back to Case 1. If $\mu_1>0, \nu_1=0, \nu_2=0$

$$\begin{aligned} -\lambda_1+\lambda_2 R_t+\mu_1 &= 0 \\ \frac{\beta\zeta}{h_{t+1}^r+h_{t+1}^t}-\lambda_1 p_t+\lambda_2 p_{t+1}+\mu_1(1-\theta)\,p_t &= 0 \\ \frac{\beta\zeta}{h_{t+1}^r+h_{t+1}^t}-\lambda_1 p_t^r &= 0 \end{aligned}$$

Hence, the condition for R_t is

$$R_t<\frac{p_{t+1}}{p_t-p_t^r}<\frac{\lambda_1}{\lambda_2}=\frac{p_{t+1}-(1-\theta)\,R_t p_t}{\theta p_t-p_t^r} \tag{12}$$

Because

$$\begin{aligned} c_t^t &= (1-\tau)\,y_t^t-\theta p_t h_{t+1}^t+p_t^r h_{t+1}^t-p_t^r\left(h_{t+1}^t+h_{t+1}^r\right) \\ c_{t+1}^t &= \tau(1+n)(1+g)\,y_t^t+(p_{t+1}-R_t(1-\theta)\,p_t)\,h_{t+1}^t \end{aligned}$$

Then we have

$$1+\beta\zeta=\lambda_1(1-\tau)\,y_t^t-\lambda_1 h_{t+1}^t(\theta p_t-p_t^r)$$

and

$$\beta(1-\zeta)=\lambda_2\tau(1+n)(1+g)\,y_t^t+\lambda_2(p_{t+1}-R_t(1-\theta)\,p_t)\,h_{t+1}^t$$

Combine the above two equations and let $\frac{\lambda_1}{\lambda_2}\equiv\gamma_t$, then we have

$$(1+\beta)\,c_t^t=\frac{\tau(1+n)(1+g)\,y_t^t}{\gamma_t}+(1-\tau)\,y_t^t$$

If we know γ_t, then we can express $c_t^t, c_{t+1}^t, h_{t+1}^t$ in terms of γ_t

$$\begin{aligned}1+\beta &= \frac{(1-\tau)\, y_t^t}{(1-\tau)\, y_t^t - \theta p_t h_{t+1}^t - p_t^r h_{t+1}^r} \\ &+\beta\,(1-\zeta)\, \frac{\tau\,(1+n)\,(1+g)\, y_t^t}{\tau\,(1+n)\,(1+g)\, y_t^t + (p_{t+1} - R_t\,(1-\theta)\, p_t)\, h_{t+1}^t}\end{aligned}$$

Use $\frac{1+\beta\zeta}{(1-\tau)y_t^t-(\theta p_t - p_t^r)h_{t+1}^t} = \lambda_1 = \frac{1}{c_t^t}$, the above equation can be simplified into

$$\begin{aligned}1+\beta &= \frac{(1-\tau)\,(1+\beta\zeta)\, y_t^t}{(1-\tau)\, y_t^t - (\theta p_t - p_t^r)\, h_{t+1}^t} \\ &\quad +\beta\,(1-\zeta)\, \frac{\tau\,(1+n)\,(1+g)\, y_t^t}{\tau\,(1+n)\,(1+g)\, y_t^t + (p_{t+1} - R_t\,(1-\theta)\, p_t)\, h_{t+1}^t}\end{aligned}$$

This is a quadratic equation for $p_t h_{t+1}^t$. Let

$$\begin{aligned} x &= p_t h_{t+1}^t \\ \hat{\theta} &= \theta - \frac{p_t^r}{p_t} \\ \varphi &= \frac{p_{t+1}}{p_t} - (1-\theta)\, R_t \\ a &= (1-\tau)\, y_t^t \\ b &= \tau\,(1+n)\,(1+g)\, y_t^t \end{aligned}$$

$$1+\beta = \frac{(1+\beta\zeta)\, a}{a - \hat{\theta} x} + \frac{\beta\,(1-\zeta)\, b}{b + \varphi x}$$

with one solution is zero, the other solution is

$$x = \frac{a\varphi\beta\,(1-\zeta) - b\hat{\theta}\,(1+\beta\zeta)}{\hat{\theta}\varphi\,(1+\beta)}$$

We can still define γ_t

$$\begin{aligned}\gamma_t &= \frac{\lambda_1}{\lambda_2} = \frac{c_{t+1}^t}{\beta\,(1-\zeta)\, c_t^t} = \frac{(b+\varphi x)\,(1+\beta\zeta)}{\beta\,(1-\zeta)\left(a - \hat{\theta} x\right)} \\ &= \frac{\varphi}{\hat{\theta}} = \frac{p_{t+1} - (1-\theta)\, R_t p_t}{\theta p_t - p_t^r}\end{aligned}$$

which gives

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta(1-\zeta)\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\
p_t h_{t+1}^t &= \frac{p_t}{\theta p_t - p_t^r}\left[(1-\tau)y_t^t-(1+\beta\zeta)c_t^t\right] \\
h_{t+1}^r &= \frac{(1-\tau)y_t^t-\theta p_t h_{t+1}^t - c_t^t}{p_t^r} \\
a_{t+1}^t &= -(1-\theta)p_t h_{t+1}^t
\end{aligned}$$

5. $a_{t+1}^t+(1-\theta)p_t h_{t+1}^t=0$, $h_{t+1}^t>0, h_{t+1}^r=0$, then $\mu_1\geq 0,\nu_1=0,\nu_2\geq 0$.If $\mu_1=\nu_1=\nu_2=0$, then we go back to Case 1. If $\mu_1=0,\nu_1=0,\nu_2>0$, then we go back to Case 2. If $\mu_1>0,\nu_1=0,\nu_2=0$, then we go back to Case 4. If $\mu_1>0,\nu_1=0,\nu_2>0$, then the solution is the same as the benchmark model without rental market.

6. $a_{t+1}^t+(1-\theta)p_t h_{t+1}^t=0$, $h_{t+1}^t=0, h_{t+1}^r>0$, then $\mu_1\geq 0,\nu_1\geq 0,\nu_2=0$. If $\mu_1=\nu_1=\nu_2=0$, then we go back to Case 1. If $\mu_1>0,\nu_1=0,\nu_2=0$, then we go back to Case 4. If $\mu_1=0,\nu_1>0,\nu_2=0$, then we go back to Case 3. If $\mu_1>0,\nu_1>0,\nu_2=0$, then

$$\begin{aligned}
-\lambda_1+\lambda_2 R_t+\mu_1 &= 0 \\
\frac{\beta\zeta}{h_{t+1}^r}-\lambda_1 p_t+\lambda_2 p_{t+1}+\mu_1(1-\theta)p_t+\nu_1 &= 0 \\
\frac{\beta\zeta}{h_{t+1}^r}-\lambda_1 p_t^r &= 0
\end{aligned}$$

Either

$$\frac{\lambda_1}{\lambda_2}>R_t>\frac{p_{t+1}}{p_t}>\frac{p_{t+1}-(1-\theta)R_t p_t}{\theta p_t-p_t^r}$$

or

$$\frac{\lambda_1}{\lambda_2}>\frac{p_{t+1}-(1-\theta)R_t p_t}{\theta p_t-p_t^r}>\frac{p_{t+1}}{p_t}>R_t$$

$$
\begin{aligned}
a_{t+1}^t &= 0 \\
h_{t+1}^t &= 0 \\
c_{t+1}^t &= \tau(1+n)(1+g)\,y_t^t \\
c_t^t &= \frac{1}{1+\beta\zeta}(1-\tau)\,y_t^t \\
p_t^r h_{t+1}^r &= \frac{\beta\zeta}{1+\beta\zeta}(1-\tau)\,y_t^t
\end{aligned}
$$