Manufacturing Risk-free Government Debt*

Zhengyang Jiang
Northwestern Kellogg

Hanno Lustig
Stanford GSB, NBER

Stijn Van Nieuwerburgh
Columbia Business School, NBER, and CEPR

Mindy Z. Xiaolan
UT Austin McCombs

March 5, 2021

Abstract

Governments face a trade-off between insuring bondholders and taxpayers. If the government fully insures bondholders by manufacturing risk-free zero-beta debt, then it cannot also insure taxpayers against permanent macroeconomic shocks over long horizons. Instead, taxpayers will pay more in taxes in bad times. Conversely, if the government fully insures taxpayers against adverse macro shocks, then the debt becomes risky, at least as risky as unlevered equity claim. As the world's safe asset supplier, the U.S. appears to have escaped this trade-off thus far, whereas the U.K. has not.

Key words: fiscal policy, term structure, debt maturity, convenience yield

*First version: 07/08/2020. Jiang: Finance Department, Kellogg School of Management, Northwestern University; zhengyang.jiang@kellogg.northwestern.edu. Lustig: Department of Finance, Stanford Graduate School of Business, Stanford CA 94305; hlustig@stanford.edu; https://people.stanford.edu/hlustig/. Van Nieuwerburgh: Department of Finance, Columbia Business School, Columbia University, 3022 Broadway, New York, NY 10027; svnieuwe@gsb.columbia.edu. Xiaolan: McCombs School of Business, the University of Texas at Austin; mindy.xiaolan@mccombs.utexas.edu. We are grateful to Adrien Auclert, Jonathan Berk, Saki Bigio, Luigi Boccola, Lorenzo Bretscher, Pierre Collin-Dufresne, Peter DeMarzo, Sebastian DiTella, Francois Gourio, Patrick Kehoe, Ralph Koijen, Arvind Krishnamurthy, Matteo Maggiori, Ellen McGrattan and seminar participants at Stanford University (macro and finance), Columbia University (macro and finance), the University of Lausanne, and the European Winter Meetings of the Econometric Society for helpful comments. Yuan Tian provided outstanding research assistance. We gratefully acknowledge financial support from NSF award 2049260.

In this paper, we show that governments face a trade-off between insuring its bondholders by making its debt risk-free and insuring its taxpayers and transfer recipients against adverse macroeconomic shocks. If a government provides more insurance to bondholders, who require lower risk premia on the government debt as a result, then it can provide less insurance to taxpayers. In other words, making government debt safer requires raising more tax revenue as a fraction of GDP from taxpayers in bad times. The larger the sovereign debt burden, the steeper this trade-off becomes.

A country's government debt is risk-free if the government debt portfolio has a zero beta, meaning that its valuation is immune to fluctuations in the economy and financial markets. Default-free debt is not necessarily risk-free debt, as its valuation can still fluctuate before its expiration. Manufacturing risk-free debt in the presence of permanent output shocks requires a non-trivial feat of financial engineering. The Treasury's bond portfolio is backed by a long position in a claim to tax revenue and a short position in a claim to government spending. Both claims are exposed to output risk, and we define these exposures as their respective betas. The Treasury's long position in the tax claim exceeds the short position in the spending claim by the value of outstanding debt. To render the entire Treasury portfolio risk-free, the claim to tax revenues needs to have a lower beta than the spending claim to ensure that the beta of the Treasury portfolio is exactly zero.

Recast in the language of Modigliani-Miller, the claim to tax revenue can be regarded as the government's unlevered asset, which is divided into the government debt and the claim to government spending. To manufacture risk-free debt, the spending claim has to be a levered version of the government's asset. Therefore, just as the equity has to be riskier than a typical firm's asset in order to generate risk-free debt, the government's spending beta has to be higher than its tax beta to ensure a zero debt beta.

The tax claim has a low beta if the present discounted value (PDV) of future tax revenues increases in bad times, times in which the investor's marginal utility is high. Since the taxpayers pay the taxes, they take a short position on the tax revenue claim. From their perspective, a low-beta tax claim is a risky tax liability. As a result, the government cannot insure taxpayers when it insures bondholders by keeping the debt risk-free. The larger the amount of outstanding debt, the more levered the government becomes, and the larger the gap between the tax beta and the spending beta needs to be to keep the debt risk-free. As the debt grows, the beta of the tax claim has to go to zero. The trade-off between insuring taxpayers and bondholders steepens.

Conversely, if the government insists on insuring the taxpayers by lowering tax rates in bad times, then the tax beta is high and the government debt becomes risky for bondholders. The bondholders now bear the macroeconomic risk. The empirical properties of tax now revenue and spending are consistent with taxpayer insurance, and hence inconsistent with risk-free debt.

Our paper is the first one to analytically characterize the trade-off between insuring taxpayers and bondholders at different horizons. We start with a special case where the government keeps the debt/output ratio constant. In this case, there is no scope for insurance of taxpayers. The tax process has to be safer than the spending process at all horizons –short, intermediate and long horizons. Next, we assume that the government can issue more debt in response to a negative GDP growth shock rather than raise taxes. That is, it can run a deficit in recessions. In this case, the tax claim is riskier than the spending claim over short horizons. Over intermediate and longer horizons, the tax revenue claim has to become sufficiently safe for investors (risky for taxpayers). When output shocks are permanent, the government can only escape the trade-off over short horizons. This long-run risk in debt arises from the long-run risk in output, as along as debt and output are co-integrated. This trade-off survives even when the risk-free rate is lower than the average growth rate of the economy. How long the government can escape the trade-off depends on the persistence of the debt/output ratio.

We quantify the trade-off under the assumption that the government commits to a counter-cyclical debt issuance policy with AR(2) dynamics. In annual data, the U.S. government's debt/GDP dynamics over the post-war sample are well described by an AR(2) process with a negative exposure to output shocks. We derive the restrictions that risk-free debt impose on the properties of the surplus/output ratio. Given those AR(2) dynamics for debt, risk-free debt dictates that the surpluses quickly revert back to the mean. The impulse response function (IRF) of the surplus/output ratio can only remain negative for two years in response to a negative output shock. After two years of deficits, the government must run a surplus for the next several years. The fast mean-reversion of surpluses implied by risk-free debt is at odds with the observed persistence of surpluses in the U.S. data.

In the post-war data, the U.S. appears to escape the trade-off we have derived: despite its low government debt risk premium, the U.S. insures its taxpayers against output growth risk. The sensitivity of federal government spending to GDP growth is much lower than the sensitivity of federal tax revenues over horizons between one and ten years in the U.S. data. Tax revenues rise and fall strongly with GDP growth. Spending moves inversely with GDP growth at short horizons and mildly positively at longer horizons.[1] The U.S. government appears to insure taxpayers at all horizons by lowering their tax rates in recessions as well as by increasing the spending-to-output ratio in recessions. In the language of asset pricing, the claim to tax revenues has a higher cash flow beta than the claim to spending.

Over the last two decades, the beta of U.S. government debt has even turned negative (Campbell, Pflueger, and Viceira, 2020). A negative debt beta strictly worsens the trade-off faced by the U.S. government relative to the zero-beta case we consider. It deepens the puzzle.

We find that the U.K. looks quite different from the U.S. The U.K. government has not provided nearly as much insurance to its taxpayers as the U.S. government. The U.K.'s beta of tax revenue

[1] The government spending/GDP ratio is well described by an AR(1) model with negative exposure to output shocks.

with GDP growth is 0.5, compared to 3 for the U.S. In the post-war era, the U.K. tax claim has a smaller beta than the spending claim at longer horizons, consistent with risk-free debt. Prior to WW-II, the U.K.'s spending policy was even pro-cyclical, and the tax claim had a smaller beta than the spending claim at all horizons.

How can we reconcile the observed insurance of taxpayers with the low risk premia on government debt in the U.S. data? As the world's safe asset supplier, unlike the U.K., the U.S. government may have been able to temporarily escape the trade-off if government debt earns large and counter-cyclical convenience yields. We show how counter-cyclical convenience yields may help to keep the debt risk-free, even when the government insures its taxpayers. However, the demand curve for safe assets is downward sloping; as the supply increases, the convenience yields are likely to disappear (Krishnamurthy and Vissing-Jorgensen, 2012). Using a no-arbitrage dynamic asset pricing model, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019b) conclude that the seigniorage from convenience yields is not quantitatively large enough to fully rationalize the valuation of the U.S. public debt.

Related Literature Regarding the riskiness of government debt, the focus in the literature has been mostly on countries' willingness and ability to repay (see, e.g., Eaton and Gersovitz, 1981; Bulow and Rogoff, 1989; Aguiar and Gopinath, 2006; Arellano, 2008; DeMarzo, He, and Tourre, 2019, for examples). The trade-off we focus on between bondholder and taxpayer insurance applies regardless of whether a country contemplates default and regardless of which securities the country decides to issue (e.g., duration). By changing the maturity composition of debt, the government may be able to get closer to the optimal tax policy when markets are incomplete, essentially by making the debt riskier (Angeletos, 2002; Buera and Nicolini, 2004; Lustig, Sleet, and Yeltekin, 2008; Arellano and Ramanarayanan, 2012; Bhandari, Evans, Golosov, and Sargent, 2017; Aguiar, Amador, Hopenhayn, and Werning, 2019), and shifting risk from taxpayers to bondholders. Our work is not focused on how the maturity choice of the government informs the riskiness of debt, but instead focuses directly on the fundamental cash flow determinants of the riskiness of the government's balance sheet. The fact that long-term government debt has a negative beta in recent decades only reinforces our conclusions.

Our paper applies a basic insight from the asset pricing literature to the fiscal policy literature. Modern asset pricing has consistently found that permanent shocks to output and consumption account for most of the variance of the pricing kernel, and receive a high price of risk in securities market (e.g., Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Bansal and Yaron, 2004; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). Models without large permanent shocks produce bond risk premia that exceed equity risk premia.

If GDP growth has a permanent component, which modern macro and econometrics recognizes to be the case, then the surplus process in levels S_t inherits that permanent component from

Y_t. Surpluses have long-run risk. Because of the exposure of the surplus to long-run GDP risk, the claim to current and future surpluses will typically have a substantial risk premium. Since the value of the surplus claim equals the market value of outstanding debt, the portfolio of government debt is generally a risky asset. The properties of the stationary surplus/output ratios, which the literature focuses on, are irrelevant for the long-run discount rates of surpluses. For long-run discount rates, only long-run risk matters (Backus et al., 2018). Therefore, even when the entire debt portfolio is risk-free—in the sense that there is no news about current or future surpluses—the risk-free rate is not the right discount rate for surpluses in the presence of permanent output risk. In an economy with permanent GDP growth risk, comparing the risk-free rate to the average growth rate of the economy, as in Blanchard (2019), sheds no light on the fiscal cost of deficits. More recently, Barro (2020) makes a related point about dynamic efficiency, correctly pointing out that the comparison of the risk-free rate to the average growth rate of the economy is not informative about dynamic efficiency in a economy with growth risk.

There is an extensive literature which tests the government's inter-temporal budget constraint. Hansen, Roberds, and Sargent (1991); Hamilton and Flavin (1986); Trehan and Walsh (1988, 1991); Bohn (1998, 2007) derive time-series restrictions on the government revenue and spending processes that enforce the government's inter-temporal budget constraint. This literature uses the risk-free rate as the discount rate. This is the right discount rate only when the shocks to output are temporary and the risk-free rate exceeds the growth rate of the economy.[2]

We derive restrictions on the surplus/output process that are compatible with the knife-edge case of risk-free debt. The answer depends crucially on whether GDP has a permanent component or not. In the realistic case where it does, the surplus/output ratio cannot be sufficiently persistent to match the dynamics of the U.S. surpluses in the data. Further, we show analytically that the substantial S-shaped impulse-responses of the surplus/output ratio discussed by Bohn (1998); Canzoneri, Cumby, and Diba (2001); Cochrane (2019, 2020) are not consistent with risk-free debt. Those require a debt/output ratio that has dynamics of a higher order than those observed for debt/output in U.S. data.

The U.S. government debt earns returns close to the risk-free rate, but the cash flow dynamics do not bear this out: the surpluses are too persistent, not predicted by the debt/GDP ratio and too risky. We call this the U.S. government risk premium puzzle. The U.S. government debt risk premium puzzle we document in this paper is distinct from, but related to the government debt valuation puzzle discussed by Jiang et al. (2019b), because the risk premium puzzle does not pertain to the first moments of future surpluses.

Our results may help explain why emerging economies with more sovereign risk typically have more pro-cyclical fiscal policies (Bianchi, Ottonello, and Presno, 2019). These countries

[2] These authors really test the *joint hypothesis* that both the government budget constraint and the measurability condition—to render the debt risk-free—are satisfied.

do not benefit from the convenience yields, and hence cannot escape the trade-off. In international economics, there is a growing literature that emphasizes the U.S. role as the world's safe asset supplier (see Gourinchas and Rey, 2007; Caballero, Farhi, and Gourinchas, 2008; Caballero and Krishnamurthy, 2009; Maggiori, 2017; He, Krishnamurthy, and Milbradt, 2018; Gopinath and Stein, 2018; Krishnamurthy and Lustig, 2019; Jiang, Krishnamurthy, and Lustig, 2018a, 2019a; Liu, Schmid, and Yaron, 2019; Koijen and Yogo, 2019). Liu et al. (2019) provide a structural model of convenience yields and fiscal policy. Government debt can only earn safe asset convenience yields if it is in fact risk-free. Our paper is the first to show how convenience yields may have helped to keep the U.S. government debt risk-free.

Our paper contributes to the normative literature on optimal government taxation and debt management, starting with Barro (1979)'s seminal work on tax smoothing. In the literature after Barro (1979), starting with Lucas and Stokey (1983), the risk-return trade-off we highlight is present in the background, but is not explicitly analyzed. Importantly, most of these models do not have plausible asset pricing implications because they do not have permanent output risk. When markets are complete, the planner favors shifting the risk from taxpayers to bond investors (Lucas and Stokey, 1983). We do not derive the optimal tax rate, but show that, for any tax policy, the government can only truly insure taxpayers over short horizons, while keeping the debt risk-free. When the government accumulates assets rather than have debt, it can implement the complete markets Ramsey allocation, as shown by Aiyagari, Marcet, Sargent, and Seppälä (2002). Insuring taxpayers at all horizons against adverse macro shocks always comes at a large debt service cost to the Treasury in a model with plausible asset prices.

Brunnermeier, Merkel, and Sannikov (2020) link incomplete risk sharing between agents to a bubble component in debt. Other equilibrium models that generate violations of the TVC are Samuelson (1958); Diamond (1965); Blanchard and Watson (1982). Such violations typically show up in all long-lived assets, including stocks, not just government debt. As our paper shows, it is not easy to generate violations from transversality when there is enough permanent output risk in the economy to match the equity risk premium. Put differently, violations of the TVC for government debt may also result in violations of the TVC for the stock market.

In recent work, Mian, Straub, and Sufi (2020a,b) examine the distributional implications of government debt issuance, pointing out that the wealthy buy a large share of government (and private) debt. To the extent that the Gini coefficient of government debt holdings exceeds that of taxes, the government is trading off insuring the rich versus insuring the middle class.

The paper is organized as follows. Section 1 derives the general trade-off between the insurance of bondholders and taxpayers. When the government commits to plausible spending and tax revenue policies, the debt will generally be risky. We characterize these risk premia in closed form. Section 2 develops a simple model with permanent shocks to output and to the investor's marginal utility. The governments commits to a spending policy and a debt policy. We solve for the tax policy that keeps the debt risk-free. We start with the case of constant debt/output ratios. Section 3 introduces time-varying debt/output ratios. Section 4 characterizes the trade-off faced by the government at different horizons. Section 5 explores convenience yields as a resolution of the U.S. puzzle. Appendix A contains the proofs. Appendix B develops a model without permanent shocks. With only transitory shocks to output and marginal utility, the government is able to insure taxpayers over longer horizons. However, this model has counterfactual asset pricing implications. The only model in which the government can insure taxpayers at all horizons is one in which the output shocks are transitory, but they are priced as if they are permanent.

1 The General Trade-off between Insuring Bondholders and Taxpayers

We use T_t to denote government revenue, and G_t to denote government spending. M_t denotes the stochastic discount factor. We assume that debt is fairly priced and does not earn any convenience yields. Let B_t denote the market value of outstanding government debt at the beginning of period t, before expiring debt is paid off and new debt is issued. The debt can be long-term or short-term, and it can be nominal or real. In fact, it can be any contingent claim. Jiang et al. (2019b) show that the value of the government debt equals the sum of the expected present values of future tax revenues minus future government spending:

$$B_t \;=\; \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j}-G_{t+j})\right], \tag{1}$$

provided that there are no arbitrage opportunities in the bond market and a transversality condition holds $\lim_{k\to\infty}\mathbb{E}_t M_{t,t+k}B_{t+k}=0$. This result does not rely on complete markets, and still applies even when the government can default on the debt. Let $P_t^T=\mathbb{E}_t\left[\sum_{j=0}^{\infty}M_{t,t+j}T_{t+j}\right]$ and $P_t^G=\mathbb{E}_t\left[\sum_{j=0}^{\infty}M_{t,t+j}G_{t+j}\right]$ denote the present values of the "cum-dividend" tax claim and spending claim. Value additivity then implies that $B_t=P_t^T-P_t^G$. The value of a claim to surpluses equals the value of a claim to taxes minus the value of a claim to spending.

1.1 The Government Debt Risk Premium

For notational convenience, let $D_t=B_t-S_t$ denote the difference between the market value of outstanding government debt and the government surplus. By the government budget condition, D_t is the market value of outstanding government debt at the end of period t, after expiring debt is paid off and new debt is issued.

Let R_{t+1}^D, R_{t+1}^T and R_{t+1}^G denote the holding period returns on the bond portfolio, the tax claim, and the spending claim, respectively:

$$R^D_{t+1} = \frac{B_{t+1}}{B_t - S_t}, \qquad R^T_{t+1} = \frac{P^T_{t+1}}{P^T_t - T_t}, \qquad R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t}.$$

In Jiang et al. (2019b), we show that the government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] \;=\; \frac{P^T_t - T_t}{D_t}\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] - \frac{P^G_t - G_t}{D_t}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right]. \tag{2}$$

This result only relies on equation (1) and additivity.

The government bond risk premium varies dramatically across countries. In some countries, such as the U.S., this risk premium $\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right]$ is small. Hall and Sargent (2011) compute a real return of 168 basis points on all U.S. Treasuries. Jiang et al. (2019b) update this calculation and compute a risk premium of 111 basis points. The returns on debt issued by peripheral or developing countries are estimated to be much higher. Using EMBI indices on a short sample, Borri and Verdelhan (2011) estimate annual excess returns between 4% and 15%. On a much longer sample going back to the 19th century, Meyer, Reinhart, and Trebesch (2019) estimate excess returns of around 4% above U.S. and U.K bond returns, taking into account defaults.

1.2 Characterizing the Trade-Off with Return Betas

By rearranging equation (2), we derive the following expression for the risk premium on the tax claim:

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{P^G_t - G_t}{D_t + (P^G_t - G_t)}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] + \frac{D_t}{D_t + (P^G_t - G_t)}E_t\left[R^D_{t+1} - R^f_t\right]. \tag{3}$$

Governments typically want a counter-cyclical spending claim, i.e. they want to spend more in recessions. On the other hand, they also want a risky tax claim, because they want to reduce the tax burden in recessions. As a result, the tax claim's risk premium $\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right]$ is high and the spending claim's risk premium $\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right]$ is low. When the debt value D_t is positive, the fraction $\frac{P^G_t - G_t}{D_t + (P^G_t - G_t)}$ is between 0 and 1. Then, for equation (3) to hold, it requires a high risk premium $\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right]$ on the government debt portfolio. As the debt risk premium is a measure of the risk premium or insurance premium charged by bondholders, the government's debt portfolio has to be risky.

According to equation (3), the tax revenue claim is the unlevered version of the spending claim, or, equivalently, the spending claim is the levered version of the tax claim. This result is analogous to the Miller-Modigliani relation between the unlevered return on equity (the return on the tax claim) and the levered return on equity (the return on the spending claim).

We define the beta of an asset i as:

$$\beta_t^i = \frac{-cov_t\left(M_{t+1}, R_{t+1}^i\right)}{var_t(M_{t+1})}.$$

By the investor's Euler equation, $\beta_t^i \lambda_t$ determines the conditional risk premium of this asset

$$\mathbb{E}_t\left[R_{t+1}^i - R_t^f\right] = \beta_t^i \cdot \lambda_t,$$

where the market price of risk is $\lambda_t = R_t^f \cdot var_t(M_{t+1})$.

Let β_t^D, β_t^T and β_t^G denote the beta of the bond portfolio, the tax claim, and the spending claim, respectively. We assume $\beta_t^Y > 0$, so that the output claim has a positive risk premium. The following proposition characterizes the relationship of these risk exposures.

Proposition 1. *The beta on the tax claim is a weighted average of the beta of the spending claim and the beta of the debt:*

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)}\beta_t^G + \frac{D_t}{D_t + (P_t^G - G_t)}\beta_t^D.$$

The proof is in Appendix A.2. Governments want to provide insurance to transfer recipients by choosing $\beta_t^G < \beta_t^Y$, but they also want to provide insurance to taxpayers by choosing $\beta_t^T > \beta_t^Y$. However, the following corollary states that $\beta_t^G < \beta_t^Y < \beta_t^T$ is impossible if the government debt is risk-free.

Corollary 1. *In order for debt to be risk-free ($\beta_t^D = 0$), the beta of the tax claim needs to equal the unlevered beta of the spending claim:*

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)}\beta_t^G.$$

If the government has a positive amount of risk-free debt $D_t > 0$, there is no scope to insure taxpayers. Instead, the taxpayers provide insurance to the rest of the economy.

Consider the first case in which the spending claim has a positive beta ($\beta_t^G > 0$). Then, the government engineers risk-free debt by lowering the beta of the tax claim relative to that of the spending claim: $\beta_t^T < \beta_t^G$. A low beta for the tax claim means that tax revenue must fall by less than GDP in a recession. Tax rates must rise in recessions. The more debt there is outstanding, the lower the beta of the tax claim needs to be relative to that of the spending claim. With more debt, the trade-off between insuring bondholders and taxpayers becomes steeper. The restriction on the betas holds true regardless of the specific dynamics of the tax and spending process. In the next sections, we will derive restrictions on the underlying cash flows by committing to particular processes for debt/output and spending/output.

The only way the government can provide insurance to debt holders, while keeping the debt risk-free, is by saving—choosing $D_t < 0$. In other words, the government can only insure taxpayers at the expense of bondholders.[3]

Consider the second case in which the spending claim has a negative beta ($\beta_t^G < 0$). To ensure risk-free debt, the tax claim must also have a negative beta when $D_t > 0$ ($\beta_t^T < 0$). The taxpayers have large tax payments during recessions; they are insuring the bondholders.

This discussion implicitly assumes that taxpayers are long-lived households who value a dollar in each aggregate state in the same way as the marginal investor in Treasury markets. When markets are incomplete, agents may have different IMRS. However, even when markets are incomplete, the aggregate component of households' IMRS will be common.[4] The trade-off we analyze applies equally to incomplete markets settings.

1.3 Characterizing the Trade-Off with Cash Flow Betas

Thus far, we have characterized the return betas of the tax and spending claims. We can get further insight on what restrictions risk-free debt imposes on surplus dynamics by studying cash-flow betas for the surplus claim.

Proposition 2. *When the debt is risk-free, the present discounted value of future government surpluses is measurable in the previous period:*

$$(\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} M_{t+1,t+j} S_{t+j} = 0.$$

The proof is in Appendix A.3. This expression implies a similar restriction of the "cash flow betas" of future discounted surpluses:

$$-cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} M_{t+1,t+j} S_{t+j} \right) = 0.$$

That is, for the government debt to be risk-free, the cash flow beta of the entire discounted surplus

[3] Aiyagari et al. (2002) show that it is optimal for a government issuing only risk-free one period debt to accumulate savings $D_t \ll 0$ in the limit. This makes perfect sense, because that allows the government to choose $\beta_t^T \gg \beta_t^G$ and insure taxpayers against macro shocks. In the limit, by accumulating sufficient assets, the government can implement the Lucas and Stokey (1983) complete markets allocation.

[4] Krueger and Lustig (2010); Werning (2015) show that risk premia are identical to those in the equivalent representative agent economy, as long as the conditional distribution of idiosyncratic risk does not depend on the aggregate state of the economy. The only effect from incomplete markets is that the risk-free rate is lower due to a precautionary savings effect.

stream must be zero. This result follows directly from the fact that the debt D_t equals the present discounted value of all future surpluses, and that risk-free debt imposes a zero return beta of debt ($\beta_t^D = 0$). For debt to be risk-free, the discounted sum of surpluses must not respond to any future shock.

This proposition implies a restriction on the dynamics of future surpluses in response to any shock that arrive at time $t+1$. If this shock raises the future surpluses in the short term, either the future surpluses in the long term or the discount rates have to adjust. Below, we consider models with simple discount rate dynamics, and focus on the joint restriction on future surpluses.

2 The Insurance Trade-off in a Benchmark Economy

We characterize the trade-off between insuring debtholders and taxpayers in a canonical macro-finance model in the tradition of Breeden (2005); Lucas (1978); Rubinstein (1974). We reverse-engineer the revenue process T that keeps the debt risk-free. We do so under simple spending and debt policies in this section and more complex policies in the next section.

2.1 Characterizing the Trade-off

Consider an economy with permanent output shocks and a homoscedastic stochastic discount factor (SDF):

Assumption 1. *(a) Let Y_t and $y_t = \log Y_t$ denote output and its log. All output shocks are i.i.d. and permanent:*

$$y_{t+1} = \mu + y_t + \sigma\varepsilon_{t+1},$$

where ε_{t+1} denotes the innovation to output growth that is i.i.d. normally distributed with mean zero and standard deviation one.
(b) The log SDF is given by:

$$m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}.$$

(c) The government only issues one-period real risk-free debt.

Note that the one-period risk-free rate in this model is constant and equal to ρ.

To build intuition for the general trade-off between insurance of bondholders and taxpayers, we start by considering the simplest case of constant spending/output and debt/output ratio policies.

Assumption 2. *(a) The government commits to a constant spending/output ratio $x = G_t/Y_t$.*
(b) The government commits to a constant debt/output ratio $d = D_t/Y_t$.

Under Assumption 2, the government budget constraint implies a counter-cyclical process for tax revenue-to-GDP (the tax rate):

$$\frac{T_t}{Y_t} \quad = \quad \frac{G_t}{Y_t} - \frac{D_t}{Y_t} + R^f_{t-1}\frac{D_{t-1}}{Y_{t-1}} = x - d\left(1 - \exp\left\{-(\mu - \rho + \sigma\varepsilon_t)\right\}\right).$$

To perfectly insure the bondholders by keeping the debt risk-free, the government must make the tax revenue claim counter-cyclical: $\partial(T/Y)\ \partial\varepsilon < 0$. When the growth rate of output is low ($\varepsilon < 0$), tax revenue needs to increase as a fraction of GDP. Tax rates must rise in recessions. The magnitude of the counter-cyclical exposure is increasing in the debt-to-GDP ratio d.

Similarly, the primary surplus/output ratio is counter-cyclical:

$$s_t = \frac{S_t}{Y_t} = \frac{T_t - G_t}{Y_t} = -d\left(1 - \exp\left\{-(\mu - \rho + \sigma\varepsilon_t)\right\}\right). \tag{4}$$

We have that $\partial s_t/\partial\varepsilon_t < 0$. When the unconditional growth rate of output exceeds the risk-free rate ($\mu > \rho$), the government runs a primary deficit on average. But when shocks are negative enough ($\mu - \rho < -\sigma\varepsilon$), the government must run a primary surplus.

This simple model places tight restrictions on the persistence of surpluses. The conditional auto-covariance of the surplus/output ratio is zero: $cov_t(s_t, s_{t-1}) = 0$. The government cannot run persistent deficits. When $\sigma \to 0$, the government always runs deficits. But $\mu > \rho$ now implies a violation of the TVC, as we show below. This result is more general. With risk-free debt, the autocorrelation of surpluses tends to zero as the persistence of the debt/output ratio tends to one.

The restrictions on the surplus and tax processes described above were independent on the SDF model. Next, we turn to valuing the debt as the expected present-discounted value of future surpluses.

Proposition 3. *Under Assumptions 1 and 2, if the transversality condition holds and the primary surplus satisfies (4), the government debt value is the sum of the values of the surplus strips:*

$$D_t \quad = \quad \mathbb{E}_t\left[\sum_{k=1}^{\infty} M_{t,t+k}S_{t+k}\right] = dY_t.$$

The proof is in Appendix A.4. This proposition confirms that the (ex-dividend) value of outstanding debt in period t is indeed a constant fraction of output. The proof solves for the price of a claim to a single future surplus realization (a surplus strip), and adding up the surplus strip prices at all horizons. The result implies that there is no news about the present discounted value of future surpluses since output is already known at time t.[5]

[5] Hansen et al. (1991) discuss a version of this condition that uses the risk-free rate when devising an econometric

Note that in this equation, the government surpluses are not discounted at the risk-free rate even though the debt is risk-free. To see why, consider the valuation equation for debt as a function of surplus/output ratios:

$$D_t = \mathbb{E}_t \left[\sum_{j=0}^{T} M_{t,t+j} Y_{t+j} s_{t+j} \right] + \mathbb{E}_t \left[M_{t,t+T} Y_{t+T} \frac{D_{t+T}}{Y_{t+T}} \right].$$

The debt/output ratio $\frac{D_{t+T}}{Y_{t+T}} = d$ in the second term is constant. The correct TVC for government debt in this model is given by:

$$\lim_{T\to\infty} \mathbb{E}_t \left[M_{t,t+T} D_{t+T} \right] = \lim_{T\to\infty} \exp \left\{ T(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma) \right\} dY_t. \tag{5}$$

This TVC is satisfied if and only if $-\rho + \mu + \frac{1}{2}\sigma^2 - \gamma\sigma < 0$. The textbook condition $\rho < \mu$ is neither necessary nor sufficient for a TVC violation. A necessary and sufficient condition is that there is enough permanent, priced risk in output: $\gamma\sigma > \mu - \rho + \frac{1}{2}\sigma^2$. The output risk premium (unlevered equity risk premium) must be high enough. This ensures that This term $\mathbb{E}_t \left[M_{t,t+T} Y_{t+T} \right] \to 0$ as $T \to \infty$. So, it is not the case that the government can always run deficits when $\rho < \mu$, at least not without violating the TVC.[6] Note that $\rho < \mu$ implies a violation of TVC only as $\sigma \to 0$. In general, the output risk premium matters even when debt is risk-free. The risk-free rate is not the correct discount rate for surpluses even when the debt is risk-free, in the presence of permanent output shocks.

Next, we turn to the main result characterizing the expected return and beta of the tax claim.

Proposition 4. *(a) The ex-dividend values of the spending and revenue claims are given by:*

$$\begin{aligned} P_t^G - G_t &= x\frac{\xi_1}{1-\xi_1} Y_t, \\ P_t^T - T_t &= \left(d + x\frac{\xi_1}{1-\xi_1} \right) Y_t, \end{aligned}$$

with $\xi_1 = \exp\left\{-\rho - \gamma\sigma + \mu + \frac{1}{2}\sigma^2\right\}$.

(b) The risk premia and betas on the tax claim and the spending claim satisfy:

$$\mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right], \tag{6}$$

approach to testing the budget constraint: $(\mathbb{E}_{t+1} - \mathbb{E}_t) \left[\sum_{k=1}^{\infty} \exp(-r_{t,t+k}^f) S_{t+k} \right] = 0$. However, this condition is equivalent to the one in the Proposition, only if the risk-free rate exceeds the growth rate of the economy. If not, this equation may fail even when the condition in Proposition 3 holds.

[6] See Bohn (1995) for an early reference on why discounting at the risk-free may fail. However, Bohn (1995) refers to this case as one in which the government runs persistent deficits, while the deficits really are uncorrelated over time. In recent work, Barro (2020) points out that comparing μ to ρ is not informative about dynamic efficiency, consistent with our result, unless $\sigma = 0$.

$$\beta^T = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\beta^G < \beta^G. \tag{7}$$

The proof is in Appendix A.5. The constant ξ_1 is the price/dividend ratio of a one-period output strip, a claim to GDP next year. The expected return on this output strip is given by $E_t\left[R^Y_{t+1}\right] = \frac{\exp(\mu+0.5\sigma^2)}{\exp(-\rho-\gamma\sigma+\mu+0.5\sigma^2)} = \exp(\rho+\gamma\sigma)$. Hence, the (log of the multiplicative) output risk premium is constant and equal to $\gamma\sigma$. Since spending is a constant fraction of output, the risk premium on the spending claim equals that of the output claim: $\mathbb{E}[R^G - R^f] = \mathbb{E}[R^Y - R^f]$. The beta of the spending claim equals the beta of the output claim: $\beta^G = \beta^Y > 0$.

The investor in government debt is long in a tax revenue claim and short in a spending claim. To make the debt risk-free, as long as the debt/output ratio d is positive, we need to render the government tax revenue process safer than the spending process. A positive d implies the fraction $\frac{x\frac{\xi_1}{1-\xi_1}}{d+x\frac{\xi_1}{1-\xi_1}}$ is between 0 and 1, which requires the return on the tax claim to be less risky than the return on the output claim: $0 < \beta^T < \beta^Y$. When output falls, tax revenues must fall by less. The tax rate increases. In other words, there is no scope to insure taxpayers. As the debt/output ratio d increases, the government needs to make the tax revenue increasingly safe. The tax claim is really a portfolio of a claim to government spending and risk-free debt. The larger the debt/output ratio d, the safer the tax claim needs to be. As the debt/output ratio approaches infinity, the beta of the tax claim tends to 0.

2.2 Quantifying the Trade-off

Panel A of Table 1 proposes a calibration of the model that matches basic features of post-war U.S. data. We set γ to 1. This parameter measures the maximum Sharpe ratio in the economy. A long asset pricing literature suggests that this is a reasonable value given high average excess returns on a broad set of risky assets. The standard deviation of output is set to $\sigma = 0.05$. The growth rate of real GDP is set to its observed value: $\mu = 3.1\%$. The real risk-free rate ρ is set to 2%. Spending accounts for 10% of GDP in post-war data: $x = 0.10$.

Note that this calibration features a risk-free rate below the growth rate of output. However, per our discussion above, the TVC is satisfied because $-\rho + \mu + \frac{1}{2}\sigma^2 - \gamma\sigma = \log(\xi_1) = -0.0418 < 0$. The government cannot simply roll over the debt. The surpluses need to satisfy tight restrictions.

Figure 1 plots the risk premia on the tax and the spending claim as we vary the debt/output ratio d. The risk premium on the spending claim is 5% per annum. This is also the output risk premium, which we can think of as an unlevered equity premium. By Corollary 4, the risk premium on the tax claim is given by (6). The risk premium on the tax claim is 5% when $d = 0$. It falls to 4% when $d = 1$, and close to 3% when $d = 2$. As the government becomes more levered,

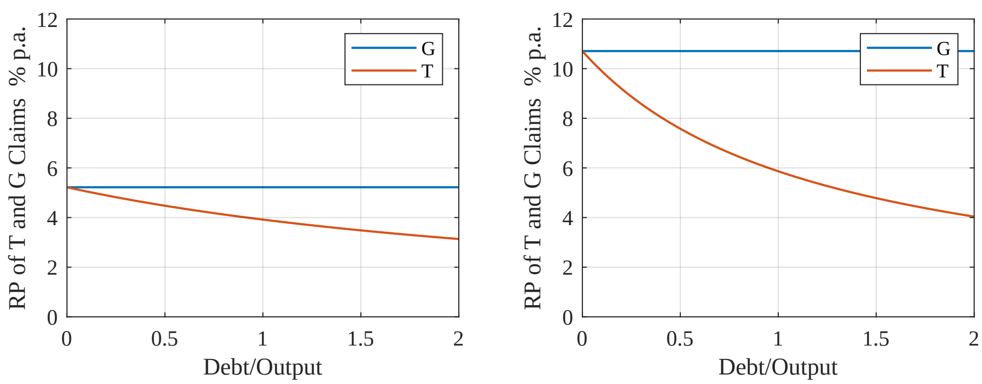
Table 1: Benchmark Calibration for U.S.

Panel A: Preferences and Output Dynamics		
γ	1	maximum annual Sharpe ratio
ρ	2.0%	real risk-free rate
μ	3.1%	mean of growth rate of output
σ	5.0%	std. of growth rate of output
Panel B: Debt/Output Ratio Dynamics		
λ	$1.94 \times \sigma$	sensitivity of debt/output to output innovations
$d = \exp\{\phi_0/(1-\phi_1-\phi_2)\}$	0.43	mean of debt/output
ϕ_1	1.40	AR(1) coeff of debt/output
ϕ_2	-0.48	AR(2) coeff of debt/output
Panel C: Government Spending/Output Ratio Dynamics		
β^g	$1.53 \times \sigma$	sensitivity of spending/output to output innovations
φ_1^g	0.88	AR(1) coeff of spending/output
$x = \exp\left\{\varphi_0^g/(1-\varphi_1^g)\right\}$	0.10	mean of govt. spending/output

the tax claims needs to become safer for debt to remain risk-free. The scope for taxpayer insurance disappears. This trade-off steepens when we increase the maximum Sharpe ratio γ from 1 to 2. When $\gamma = 2$, the risk premium on the spending claim is 10% per annum. The risk premium on the tax claim falls to 6% when $d = 1$ and close to 4% when $d = 2$.

Figure 1: Risk Premium of T and G Claims with $\gamma = 1$ or 2

The figure plots the implied risk premium of the T and G claims when the debt/output ratio and spending/output ratio are constant. The figure plots two values for the maximum Sharpe ratio γ of 1 (left panel) and γ of 2 (right panel). The other parameters are given in Table 1.



3 Model with State-Contingent Debt/Output

The previous section showed that there is no scope for insuring taxpayers at any horizon in the presence of permanent output shocks when the debt/output ratio is constant. Next, we allow the government to introduce state-contingent variation in the debt/output ratio. This will create limited opportunities for the government to temporarily insure taxpayers over short horizons.

3.1 Characterizing the Trade-Off with Counter-cyclical Debt/Output

We allow the government to vary the debt/output ratio counter-cyclically.

Assumption 3. *The government commits to a policy for the debt/output ratio $d_t = D_t/Y_t$ given by:*

$$\log d_t = \sum_{p=1}^{P} \phi_p \log d_{t-p} + \phi_0 - \lambda \varepsilon_t - \frac{1}{2}\lambda^2,$$

where $\lambda > 0$ so that the debt-output ratio increases in response to a negative output shock ε_t.

The results in Section 1 still apply and are a straightforward generalization of the results from the simple benchmark model of Section 2. The value of the spending is unchanged and the value of the tax claim now depends on the time-varying debt/output ratio d_t:

$$P_t^G - G_t = x\frac{\xi_1}{1-\xi_1}Y_t, \qquad P_t^T - T_t = \left(d_t + x\frac{\xi_1}{1-\xi_1}\right)Y_t.$$

The tax claim's conditional beta satisfies:

$$\beta_t^T \quad = \quad \frac{x\frac{\xi_1}{1-\xi_1}}{d_t + x\frac{\xi_1}{1-\xi_1}}\beta_t^G.$$

Can the government systematically issue more risk-free debt, instead of raising taxes, when the economy is hit by a permanent, adverse shock, in order to break the restriction on insurance of taxpayers? We consider two special cases for the debt/output dynamics.

Case 1: AR(1) Assume that the debt/output ratio evolves according to an $AR(1)$-process:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2.$$

There are two sub-cases. First, when $0 < \phi_1 < 1$, the debt/output process is stationary. Second, when $\phi_1 = 1$ and $\phi_0 = 0$, the debt/output process is a martingale (non-stationary). In both cases, a positive λ means that the debt/output ratio increases when the shock ε_t is negative, implying a counter-cyclical debt policy.

First, we need to make sure the transversality (TVC) is satisfied. How persistent can debt be without violating TVC?

Proposition 5. *Under Assumptions 1 and 3 with the maximal lag $P = 1$, (a) when $0 < \phi_1 < 1$, the TVC condition is satisfied if and only if:*

$$\log(\xi_1) = -\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0.$$

(b) When $\phi_1 = 1$ and $\phi_0 = 0$, then the TVC condition is satisfied if and only if:

$$\log(\xi_1) + \lambda(\gamma - \sigma) = -\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) + \lambda(\gamma - \sigma) < 0.$$

The proof is in Appendix A.6. For the case of $0 < \phi_1 < 1$, the TVC is satisfied whenever the price-dividend ratio of a claim to next period's output is less than one. That is, when investors are willing to pay less than Y_t today for a claim to Y_{t+1}. This requires the discount rate to exceed the growth rate of GDP (modulo a Jensen adjustment). This condition can be satisfied even when $\rho < \mu$, as long as the risk premium $\gamma\sigma$ is large enough.

For the random walk case in which $\phi_1 = 1$, the same condition ensures that the TVC is satisfied when the government does not pursue counter-cyclical stabilization ($\lambda = 0$). If the government does pursue counter-cyclical stabilization ($\lambda > 0$), then the TVC is only satisfied if

$$\gamma\sigma - \lambda(\gamma - \sigma) > -\rho + \mu + \frac{1}{2}\sigma^2 \Leftrightarrow \lambda < \frac{\rho + \gamma\sigma - \mu - \frac{1}{2}\sigma^2}{\gamma - \sigma}.$$

The left-hand side of the first inequality is now lower than before when the Sharpe ratio of the economy exceeds the volatility of output ($\gamma > \sigma$). When debt issuance is sufficiently counter-cyclical, $\lambda > \sigma$, the expression on the left-hand side is decreasing in the economy's maximum Sharpe ratio γ. For high enough γ, the TVC is violated. Intuitively, when investors are risk averse enough, the insurance provided by the counter-cyclical debt issuance policy is so valuable that the price of a claim to the debt outstanding in the distant future $d_{t+T}Y_{t+T}$ fails to converge to zero. This claim is a terrific hedge. This is the first important insight contributed by asset pricing theory. If output is subject to permanent, priced risk and we want to rule out arbitrage opportunities then there have to be limits to the government's ability to pursue counter-cyclical debt issuance. This bound on λ is shown in the second inequality. When the government exceeds this bound, it has granted itself an arbitrage opportunity.

Case 2: AR(2) As we show below, a better description of the debt/output ratio is the data is an AR(2) process:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2. \tag{8}$$

When the roots of the characteristic equation $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle, the debt/output process is mean-reverting. The result of part (a) of Proposition 5 applies. If one or both roots are smaller than one, the result in part (b) of Proposition 5 applies.

Response of the Surplus to Adverse Shock We can compute the impulse-response functions (IRF) of the surpluses with respect to an output shock in closed form when the government issues risk-free debt. These moments are particularly powerful because they do not depend on the properties of the SDF.

We start from the expression for the surplus/output ratio in period $t+j$ for $j \geq 1$:

$$s_{t+j} = \frac{S_{t+j}}{Y_{t+j}} = d_{t+j-1} \exp(\rho - \mu - \sigma \varepsilon_{t+j}) - d_{t+j}.$$

If we assume that the risk-free rate equals the growth rate of the economy ($\mu = \rho$), we obtain closed-form expression for the IRF of the surplus with respect to an output shock. Specifically, the IRF is evaluated at $\varepsilon_\tau = 0$ for all τ, and hence $d_t = \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) = d$.

Proposition 6. *If Assumptions 1 and 3 hold, the TVC is satisfied, and $\rho = \mu$,*
(a) when the debt/output ratio follows an $AR(1)$ process, the IRF of the surplus/output ratio is given by:

$$\begin{aligned} \frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)d, \text{ for } j = 1, \\ &= \lambda \phi_1^{j-1}(\phi_1 - 1)d, \text{ for } j > 1. \end{aligned}$$

(b) when the debt/output ratio follows an $AR(2)$ process, the IRF of the surplus output ratio is given by:

$$\begin{aligned} \frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)d, \text{ for } j = 1, \\ &= \lambda(\phi_1 - 1)d, \text{ for } j = 2, \\ &= \lambda(\psi_{j-1} - \psi_{j-2})d, \text{ for } j > 2. \end{aligned}$$

where $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2}, j > 2; \psi_2 = \phi_2 + \phi_1 \psi_1; \psi_1 = \phi_1$.

(c) When the debt/output ratio follows an AR(3) process, the IRF of the surplus output ratio is given by:

$$\begin{aligned}\frac{\partial \frac{S_{t+j}}{Y_t}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)d, \textit{ for } j = 1,\\ &= \lambda(\phi_1 - 1)d, \textit{ for } j = 2,\\ &= \lambda(\psi_2 - \psi_1)d, \textit{ for } j = 3,\\ &= \lambda(\psi_{j-1} - \psi_{j-2})d, \textit{ for } j > 3.\end{aligned}$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2} + \phi_3\psi_{j-3}, j > 3; \psi_3 = \phi_3 + \phi_2\psi_1 + \phi_1\psi_2; \psi_2 = \phi_2 + \phi_1\psi_1; \psi_1 = \phi_1.$

The proof is in Appendix A.7. For an $AR(1)$, the initial response of the surplus is positive in the empirically relevant case where $\lambda > \sigma$. That is, a negative shock to output is countered with a large enough government debt issuance that the surplus in the initial period can be negative without jeopardizing the risk-free nature of the debt. However, the deficit must turn to a surplus starting in the second year since $\phi_1 < 1$. Surpluses remain in the years that follow. As the persistence of the debt/output process ϕ_1 increases, the response of the surplus/output ratio converges to zero in year 2 and beyond.

For an $AR(2)$, by choosing $\phi_1 > 1$, the government can run a deficit in the year of the shock (year 1) as well as in the following year. In year 3, the IRF equals $\lambda(\psi_2 - \psi_1) = \lambda(\phi_2 + \phi_1(\phi_1 - 1))$. This expression can be positive or negative depending on parameter values but is smaller than the response in year 2. In other words, the government's ability to run a third year of deficits in response to the negative output shock is either limited or gone. The IRF flips sign in year 3 or 4. The government must revert to running surpluses as the ACFs decline: $\psi_{j-1} < \psi_{j-2}$.

With higher-order $AR(p)$ models for debt/output, the government is able to run deficits for longer before a reversal. For example, for an AR(3), there is an additional year of deficits possible while keeping debt risk-free. These deficits must be made up by several years of surpluses afterwards. The surplus dynamics can display more pronounced hump-shaped IRFs. However, as shown below, there is limited empirical support for higher-order $AR(p)$ dynamics (i.e., $p > 2$) in the observed US debt/output process.

Persistence of the Surplus The auto-covariance of the surplus/output ratio is defined as follows:

$$cov_t(s_{t+1}, s_{t+j}) = \mathbb{E}_t[s_{t+1}s_{t+j}] - \mathbb{E}_t[s_{t+1}]\mathbb{E}_t[s_{t+j}].$$

In the case of an $AR(1)$ for debt/output, we can show that the conditional autocovariance declines to zero as the persistence of the debt/output process grows: $\lim_{\phi_1 \to 1} cov_t(s_{t+1}, s_{t+j}) = 0$. Recall that in the case of a constant debt/output ratio considered in the previous section, surplus/output ratios were uncorrelated at all horizons. This result is a natural extension.

Predictability of the Surplus When debt is risk-free, an increase in debt today needs to be followed by higher future surpluses. For the realistic case of an AR(2) process for debt/output, those surpluses must begin 2–3 years after the initial increase in debt. In the model, the debt/output ratio is a strong predictor of future surpluses. Below, we study this predictability relationship in the model and contrast it to that in the data.

In sum, insisting on debt to be risk-free imposes tight constraints on (i) how much and how long the government can run deficits in response to an adverse shock, (ii) on the persistence of the surplus/output ratio, and (iii) on the predictability of future surplus/output ratios by the current debt/output ratio.

3.2 Quantifying the Trade-Off with Counter-cyclical Debt/Output

Persistence of Fiscal Processes in the Data Panel A of Figure 2 plots the sample autocorrelation function (ACF) of the log government debt/output ratio as a function of the number of annual lags. The top right panel plots the partial autocorrelation function (PACF). They are estimated on the post-war U.S. sample (1947–2019). The PACF function indicates that an AR(2) process fits the data well. Lags beyond two years in the PACF are not statistically different from zero. The point estimates for ϕ_1 and ϕ_2 are 1.40 and -0.48, respectively. Both roots lie outside the unit circle (1.66 and 1.25), so that the debt/output process is stationary. While the AR(2) is our preferred specification, if we were to fit an AR(1), the point estimate for ϕ_1 would be 0.986.

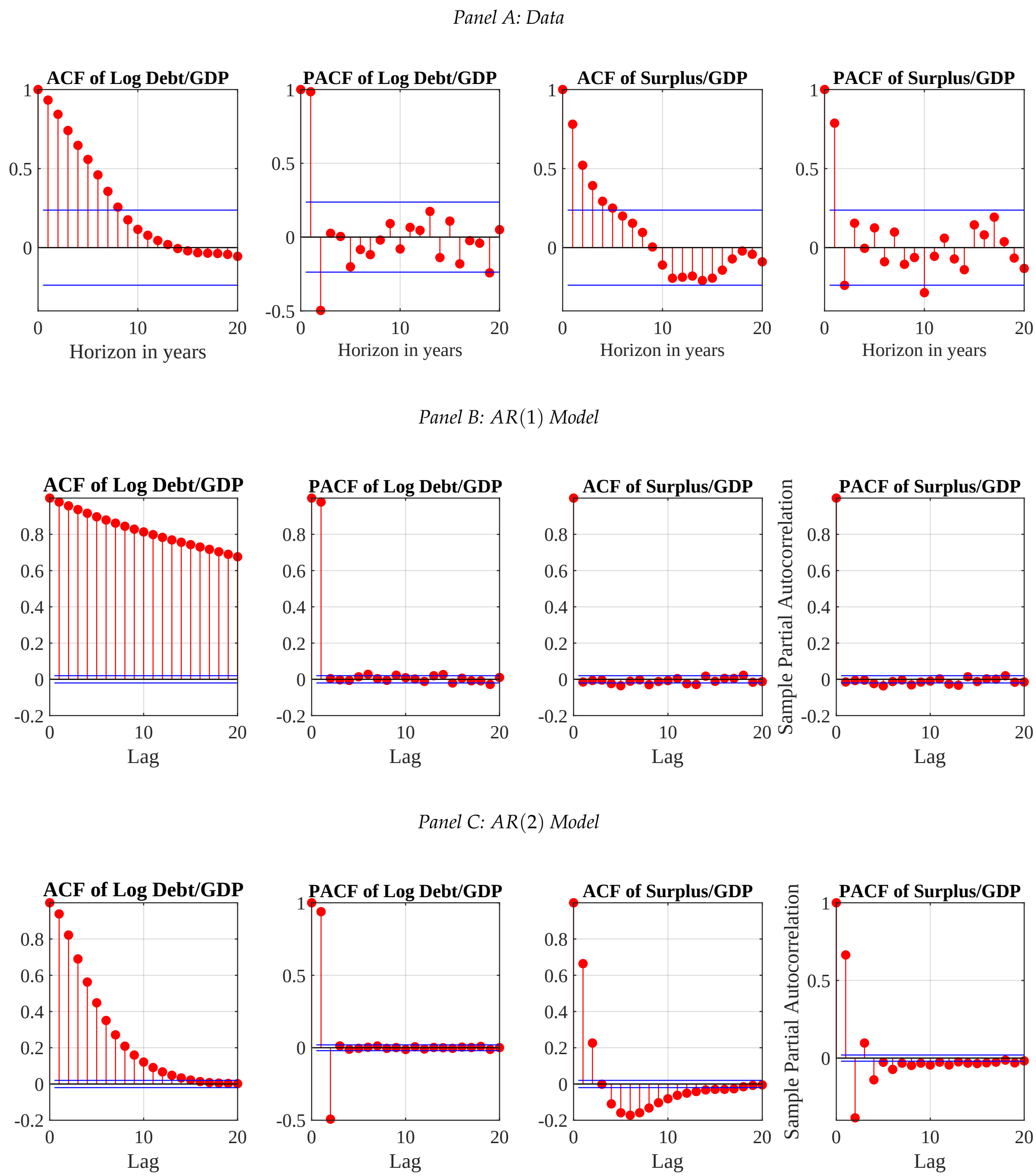
We set ϕ_0 to match the unconditional mean of the debt/output ratio of 0.43. Finally, we set $\lambda = 1.953 \times \sigma$ equal to match the slope coefficient in a regression of the debt/output ratio innovations on GDP growth in the post-war U.S. sample. A one percentage point increase in GDP growth lowers the debt/output ratio by 1.95 percentage points. The calibration is reported in Table 1.

Given our values of $\sigma = 0.05$, $\gamma = 1$, and $\lambda = 1.96\sigma = 0.098$, we have $\gamma\sigma - \lambda(\gamma - \sigma) < 0$. If debt/output were non-stationary (have roots inside the unit circle), then this much counter-cyclicality would result in a violation of the TVC condition. The coefficient λ would need to remain below 0.85σ, which is only half of its empirical value, for the TVC to be satisfied in this case. Once we exceed this upper bound, the value of outstanding debt explodes. To be clear, the data suggest that the debt/output ratio is stationary, in which case the TVC is satisfied irrespective of the value for λ. Indeed, the parameter restriction in part (a) of Proposition 5 is satisfied. This is the case despite the risk-free interest rate being below the growth rate of output, because the risk premium $\gamma\sigma$ is large enough.

Panel A of Figure 2 also plots the sample ACF and PACF for the primary surplus/output ratio in the data. The dynamics of surplus/output are well described by an AR(1). The surplus is quite persistent, with an $AR(1)$ coefficient around 0.81.

Figure 2: Autocorrelation in Debt/Output and Surplus/Output

Panel A plots the sample autocorrelation of the U.S. log government debt/output ratio, the U.S. government surplus/output ratio, the tax/output ratio and the spending/output ratio against GDP. Sample is annual, 1947—2019. Panel B plots the ACF and PACF of S/Y and D/Y for an $AR(1)$ with parameters $\phi_1 = 0.985$ and $\phi_2 = 0$. Panel C plots the ACF and PACF of S/Y and D/Y for an $AR(2)$. The parameters are listed in Table 1.



Persistence of Fiscal Processes in Model with Risk-free Debt We now show that the risk-free debt model cannot simultaneously match the high persistence of the debt/output ratio and that of the surplus/output ratio. Figure 2 also plots the ACF and PACF of the debt/output and surplus/output ratios implied by the model of risk-free debt. Panel B is for the case where debt/output follows an AR(1) with the estimated persistence $\phi_1 = 0.985$. Panel C is for the case where debt/output follows an AR(2) with the estimated coefficients $\phi_1 = 1.4$ and $\phi_2 = -0.48$. The ACF and PACF for debt/output match the data by construction. As argued above, the AR(2) fits the ACF and PACF of the observed debt/output ratio the closest.

The key observation is that insisting on risk-free debt produces counter-factual ACF and PACF for the surplus/output ratio. In particular, the model cannot replicate the strong autocorrelation in surplus/output observed in the data. In the case of the AR(1), the ACF is zero from horizon 1 onwards. In the case of the AR(2), the ACF converges much too quickly to zero, compared to the observed one plotted in Panel A of Figure 2. The ACF is no longer different from zero past two years, while in the data the ACF remains significantly positive for five years. Furthermore, the model produces a PACF(2) coefficient of -0.5, which is larger in absolute value than the one estimated in the data.

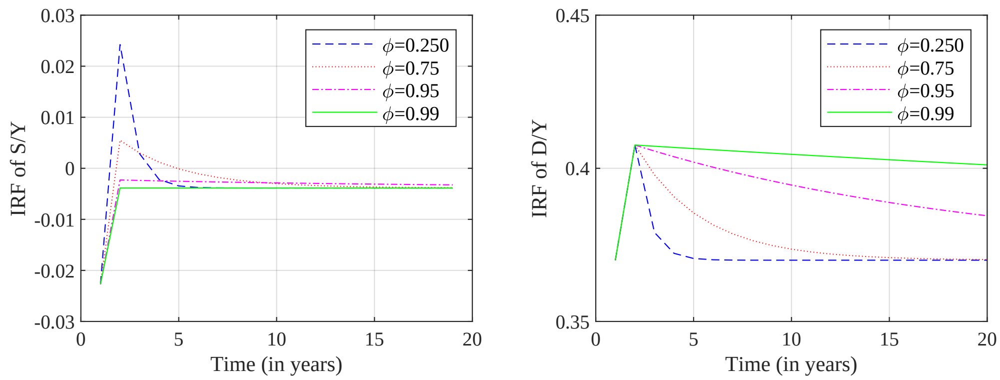
Impulse Responses in the Model The top left panel of Figure 3 plots the response of the surplus/output ratio to a negative shock to output, when debt/output follows an AR(1) process. Each line corresponds to a different autocorrelation, with ϕ_1 ranging from 0.25 to 0.99. The top right panel plots the response of the debt/output ratio. Upon impact, the debt/output ratio increases by about 4% from its mean. After that, the rate of mean-reversion is governed by ϕ_1. In the least persistent case ($\phi = 0.25$), the government runs a large surplus after the initial period deficit to bring the debt back down quickly. In the most persistent debt case ($\phi_1 = 0.99$), the initial deficit is followed by a reversal in the next period as the surplus jumps to just above its long-run value of $\bar{s} = 0$ and then slowly converges to $\bar{s}$ from above. Note that when $\rho < \mu$, the government can run a small steady-state deficit $\bar{s} = -d\,(1 - \exp(\rho - \mu)) < 0$. In sum, when the debt/output ratio follows an $AR(1)$ and the debt is risk-free, there can be no S-shaped response of the surplus/output ratio to the output shock.

Panel B of Figure 3 plots the IRF when debt/output follows an AR(2), our preferred empirical specification. We vary ϕ_1 from 1.1 to 1.4 and choose ϕ_2 to match the first-order autocorrelation of debt/output. With $\phi_1 = 1.1$, the IRF looks similar to the $AR(1)$ case with ϕ_1 close to 1. However, with $\phi_1 = 1.4$ and $\phi_2 = -0.48$, the point estimates from the data, the IRF for the surplus/output ratio displays a hump-shaped pattern. Consistent with the results in Proposition 6, a state-contingent and persistent debt issuance policy enables the government to delay the fiscal adjustment. The deficit/output ratio in the year of the shock is followed by an even larger deficit in year 2. However, the deficit must shrink dramatically in year 3 and turn into a surplus starting in year 4 and beyond. The surplus eventually converges back to $\bar{s}$ from above. Keeping debt risk-free still imposes severe restrictions on the size of the S-shaped surplus dynamics. Running sizeable deficits for more than two years is incompatible with risk-free debt.

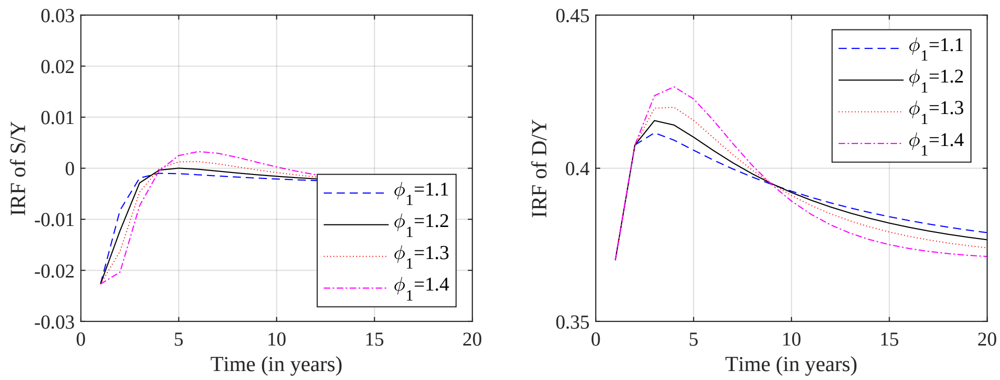
Figure 3: IRF of Surplus/Output and Debt/Output in Model

The figure plots the IRF of S/Y and D/Y for an $AR(1)$ (top panel) and an $AR(2)$ (bottom panel). In Panel B, ϕ_2 is chosen to match the first-order autocorrelation. The other parameters are given in Table 1.

Panel A: $AR(1)$



Panel B: $AR(2)$



Bohn (1998); Canzoneri et al. (2001); Cochrane (2019, 2020) find evidence of S-shaped dynamics in the U.S. surplus/output ratios. These authors argue that such surplus dynamics are consistent with budget balance. Our results show that the S-shaped surplus dynamics in the data violate the risk-free debt condition. Governments cannot defer running a surplus for more than 2 years after output declines, if they want to keep the debt risk-free.

Predictability of Surplus/Output Table 2 reports the results for the predictability regressions;

$$
\begin{aligned}
s_{t+j} &= a_j + b_j d_t + e_{t+j} \\
s_{t+j} &= a_j + b_j d_t + c_j s_t + e_{t+j}
\end{aligned}
$$

Table 2: Forecasting Surplus/Output Ratios

Panel A is for post-war annual U.S. data (1947-2019). Panel B is for a 10,000 period simulation of the AR(2) model with parameters given in Table 1. We forecast the primary surplus/output ratios two to five years hence with the current debt/output ratio and the current surplus/output ratio. The table reports the regression coefficients and R^2 statistics for $s_{t+j} = a_j + b_j d_t + c_j s_t + e_{t+j}$.

Horizon j	1	2	3	4	5
Panel A: U.S. Data					
Specification 1					
b_j	-0.031	-0.0099	0.013	0.023	0.028
[$s.e.$]	[0.023]	[0.025]	[0.03]	[0.031]	[0.03]
R^2	0.043	0.0041	0.006	0.018	0.024
Specification 2					
b_j	0.0085	0.018	0.036	0.042	0.044
[$s.e.$]	[0.01]	[0.015]	[0.019]	[0.02]	[0.021]
c_j	0.81	0.57	0.47	0.37	0.33
[$s.e.$]	[0.087]	[0.13]	[0.12]	[0.11]	[0.10]
R^2	0.64	0.30	0.21	0.15	0.13
Panel B: AR(2) Model with Risk-free Debt					
Specification 1					
b_j	0.0629	0.117	0.132	0.127	0.114
R^2	0.0781	0.271	0.342	0.316	0.254
Specification 2					
b_j	0.0701	0.12	0.132	0.126	0.112
c_j	0.695	0.265	0.045	-0.055	-0.11
R^2	0.560	0.342	0.345	0.319	0.266

for horizons $j = 1$ through $j = 5$. The results for the data are in Panel A while the results for the AR(2) model are in Panel B.

In the data, the lagged debt/output ratio has little to no predictive power for the future surplus/output ratio. Lagged surpluses are much better predictors and substantially increase the regression R^2. In contrast, in the model, the debt/GDP ratio has strong forecasting power for future surplus/output ratios, with a positive sign. This predictability is changed little even when we control for lagged surplus/output ratios. The R^2 is 25%–26% at the five-year horizon. At horizons up to 2 years, the lagged surplus/output ratio also forecasts future surplus/output ratios with a positive sign. After 2 years, the lagged surplus/output ratio has no incremental forecasting power over and above that of the lagged debt/output ratio. In sum, the risk-free debt model is unable to generate the predictability patterns in the data. It generates too much predictability by debt/output and too little predictability by the lagged surplus/output ratio. The latter is consistent with our earlier finding that a model with risk-free debt generates a much faster decay in the ACF of surplus/output than what we see in the data.

Covariance of Tax and Spending with GDP Finally, we compare the covariance of tax revenue/output with output growth in model and data. Given a process for spending/output, the surplus/output ratio implies a tax revenue/output ratio from the government's budget constraint. To make the model's implications for tax revenues as comparable to the data as possible, we posit a realistic process for spending/output. Specifically, we assume that the government commits to a policy for the spending/output ratio $x_t = G_t/Y_t$ given by:

$$\log x_t = \varphi_0^g + \varphi_1^g \log x_{t-1} - b_g \varepsilon_t - \frac{1}{2} b_g^2. \tag{9}$$

When $b_g > 0$, the spending/output ratio rises in response to a negative output shock. We estimate $(\varphi_0^g, \varphi_0^g, \beta^g)$ from the post-war U.S. data. The parameter estimates are reported in Panel C of Table 1. Spending/output is counter-cyclical in the data. A 1% point decline in output coincides with a 1.53% point increase in the spending/output ratio. The persistence of spending/output matches that in the data with an AR(1) coefficient of 0.88. With this spending process in hand, we compute the model-implied tax revenue/output.

Figure 4: GDP Growth Betas of U.S. Tax Revenue and Spending

This figure reports the betas in regression of log U.S. spending G growth and log tax revenue T growth over horizon j on the concurrent GDP growth over horizon j. The blue curve plots the estimate from the data. Sample is annual, 1947—2019. The plot also shows 2 standard error bands, generated by 30,000 bootstrapped samples by drawing jointly with replacement from the 4×1 vector of innovations in the AR models for $(\log d_t, x_t, \log \tau_t, \log g_t)$. The red curve plots the coefficients implied from the model with risk-free Debt. Benchmark calibration in Table 1.

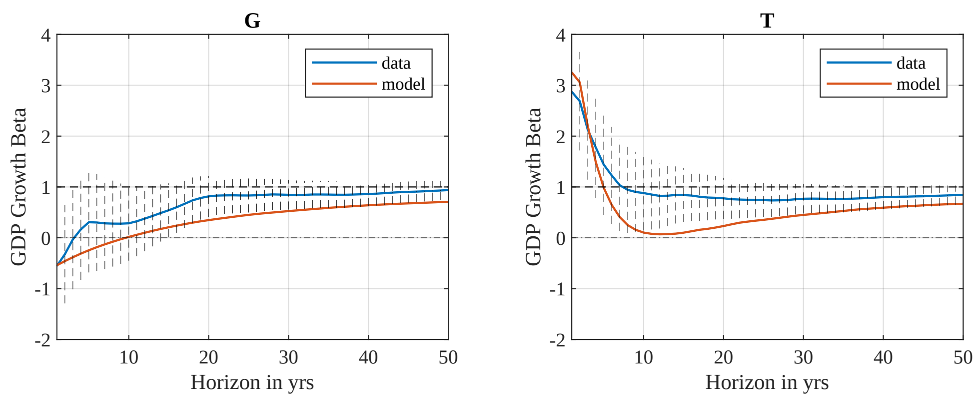


Figure 4 plots the tax revenue beta, namely the covariance of tax revenue/output with output growth divided by the variance of output growth, in the model. These betas do not depend on the properties of the SDF. They are estimated from a 10,000 period simulation of the $AR(2)$-model for the debt/output ratio, in which we calculate the beta over horizons ranging from 1 to 50 years. The model implies that the tax betas drop below the spending betas at longer horizons to ensure that the debt is risk-free.

This property of tax revenues is contrary to what we see in the data. In post-war U.S. data, the tax revenue beta is always above the spending beta at each horizon, as shown in Figure 4. In the data, tax revenues are too risky at longer horizons for the debt to be risk-free. Note that the model-implied tax betas are outside of the 2-standard-error bands around the point estimates in the data at horizons between 10 and 30 years.

3.3 Counter-cyclical Tax in the U.K.

In sharp contrast with the U.S. results, the U.K. Treasury is more conservative in trading off the insurance of taxpayers and bondholders. In fact, we find that the long-run tax and revenue betas in the U.K. are consistent with risk-free debt.

As we did for the U.S., we calibrate processes for the U.K. debt/output ratio and spending/output ratio, and compute the required tax process that makes government debt risk-free under the government's budget constraint. Table 3 reports our calibration for the U.K., with data sources listed in Appendix C. The U.K. debt issuance is substantially less counter-cyclical than in the U.S., and the spending is less counter-cyclical as well. The U.K. spending/output ratio $x_t = G_t/Y_t$ is well-described by an AR(2):

$$\log x_t = \varphi_0^g + \varphi_1^g \log x_{t-1} + \varphi_2^g \log x_{t-2} - \beta_g \varepsilon_t - \frac{1}{2}\beta_g^2. \tag{10}$$

We estimate $(\varphi_0^g, \varphi_1^g, \beta^g)$ from the post-war U.K. data. The parameter estimates are reported in Panel C of Table 3. Spending/output is counter-cyclical in the data. A 1% point decline in output coincides with a 0.88% point increase in the spending/output ratio.

Figure 5 reports the spending and tax betas with respect to the output growth in the U.K. We find that the short-horizon tax beta in the U.K. is much smaller than in the U.S., while the spending beta is similar. The U.S. tax process is much riskier (the tax liability safer for taxpayers). The U.K. tax beta is around 0.5 at the one-year horizon, compared to 3 for the U.S beta. Put differently, the U.S. provides much more insurance to its taxpayers, whereas the U.K. raises taxes when output

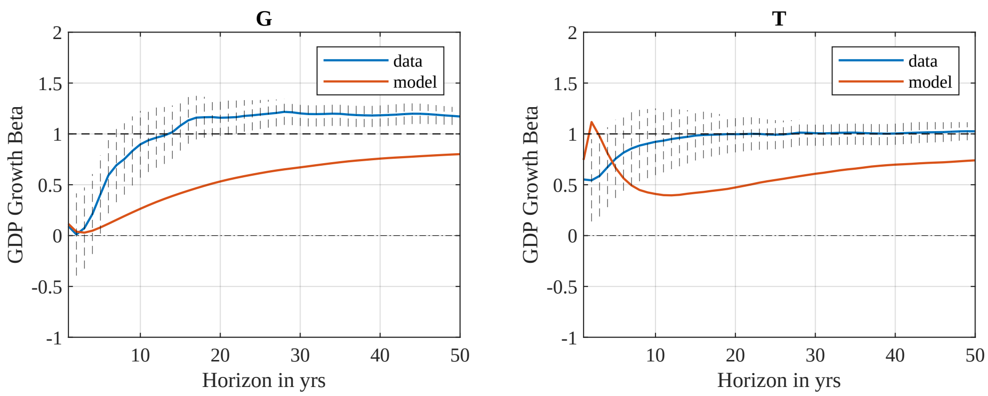
Table 3: Benchmark Calibration for U.K

Panel A: Preferences and Output Dynamics		
γ	1	maximum annual Sharpe ratio
ρ	4.16%	real rate
μ	2.2%	mean of growth rate of output
σ	5.0%	std. of growth rate of output
Panel B: Debt/Output Ratio Dynamics		
λ	$1.27 \times \sigma$	sensitivity of debt/output to output innovations
$d = \exp\{\phi_0/(1-\phi_1-\phi_2)\}$	0.73	mean of debt/output
ϕ_1	1.50	AR(1) coeff of debt/output
ϕ_2	-0.53	AR(2) coeff of debt/output
Panel C: Government Spending/Output Ratio Dynamics		
β^g	$0.88 \times \sigma$	sensitivity of spending/output to output innovations
φ_1^g	1.17	AR(1) coeff of spending/output
φ_2^g	-0.26	AR(2) coeff of spending/output
$x = \exp\left\{\varphi_0^g/(1-\varphi_1^g-\varphi_2^g)\right\}$	0.3069	mean of govt. spending/output

declines. Even for horizons in excess of ten years, the U.K. tax beta remains below the spending beta. According to Proposition 1, U.K. tax revenues are potentially safe (counter-cyclical) enough to keep government debt risk-free. This result stands in sharp contrast wit that for the U.S.

Figure 5: Post-war GDP Growth Betas of U.K. Tax Revenue and Spending

This figure reports the betas in regression of log U.K. spending G growth and log tax revenue T growth over horizon j on the concurrent GDP growth over horizon j. The blue curve plots the estimate from the data. Sample is annual, 1946—2015. The plot also shows 2 standard error bands, generated by 30,000 bootstrapped samples by drawing jointly with replacement from the 4×1 vector of innovations in the AR models for $(\log d_t, x_t, \log \tau_t, \log g_t)$. The red curve plots the coefficients implied from the model with risk-free Debt. Benchmark calibration in Table 3.



4 The Riskiness of the Surplus Claim Across Maturities

How much smoothing can the U.S. government achieve by issuing more debt in response to bad shocks? It depends on the horizon. This section characterizes the trade-off at different horizons using the cash-flow betas of the surplus and tax revenues. In the presence of permanent shocks, the government can only insure taxpayers over a limited period of time. This period can be extended by imputing more persistence or higher-order dynamics into the debt/output process.

We define the conditional beta of a generic stream of discounted cash flows Z as:

$$\beta_t^{Z,CF}(h) \equiv -cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{h+1} M_{t+1,t+j} Z_{t+j} \right).$$

We refer to this object as the cash-flow beta for short. The cash-flow beta of the surplus process over the next h periods is a sufficient statistic for how much insurance the government can provide to taxpayers over the next h periods.

Proposition 7. *Under Assumptions 1 and 3, when debt is risk-free and debt/output follows an AR(2) as in (8), the cash-flow beta of the discounted surpluses over h periods is given by the beta of debt h periods from* *now:*

$$\begin{aligned}
\beta_t^{S,CF}(h) &\equiv -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{j=1}^{h} M_{t+1,t+j}S_{t+j}\right) \\
&= cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)M_{t+1,t+h}D_{t+h}\right) \\
&= \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+h}d_{t+h}Y_{t+h}](\exp\{\gamma(\psi_{h-1}\lambda-\sigma)\}-1). \\
sign\left(\beta_t^{S,CF}(h)\right) &= sign\left(\gamma(\psi_{h-1}\lambda-\sigma)\right)
\end{aligned}$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}, j > 2; \psi_2 = \phi_2 + \phi_1\psi_1; \psi_1 = \phi_1; \psi_0 = 1.$

The proof is in Appendix A.8. The risk properties of the government surpluses over a given horizon are completely determined by riskiness of the debt issuance process, as long as the debt is risk-free. The cash-flow beta of the surplus at various horizons does not depend on the spending and tax revenue dynamics.

Analogously, we define the cash-flow beta of discounted government spending and of tax revenues.

Corollary 2. *Under Assumptions 1 and 3, and when debt is risk-free and debt/output follows an* $AR(2)$*, the cash flow beta of spending and tax revenues have to satisfy the following restrictions:*

$$\begin{aligned}
\beta_t^{G,CF}(h) &\equiv -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{j=1}^{h} M_{t+1,t+j}G_{t+j}\right) \\
&= \sum_{j=1}^{h} E_t[M_{t+1}]E_t[M_{t+1,t+h}xY_{t+h}](\exp\left\{\gamma(\varphi_g^{h-1}b_g-\sigma)\right\}-1). \\
\beta_t^{T,CF}(h) &\equiv -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{j=1}^{h} M_{t+1,t+j}T_{t+j}\right) \\
&= E_t[M_{t+1}]E_t[M_{t+1,t+h}d_{t+h}Y_{t+h}](\exp\{\gamma(\psi_{h-1}\lambda-\sigma)\}-1) \\
&+ \sum_{j=1}^{h} E_t[M_{t+1}]E_t[M_{t+1,t+h}xY_{t+h}](\exp\left\{\gamma(\varphi_g^{h-1}b_g-\sigma)\right\}-1).
\end{aligned}$$

The properties of the $\beta_t^{G,CF}(h)$ depend on the persistence and cyclicality of the exogenous spending/GDP process in equation (9). The properties of $\beta_t^{T,CF}(h)$ depend on the risk properties of both the debt claim and the spending claim.

4.1 Constant debt/output

When debt/output is constant, $\lambda = 0$, and proposition 7 simplifies to:

$$\beta_t^{S,CF}(h) = \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+h}dY_{t+h}](\exp\{-\gamma\sigma\}-1).$$

The cash-flow beta of the surplus is negative at all horizons since $\gamma\sigma > 1$. In bad times, the surplus/output ratio goes up. When spending/output is constant (or also goes up), tax revenues/output must go up. The government cannot insure taxpayers against adverse output shocks. Rather, the taxpayers insure the bondholders.

The left panel of Figure 6A plots the cash-flow beta of the cumulative discounted surplus, $\beta_t^{S,CF}(h)$, divided by $\mathbb{E}_t[M_{t+1}]$, as a function of the horizon h. This ratio can be interpreted as the risk premium on a claim to cumulative surpluses over the next h periods. The negative risk premium indicates that surpluses are a hedge. Since taxpayers are short the surplus claim, their tax-minus-transfer liability is risky. When debt/output is constant and there is no possibility to raise the debt in response to an adverse shock, the surplus/output ratio must rise on impact. This makes the one-period surplus claim a hedge, with a negative risk premium of -4%. The year-2 surplus claim in contrast earns a small positive risk premium, reflecting the underlying output risk. The cumulative risk premium at horizon h is the sum of the individual strip risk premia up until horizon h.

This risk premium on cumulative surpluses is inversely related to the risk premium on a debt strip, which is positive and constant at all horizons. This debt strip risk premium is plotted in the right panel of Figure 6A. When the debt/output ratio is constant, the debt strip has the same risk as the output strip at all horizons. To offset the output risk in debt, the risk premium on the surplus has to be negative.

As $h \to \infty$, the sum of discounted surpluses converges to the current value of debt D_t. Insisting on risk-free debt ($\beta_t^D = 0$) implies that $\beta_t^{S,CF}(h) \to 0$. The red line in the left panel converges to zero from below for large h.

The solid black line in the left panel plot the cash-flow beta of the spending claim scaled by $\mathbb{E}_t[M_{t+1}]$. It is the risk premium on a claim to cumulative spending. Since the spending/output dynamics are exogenously given, the spending beta does not depend on the debt policy. The countercyclical nature of spending/output makes the risk premium negative at short horizons. At longer horizons, the spending risk premium turns positive reflecting the long-run output risk in the spending claim, since the spending/output ratio is stationary.

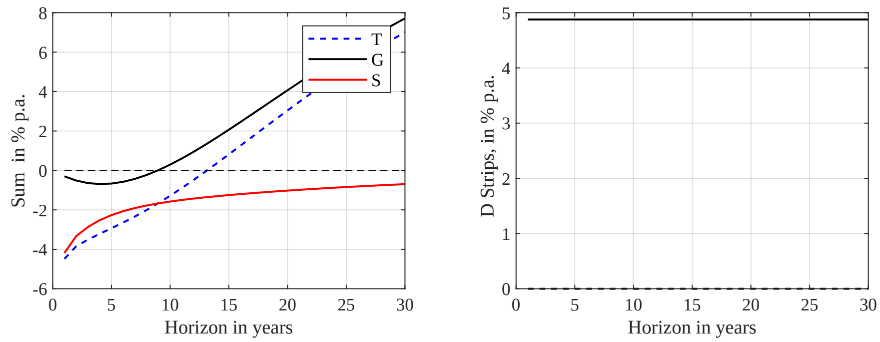
The extent of taxpayer insurance is captured by $\beta_t^{T,CF}(h)$. The blue dashed line in Figure 6A plots $\beta_t^{T,CF}(h)$ scaled by $\mathbb{E}_t[M_{t+1}]$. It is the risk premium on a claim to the next h periods of tax revenue. When this risk premium is negative, taxpayers are providing insurance to the government rather than receiving insurance. The risk premium is negative until year 13 for our parameters. It then turns positive. The positive risk premium on longer-dated tax strips reflects cointegration between tax revenues and output and a positive risk premium for output risk.

Note that the tax beta $\beta_t^{T,CF}(h)$ in the left panel is below the spending beta $\beta_t^{G,CF}(h)$ at all horizons. As $h \to \infty$, these cash-flow betas converge to the return betas β_t^T and β_t^G. As we discussed in Corollary 1, $\beta_t^T < \beta_t^G$ was the condition to keep the debt risk-free.

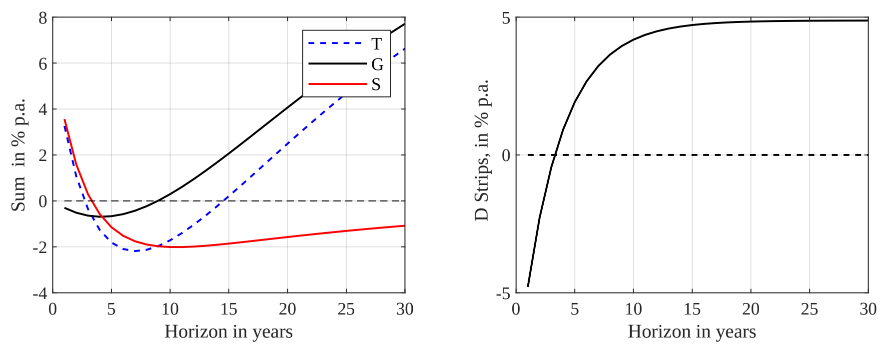
Figure 6: Risk Premia Across Horizons

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{i,CF}(h)/\mathbb{E}_t[M_{t+1}]$, in the left panel against the horizon h. The right panel plots the risk premium on the debt strips: $1 - \exp\left\{\gamma(\phi^{h-1}\lambda - \sigma)\right\}$. The parameters are given in Table 1, except for the debt dynamics in the first two panels.

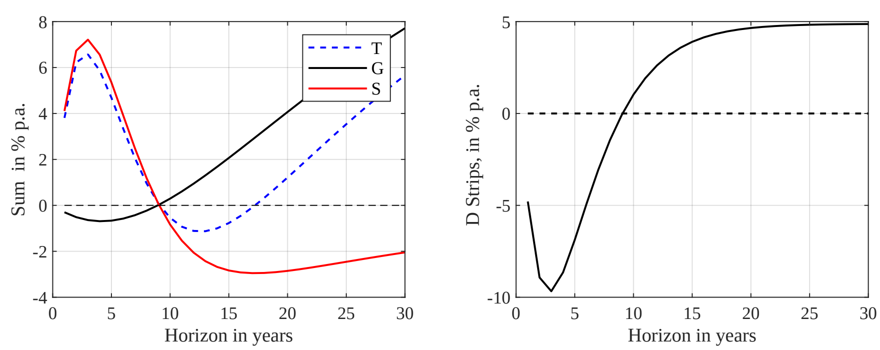
Panel A: Constant Debt/Output ($\lambda = 0$)



Panel B: $AR(1)$ Debt/Output ($\phi_1 = 0.75$)



Panel C: $AR(2)$ Debt/Output



4.2 AR(1) for debt/output

The sign and magnitude of the cash-flow beta of the surplus is now governed by $\gamma(\phi_1^{h-1}\lambda - \sigma)$. This term has a natural economic interpretation. $1 - \exp\left\{\gamma(\phi^{j-1}\lambda - \sigma)\right\}$ is the risk premium of a h-period debt strip with payoff $Y_{t+h}d_{t+h}$. The cumulative surplus can be risky over a horizon h only if this is offset by the safety of debt issuance at time $t+h$.

If $\lambda \leq 0$, and debt/output is pro-cyclical, $\beta_t^{S,CF}(h) < 0$ at all horizons. We are back in the previous case. In other words, the government cannot insure taxpayers by running deficits in bad times over any horizon.

In the empirically relevant case of $\lambda > \sigma > 0$, the initial $\beta_t^{S,CF}(1) > 0$. By issuing more debt in response to an adverse shock, the government prevents the tax rate and the surplus from going up. This provides insurance to the taxpayers $\beta_t^{T,CF}(1) > 0$. The one-period debt strip has a negative risk premium due to the counter-cyclical nature of debt issuance, as shown in the right panel of Figure 6B.

However, due to its AR(1) nature, the debt/output ratio starts to revert back to its mean the very next period. The cumulative two-period surplus risk premium depends on $\gamma(\phi_1\lambda - \sigma)$ which is still positive but not as large as the one-period risk premium since $\phi_1 < 1$. Conversely, the cumulative two-period debt strip risk premium is not as negative as the one-period debt strip risk premium. The risk premium on the strip that pays the annual surplus two years from now is negative. The same is true for the two-year tax strip.

The surplus beta $\beta_t^{S,CF}(h)$ inherits the dynamics of the $AR(1)$ process for the debt/output ratio and starts to decline right away. As h increases, the surplus beta eventually switches signs. This occurs at the first time h that $\phi^{h-1}\lambda < \sigma$. If the rate of mean-reversion in debt is high (ϕ_1 is small), this switch occurs sooner. If the debt/output ratio is more persistent, the sign switch occurs later.

Given the counter-cyclical nature of government spending, the tax beta $\beta_t^{T,CF}(h)$ must cross over into negative territory sooner than the surplus beta. There is only a very limited amount of taxpayer insurance that the government can provide when debt is risk-free and follows AR(1) dynamics. This insurance is further curtailed due to the counter-cyclical nature of spending.

As the right panel shows, the risk premium on the debt strip increases with the horizon. As $h \to \infty$, it converges to the risk premium on a long-dated output strip. Again, this reflects the fact that debt is co-integrated with output. It is common in the literature to assume that this risk premium is zero at long horizons, because this allows discounting at the risk-free rate. In the presence of permanent shocks, this is incorrect. Similarly, the risk premia on the long-dated T-strip and G-strip also converge to risk premium on the long-dated output strip of 5% as $h \to \infty$.

When output shocks are i.i.d. and permanent, far-out surpluses are risky as they inherit the permanent output risk. Medium-term surpluses must be safe and have negative risk premia to offset both the positive risk premium of the short-run surpluses (insurance provision) and the positive risk premium of the long-run surpluses (output risk). Equivalently, the cash-flow betas of the tax strip must be below those of the spending strip at medium horizons. The cash-flow beta at $h = \infty$ equals the return beta, and so $\beta_t^T < \beta_t^G$ ensures that $\beta_t^D = 0$. Permanent output risk rules out insurance provision to taxpayers over long horizons.

4.3 AR(2) for debt/output

In our preferred case of an AR(2) for debt/output, the sign of the cash flow beta of the surplus is determined by $\gamma(\psi_{j-1}\lambda - \sigma)$. If $\lambda > \sigma$, the initial surplus beta is positive. The second beta is larger since $\psi_1 = \phi_1 > 1$. The third beta remains positive and is larger than the second beta if $\psi_2 > \psi_1$ or $\phi_1(\phi_1 - 1) + \phi_2 > 0$. This condition is satisfied for $\phi_1 = 1.40$ and $\phi_2 = -0.48$. For these parameter values, the fourth beta is lower than the third, the fifth lower than the fourth, etc. Eventually this beta crosses over into negative territory. The left panel of Figure 6C shows this occurs around year 9. The cash-flow beta for tax revenue follows a similar pattern. The cash-flow betas inherit the hump-shaped pattern from the debt/output ratio, plotted in the right panel.

What allows the government to provide temporary insurance to taxpayers is a debt issuance policy with more history dependence. Risk premia on debt strips, shown in the right panel, are more negative than in the AR(1) model and remain negative for longer (9 versus 3 years). The slow expansion and repayment of the debt in response to an adverse shock allows the government to postpone fiscal rectitude. But as h increases, the expression $\gamma(\sigma - \psi_{j-1}\lambda)$ turns positive and converges to $\gamma\sigma$, the risk premium on the output strip.

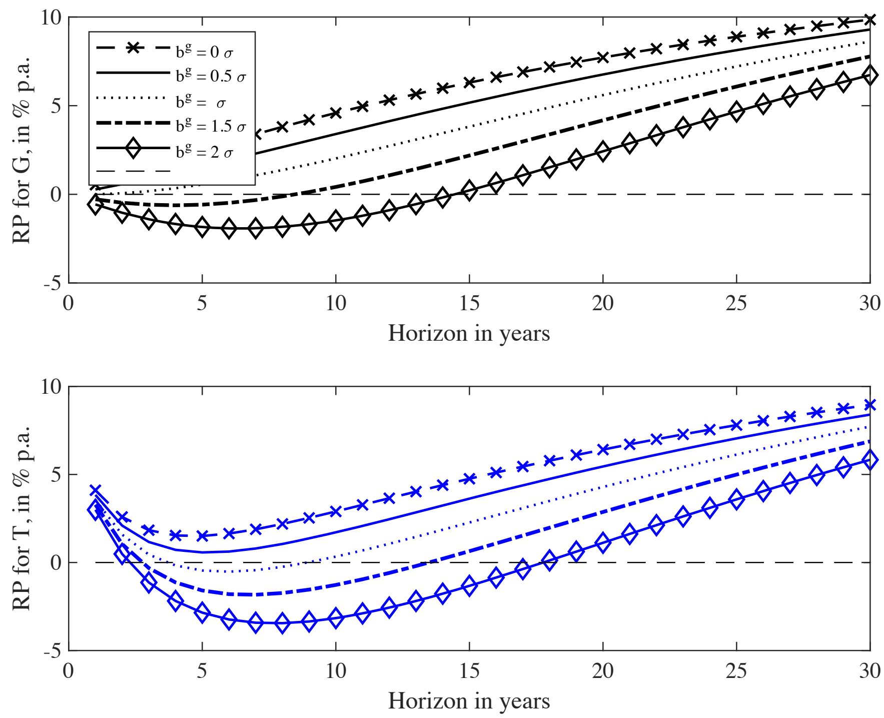
4.4 Counter-cyclical Spending

The government insures transfer recipients by spending a larger fraction of GDP in recession. The counter-cyclical nature of spending further constraints the government in navigating the trade-off between insurance of bondholders and taxpayers. We analyze the sensitivity of our results to the cyclicality of spending.

Figure 7 plots scaled cash-flow betas for spending, $\beta_t^{G,CF}(h)/\mathbb{E}_t[M_{t+1}]$, in the top panel, and the implied tax revenue betas, $\beta_t^{T,CF}(h)/\mathbb{E}_t[M_{t+1}]$, in the bottom panel, for a range of values of the cyclicality of spending b_g. As government spending becomes more counter-cyclical, the risk premium on the spending claim declines. The risk premium on the tax claim has to decline as well in order to keep the government debt risk-free. As the tax claim becomes safer, taxpayers face a riskier tax liability proposition. As the governments provides more insurance to transfer recipients, this reduces the scope for insurance of taxpayers. When spending is acyclical ($b_g = 0$), the tax claim inherits the risk properties of the surplus claim.

Figure 7: Varying the Counter-cyclicality of Spending

This figure plots the scaled cash-flow beta of spending $\beta_t^{G,CF}(h)/\mathbb{E}_t[M_{t+1}]$ in the top panel and the implied tax revenue betas, $\beta_t^{T,CF}(h)/\mathbb{E}_t[M_{t+1}]$, in the bottom panel for a range of values of the cyclicality of spending b_g.



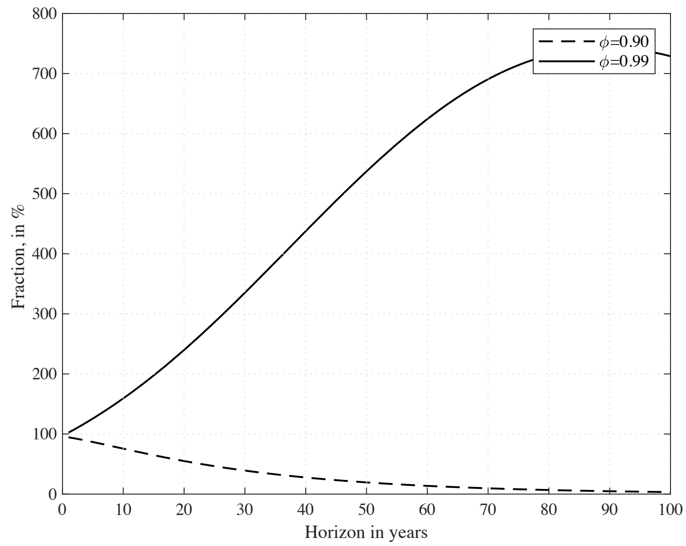
4.5 Debt Persistence

To provide more intertemporal smoothing, the government can increase the persistence of the debt/output process. This allows it to spread out the adjustment to a negative shock further over time. Assuming AR(1) dynamics for debt/output, Figure 8 plots the value of debt h periods from now relative to debt today: $\mathbb{E}_t[M_{t+h}D_{t+h}]/D_t$. When $\phi = 0.90$, the present value of debt in the future is a vanishing share of the current debt value as the horizon grows large. When the persistence of the debt/output process is 0.99, the government imputes a near-unit root into the debt/output ratio. The tail (or bubble) component of the debt one hundred years from now is seven times larger than current debt. This allows the government to run cumulative deficits over the next one hundred years with an expected present-discounted value of 6 times current debt. Under the risk-neutral measure, investors expect the debt to increase faster than the risk-free rate; the government increases the debt/output ratio along paths characterized by adverse aggregate histories, because $\lambda > 0$. The insurance provided by such persistent debt issuance policy is so valuable to risk averse agents (γ is large enough to match the equity risk premium), that the government creates itself a near-arbitrage opportunity. Recall that the actual debt/GDP ratio in the post-war U.S. data displays substantially less persistence than an AR(1) process with

$\phi_1 = 0.99$.

Figure 8: Debt Persistence

The figure plots the tail value at t of the debt expected at $t+h$ as a fraction of debt today: $\mathbb{E}_t[M_{t+h}D_{t+h}]/D_t$. Debt/gdp follows an AR(1) with ϕ equal to 0.90 (dashed line) or 0.99 (solid line). All other parameters are as listed in Table 1.



Finally, Appendix B develops a version of the model without permanent shocks. This model produces starkly different implications for the trade-off between insuring taxpayers and bondholders. However, it has counterfactual asset pricing implications. In models with only transitory shocks, long-term bonds are the riskiest assets.

5 Characterizing the Trade-off with Convenience Yields

While the U.K. government's behavior is consistent with our trade-off, the U.S. government's behavior seems to violate this constraint. The convenience yields earned by the U.S. can relax the trade-off between insuring bondholders and taxpayers. Some governments are endowed with the ability to issue government bonds at prices that exceed their fair market value. The convenience yield λ_t is defined as a wedge in the investors' Euler equation for government bonds:

$$\mathbb{E}_t\left[M_{t,t+1}R_t^D\right] = \exp(-\lambda_t) \tag{11}$$

Typically, the debt then serves the role of a special, safe asset for domestic or foreign investors. U.S. Treasuries currently fill the role of the world's safe asset.Krishnamurthy and Vissing-Jorgensen (2012) estimate convenience yields on U.S. Treasurys of around 75 bps.[7]

[7]Using the deviations from CIP in Treasury markets, Jiang et al. (2018a); Jiang, Krishnamurthy, and Lustig (2018b); Koijen and Yogo (2019) estimate convenience yields that foreign investors derive from their holdings of dollar safe assets; these estimates exceed 200 bps.

Jiang et al. (2019b) show that, in the presence of these convenience yields, the value of the government debt equals the sum of the expected present values of future tax revenues plus future seigniorage revenues minus future government spending:

$$B_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} + (1 - e^{-\lambda_{t+j}}) D_{t+j} - G_{t+j}) \right] = P_t^T + P_t^K - P_t^G,$$

provided that a transversality condition holds. The periodic seigniorage revenue is $K_{t+j} = (1 - e^{-\lambda_{t+j}}) D_{t+j}$, which is the amount of interest the government does not need to pay thanks to the convenience yield. The current value of government debt reflects the present value of all convenience yields earned on future debt. We refer to this value as the Treasury's seigniorage revenue:

$$P_t^K \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (1 - e^{-\lambda_{t+j}}) D_{t+j} \right].$$

The government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\mathbb{E}_t \left[R_{t+1}^D - R_t^f \right] \quad = \quad \frac{P_t^T - T_t}{B_t - S_t} \mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] + \frac{P_t^K - T_t}{B_t - S_t} \mathbb{E}_t \left[R_{t+1}^K - R_t^f \right] - \frac{P_t^G - G_t}{B_t - S_t} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right],$$

where R_{t+1}^D, R_{t+1}^T,R_{t+1}^K and R_{t+1}^G are the holding period returns on the bond portfolio, the tax claim, the seigniorage claim, and the spending claim, respectively. We take government spending process, and the debt return process as given, and explore the implications for the properties of the tax claim.

For simplicity, assume that the spending/output ratio is constant: $\beta^Y = \beta^G$. Suppose that the (convenience yield) seigniorage process has a zero beta: $\beta^K = 0$. If the government wants to manufacture risk-free debt, then the implied beta of the tax revenue process must satisfy:

$$\beta_t^T \quad = \quad \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \beta^G > \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)} \beta^G,$$

The tax beta with convenience yields exceeds the tax beta without convenience. Hence, more taxpayer insurance becomes feasible.

If the seigniorage revenue/output is sufficiently counter-cyclical ($\beta^K < 0$), then the government can potentially fully insure both taxpayers and bondholders at the same time. In this case, the government surplus is procyclical while the debt is risk-free, i.e., zero beta.

The fiscal relief enjoyed by the U.S. government from convenience yields may be temporary. Krishnamurthy and Vissing-Jorgensen (2012) have demonstrated that the convenience yields earned on U.S. government debt decline as the supply of debt increases relative to output. This ultimately would restore the standard trade-off we have described.

6 Conclusion

The government engineers risk-free debt by choosing the exposure of the tax claim to output risk judiciously. The more debt there is outstanding, the lower this exposure must become, and hence the more output risk must be borne by the taxpayer. There is no scope for insurance of both taxpayers and debt holders over long horizons in the presence of permanent, priced shocks to output. The only way the government can provide insurance to taxpayers over all horizons while keeping the debt risk-free is by saving or by earning a large convenience yield on its debt.

References

Aguiar, M., M. Amador, H. Hopenhayn, and I. Werning (2019): "Take the short route: Equilibrium default and debt maturity," *Econometrica*.

Aguiar, M. and G. Gopinath (2006): "Defaultable debt, interest rates and the current account," *J. Int. Econ.*, 69, 64–83.

Aiyagari, S. R., A. Marcet, T. J. Sargent, and J. Seppälä (2002): "Optimal taxation without state-contingent debt," *Journal of Political Economy*, 110, 1220–1254, URL https://doi.org/10.1086/343744.

Alvarez, F. and U. Jermann (2005): "Using asset prices to measure the measure the persistence of the marginal utility of wealth." *Econometrica*, 1977–2016.

Angeletos, G.-M. (2002): "Fiscal policy with noncontingent debt and the optimal maturity structure*," *The Quarterly Journal of Economics*, 117, 1105–1131, URL http://dx.doi.org/10.1162/003355302760193977.

Arellano, C. (2008): "Default risk and income fluctuations in emerging economies," *Am. Econ. Rev.*, 98, 690–712.

Arellano, C. and A. Ramanarayanan (2012): "Default and the maturity structure in sovereign bonds," *J. Polit. Econ.*, 120, 187–232.

Backus, D., N. Boyarchenko, and M. Chernov (2018): "Term structures of asset prices and returns," *J. financ. econ.*, 129, 1–23.

Bansal, R. and A. Yaron (2004): "Risks for the long run: A potential resolution of asset prizing puzzles," *The Journal of Finance*, 59, 1481–1509.

Barro, R. J. (1979): "On the determination of the public debt," *J. Polit. Econ.*, 87, 940–971.

Barro, R. J. (2020): "r minus g," Technical Report w28002, National Bureau of Economic Research.

Bhandari, A., D. Evans, M. Golosov, and T. J. Sargent (2017): "Fiscal policy and debt management with incomplete markets," *Q. J. Econ.*, 132, 617–663.

Bianchi, J., P. Ottonello, and I. Presno (2019): "Fiscal stimulus under sovereign risk," .

Blanchard, O. (2019): "Public debt and low interest rates," *American Economic Review*, 109, 1197–1229.

Blanchard, O. J. and M. Watson (1982): "Bubbles, rational expectations, and financial markets," in P. Wachtel, ed., *Crises in the Economic and Financial Structure*, Lexington Books.

Bohn, H. (1995): "The sustainability of budget deficits in a stochastic economy," *J. Money Credit Bank.*, 27, 257–271.

Bohn, H. (1998): "The behavior of us public debt and deficits," *Quarterly Journal of Economics*, 113, 949–963.

Bohn, H. (2007): "Are stationarity and cointegration restrictions really necessary for the intertemporal budget constraint?" *Journal of Monetary Economics*, 54, 1837 – 1847, URL http://www.sciencedirect.com/science/article/pii/S0304393206002534.

Borovička, J., L. P. Hansen, and J. Scheinkman (2016): "Misspecified recovery," *Journal of Finance*, 71, 2493–2544.

Borri, N. and A. Verdelhan (2011): "Sovereign risk premia," .

Breeden, D. T. (2005): "An intertermporal asset pricing model with stochastic consumption and investment opportunities," in *Theory of Valuation*, WORLD SCIENTIFIC, 53–96.

Brunnermeier, M., S. Merkel, and Y. Sannikov (2020): "The fiscal theory of price level with a bubble," .

Buera, F. and J. P. Nicolini (2004): "Optimal maturity of government debt without state contingent bonds," *Journal of Monetary Economics*, 51, 531 – 554, URL http://www.sciencedirect.com/science/article/pii/S0304393204000133.

Bulow, J. and K. Rogoff (1989): "Sovereign debt: Is to forgive to forget?" *Am. Econ. Rev.*, 79, 43–50.

Caballero, R. J., E. Farhi, and P.-O. Gourinchas (2008): "An equilibrium model of "global imbalances" and low interest rates," *American Economic Review*, 98, 358–93, URL http://www.aeaweb.org/articles?id=10.1257/aer.98.1.358.

Caballero, R. J. and A. Krishnamurthy (2009): "Global imbalances and financial fragility," *American Economic Review*, 99, 584–88, URL http://www.aeaweb.org/articles?id=10.1257/aer.99.2.584.

Campbell, J. Y., C. Pflueger, and L. M. Viceira (2020): "Macroeconomic drivers of bond and equity risks," *J. Polit. Econ.*, 128, 3148–3185.

Canzoneri, M. B., R. E. Cumby, and B. T. Diba (2001): "Is the price level determined by the needs of fiscal solvency?" *Am. Econ. Rev.*, 91, 1221–1238.

Capie, F. and A. Webber (1985): *A Monetary History of the United Kingdom, 1870-1982: Data, sources, methods*, A Monetary History of the United Kingdom, 1870-1982, Allen & Unwin, URL https://books.google.com/books?id=UsobAQAAMAAJ.

Cochrane, J. H. (2019): "The fiscal roots of inflation," .

Cochrane, J. H. (2020): "A fiscal theory of monetary policy with Partially-Repaid Long-Term debt," NBER Working Paper No. 26745.

DeMarzo, P., Z. He, and F. Tourre (2019): "Sovereign debt ratchets and welfare destruction," in *Midwest Finance Association 2019 Annual Meeting*, pbcsf.tsinghua.edu.cn.

Diamond, P. A. (1965): "National debt in a neoclassical growth model," *Am. Econ. Rev.*, 55, 1126–1150.

Eaton, J. and M. Gersovitz (1981): "Debt with potential repudiation: Theoretical and empirical analysis," *Rev. Econ. Stud.*, 48, 289–309.

Gopinath, G. and J. C. Stein (2018): "Banking, trade, and the making of a dominant currency," Technical report, Harvard University.

Gourinchas, P.-O. and H. Rey (2007): "International financial adjustment," *Journal of political economy*, 115, 665–703.

Hall, G. J. and T. J. Sargent (2011): "Interest rate risk and other determinants of Post-WWII US government Debt/GDP dynamics," *American Economic Journal: Macroeconomics*, 3, 192–214.

Hamilton, J. D. and M. A. Flavin (1986): "On the limitations of government borrowing: A framework for empirical testing," *The American Economic Review*, 76, 808–819, URL http://www.jstor.org/stable/1806077.

Hansen, L. P., W. Roberds, and T. J. Sargent (1991): *Time Series Implications of Present Value Budget Balance*, Westview Press, volume Rational Expectations Econometrics, chapter 5.

Hansen, L. P. and J. Scheinkman (2009): "Long-term risk: An operator approach," *Econometrica*, 77 (1), 177–234.

He, Z., A. Krishnamurthy, and K. Milbradt (2018): "A model of safe asset determination," *American Economic Review*.

Heim, C. E. and P. Mirowski (1987): "Interest rates and crowding-out during britain's industrial revolution," *The Journal of Economic History*, 47, 117–139, URL http://www.jstor.org/stable/2121942.

Jiang, Z., A. Krishnamurthy, and H. Lustig (2018a): "Foreign safe asset demand and the dollar exchange rate," .

Jiang, Z., A. Krishnamurthy, and H. Lustig (2018b): "Foreign safe asset demand for US treasurys and the dollar," *AEA Papers and Proceedings*, 108, 537–541.

Jiang, Z., A. Krishnamurthy, and H. Lustig (2019a): "Dollar safety and the global financial cycle," *Available at SSRN 3328808*.

Jiang, Z., H. Lustig, S. Van Nieuwerburgh, and M. Z. Xiaolan (2019b): "The U.S. public debt valuation puzzle," URL https://www.nber.org/papers/w26583, nBER Working Paper 26583.

Klovland, J. T. (1994): "Pitfalls in the estimation of the yield on british consols, 1850-1914," *The Journal of Economic History*, 54, 164–187, URL http://www.jstor.org/stable/2122695.

Koijen, R. S. J. and M. Yogo (2019): "Exchange rates and asset prices in a global demand system," .

Krishnamurthy, A. and H. N. Lustig (2019): "Mind the gap in sovereign debt markets: The U.S. treasury basis and the dollar risk factor," .

Krishnamurthy, A. and A. Vissing-Jorgensen (2012): "The aggregate demand for treasury debt," *Journal of Political Economy*, 120, 233–267.

Krueger, D. and H. Lustig (2010): "When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)?" *J. Econ. Theory*, 145, 1–41.

Liu, Y., L. Schmid, and A. Yaron (2019): "The risks of safe assets," in *2019 Meeting Papers*.

Lucas, R. E. (1978): "Asset prices in an exchange economy," *Econometrica*, 46, 1429–1445.

Lucas, R. E. and N. L. Stokey (1983): "Optimal fiscal and monetary policy in an economy without capital," *Journal of Monetary Economics*, 12, 55 – 93, URL http://www.sciencedirect.com/science/article/pii/0304393283900491.

Lustig, H., C. Sleet, and Ş. Yeltekin (2008): "Fiscal hedging with nominal assets," *Journal of Monetary Economics*, 55, 710 – 727, URL http://www.sciencedirect.com/science/article/pii/S030439320800072X.

Maggiori, M. (2017): "Financial intermediation, international risk sharing, and reserve currencies," *American Economic Review*, 107, 3038–71, URL http://www.aeaweb.org/articles?id=10.1257/aer.20130479.

Meyer, J., C. M. Reinhart, and C. Trebesch (2019): "Sovereign bonds since waterloo," .

Mian, A. R., L. Straub, and A. Sufi (2020a): "Indebted demand," .

Mian, A. R., L. Straub, and A. Sufi (2020b): "The saving glut of the rich and the rise in household debt," .

Mitchell, B. and B. Mitchell (1988): *British Historical Statistics*, Cambridge University Press, URL https://books.google.com/books?id=Oyg9AAAAIAAJ.

Neal, L. (1993): *The Rise of Financial Capitalism: International Capital Markets in the Age of Reason*, Studies in Macroeconomic History, Cambridge University Press, URL https://books.google.com/books?id=ey8h1WC5w7kC.

Nishimura, S. (1971): *The Decline of Inland Bills of Exchange in the London Money Market 1855-1913*, Cambridge University Press, URL https://books.google.vu/books?id=Xe88AAAAIAAJ.

Odlyzko, A. (2016): "Economically irrational pricing of nineteenth-century British government bonds," *Financial History Review*, 23, 277–302, URL https://ideas.repec.org/a/cup/fihrev/v23y2016i03p277-302_00.html.

Rubinstein, M. (1974): "An aggregation theorem for securities markets," *J. financ. econ.*, 1, 225–244.

Samuelson, P. A. (1958): "An exact Consumption-Loan model of interest with or without the social contrivance of money," *J. Polit. Econ.*, 66, 467–482.

Trehan, B. and C. E. Walsh (1988): "Common trends, the government's budget constraint, and revenue smoothing," *J. Econ. Dyn. Control*, 12, 425–444.

Trehan, B. and C. E. Walsh (1991): "Testing intertemporal budget constraints: Theory and applications to u. s. federal budget and current account deficits," *J. Money Credit Bank.*, 23, 206–223.

Weiller, K. J. and P. Mirowski (1990): "Rates of interest in 18th century england," *Explorations in Economic History*, 27, 1 – 28, URL http://www.sciencedirect.com/science/article/pii/001449839090002G.

Werning, I. (2015): "Incomplete markets and aggregate demand," .

A Proofs

A.1 Return Betas and Cash Flow Betas

What is the relationship between return betas and cash flow betas? In the simple case with constant debt/output and spending/output ratios, there is a one-to-one mapping:

Corollary 3. *The expected returns can be expressed as a function of the cash flow betas:*

$$\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= \frac{x}{d(1-\xi_1)+x\xi_1}\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})}, \\
&= \frac{x}{d(1-\xi_1)+x\xi_1}\exp(\mu+\frac{1}{2}\sigma^2)(1-\exp(-\gamma\sigma)) \\
\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] &= \frac{1}{\xi_1}\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})} \\
&= \frac{1}{\xi_1}\exp(\mu+\frac{1}{2}\sigma^2)(1-\exp(-\gamma\sigma)),
\end{aligned}$$

where $\xi_1 = \exp(-\rho - \gamma\sigma + \mu + 0.5\sigma^2)$.

Proof. From $R^f_{t+1} = \rho\exp(\rho)$ and $\frac{T_t}{Y_t} = x - d\left(1 - R^f_{t-1}\frac{Y_{t-1}}{Y_t}\right)$, we have that the return on the tax claim can be stated as:

$$\begin{aligned}
R^T_{t+1} &= \frac{P^T_{t+1}}{P^T_t - T_t} = \frac{(d + x\frac{\xi_1}{1-\xi_1})Y_{t+1} + (x - d\left(1 - R^f_t\frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} \\
&= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} + \frac{d\exp(\rho)}{(d + x\frac{\xi_1}{1-\xi_1})}.
\end{aligned}$$

Similarly, we have an expression for the return on the spending claim:

$$R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t} = \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}.$$

As a result, we can state the risk premium as follows:

$$\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= -\frac{cov\left(M_{t+1}, R^T_{t+1}\right)}{E_t(M_{t+1})} = \frac{x}{d(1-\xi_1)+x\xi_1}\frac{-cov\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})}, \\
\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] &= -\frac{cov\left(M_{t+1}, R^G_{t+1}\right)}{E_t(M_{t+1})} = \frac{1}{\xi_1}\frac{-cov\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})},
\end{aligned}$$

where we have used that $\xi_1 = \exp(-\rho - \frac{1}{2}\gamma^2 + g + \frac{1}{2}(\gamma-\sigma)^2) = \exp(-\rho - \gamma\sigma + g + \frac{1}{2}\sigma^2)$.

Then plug in

$$\begin{aligned}
\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})} &= \frac{-cov_t\left(\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}), \exp(g + \sigma\varepsilon_{t+1})\right)}{E_t(\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}))} \\
&= \frac{-cov_t\left(\exp(-\gamma\varepsilon_{t+1}), \exp(\sigma\varepsilon_{t+1})\right)}{\exp(-\rho)}\exp(-\rho - \frac{1}{2}\gamma^2 + g) \\
&= -(\exp(\frac{1}{2}(\gamma^2+\sigma^2))(\exp(-\gamma\sigma)-1))\exp(-\frac{1}{2}\gamma^2 + g) \\
&= \exp(g + \frac{1}{2}\sigma^2)(1-\exp(-\gamma\sigma))
\end{aligned}$$

□

A.2 Proof of Proposition 1

Proof. From the investor's Euler equation $\mathbb{E}_t[M_{t+1}(R^i_{t+1} - R^f_t)]$=0, we know that the expected excess return on the tax claim, spending claim, and debt claims are given by

$$\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= \frac{-cov_t\left(M_{t+1}, R^T_{t+1}\right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t\left(M_{t+1}, R^T_{t+1}\right)}{var_t[M_{t+1}]}\frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]} = \beta^T_t\lambda_t, \\
\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] &= \frac{-cov_t\left(M_{t+1}, R^G_{t+1}\right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t\left(M_{t+1}, R^G_{t+1}\right)}{var_t[M_{t+1}]}\frac{var_t[M_{t+1}]}{E_t[M_{t+1}]} = \beta^G_t\lambda_t, \\
\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] &= \frac{-cov_t\left(M_{t+1}, R^D_{t+1}\right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t\left(M_{t+1}, R^D_{t+1}\right)}{var_t[M_{t+1}]}\frac{var_t[M_{t+1}]}{E_t[M_{t+1}]} = \beta^D_t\lambda_t.
\end{aligned}$$

□

A.3 Proof of Proposition 2

Proof. We start from the one-period budget constraint:

$$T_t = G_t - (D_t - R^f_{t-1}D_{t-1}).$$

With TVC,

$$R^f_{t-1}D_{t-1} = S_t + D_t = S_t + \mathbb{E}_t[M_{t,t+1}R^f_tD_t] = S_t + \mathbb{E}_t[M_{t,t+1}(S_{t+1} + \exp(m_{t+1,t+2})R^f_{t+1}D_{t+1})] = \mathbb{E}_t[\sum_{k=0}^{\infty} M_{t,t+k}S_{t+k}].$$

Replace the time index t by $t+1$,

$$R^f_tD_t = \mathbb{E}_{t+1}[\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j}].$$

Since the left-hand side is known at time t,

$$(\mathbb{E}_{t+1} - \mathbb{E}_t)[\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j}] = 0.$$

□

A.4 Proof of Proposition 3

Proof. To verify the expression, first conjecture the pricing of the surplus strip is

$$\mathbb{E}_t\left[M_{t,t+k}Y_{t+k}\right] = \xi_k Y_t$$

for $k \geq 0$. Then $\xi_0 = 1$ and

$$\begin{aligned}
\xi_k Y_t = \mathbb{E}_t\left[M_{t,t+k}Y_{t+k}\right] &= \mathbb{E}_t\left[M_{t,t+1}\xi_{k-1}Y_{t+1}\right] \\
&= \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-1)^2)Y_t, \\
\xi_1 Y_t &= \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\xi_{k-1}Y_t,
\end{aligned}$$

which verifies the conjecture and implies

$$\xi_k = \xi_{k-1}\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2).$$

Similarly, we define a k-period surplus strip as a claim to S_{t+k}, with price given by

$$\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = \chi_k Y_t.$$

The pricing of the first surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+1}S_{t+1}\right] &= \mathbb{E}_t\left[M_{t,t+1}\{-dY_{t+1}\left(1 - R_t^f\exp[-(\mu+\varepsilon_{t+1})]\right)\}\right], \\
&= -d\mathbb{E}_t\left[M_{t,t+1}Y_{t+1}\right] + dY_t R_t^f \mathbb{E}_t\left[M_{t,t+1}\right], \\
&= -d\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)Y_t + dY_t, \\
&= \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right] dY_t. \\
\chi_1 &= \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right] d.
\end{aligned}$$

Then, the pricing of the second surplus strip is

$$\begin{aligned}
\chi_k Y_t = \mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \mathbb{E}_t\left[M_{t,t+1}\mathbb{E}_{t+1}[M_{t+1,t+k}S_{t+k}]\right], \\
&= \mathbb{E}_t\left[M_{t,t+1}\chi_{k-1}Y_{t+1}\right], \\
&= \chi_{k-1}\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)Y_t.
\end{aligned}$$

Note that this calculation also implies that we cannot simply price these strips off the risk-free yield curve, even though the entire debt is risk-free. The solution is

$$\begin{aligned}
\chi_1 &= d\left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right] \\
\chi_k &= \chi_{k-1}\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2).
\end{aligned}$$

which implies that

$$\lim_{j\to\infty}\mathbb{E}_t\left[M_{t,t+j}S_{t+j}\right] = \sum_{k=1}^{\infty}\chi_k Y_t = \chi_1(1 + \xi_1 + \xi_1^2 + \ldots)Y_t = dY_t,$$

where $\xi_1 = \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)$.

$\square$

A.5 Proof of Proposition 4

Proof. From the gross risk-free rate xpression $R_{t+1}^f = \exp(\rho)$ and the one-period government budget constraint, we get that:

$$\frac{T_t}{Y_t} = x - d\left(1 - R_{t-1}^f\frac{Y_{t-1}}{Y_t}\right),$$

we have that the return on the tax claim can be stated as:

$$
\begin{aligned}
R^T_{t+1} &= \frac{P^T_{t+1}}{P^T_t - T_t} = \frac{(d + x\frac{\xi_1}{1-\xi_1})Y_{t+1} + (x - d\left(1 - R^f_t \frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t}, \\
&= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} + \frac{d\exp(\rho)}{(d + x\frac{\xi_1}{1-\xi_1})}.
\end{aligned}
$$

Similarly, the return on the spending claim can be stated as:

$$
\begin{aligned}
R^G_{t+1} &= \frac{P^G_{t+1}}{P^G_t - G_t} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}, \\
&= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}.
\end{aligned}
$$

Armed with these expressions, we get the following expression for the covariance:

$$
cov(R^T_{t+1}, M_{t,t+1}) = \frac{x\frac{\xi_1}{1-\xi_1}}{(d + x\frac{\xi_1}{1-\xi_1})} cov(R^G_{t+1}, M_{t,t+1}),
$$

which also translates to

$$
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}} \mathbb{E}_t\left[R^Y_{t+1} - R^f_t\right].
$$

□

A.6 Proof of Proposition 5

A.6.1 Case of $AR(1)$

Proof. From

$$
T_t = G_t - (D_t - R^f_{t-1}D_{t-1}).
$$

The surplus process is given by:

$$
\frac{S_t}{Y_t} = -\left(d_t - R^f_{t-1}d_{t-1}\frac{Y_{t-1}}{Y_t}\right) = d_{t-1}R^f_{t-1}\exp[-(\mu + \sigma\varepsilon_t)] - d^{\phi_1}_{t-1}\exp(\phi_0 - \lambda\varepsilon_t - \frac{1}{2}\lambda^2).
$$

We conjecture that the price of the surplus strips is given by:

$$
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = (\chi_{k,t} - \psi_{k,t})Y_t.
$$

The pricing of the first surplus strip is

$$
\begin{aligned}
\mathbb{E}_t\left[M_{t,t+1}S_{t+1}\right] &= \mathbb{E}_t\left[M_{t,t+1}\{-Y_{t+1}\left(d_{t+1} - R^f_t d_t\exp[-(\mu + \sigma\varepsilon_{t+1})]\right)\}\right], \\
&= \mathbb{E}_t\left[-\exp(\phi_1\log d_t + m_{t,t+1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}\right] + d_tY_t, \\
&= -\exp(\phi_1\log d_t + \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t + d_tY_t,
\end{aligned}
$$

$$(\chi_{1,t} - \psi_{1,t})Y_t \quad = \quad \left[d_t - \exp(\phi_0 + \phi_1 \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)\right] Y_t.$$

So, we define:

$$\begin{aligned}
(\chi_{1,t})Y_t &= d_t Y_t, \\
(\psi_{1,t})Y_t &= \exp(\phi_0 + \phi_1 \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t.
\end{aligned}$$

Similarly the pricing of the k-th surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \mathbb{E}_t\left[M_{t,t+1}\mathbb{E}_{t+1}[M_{t+1,t+k}S_{t+k}]\right], \\
(\chi_{k,t} - \psi_{k,t})Y_t &= \mathbb{E}_t\left[M_{t,t+1}(\chi_{k-1,t+1} - \psi_{k-1,t+1})Y_{t+1}\right],
\end{aligned}$$

where the χ's are defined by the following recursion:

$$\begin{aligned}
\chi_{2,t}Y_t &= \mathbb{E}_t\left[M_{t,t+1}\chi_{1,t+1}Y_{t+1}\right], \\
\chi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(-\lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)\exp(\mu + \sigma\varepsilon_{t+1})\right]\exp(\phi_1 \log d_t + \phi_0), \\
&= \exp(\phi_0 + \phi_1 \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) = \psi_{1,t}
\end{aligned}$$

and the ψ's are defined by the following recursion:

$$\begin{aligned}
\psi_{2,t}Y_t &= \mathbb{E}_t\left[M_{t,t+1}\psi_{1,t+1}Y_{t+1}\right], \\
\psi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi_0 + \phi_1 \log d_{t+1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + g + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + \sigma\varepsilon_{t+1})\right], \\
\psi_{2,t} &= \exp(-2\rho + \phi_0 + \phi_1\phi_0 + \phi_1^2 \log d_t - \frac{1}{2}(\gamma^2 + \phi_1\lambda^2), \\
&- \frac{1}{2}(\gamma^2 + \lambda^2) + 2g + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \frac{1}{2}(\gamma + \lambda\phi_1 - \sigma)^2), \\
&= \psi_{1,t}\exp(-\rho + \phi_1\phi_0 + (\phi_1^2 - \phi_1)\log d_t - \frac{1}{2}(\gamma^2 + \phi_1\lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda\phi_1 - \sigma)^2).
\end{aligned}$$

More generally, we note that $\chi_{k+1,t} = \psi_{k,t}$, so that

$$\sum_{k=1}^{\infty}\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] \quad = \quad \chi_{1,t}Y_t = D_t.$$

For some $0 < \phi_1 < 1$,

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+1}D_{t+1}] &= \mathbb{E}_t[M_{t,t+1}Y_{t+1}d_{t+1}], \\
&= d_t^{\phi_1}\mathbb{E}_t[\exp(m_{t,t+1} - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}], \\
&= d_t^{\phi_1}\exp(\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t, \\
&= \exp(\kappa_1)\exp(\phi_1 \log d_t)Y_t,
\end{aligned}$$

where $\kappa_1 = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2$.

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+2}D_{t+2}] &= \mathbb{E}_t[M_{t,t+1}\mathbb{E}_{t+1}[\exp(m_{t+1,t+2})D_{t+2}]], \\
&= \mathbb{E}_t[M_{t,t+1}\exp(\kappa_1)\exp(\phi_1 \log d_{t+1})Y_{t+1}],
\end{aligned}$$

$$= \mathbb{E}_t[M_{t,t+1}\exp(\kappa_1)\exp(\phi_1^2\log d_t + \phi_1\phi_0 - \phi_1\lambda\varepsilon_{t+1} - \frac{1}{2}\phi_1\lambda^2)\exp(\mu+\sigma\varepsilon_{t+1})]Y_t,$$
$$= \exp(\kappa_1+\kappa_2)\exp(\phi_1^2\log d_t)Y_t,$$

where $\kappa_2 = \phi_1\phi_0 - \rho - \frac{1}{2}(\gamma^2+\phi_1\lambda^2) + g + \frac{1}{2}(\gamma+\phi_1\lambda-\sigma)^2$. Then, by induction,

$$\lim_{j\to\infty}\mathbb{E}_t[M_{t,t+j}D_{t+j}] = \lim_{j\to\infty}\exp(\sum_{k=1}^{j}\kappa_k)\exp(\phi_1^j\log d_t)Y_t,$$
$$= \lim_{j\to\infty}\exp(\frac{\phi_0}{1-\phi_1} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1}) + \mu j + \sum_{k=1}^{j}\frac{1}{2}(\gamma+\lambda\phi_1^{k-1}-\sigma)^2)Y_t,$$
$$= \lim_{j\to\infty}\exp(\frac{\phi_0}{1-\phi_1} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1}) + \mu j + j\frac{1}{2}(\gamma-\sigma)^2 + C)Y_t,$$

which is 0 if and only if $-\rho+\mu+\frac{1}{2}\sigma(\sigma-2\gamma) < 0$. This equality does not depend on ϕ_1 and λ.

□

A.6.2 Case of Random Walk

Proof. Now, assume $\phi_1 = 1$ and $\phi_0 = 0$. Then $\kappa_j = \phi_0 - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2$.

The TVC is

$$\lim_{j\to\infty}\mathbb{E}_t[M_{t,t+j}D_{t+j}] = \lim_{j\to\infty}\exp(\sum_{k=1}^{j}\kappa_k)\exp(\log d_t)Y_t,$$

which is 0 if and only if $-\rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2 < 0$. □

A.6.3 Case of $AR(2)$

Proof. From

$$T_t = G_t - (D_t - R^f_{t-1}D_{t-1}).$$

This implies that:

$$\begin{aligned} S_t &= -\left(d_tY_t - R^f_{t-1}d_{t-1}Y_{t-1}\right), \\ &= d_{t-1}R^f_{t-1}Y_{t-1} - \exp(\phi_0 + \phi_1\log d_{t-1} + \phi_2\log d_{t-2} - \lambda\varepsilon_t - \frac{1}{2}\lambda^2)Y_t. \end{aligned}$$

Conjecture the price of the surplus strips is given by

$$\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = (\chi_{k,t} - \psi_{k,t})Y_t.$$

The pricing of the first surplus strip is

$$\begin{aligned} \mathbb{E}_t\left[M_{t,t+1}S_{t+1}\right] &= \mathbb{E}_t\left[M_{t,t+1}\{-Y_{t+1}\left(d_{t+1} - R^f_t d_t\exp[-(\mu+\sigma\varepsilon_{t+1})]\right)\}\right], \\ &= \mathbb{E}_t\left[-\exp(\phi_1\log d_t + \phi_2\log d_{t-1} + m_{t,t+1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}\right] + d_tY_t, \\ &= -\exp(\phi_1\log d_t + \phi_2\log d_{t-1} + \phi_0 - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2)Y_t + d_tY_t, \\ (\chi_{1,t} - \psi_{1,t})Y_t &= \left[d_t - \exp(\phi_0 + \phi_1\log d_t + \phi_2\log d_{t-1} - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2)\right]Y_t. \end{aligned}$$

We define

$$
\begin{aligned}
(\chi_{1,t})Y_t &= d_t Y_t,\\
(\psi_{1,t})Y_t &= \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2)Y_t.
\end{aligned}
$$

Similarly the pricing of the k-th surplus strip is

$$
\begin{aligned}
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \mathbb{E}_t\left[M_{t,t+1}\mathbb{E}_{t+1}[M_{t+1,t+k}S_{t+k}]\right],\\
(\chi_{k,t}-\psi_{k,t})Y_t &= \mathbb{E}_t\left[M_{t,t+1}(\chi_{k-1,t+1}-\psi_{k-1,t+1})Y_{t+1}\right],
\end{aligned}
$$

where the χ's are defined by the following recursion:

$$
\begin{aligned}
\chi_{2,t}Y_t &= \mathbb{E}_t\left[M_{t,t+1}\chi_{1,t+1}Y_{t+1}\right],\\
\chi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho-\frac{1}{2}\gamma^2-\gamma\varepsilon_{t+1})\exp(-\lambda\varepsilon_{t+1}-\frac{1}{2}\lambda^2)\exp(\mu+\sigma\varepsilon_{t+1})\right]\exp(\phi_1\log d_t+\phi_2\log d_{t-1}+\phi_0),\\
&= \exp(\phi_0+\phi_1\log d_t+\phi_2\log d_{t-1}-\rho-\frac{1}{2}(\gamma^2+\lambda^2)+\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2).
\end{aligned}
$$

and the ψ's are defined by the following recursion:

$$
\begin{aligned}
\psi_{2,t}Y_t &= \mathbb{E}_t\left[M_{t,t+1}\psi_{1,t+1}Y_{t+1}\right],\\
\psi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho-\frac{1}{2}\gamma^2-\gamma\varepsilon_{t+1}+\phi_0+\phi_1\log d_{t+1}+\phi_2\log d_t-\rho-\frac{1}{2}(\gamma^2+\lambda^2)+\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2\;+\;\mu+\sigma\varepsilon_{t+1})\right],\\
\psi_{2,t} &= \exp(-2\rho+\phi_0+\phi_1\phi_0+(\phi_1^2+\phi_2)\log d_t+\phi_1\phi_2\log d_{t-1}-\frac{1}{2}(\gamma^2+\phi_1\lambda^2),\\
&- \frac{1}{2}(\gamma^2+\lambda^2)+2\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2+\frac{1}{2}(\gamma+\lambda\phi_1-\sigma)^2).
\end{aligned}
$$

We note that $\chi_{k+1,t}=\psi_{k,t}$, so this expression can be simplified as follows:

$$
\sum_{k=1}^{\infty}\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = \chi_{1,t}Y_t = D_t
$$

$$
d_t = \exp(\phi_1\log d_{t-1}+\phi_2\log d_{t-2}+\phi_0-\lambda\varepsilon_t-\frac{1}{2}\lambda^2).
$$

$$
\begin{aligned}
\mathbb{E}_t[M_{t,t+1}D_{t+1}] &= \mathbb{E}_t[M_{t,t+1}Y_{t+1}d_{t+1}],\\
&= d_t^{\phi}\mathbb{E}_t[\exp(m_{t,t+1}-\lambda\varepsilon_{t+1}-\frac{1}{2}\lambda^2)Y_{t+1}],\\
&= d_t^{\phi}\exp(\phi_0-\rho-\frac{1}{2}(\gamma^2+\lambda^2)+\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2)Y_t,\\
&= \exp(\kappa_1)\exp(\phi_1\log d_t+\phi_2\log d_{t-1})Y_t,
\end{aligned}
$$

Define $\kappa_1=\phi_0-\rho+-\frac{1}{2}(\gamma^2+\lambda^2)+\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2$.

$$
\begin{aligned}
\mathbb{E}_t[M_{t,t+2}D_{t+2}] &= \mathbb{E}_t[M_{t,t+1}\mathbb{E}_{t+1}[\exp(m_{t+1,t+2})D_{t+2}]],\\
&= \mathbb{E}_t[M_{t,t+1}\exp(\kappa_1)\exp(\phi_1\log d_{t+1}+\phi_2\log d_t)Y_{t+1}],\\
&= \mathbb{E}_t[M_{t,t+1}\exp(\kappa_1)\exp((\phi_1^2+\phi_1\phi_2+\phi_2)\log d_t+\phi_1\phi_0-\phi_1\lambda\varepsilon_{t+1}-\frac{1}{2}\phi_1\lambda^2)\exp(\mu+\sigma\varepsilon_{t+1})]Y_t,\\
&= \exp(\kappa_1+\kappa_2)\exp((\phi_1^2+\phi_1\phi_2+\phi_2)\exp(\log d_t)Y_t.
\end{aligned}
$$

Define $\kappa_2 = \phi_1\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \phi_1\lambda^2) + \mu + \frac{1}{2}(\gamma + \phi_1\lambda - \sigma)^2$.

$$\begin{aligned}
\lim_{j\to\infty} \mathbb{E}_t[M_{t,t+j}D_{t+j}] &= \lim_{j\to\infty} \exp(\sum_{k=1}^{j}\kappa_k)\exp(\psi_j \log d_t)Y_t, \\
&= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi_1-\phi_2} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1-\phi_2}) + \mu j + \sum_{k=1}^{j}\frac{1}{2}(\gamma + \lambda\psi_{k-1} - \sigma)^2)Y_t, \\
&= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi_1-\phi_2} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1-\phi_2}) + \mu j + j\frac{1}{2}(\gamma-\sigma)^2 + C)Y_t,
\end{aligned}$$

which is 0 if and only if $-\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0$. This equality does not depend on ϕ and λ. So this case is similar to the i.i.d. debt case $\phi = 0$. More extremely, when $\lambda = 0$, $d_t = \exp(\phi_0)$ is a constant. Now, assume $\phi = 1$. Then

$$\kappa_j = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2,$$

and $\lim_{j\to\infty} \mathbb{E}_t[M_{t,t+j}D_{t+j}] = \lim_{j\to\infty} \exp(\sum_{k=1}^{j}\kappa_k)\exp(\log d_t)Y_t$, which is 0 if and only if $\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 < 0$. □

A.7 Proof of Proposition 6

A.7.1 Case of $AR(1)$

Proof. When the log of the debt/output process follows an $AR(1)$, the surplus/output ratio is given by:

$$\begin{aligned}
\frac{S_{t+1}}{Y_{t+1}} &= d_t R_t^f \exp[-(\mu + \sigma\varepsilon_{t+1})] - d_t^{\phi_1}\exp(\phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) \\
&= \exp(r_t^f - \mu - \sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) - \exp(\phi_1(-\sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2).
\end{aligned}$$

We assume that $r_t^f = \mu$. This expression for the surplus/output ratio can be restated as:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) - \exp(-\sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$:

$$\frac{\partial\frac{S_{t+1}}{Y_{t+1}}}{\partial\varepsilon_{t+1}} = (\lambda)\exp(g + \sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) - \sigma\exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial\frac{S_{t+1}}{Y_{t+1}}}{\partial\varepsilon_{t+1}} = + \quad (\lambda - \sigma)\exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}).$$

Next, we compute the derivative of the surplus/output ratio at $t+2$, given by

$$\frac{\partial\frac{S_{t+2}}{Y_{t+2}}}{\partial\varepsilon_{t+1}} = -\lambda\exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) + \lambda\phi_1\exp(-\sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+2-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) + \lambda\phi_1 \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}).$$

This generalizes to the following expression. For $j \geq 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda\phi_1^{j-1} \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) + \lambda\phi_1^{j} \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}).$$

Assume $r^f = g$. Then we obtain the IRF:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = \lambda\phi_1^{j-1}(\phi_1 - 1)d, j > 1,$$
$$\frac{\partial \frac{S_{t+1}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = (\lambda - \sigma)d, j = 1.$$

□

A.7.2 Case of $AR(2)$

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. We assume that $r_t^f = g$. When the log of the debt/output process follows an $AR(2)$, the surplus/output ratio is given by:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\rho^j\lambda\varepsilon_{t-j} + \bar{d}) - \exp(+\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-1-j}) - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$, and we evaluate this derivative at $\varepsilon_{t+j} = 0$:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = + \quad (\lambda - \sigma)\exp(\bar{d})).$$

The surplus/output ratio at $t+2$ is given by:

$$\frac{S_{t+2}}{Y_{t+2}} = \exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty}\rho^j\lambda\varepsilon_{t+1-j} + \bar{d}) - \exp(+\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) - \lambda\varepsilon_{t+2} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda\exp(\bar{d}) + \lambda(\phi_1)\exp(\bar{d})).$$

The surplus/output ratio at $t+3$ is given by:

$$\frac{S_{t+3}}{Y_{t+3}} = \exp(-\sigma\varepsilon_{t+3} - \sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j} + \bar{d}) - \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j}) - \lambda\varepsilon_{t+3} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+3}}{Y_{t+3}}}{\partial \varepsilon_{t+1}} = -\psi_1\lambda\exp(\bar{d}) + \lambda(\phi_1\psi_1 + \phi_2)\exp(\mu + \bar{d}).$$

This generalizes to the following expression. For $j > 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda \psi_{j-1} \exp(+\overline{d})) + \lambda \psi_j \exp(\overline{d})).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\begin{aligned}
\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma) \exp(\overline{d}), \textit{ for } j = 1, \\
&= \lambda(\phi_1 - 1) \exp(\overline{d}), \textit{ for } j = 2, \\
&= \lambda(\psi_{j-1} - \psi_{j-2}) \exp(\overline{d}), \textit{ for } j > 2.
\end{aligned}$$

□

A.7.3 Case of AR(3)

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. We assume that the risk-free rate equals the growth rate of the economy. When the log of the debt/output process follows an AR(3), the surplus/output ratio is given by:

$$\begin{aligned}
\frac{S_{t+1}}{Y_{t+1}} &= \exp(-\sigma \varepsilon_{t+1} - \sum_{j=0}^{\infty} \rho^j \lambda \varepsilon_{t-j} + \overline{d}) \\
&- \exp(+\overline{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-1-j} + \phi_3(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-2-j}) - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2).
\end{aligned}$$

Next, we compute the derivative of the surplus/output ratio at $t+1$, and we evaluate this derivative at $\varepsilon_{t+j} = 0$:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = + (\lambda - \sigma) \exp(\overline{d})).$$

The surplus/output ratio at $t+2$ is given by:

$$\begin{aligned}
\frac{S_{t+2}}{Y_{t+2}} &= \exp(-\sigma \varepsilon_{t+2} - \sum_{j=0}^{\infty} \rho^j \lambda \varepsilon_{t+1-j} + \overline{d}) \\
&- \exp(\overline{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+1-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) + \phi_3(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-1-j}) - \lambda \varepsilon_{t+2} - \frac{1}{2}\lambda^2).
\end{aligned}$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda \exp(\overline{d})) + \lambda(\phi_1) \exp(\overline{d})).$$

The surplus/output ratio at $t+3$ is given by:

$$\begin{aligned}
\frac{S_{t+3}}{Y_{t+3}} &= \exp(-\sigma \varepsilon_{t+3} - \sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+2-j} + \overline{d}) \\
&- \exp(\overline{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+2-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+1-j} + \phi_3(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) - \lambda \varepsilon_{t+3} - \frac{1}{2}\lambda^2).
\end{aligned}$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+3}}{Y_{t+3}}}{\partial \varepsilon_{t+1}} = -\psi_1 \lambda \exp(\bar{d}) + \lambda(\phi_1 \psi_1 + \phi_2) \exp(\mu + \bar{d}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+4}}{Y_{t+4}}}{\partial \varepsilon_{t+1}} = -\rho_2 \lambda \exp(\bar{d}) + \lambda(\phi_1 \rho_2 + \phi_2 \psi_1 + \phi_3) \exp(\mu + \bar{d}).$$

This generalizes to the following expression. For $j > 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda \psi_{j-1} \exp(\bar{d}) + \lambda \psi_j \exp(\mu + \bar{d}).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\begin{aligned}
\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma) \exp(\bar{d}), \textit{ for } j = 1, \\
&= \lambda(\phi_1 - 1) \exp(\bar{d}), \textit{ for } j = 2, \\
&= \lambda(\phi_1 \psi_1 + \phi_2 - \psi_1) \exp(\bar{d}), \textit{ for } j = 3, \\
&= \lambda(\psi_{j-1} - \psi_{j-2}) \exp(\bar{d}), \textit{ for } j > 3.
\end{aligned}$$

□

A.8 Proof of Proposition 7

A.8.1 Case of *AR(1)*

Proof. As a result, we can solve for an expression of the log debt/output ratio as a function of the past shocks:

$$\log d_t = -\sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1 - \phi}.$$

Consider a government that only issues risk-free debt. Note that the surplus at $t + 1$ is given by:

$$S_{t+1} = d_t Y_t \exp(r_t^f) - \exp(\phi \log d_t + \phi_0 - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}.$$

We get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1} Y_{t+1}) \\
&= -E_t[M_{t+1} d_{t+1} Y_{t+1}] + E_t[M_{t+1}] E_t[d_{t+1} Y_{t+1}] \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho) \exp(\frac{1}{2}(\lambda - \sigma)^2 + \mu + y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1) E_t[M_{t+1}] E_t[d_{t+1} Y_{t+1}] \\
&= -E_t[M_{t+1}] E_t[d_{t+1} Y_{t+1}] (\exp(-\gamma(\sigma - \lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over two periods:

$$
\begin{aligned}
& cov_t(M_{t+1}, S_{t+1} + E_{t+1}[M_{t+1,t+2}S_{t+2})] \\
= \;& cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}]) \\
= \;& -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}](\exp(-\gamma(\sigma - \phi\lambda)) - 1)
\end{aligned}
$$

Check the proof of Prop. 2 to see why the sum of the discounted surpluses drop out, and only the debt issuance term remains. We get the following expression for the covariance of the discounted surpluses over j periods:

$$
\begin{aligned}
& cov_t(M_{t+1}, \sum_{k=1}^{j} E_{t+1}[M_{t+1,t+j}S_{t+j}]) \\
= \;& cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
= \;& -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma - \phi^{j-1}\lambda)) - 1).
\end{aligned}
$$

□

A.8.2 Case of *AR(2)*

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. As a result, we can solve for an expression of the log debt/output ratio as a function of the past shocks:

$$\log d_t = -\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1 - \phi_1 - \phi_2}.$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}$. Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by:

$$S_{t+1} = d_t Y_t \exp(r_t^f) - \exp(+\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}.$$

As a result, we get the following expression for the covariance:

$$
\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\
&= -E_t[M_{t+1}d_{t+1}Y_{t+1}] + E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + y_t + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho)\exp(\frac{1}{2}(\lambda - \sigma)^2 + \mu + y_t + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1)E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma - \lambda)) - 1).
\end{aligned}
$$

By the same token, we get the following expression for the covariance of the discounted surpluses over two periods:

$$
\begin{aligned}
& cov_t(M_{t+1}, S_{t+1} + E_{t+1}[M_{t+1,t+2}S_{t+2})] \\
= \;& cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}]) \\
= \;& -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}](\exp(-\gamma(\sigma - \psi_1\lambda)) - 1)
\end{aligned}
$$

Check the proof of Prop. 2 to see why the sum of the discounted surpluses drop out, and only the debt issuance term remains. And we get the following expression for the covariance of the discounted surpluses over j periods:

$$\begin{aligned}
& cov_t(M_{t+1}, \sum_{k=1}^{j} E_{t+1}[M_{t+1,t+j}S_{t+j}]) \\
= \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
= \; & -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1).
\end{aligned}$$

□

A.9 Proof of Corollary 2

A.9.1 Case of *AR(1)*

Proof. Start from the restriction:

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1) \\
+ \; & x cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}Y_{t+k}\right) \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1) \\
+ \; & x\sum_{k=1}^{j} E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma\sigma)-1)
\end{aligned}$$

We substitute for the price of debt strips:

$$\begin{aligned}
& \mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \\
= \; & \exp(\sum_{k=1}^{j}\kappa_k)\exp(\phi^j\log d_t)Y_t \\
= \; & \exp(\frac{\phi_0(1-\phi^j)}{1-\phi}-\rho j-\frac{1}{2}(\gamma^2 j+\frac{\lambda^2(1-\phi^j)}{1-\phi})+\mu j+\sum_{k=1}^{j}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2)\exp(\phi^j\log d_t)Y_t
\end{aligned}$$

For $j>1$, we obtain the following expression:

$$\begin{aligned}
& \mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & \exp(\frac{\phi_0(1-\phi^{j-1})}{1-\phi}-\rho(j-1)-\frac{1}{2}(\gamma^2(j-1)+\frac{\lambda^2(1-\phi^{j-1})}{1-\phi})+\mu(j-1)+\sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2) \\
& \exp(\phi^{j-1}\log d_{t+1})Y_{t+1},
\end{aligned}$$

and, for $j=1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]=\exp(\frac{\phi_0}{1-\phi})\exp(\log d_{t+1})Y_{t+1}.$$

For $j>1$, this simplifies to the following expression:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & \exp(\frac{1-\phi^j}{1-\phi}(\phi_0-\frac{1}{2}\lambda^2)-\rho(j-1)-\frac{1}{2}\gamma^2(j-1)+\mu j+\sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2)
\end{aligned}$$

$$\exp(\phi^j \log d_t + \frac{1}{2}(-\phi^{j-1}\lambda + \sigma)^2)Y_t.$$

Note that by a similar logic, the price of the output strips is given by:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}] \\
= \quad & \exp(-\rho(j-1) - \frac{1}{2}\gamma^2(j-1) + \mu j + (j-1)\frac{1}{2}(\gamma-\sigma)^2 + \frac{1}{2}(\sigma)^2)Y_t
\end{aligned}$$

To summarize, for $j > 1$, this implies that we have the following expression:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \quad & \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}]\exp(\frac{1-\phi^j}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}\frac{1}{2}((\lambda\phi^{k-1})^2 + 2(\gamma-\sigma)\lambda\phi^{k-1})) \\
& \exp(\phi^j \log d_t + \frac{1}{2}((\phi^{j-1}\lambda)^2 - 2\sigma\phi^{j-1}\lambda)).
\end{aligned}$$

and for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp(\frac{\phi_0}{1-\phi})\exp(\phi \log d_t)\exp(\mu + \frac{1}{2}\sigma^2)Y_t.$$

□

A.9.2 Case of *AR(2)*

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. Start from the restriction:

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= \quad & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma - \psi_{j-1}\lambda)) - 1) \\
+ \quad & x cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}Y_{t+k}\right) \\
= \quad & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma - \psi_{j-1}\lambda)) - 1) \\
+ \quad & x\sum_{k=1}^{j} E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma\sigma) - 1)
\end{aligned}$$

We substitute for the price of debt strips:

$$\begin{aligned}
& \mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \\
= \quad & \exp(\sum_{k=1}^{j}\kappa_k)\exp(\psi_j \log d_t)Y_t \\
= \quad & \exp(\sum_{k=1}^{j}\psi_{k-1}\phi_0 - \rho j - \frac{1}{2}(\gamma^2 j + \sum_{k=1}^{j}\psi_{k-1}\lambda^2) + gj + \sum_{k=1}^{j}\frac{1}{2}(\gamma + \lambda\psi_{k-1} - \sigma)^2)\exp(\psi_j \log d_t)Y_t
\end{aligned}$$

For $j > 1$, we obtain the following expression:

$$\begin{aligned}
& \mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \quad & \exp(\sum_{k=1}^{j-1}\psi_{k-1}\phi_0 - \rho(j-1) - \frac{1}{2}(\gamma^2(j-1) + \sum_{k=1}^{j-1}\psi_{k-1}\lambda^2) + \mu(j-1) + \sum_{k=1}^{j-1}\frac{1}{2}(\gamma + \lambda\psi_{k-1} - \sigma)^2) \\
& \exp(\psi_{j-1}\log d_{t+1})Y_{t+1},
\end{aligned}$$

and, for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp(\frac{\phi_0}{1-\phi_1-\phi_2})\exp(\log d_{t+1})Y_{t+1}.$$

For $j > 1$, this simplifies to the following expression:

$$\begin{aligned} & \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\ = \quad & \exp(\sum_{k=1}^{j}\psi_{k-1}(\phi_0 - \frac{1}{2}\lambda^2) - \rho(j-1) - \frac{1}{2}\gamma^2(j-1) + \mu j + \sum_{k=1}^{j-1}\frac{1}{2}(\gamma + \lambda\psi_{k-1} - \sigma)^2) \\ & \exp(\rho_j \log d_t + \frac{1}{2}(-\psi_{j-1}\lambda + \sigma)^2)Y_t. \end{aligned}$$

Note that by a similar logic, the price of the output strips is given by:

$$\begin{aligned} & \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}] \\ = \quad & \exp(-\rho(j-1) - \frac{1}{2}\gamma^2(j-1) + \mu j + (j-1)\frac{1}{2}(\gamma-\sigma)^2 + \frac{1}{2}(\sigma)^2)Y_t. \end{aligned}$$

To summarize, for $j > 1$, this implies that we have the following expression:

$$\begin{aligned} & \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\ = \quad & \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}]\exp(\sum_{k=1}^{j}\psi_{k-1}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}\frac{1}{2}((\lambda\psi_{k-1})^2 + 2(\gamma-\sigma)\lambda\psi_{k-1})) \\ & \exp(\psi_j \log d_t + \frac{1}{2}((\psi_{j-1}\lambda)^2 - 2\sigma\psi_{j-1}\lambda)), \end{aligned}$$

and for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp((\frac{\phi_0}{1-\phi_1-\phi_2})\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1})\exp(g + \frac{1}{2}\sigma^2)Y_t.$$

□

A.10 Model with Convenience Yields

The government debt portfolio return equals the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] = \frac{P^T_t - T_t}{B_t - S_t}\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] + \frac{P^K_t - T_t}{B_t - S_t}\mathbb{E}_t\left[R^\lambda_{t+1} - R^f_t\right] - \frac{P^G_t - G_t}{B_t - S_t}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right],$$

where R^D_{t+1}, R^T_{t+1},R^K_{t+1} and R^G_{t+1} are the holding period returns on the bond portfolio, the tax claim, and the spending claim, respectively. We take government spending process, and the debt return process as exogenously given, and we explore the implications for the properties of the tax claim.

Proposition 8. *In the absence of arbitrage opportunities, if the TVC holds, the expected excess return on the tax claim is the unlevered return on the spending claim and the debt claim:*

$$\begin{aligned} \mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] \quad = \quad & \frac{P^G_t - G_t}{D_t + (P^G_t - G_t) - (P^K_t - K_t)}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] \\ & + \frac{D_t}{D_t + (P^G_t - G_t) - (P^K_t - K_t)}\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] \end{aligned}$$

$$-\frac{P_t^K - K_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \mathbb{E}_t \left[R_{t+1}^K - R_t^f \right]$$

If we want the debt to be risk-free, then the following equation holds for expected returns:

$$\begin{aligned}
\mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] &= \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right] \\
&\quad - \frac{P_t^K - K_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \mathbb{E}_t \left[R_{t+1}^K - R_t^f \right]
\end{aligned}$$

$$\begin{aligned}
\beta_t^T &= \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \beta_t^G \\
&\quad - \frac{P_t^K - K_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \beta_t^K.
\end{aligned}$$

Suppose we consider the case of a constant spending ratio and a constant convenience yield ratio. Then this implies that the beta of the tax revenue process is given by:

$$\beta_t^T = \frac{(P_t^G - G_t) - (P_t^K - K_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \beta^G$$

On the other hand, suppose that the convenience yield seigniorage process has a zero beta. Then the implied beta of the tax revenue process

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \beta^G,$$

which exceeds the beta of the tax revenue without seigniorage: $\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)}$. If the seigniorage revenue is sufficiently counter-cyclical, then the government can insure both taxpayers and bondholders at the same time. For example, consider the case in which the government runs zero primary surpluses in all future states of the world. Then the beta of the tax revenue is one:$\beta_t^T = 1$, where $D_t = P_t^\lambda - K_t$. In this case, the average tax rate is constant: $\Delta \log \tau_{t+1} = 0$.

B Quantifying the Trade-off in Model with Transitory Output Shocks

Next, we consider the impact of transitory shocks to the level of output, but we, in a first pass, we keep our original pricing kernel with permanent shocks to the level of marginal utility. We call this the goldilocks economy. In this setting, the government can insure taxpayers at all horizons while keeping the debt risk-free.

B.1 Permanent Shocks to Marginal Utility

Assumption 4. *(a) The shocks to output are transitory:*

$$y_{t+1} = \xi_0 + \xi y_t + \sigma \varepsilon_{t+1}$$

where ε_{t+1} still denotes the innovation to output growth that is normally distributed and i.i.d.

(b) The log pricing kernel is

$$m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \gamma \varepsilon_{t+1}.$$

(c) The government commits to a policy for the debt/output ratio $d_t = D_t/Y_t$ given by:

$$\log d_t = \phi_1 \log d_{t-1} + \phi_0 - \lambda \varepsilon_t - \frac{1}{2}\lambda^2,$$

where $\lambda > 0$ so that the debt-output ratio increases in response to a negative output shock ε_t.

This asset pricing model is fundamentally misspecified. This pricing kernel does not reflect the mean-reversion in output and hence cannot be micro-founded. However, we use this model as an expositional device. In this setting, the government faces no trade-off between insuring taxpayers and bondholders. When there are no permanent shocks to output, but the pricing kernel does not reflect this, then the government can insure taxpayers over all horizons.

Proposition 9. *The cash flow beta of the surpluses over j periods is given by:*

$$\begin{aligned} & cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \textstyle\sum_{k=1}^{j} M_{t+1,t+k} S_{t+k} \right) \\ = \quad & -E_t[M_{t+1}] E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}] (\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1) \end{aligned}$$

when $j \geq 2$. The sign of the cash flow covariance is sign $\left(\xi^{j-1}\sigma - \phi^{j-1}\lambda\right)$.

Hence, the sign of the cash flow covariance is determined by the sign of $\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda)$. As before, this is the risk premium on a debt strip, and compensates investors for output risk. Because the innovations are temporary, the output component of this risk premium converges to zero. The transitory nature of output risk broadens the scope for insurance of taxpayers. As we consider $\xi \to 1$, we revert back to the expression derived in the benchmark model: $\gamma(\sigma - \phi^{k-1}\lambda)$. If $\lambda > \sigma$, the initial covariance is negative. If the rate of mean-reversion in output is higher than in the debt/output ratio, $\phi > \xi$, the covariance stays negative for all j. As a result, the government can now insure taxpayers at all horizons. This was not feasible in the case of permanent innovations.

Corollary 4. *The cash flow beta of taxes have to satisfy the following restriction.*

$$\begin{aligned} & cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \textstyle\sum_{k=1}^{j} M_{t+1,t+k} T_{t+k} \right) \\ = \quad & -E_t[M_{t+1}] E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}] (\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1) \\ + \quad & cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \textstyle\sum_{k=1}^{j} M_{t+1,t+k} G_{t+k} \right). \end{aligned}$$

Quantitative Implications We return to our calibrated economy. Figure 9 plots the risk premium contributions of the surpluses over different horizons for the benchmark calibration:

$$
\begin{aligned}
& -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k} S_{t+k}\right) / \mathbb{E}_t[M_{t+1}] \\
= \quad & E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1)
\end{aligned}
$$

However, the output process no longer has a unit root. We start by considering the case in which $\phi = \xi$. At all horizons, the tax claim is risky, contributing positive risk premium across all horizons, because λ exceeds σ. The tax claim is also risky across all horizons. In this goldilocks scenario, the government can insure taxpayers at all horizons. $\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda)$ is positive across all horizons.

Figure 9: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. The bottom panel plots the risk premium on the debt strips: $-(\exp(-\gamma(\sigma\xi^{j-1} - \phi^{j-1}\lambda)) - 1)$. Calibration: ϕ is 0.75 and ξ is 0.75. Other parameters–Benchmark calibration in Table 1.

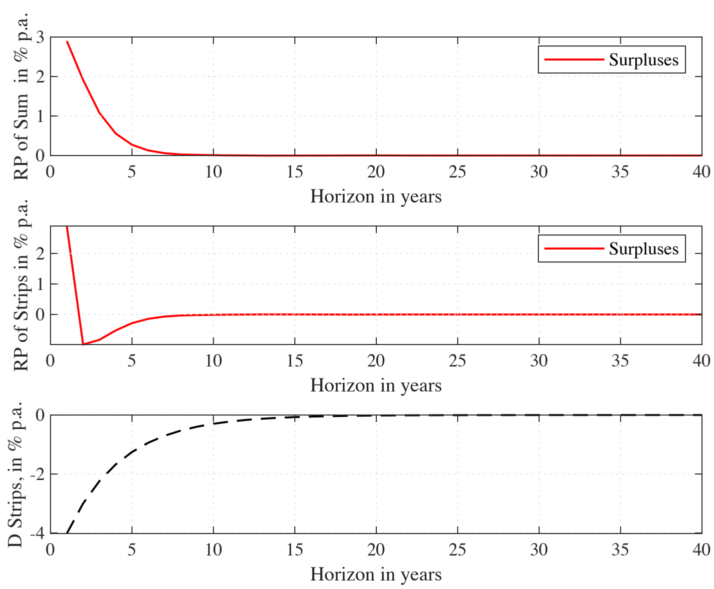


Figure 9 plots the risk premium on the debt strips, which pay off $d_{t+k}Y_{t+k}$, given by

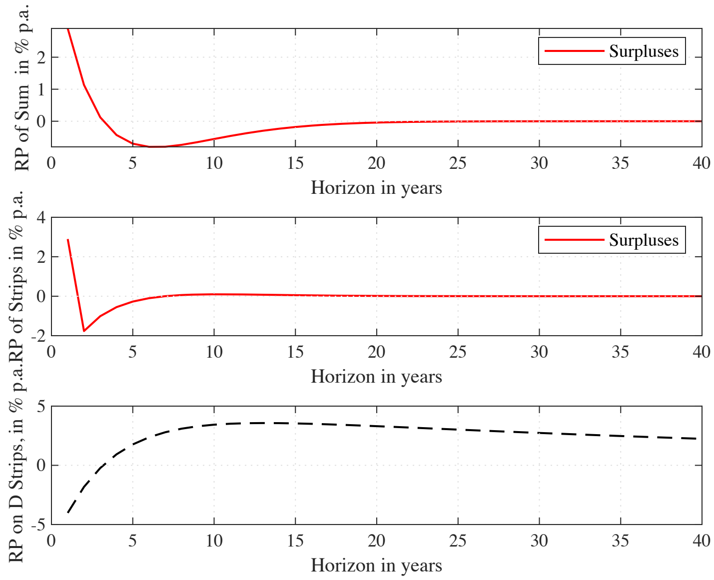
$$
\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda) \approx -(\exp(-\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda)) - 1).
$$

Given that λ exceeds σ, the risk premium on the debt strips are uniformly negative. These are the mirror image of the surplus risk premium in the top panel of Figure 9. As $j \to \infty$, this debt strip risk premium converges to the risk premium on the output strips, 0%, because the output innovations are transitory, and the pricing kernel does not have a transitory component which contributes interest rate risk. Why can the government insure taxpayers over long horizons (by delivering a risky tax claim)? Because the debt strip risk premium are negative at all horizons.

Of course, insurance of taxpayers only works if the governments commits to a debt policy that is at least as persistent as the output process ($\phi > \xi$). Figure 10 plots the risk premia contributions when the output shocks are close to a unit root, but the debt/output ratio reverts back to the mean at a faster rate. In this case, the government has to produce safer surplus claims over longer horizons.

Figure 10: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (bottom panel) against the horizon. Calibration: ϕ is 0.75 and ξ is 0.98. Other parameters–Benchmark calibration in Table 1.



B.2 Transitory Shocks to Marginal Utility

Next, we consider an internally consistent model: we shut down permanent shocks to the level of output, as well as to marginal utility.

Assumption 5. *(a) The shocks to output are transitory:*

$$y_{t+1} \quad = \quad \xi_0 + \xi y_t + \sigma\varepsilon_{t+1}$$

where ε_{t+1} still denotes the innovation to output growth that is normally distributed and i.i.d.

(b) The log pricing kernel is

$$m_{t,t+1} \quad = \quad -\rho - \frac{1}{2}\gamma^2 - \gamma\frac{\sigma\varepsilon_{t+1} + (\xi - 1)y_t}{\sigma}.$$

When shocks to output are transitory, most asset pricing models predict that there are no permanent shocks to the marginal utility of wealth. This specific modification of the pricing kernel is motivated by the fact that if the agent's consumption is equal to the output and has CRRA preference with a relative risk aversion of γ/σ, the marginal utility growth is $m_{t,t+1} = -\tilde{\rho} - \gamma/\sigma(\xi_0 + (\xi - 1)y_t + \sigma\varepsilon_{t+1})$. In this case, the marginal utility of wealth can be written as:

$$\Lambda_{t+1} = \exp(-\tilde{\rho}(t+1) - (\gamma/\sigma)y_{t+1}).$$

There are no permanent shocks to the marginal utility of wealth. Given this pricing kernel, the log of the risk-free rate is given by:

$$r_t^f \quad = \quad \rho + \gamma\frac{(\xi - 1)y_t}{\sigma}.$$

Note that this model has counterfactual asset pricing implications. In the model, the interest rate risk will make the long bond the riskiest asset in the economy. Modern asset pricing has consistently found that permanent cash flow shocks receive a high price of risk in the market (e.g., Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Bansal and Yaron, 2004; Borovička et al., 2016; Backus et al., 2018). This model has no permanent priced risk, except when $\xi = 1$. In that case, we recover the pricing kernel in our benchmark model.

When there are no permanent shocks to output and the pricing kernel, then the government can insure taxpayers over longer horizons.

Proposition 10. *The cash flow beta of the surpluses over j periods is given by:*

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k} S_{t+k}\right) \\ = \quad & -E_t[M_{t+1}]E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1) \end{aligned}$$

when $j \geq 2$. The sign of the cash flow covariance is sign $\left(\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))\right)$.

Hence, the sign of the cash flow covariance is determined by the sign of $\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{k-1}))$. As before, this is the risk premium on a debt strip. The first component, $\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)$, compensates for output risk. The second component, $\frac{\gamma}{\sigma}(1-\xi^{j-1})$, compensates for interest rate risk. Because the innovations are temporary, the output component of this risk premium converges to zero. The interest rate risk does not converge to zero; the long bond is the riskiest asset in an economy with only transitory risk. The transitory nature of output risk broadens the scope for insurance of taxpayers, but this is counteracted by interest rate risk. As we consider $\xi \to 1$, we revert back to the expression derived in the benchmark model: $\gamma(\sigma - \phi^{k-1}\lambda)$. The interest rate risk term disappears.

Corollary 5. *The cash flow beta of taxes have to satisfy the following restriction.*

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k} T_{t+k}\right) \\ = \quad & -E_t[M_{t+1}]E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1) \\ + \quad & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k} G_{t+k}\right). \end{aligned}$$

which can be restated as:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k} T_{t+k}\right) \\ = \quad & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_{t+1}[M_{t+1,t+j} Y_{t+j}] \exp(\frac{1-\phi^j}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}((\gamma-\sigma)\xi^{k-1}\phi^{k-1}\lambda + \frac{1}{2}(\phi^{k-1}\lambda)^2) \\ + \quad & \phi^j \log d_t - \phi^{j-1}\lambda((\sigma-\gamma)\xi^{j-1} + \gamma) + \frac{1}{2}(\phi^{j-1}\lambda)^2)(\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1) \\ + \quad & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k} G_{t+k}\right). \end{aligned}$$

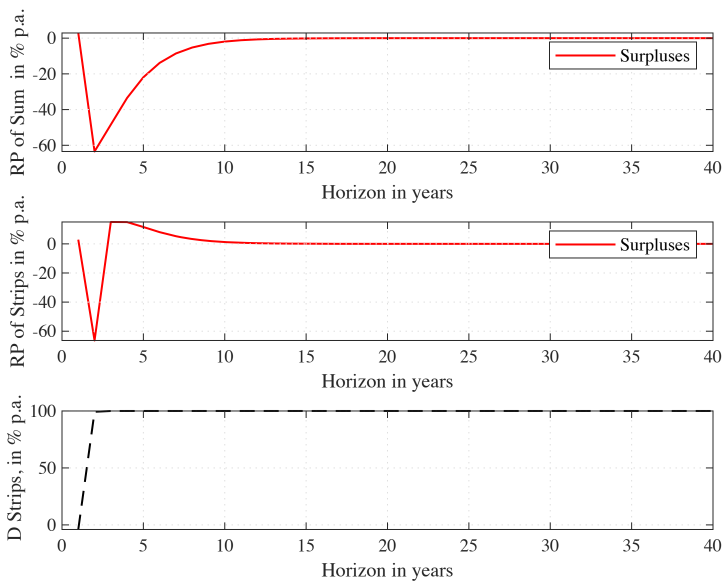
Quantitative Model Implications We return to our calibrated economy. Figure 11 plots the risk premium contributions of the surpluses over different horizons j for the benchmark calibration:

$$\begin{aligned} & -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k} S_{t+k}\right) / \mathbb{E}_t[M_{t+1}] \\ = \quad & E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1) \end{aligned}$$

However, the output process no longer has a unit root. At short horizons, the tax claim is safe, contributing negative risk premium, but the tax claim turns risky over horizons that exceed 10 years.

Figure 11: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. Calibration: ϕ is 0.75. Other parameters–Benchmark calibration in Table 1.



B.3 Proofs

B.3.1 Proof of Proposition 9

Proof. Since

$$S_{t+1} \quad = \quad d_t Y_t \exp(r_t^f) - d_{t+1} Y_{t+1},$$

we get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\
&= -E_t[M_{t+1}d_{t+1}Y_{t+1}] + E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \xi_0 + \xi y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho)\exp(\frac{1}{2}(\lambda - \sigma)^2 + \xi_0 + \xi y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1)E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma - \lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over $j \geq 2$ periods:

$$\begin{aligned}
& cov_t(M_{t+1}, E_{t+1}[\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}]) \\
= \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
= \; & -E_t[M_{t+1}M_{t+1,t+j}d_{t+j}Y_{t+j}] + E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}]
\end{aligned}$$

$$
\begin{aligned}
&= -E_t[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\ldots - \frac{\gamma(\xi-1)}{\sigma}(1+\xi+\ldots+\xi^{j-2})y_{t+1}) \\
&\quad \exp(\phi^j \log d_t - \phi^{j-1}\lambda\varepsilon_{t+1} + \ldots)\exp(\xi^j y_t + \xi^{j-1}\sigma\varepsilon_{t+1} + \ldots)] + E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
&= -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1)
\end{aligned}
$$

□

B.3.2 Proof of Corollary 4

Proof. This result simply follows from

$$
\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= \; & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right) + cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}G_{t+k}\right).
\end{aligned}
$$

□

B.3.3 Proof of Proposition 10

Proof. Since

$$
S_{t+1} = d_t Y_t \exp(r_t^f) - d_{t+1}Y_{t+1},
$$

we get the following expression for the covariance:

$$
\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\
&= -E_t[M_{t+1}d_{t+1}Y_{t+1}] + E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -\exp(-\rho - \frac{\gamma}{\sigma}(\psi - 1)y_t - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \xi_0 + \xi y_t + \phi \log d_t \\
&+ \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho - \frac{\gamma}{\sigma}(\xi - 1)y_t)\exp(\frac{1}{2}(\lambda - \sigma)^2 + \xi_0 + \xi y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1)E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma - \lambda)) - 1).
\end{aligned}
$$

By the same token, we get the following expression for the covariance of the discounted surpluses over $j \geq 2$ periods:

$$
\begin{aligned}
& cov_t(M_{t+1}, E_{t+1}[\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}]) \\
&= cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
&= -E_t[M_{t+1}M_{t+1,t+j}d_{t+j}Y_{t+j}] + E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
&= -E_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi - 1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\ldots - \frac{\gamma(\xi-1)}{\sigma}(1+\xi+\ldots+\xi^{j-2})y_{t+1}) \\
&\quad \exp(\phi^j \log d_t - \phi^{j-1}\lambda\varepsilon_{t+1} + \ldots)\exp(\xi^j y_t + \xi^{j-1}\sigma\varepsilon_{t+1} + \ldots)] + E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
&= -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda - \frac{\gamma(\xi-1)}{\sigma}\frac{1-\xi^{j-1}}{1-\xi})) - 1) \\
&= -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1)
\end{aligned}
$$

$\square$

B.3.4 Proof of Corollary 5

Proof. Start from the restriction:

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= \ & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda+\frac{\gamma}{\sigma}(1-\xi^{j-1})))-1) \\
+ \ & xcov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}Y_{t+k}\right)
\end{aligned}$$

where

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)M_{t+1,t+k}Y_{t+k}\right) \\
= \ & E_t[M_{t+1}M_{t+1,t+k}Y_{t+k}]-E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}] \\
= \ & E_t[\exp(-\rho-\frac{\gamma}{\sigma}(\xi-1)y_t-\frac{1}{2}\gamma^2-\gamma\varepsilon_{t+1})M_{t+1,t+k}\exp(\xi^k y_t+\xi^{k-1}\sigma\varepsilon_{t+1}+\dots)] \\
- \ & E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}] \\
= \ & -E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma(\xi^{k-1}\sigma+\frac{\gamma}{\sigma}(1-\xi^{k-1})))-1).
\end{aligned}$$

Next, we conjecture

$$\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] = \exp(\sum_{k=1}^{j}\tilde{\kappa}_k)\exp(\phi^j\log d_t+f_j y_t)$$

Note

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] &= \mathbb{E}_t[M_{t,t+1}\exp(\sum_{k=1}^{j-1}\kappa_k)\exp(\phi^{j-1}\log d_{t+1}+f_{j-1}y_{t+1})] \\
&= \mathbb{E}_t[\exp(-\rho-\frac{\gamma}{\sigma}(\xi-1)y_t-\frac{1}{2}\gamma^2-\gamma\varepsilon_{t+1})\exp(\sum_{k=1}^{j-1}\tilde{\kappa}_k) \\
&\quad \exp(\phi^{j-1}(\phi\log d_t+\phi_0-\lambda\varepsilon_{t+1}-\frac{1}{2}\lambda^2)+f_{j-1}(\xi_0+\xi y_t+\sigma\varepsilon_{t+1}))]
\end{aligned}$$

So we confirm the conjecture,

$$\begin{aligned}
\exp(\tilde{\kappa}_j) &= \mathbb{E}_t[\exp(-\rho-\frac{\gamma}{\sigma}(\xi-1)y_t-\frac{1}{2}\gamma^2-\gamma\varepsilon_{t+1}+\phi^{j-1}(\phi_0-\lambda\varepsilon_{t+1}-\frac{1}{2}\lambda^2)+f_{j-1}(\xi_0+\sigma\varepsilon_{t+1}))] \\
\tilde{\kappa}_j &= -\rho-\frac{1}{2}\gamma^2+\phi^{j-1}(\phi_0-\frac{1}{2}\lambda^2)+f_{j-1}\xi_0+\frac{1}{2}(-\gamma-\phi^{j-1}\lambda+f_{j-1}\sigma)^2
\end{aligned}$$

and

$$\begin{aligned}
f_j &= -\frac{\gamma}{\sigma}(\xi-1)+f_{j-1}\xi \\
&= \xi^j+\frac{\gamma}{\sigma}(1-\xi^j)=\frac{\sigma-\gamma}{\sigma}\xi^j+\frac{\gamma}{\sigma}
\end{aligned}$$

So, for $j>1$,

$$\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}]$$

$$
\begin{aligned}
&= \mathbb{E}_t[\exp(\sum_{k=1}^{j-1} \tilde{\kappa}_k) \exp(\phi^{j-1} \log d_{t+1} + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma}) y_{t+1})] \\
&= \exp((-\rho - \frac{1}{2}\gamma^2)(j-1) + \frac{1-\phi^{j-1}}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \left(\frac{1-\xi^{j-1}}{1-\xi}\frac{\sigma-\gamma}{\sigma} + \frac{\gamma}{\sigma}(j-1)\right)\xi_0 \\
&+ \sum_{k=1}^{j-1} \frac{1}{2}(-\gamma - \phi^{k-1}\lambda + ((\sigma-\gamma)\xi^{k-1} + \gamma))^2 \\
&+ \phi^{j-1}(\phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})(\xi_0 + \xi y_t) + \frac{1}{2}(-\phi^{j-1}\lambda + ((\sigma-\gamma)\xi^{j-1} + \gamma))^2)
\end{aligned}
$$

By a similar logic,

$$
\begin{aligned}
& \mathbb{E}_{t+1}[M_{t+1,t+j} Y_{t+j}] \\
&= \exp((-\rho - \frac{1}{2}\gamma^2)(j-1) + \left(\frac{1-\xi^{j-1}}{1-\xi}\frac{\sigma-\gamma}{\sigma} + \frac{\gamma}{\sigma}(j-1)\right)\xi_0 \\
&+ \sum_{k=1}^{j-1} \frac{1}{2}(-\gamma + ((\sigma-\gamma)\xi^{k-1} + \gamma))^2 + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})(\xi_0 + \xi y_t) + \frac{1}{2}(((\sigma-\gamma)\xi^{j-1} + \gamma))^2)
\end{aligned}
$$

So

$$
\begin{aligned}
& \mathbb{E}_{t+1}[M_{t+1,t+j} d_{t+j} Y_{t+j}] \\
&= \mathbb{E}_{t+1}[M_{t+1,t+j} Y_{t+j}] \exp(\frac{1-\phi^j}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}((\gamma-\sigma)\xi^{k-1}\phi^{k-1}\lambda + \frac{1}{2}(\phi^{k-1}\lambda)^2) \\
&+ \phi^j \log d_t - \phi^{j-1}\lambda((\sigma-\gamma)\xi^{j-1} + \gamma) + \frac{1}{2}(\phi^{j-1}\lambda)^2)
\end{aligned}
$$

□

C UK Data Sources

The main dataset we use is *A millennium of macroeconomic data* published by the bank of England. The dataset contains a broad set of macroeconomic and financial data for the UK. We also use other data sets as complementing the main dataset. Below we describe how we construct variables in our estimation procedure from the raw data set. The time period for all series are from 1814 to 2015. All sheets and columns refer to the excel table *A millennium of macroeconomic data* unless described otherwise.

Primary Surpluses The government expenditure G is the total government expenditure (Sheet A27, Column C) plus interest payments (Sheet A27, Column N). The government revenue T is from Sheet A27, Column N. The raw source for the data is from Mitchell and Mitchell (1988) and UK Office of National Statistics. The primary surpluses are the government revenue T minus the government spending before interest payments G. The market value of debt is the market value of central government liabilities (Sheet A30b, Column W).

GDP and Inflation: For real GDP, we use Sheet A8, Column D. For nominal GDP, we use Sheet A9, Column D. Both of the GDP series are measured based on the current definition of UK (Great Britain and Northern Ireland). We use the ratio of real GDP and nominal GDP to get the GDP deflator and the inflation series.

Short Rate: We use *Prime Commercial Bill/Paper Rate* in Sheet A31, Column F as our 1-period interest rate in our model. This series is based on spliced yield in the monthly short-term rates sheet (Sheet M9). The detailed break-down of the short rate are: 1) 1718 – 1825: rates on 6-month East India Bonds from Weiller and Mirowski (1990) and Heim and Mirowski (1987); 2) 1824 – 1870: rates on three month prime or first class bills from *Economist*; 3) 1870 – 1974: Prime Bank Bill Rate for 3 month bills from Nishimura (1971); 4) 1975 – 2005: rates on eligible bills from Bank of England; 5) 2005 –2012: end month rates on 3-month Sterling Euro-commercial paper from Bank of England, and 3 month Libor rate. All rates are annualized.

Long Rate: We use *Yield on perpetual annuities/consols* in Sheet A31, Column T as the yield for an infinitely lived bond in our model. The detailed break-down are: 1) 1703 – 1726: yields on new long-term issues from Neal (1993); 2) 1727 – 1753: yields on 3% perpetual annuities from Odlyzko (2016); 3) 1756 – 2015: yields on consols from Capie and Webber (1985) and Klovland (1994).