Growth, Automation, and the Long-Run Share of Labor†

Debraj Ray ⓡ *Dilip Mookherjee*

September 2020

Abstract. We study the implications of capital accumulation and workplace automation for long run growth and factor shares. Under a self-replication condition on the production of robot services, our theory delivers long run growth in per capita income and progressive automation, with the share of labor in national income converging to zero. The model is based on an asymmetry between physical and human capital. Individual claims on the former can be reproduced linearly and indefinitely. Because no similar claim on humans is possible, human capital accumulation instead takes the form of acquiring occupational skills, the returns to which are determined by endogenously determined wages. The displacement of human labor is gradual, and real wages could rise indefinitely. The results extend to endogenous technical change, as well as relaxations of the sharply posited human-physical asymmetry.

1. Introduction

We develop a disaggregated multi-sectoral theory of the dynamics of automation, with implications for the long run human labor share in national income. The theory features the possibility of gradual replacement of human workers by robots, as capital good prices decline. It also permits ongoing and indefinite human capital accumulation, in part as a response to the automation threat. The net replacement of humans by robots occurs under three conditions that we develop in detail: a self-replication condition on the robot production sector, an explicit consideration of the difference between human and physical capital accumulation, and a neutrality condition on the pattern of consumer demand. When these conditions are met, the economy is eventually automated under any competitive equilibrium with high discount factor and it is no longer bound by the fixity of the human labor endowment. In effect the economy behaves "as if" it has an Ak type of asymptotic aggregate production function, as in Rebelo (1990) or Jones and Manuelli (1991).

Our main interest is in the implications of such a theory for the long run income share of human labor. An asymptotic Ak model can, of course, co-exist with both a positive or an

†Mookherjee: Boston University; Ray: NYU and University of Warwick. Mookherjee thanks the Department of Economics at NYU for hosting his visit in Fall 2017 when this project was started. Ray acknowledges funding from the National Science Foundation under grant SES-1851758. We are grateful to Pascual Restrepo and Erik Madsen for useful conversations. Author names are in random order.

ever-declining labor share. Endogenous growth models such as those in Romer (1986) and Alesina and Rodrik (1993) generate a positive labor share via private diminishing returns, coupled with nondecreasing society-wide returns via externalities or infrastructure. A positive human share occurs in Ak models in which human and physical capital keep pace with each other, as in Lucas (1988) or Mankiw, Romer and Weil (1992). A stable share can also arise from parametric restrictions in aggregative models with automation, as in Aghion, Jones and Jones (2019), if automation and inelastic capital-labor substitution happen to mutually neutralize their opposing effects on the labor share.

In contrast, an aggregate production function which is exactly Ak, as in the Harrod-Domar model or its asymptotic variants, implies a zero labor share. Indeed, Jones and Manuelli (1991, fn. 2) argue that any long run labor share between 0 and 1 can be generated with suitable parametric assumptions on the class of Ak aggregate production functions they study. Without an underlying theory of the underlying *disaggregated* economy, how it evolves with progressive automation, and a consideration of more primitive forces, it is not possible to make strong predictions regarding the asymptotic human labor share. Do all sectors eventually get automated, or just a subset? And even if the former is asymptotically true, are there not sectors that are yet to be automated at any finite date — and might wages in such sectors conceivably keep pace with capital income? Additionally, what if workers can invest in human capital to “compete" with robots?

Therefore a more careful examination of the microfoundations of the Ak model is necessary. To this end, we study a model with many final goods, and additional sectors for intermediates: machine capital, robots and education. In any firm, production takes place with machine capital and one or more “labor aggregates," generated by human labor or robot services, or both. Apart from constant returns to scale in production, no substantive restrictions are placed on technology, not even convexity. Production functions vary across sectors. The relative efficiency of humans to robots can become unboundedly large over sequences of sectors. Human labor could be sector-specific, or migrate across sectors via education or training — this is the main source of human capital accumulation. Households are infinitely lived (or are connected via intergenerational altruism), and maximize infinite-horizon utility. They can accumulate financial wealth, which are claims on physical capital, and they can purchase education to move across occupations. In addition, households can be heterogenous in their tastes, discount factors or initial endowments.

Within this general skeleton, we study three features, each of which necessitates a disaggregated perspective. First, we discuss the nature of human capital and how it differs from physical capital. While individual claims to physical capital can be scaled indefinitely by accumulating wealth, the same is not true for ownership of labor. Instead, human capital accumulation takes the form of acquiring skills, within or across occupations, but always contained in *one* physical self. Moreover, if that accumulation consists in switching occupations (even within the same sector), e.g., a move from the shop floor to supervisory or managerial positions, the return to human capital will depend on relative wages across occupations. Those wages will be endogenously determined by the entire general equilibrium of the economy,[1] not just its technology. In particular — and this is the second feature we make explicit — the degree to which human capital can be persistently accumulated depends on the structure of demand for final goods or for intermediate inputs. An explicit consideration of such demand patterns requires us to go beyond the aggregative model.

The third key feature is connected to the possibility of automation in the sector that produces robot services; that is, the potential for using machines and robots to produce robots. This "self-replication" condition is *placed on the robot sector alone*.[2] And yet, when satisfied, it has economy-wide ramifications. It implies an upper bound to robot prices relative to those of physical capital, which provides firms in all other sectors with the option to automate at finite cost. Any sector yet to be automated generates a growing demand for humans with the relevant skills, driving up the wage for such workers. So absolute wages rise even as the tendency for automation does, but this tension is ultimately resolved only in one direction.

Our central result states that with sufficiently high patience among a subset of households, per capita income in the economy grows without bound, *and the share of labor in national income converges to zero*, provided consumer preferences are asymptotically homothetic, and provided that self-replication holds. At the same time, the displacement of human labor must be gradual, and real wages could rise indefinitely. In a sense, then, the "aggregate production function" — to the extent that anything of the sort exists — exhibits a capital-labor elasticity that exceeds one, even though individual sectors exhibit inelastic substitution. This part of our theoretical findings can be viewed as an aggregation result; it is not imposed as an assumption in an *ad hoc* specification of an aggregate production function.

[1]This observation is entirely in line with empirical evidence on skill premia in wages as shown by a large literature in labor economics stemming from Katz and Murphy (1994).

[2]The condition compares the elasticity of substitution between capital and (human or robot) labor with the efficiency of robots in generating the intermediate labor aggregates used in robot production. It is automatically met if the former elasticity equals or exceeds 1, but is also consistent with sub-unitary elasticities of substitution. Self-replication is like a singularity, though quite different from the notion that emerges from the work of von Neumann and others, which refers to some runaway threshold being crossed, thus generating *infinite* per-capita output at a *finite* date; see, for instance, Aghion, Jones and Jones (2019), who develop this viewpoint.

Faced with "robot creep," workers in threatened sectors can react in a farsighted way by acquiring education and moving to other, safer sectors. Therefore automation is consistent with unbounded growth in the wages of some or all workers, who manage to stay ahead of the threat by entering human-friendly occupations. Yet our distributional result holds if preferences are asymptotically homothetic — the aggregate share of such sectors must eventually shrink, and human wage growth will fail to match the growth of capital incomes.

Our theory helps identify four possible escape routes from the prediction of an asymptotically declining labor share. The first is a failure of self-replication, so that sustained automation is not guaranteed. The second is the sheer impossibility of full automation in some sectors. The third has to do with a particular preference structures. If preferences are non-homothetic in a serendipitous direction that aligns with ever greater "human-friendliness" in production, then human labor might perpetually retain a positive share of income. Humans would progressively move to these friendly sectors, and by the assumed non-homotheticity of preferences, there would be adequate demand for such products, shoring up wages there.

The fourth escape hatch is based on asymmetries in endogenous technical progress; specifically, biases in favor of human labor. We would need to *assume* those asymmetries, and their existence would be an empirical question. Acemoglu and Restrepo (2018) is a leading example of this approach. Section 5 extends our model to permit directed technical progress in machine, human and robot productivities, in response to profit opportunities created by changing prices. In our setting, technical progress operates on the intensive margin in each sector, raising the productivity of different inputs and possibly generating spillovers for the productivity of that input in other sectors. It may well be that technical progress inherently favors some inputs or sectors over others, but we maintain the neutrality of technological opportunities as an appropriate benchmark. We then show that our long run distributional implications continue to be robust in such a setting even with endogenous technical progress.

Our theory can be viewed as providing a potential explanation for the recent decline in labor shares documented by Karabarbounis and Neiman (2014) among others. As elaborated in Section 2, such a theory can be distinguished from alternative explanations based on capital-augmenting technical progress, human capital accumulation, rising markups and market concentration or declining bargaining power of labor unions. Its relevance is indicated by the fact that a substantial fraction of the decline in labor share worldwide is explained by declining capital goods prices, *even after controlling for capital-augmenting technical progress, markup rates and the skill composition of the labor force* (Karabarbounis and Neiman 2014).

Section 2 discusses related literature. Section 3 presents the model without technical progress. The main results are in Section 4, with related lines of discussion. Section 5 studies endogenous technical progress, and Section 6 concludes. Proofs are collected in an Appendix.

2. Related Literature

Existing approaches provide different explanations for a falling labor share, largely in models with a single good and a representative consumer. We classify them as follows:

(i) Theories of *automation under technical progress*. In a task-based setting, this is akin to an increasing capital share in an aggregate production function (Zeira 1998). Acemoglu and Restrepo (2018) and Aghion, Jones and Jones (2019) extend this model to study the distributional implications of automation under technical progress (endogenous in the former case, exogenous in the latter). Both emphasize the possibility of balanced growth, in which labor retains a long-run positive share despite automation.

(ii) Theories based on *sustained capital accumulation* in an aggregative model with capital-labor substitution elasticities exceeding one (e.g., Piketty 2014). But evidence from industry panel studies suggests inelastic substitution (Chirinko and Mallick 2014).

(iii) In contrast to (ii), theories based on *sustained human capital investments*, which cause effective labor to grow relative to effective capital (Grossman et al 2020). Human capital investments rise owing to a fall in the interest rate, driven in turn by an exogenous decline in rates of technical progress. Ancillary assumptions include capital-skill complementarity, a low intertemporal substitution elasticity in consumption, a closed economy, and an aggregate capital-labor elasticity of substitution below 1.

(iv) Theories based on *globalization, rising markups, rising market concentration, or fall in labor bargaining power*. Arguments include globalization, whereby labor in developed countries are devastated by cheap imports (Autor, Dorn and Hansen 2016), selection into more profitable, higher-markup firms (Autor et al 2017), or factors such as the rise of the gig economy or greater product differentiation, leading to a decline in firm competition and the bargaining power of labor (Neary 2003, Gutiérrez and Philippon 2017, Azar and Vives 2018, Eggertsson, Robbins, and Wold 2018, and Kaplan and Zoch 2020).

We provide an explanation for the secular decline in labor share that is driven by capital accumulation *alone*, one that operates even in the absence of technical progress, changing concentration or skewed bargaining power. As mentioned in the Introduction, the empirical relevance of our explanation is indicated by the results of Karabarbounis and Neiman (2014).

Our theory shares elements with (i)–(iii), but is distinct from each. The underlying driving force for the falling labor share is automation, as in (i), but also without technical progress, unlike (i). Automation is driven by the accumulation of *physical* capital, as in (ii), and yet the theory is consistent with inelastic substitution, as in (iii).[3] However, unlike (iii), our theory exhibits progressive *physical*, not human capital deepening. Quite apart from these features, our model is elementary, in the sense that it can serve as an alternative benchmark for a theory of economic change without balanced growth. This elementary nature is reflected in the substantial generality of our specification, devoid of functional form specifications on technology or utility or even restrictions on the curvature of the production functions.

Finally, at an empirical level, our theory makes distinct predictions. The relative growth of human capital and physical capital in efficiency units is inverted relative to Grossman et al (2020). Even if such evidence is not available, our theory becomes relevant if labor shares decline despite a violation of the Grossman et al assumptions, e.g., if technical progress does not decline, intertemporal consumption elasticities are high, or the economy is small in the sense that the interest rate is determined by the world capital market.

Our theory is clearly separate from the literature in (iv). The results would continue to apply in the absence of changes in monopoly power of firms or a decline in worker bargaining power. Indeed, the Karabarbounis and Neiman (2014) evidence suggests that half the decline in labor share worldwide associated with falling capital prices cannot be explained by rising monopoly power or changes in skill composition of the labor force. So while these explanations are certainly relevant, they are complementary to ours.

3. Baseline Model with No Technical Progress

3.1. Production. There is a countable collection I of consumption goods, indexed by i. The set I is infinite and so allows for unbounded occupational ladders. In addition, there are three intermediate good sectors producing education, robot services, and machine capital. The index j serves as generic notation for any of these sectors. Everything is produced, with the exception of raw human labor. That endowment is fixed (or is normalized as in the standard theory), but human *capital* evolves as individuals make educational investment decisions, thereby moving across the universe of occupations. The stock of physical capital changes over time as a result of production or depreciation.

[3]In fact, our multisectoral model cannot even be reduced to a CES specification through aggregation. In each sector, the elasticity of substitution across machines and labor could be inelastic. But human labor can be slowly replaced by robot services, sector by sector.

Physical capital has two incarnations. It is, first, a complementary input to labor; call this "machine capital" k_j. It combines with a finite vector of labor services $\boldsymbol{\ell}_j = \{\ell_j^o\}$, where $o \in O_j$ (an index set for tasks or occupations in sector j), to produce:

(1) $$y_j = f_j(k_j, \boldsymbol{\ell}_j)$$

where f_j is increasing, smooth, and linearly homogeneous, with unbounded steepness in capital input near zero, and in at least *one* of the effective labor inputs when all are near 0. Moreover, $f_j(k, 0) = f_j(0, \boldsymbol{\ell}) = 0$.[4] No curvature restrictions on f_j are imposed.

A second form of capital has the potential to displace human labor, and is represented by "robot services." We view each component ℓ_j^o of the labor variable as an aggregator, combining human input and robots services. Specifically, for each occupation $o \in O_j$,

(2) $$\ell_j^o = \ell_j^o(h_j^o, r_j^o),$$

where h_j^o is human input, r_j^o is robot services, and ℓ_j^o is increasing, smooth and linearly homogeneous with $\ell_j(0, 0) = 0$ (again, no assumption on curvature). In the baseline model we assume that no sector can *fully* protect humans; that is, $\ell_j^o(0, r) > 0$ for some $r > 0$, so that it is technologically feasible to produce tasks with robot services alone. Observe that this assumed feasibility of full automation does not imply its *economic* viability. In particular, the marginal effectiveness of human labor could be unboundedly large relative to that of robots. For instance, suppose that $\ell_j^o(h, r) = \nu r + \mu h + r^\alpha h^{1-\alpha}$ for $\alpha \in (0, 1)$, which satisfies all our conditions. Then in any equilibrium humans would be perennially employed in every occupation, no matter what factor prices are. In Section 4.7, we take an additional step back and discuss how our results are modified if full automation is not even technically feasible in all occupations.

While human labor is generally occupation-specific and additional education could be required to move across occupations, we assume for expositional ease that physical capital and robot services are perfectly homogeneous and can move freely across sectors.

3.2. Prices. Within any date, machine capital services serve as numeraire: the rental price of k is set to 1. The collection $\boldsymbol{w} = \{w_j^o\}$ for j and $o \in O_j$ is the wage system. Output prices are $(\mathbf{p}, p_r, p_e, p_k)$ for final goods, robot services, education, and capital. By constant returns to scale and the assumption of a competitive economy, all prices will equal unit costs

[4]The end-point restrictions are not needed, but we keep these to cut down on the number of cases. The assumed necessity of both inputs bounds substitution elasticities near the "axes," but not elsewhere.

of production for any sector with strictly positive output:

$$p_j \leq c_j(1, \boldsymbol{\omega}_j), \text{ with equality if } y_j > 0, \tag{3}$$

where 1 is the return to machine capital, $\boldsymbol{\omega}_j$ is the price vector of "labor" in sector j, and c_j is the *unit cost function*, dual to the function f_j.[5] The price of labor, in turn, comes from a second collection of unit cost functions $\{c_j^o\}$ for each occupation in that sector:

$$\omega_j^o = c_j^o(w_j^o, p_r). \tag{4}$$

3.3. Factor Demands and Automation. In each sector, machine capital and labor are chosen to maximize profits, satisfying familiar first-order necessary conditions when an input is positive. The mapping from prices to human and robot demand then flows through the aggregators $\boldsymbol{\ell}_j$. Define an automation index $a_j^o \equiv r_j^o/h_j^o$ for any "active" sector (with $y_j > 0$) and occupation o, where a_j^o is to be interpreted as ∞ when $h_j^o = 0$. By the linear homogeneity of ℓ_j, the set of optimal choices of a_j^o is represented by a correspondence $A_j^o(\zeta_j)$ defined only on the ratio $\zeta_j^o \equiv w_j^o/p_r$. By a standard monotonicity argument, this correspondence is nondecreasing (in the sense that whenever $\zeta_j^o \leq \zeta_j^{o\prime}$, $a_j^o \in A_j^o(\zeta_j^o)$, and $a_j^{o\prime} \in A_j^o(\zeta_j^o)$, we have $a_j^o \leq a_j^{o\prime}$). Moreover, $a_j^o \to \infty$ as $\zeta_j^o \to \infty$.[6] The *levels* of the factors h_j^o and r_j^o are then obtained by scaling quantities up or down, consistent with the selected automation ratio, so that the desired value of ℓ_j^o is produced.

3.4. Accumulation. The aggregate stock of capital $K(t)$ evolves according to

$$K(t+1) = (1-\delta)K(t) + y_k(t), \tag{5}$$

where $\delta \in [0, 1]$ is a constant, sector-independent depreciation rate for physical capital.[7] Our formalization puts all durability within the physical capital sector. Durable robots are included in this formulation, embedded in physical capital in the robot sector, where they produce services under the auspices of the robot production function f_r (with labor needed, perhaps, for maintenance).

The stock of raw human labor is given (or is normalized if it grows exogenously). But human *capital*, captured by movement across occupations, can change endogenously with individual education decisions. We take as given some initial allocation across occupations. There

[5] Our results easily extend to monopolistic competition with CES preferences, which generates a constant profit markup in all sectors. Profits would appear in that setting, so national income would be the sum of returns to capital, to workers and profits. Our distributional results would continue to apply.

[6] See Lemma 3(i)–(iii) in the Appendix for details.

[7] The model can be extended to incorporate sector specificity of capital services and depreciation rates.

could be a “null occupation" where individuals without initial skill can be placed, or can “drop out" to at zero cost. An individual can move from one occupation o to another o'; at an educational cost of $e(o, o')$ units times the price p_e, the latter endogenously determined. Human capital might depreciate; that is, it could be that $e_{oo} > 0$ for some or all o. We place no restriction on the education needed to switch occupations, so the model captures both inflexible occupation-specificity (no individuals can move at all) at one extreme, or complete flexibility (where no education is needed to switch occupations) at the other extreme, and everything in between. Observe also that our formulation allows humans to move both within and across sectors. Occupational returns will be determined by market-clearing conditions. The supply of services by different occupations may and in general will not be perfect substitutes, in which case skill premia will be endogenously determined rather than pinned down by technology.

3.5. Preferences. There is a continuum of infinitely lived individuals divided into a finite set of types, indexed by m. Each type m has a one-period increasing, continuous,[8] strictly concave utility indicator u_m, defined on vectors of final goods, and a discount factor $\beta_m \in (0, 1)$. For any pre-determined current expenditure z on final goods and price vector $\mathbf{p}$, her demand vector for goods maximizes $u_m(\mathbf{x})$, subject to $\mathbf{px} \leq z$. That generates a demand function $\mathbf{x}_m(\mathbf{p}, z)$. Denote by $v_m(z, \mathbf{p})$ the corresponding indirect utility function. We assume u_m is such that for every $\mathbf{p}$, the indirect function v_m is increasing, concave and differentiable, with unbounded steepness in z at zero.

At the start of any date, an individual has some financial wealth (representing her existing claims on capital or debt), and one unit of human labor along with a starting occupation. At date 0, her financial assets are nonnegative, and she can also work in a subsistence activity at any date to earn some small, exogenous, strictly positive income. We ignore the subsistence activity as it will get swamped in a growing economy: it is an expedient device to ensure a positive lower bound to human wages in all occupations.

At each date, an individual chooses an occupation by acquiring the necessary education. She supplies one unit of labor with no disutility, and is paid the occupational wage, provided it is no less than subsistence income. Then she decides how much to spend on different goods and how much to save financially (the rate of return on which is endogenously determined). All these decisions are made within an infinite-horizon utility maximization setting. Every

[8]The continuity of preferences or demand, here and everywhere else, will be taken relative to the pointwise or product topology on sequences of goods or price vectors.

individual makes farsighted investment decisions in human capital and financial wealth with perfect foresight about future wages and prices.

More formally, given some dated price-wage system for all goods, capital, and occupations, an individual of type m with initial (date-0) endowments of financial wealth $F_m(0) \geq 0$ and human capital (skill for occupation $o_m(-1)$) maximizes[9]

$$\sum_{t=0}^{\infty} \beta_m^t v_m(z(t), \mathbf{p}(t)), \tag{6}$$

by choosing a path of financial wealths $F_m(t)$ and occupations $o_m(t)$ at educational cost

$$E_m(t) \equiv e(o_m(t-1), o_m(t)), \tag{7}$$

along with current expenditure $z_m(t)$, subject to the date t budget constraint:

$$F_m(t) + w_{j_m(t)}(t) = z_m(t) + p_e(t)E_m(t) + \frac{F_m(t+1)}{\gamma(t)}, \tag{8}$$

and the no-Ponzi condition $\liminf_t F_m(t) \geq 0$. To accommodate imperfect capital markets, we impose $F_m(t) \geq B_m$ for all t, a borrowing limit that can be set arbitrarily high. Note that $\gamma(t)$ is the "return factor" on financial wealth at date t, and that:

$$\gamma(t) = \frac{1 + (1-\delta)p_k(t+1)}{p_k(t)}. \tag{9}$$

To understand (9), note that one unit of wealth can purchase claims to $\frac{1}{p_k(t)}$ units of physical capital at t. Each such unit generates a rental income of 1, then depreciates to yield $(1-\delta)$ units of physical capital worth $(1-\delta)p_k(t+1)$ at the next date.

A sufficient condition for the above maximization problem to be well-defined is that all utility functions are bounded. But well-known weaker conditions can be imposed, for instance, when utility functions have a well-defined tail elasticity. For our main results, we also presume that u_m is *asymptotically homothetic*:

$$\lim_{z\to\infty} \frac{\mathbf{x}_m(\mathbf{p}, z)}{z} = \mathbf{d}_m(\mathbf{p}) \text{ for some function } \mathbf{d}_m \tag{10}$$

for every $\mathbf{p} \gg 0$. We assume: (i) $\mathbf{d}_m$ is continuous on any bounded sequence of price vectors with strictly positive pointwise limit, and (ii) if there is a sequence $\{\mathbf{p}^n\}$ with some p_i^n converging to zero, then $\liminf_n d_{mi}(\mathbf{p}^n) > 0$ for at least one such i.

3.6. Equilibrium. Given initial $K(0)$, an allocation of financial claims to these $\{F_m(0)\}$ and initial human capital $\{o_m(-1)\}$ (varying across or within types), an *equilibrium* is a

[9]We avoid the additional notation, but allow for heterogenous behavior and endowments within m.

sequence of wages $\{\boldsymbol{w}(t)\}$, prices $\{\mathbf{p}(t), p_r(t), p_e(t), p_k(t)\}$ and quantities $\{F_m(t), z_m(t), E_m(t), j_m(t), k_j(t), r_j(t), h_j(t), y_j(t)\}$ for every person, sector and occupation such that:[10]

A. All individuals maximize utility as described in (6)–(9), with $F_m(0) = p_k(0)k_m(0)$ for all m, and firms maximize per-period profits at every date, with (3) holding.

B. The final goods markets clear: at every date, and for every final good i:

$$\sum_m \int x_i(z_m(t), \mathbf{p}(t)) = y_i(t), \tag{11}$$

where the integral (here and in (15) and (16) below) stands for aggregation within m.

C. The robot market clears; for each t:

$$y_r(t) = \sum_i r_i(t) + r_r(t) + r_e(t) + r_k(t). \tag{12}$$

D. The human labor market clears; for each t and each occupation o in sector j:

$$h_j^o(t) = \text{ Measure of all individuals of each type } m\text{: } o_m(t) = o. \tag{13}$$

E. The capital market clears; for each t, $K(t)$ evolves as in (5), with:

$$K(t) = \sum_i k_i(t) + k_r(t) + k_e(t) + k_k(t), \tag{14}$$

and the undepreciated capital stock plus rental income on it is willingly absorbed:

$$[1 + (1-\delta)p_k(t)]K(t) = \sum_m \int F_m(t), \tag{15}$$

F. Finally, the education market clears; that is, for every t:

$$y_e(t) = \sum_m \int E_m(t), \text{ where } \{E_m(t)\} \text{ satisfies (7).} \tag{16}$$

Per-capita national income (gross) is given by the expenditure on all final goods, plus investment in new capital goods and education:

$$Y(t) = \sum_i p_i(t)y_i(t) + p_e(t)y_e(t) + p_k(t)y_k(t). \tag{17}$$

We could just as easily work with net national income, subtracting capital depreciation.

[10]By our minimum wage assumption, all prices are bounded away from zero (see Lemma 2 in the Appendix), so without loss we ignore equilibrium with excess supply and zero price in any sector.

4. Long Run Growth, Automation and the Declining Labor Share

4.1. An Illustrative Example. There is a single occupation in each sector. There is one final good with production function $y_1 = k_1^{1/2}\ell_1^{1/2}$, a capital goods sector with $y_k = k_k^{1/2}\ell_k^{1/2}$, and a robot sector that has a CES production function with elasticity $1/2$:

$$y_r = \left[\frac{1}{2}k_r^{-1} + \frac{1}{2}\ell_r^{-1}\right]^{-1}.$$

Humans and robots are substitutable at a constant rate ν everywhere: $\ell_j = h_j + \nu r_j$ for all j. Humans move freely across sectors, so there is no education and just a single wage w. Then the price of effective labor ω is w if there is no automation, and $\nu^{-1}p_r$ if there is (partial or full) automation. In the final good and machine sectors, the unit cost function is $c_1(1,\omega) = c_k(1,\omega) = \sqrt{\omega}$, while in the robot sector it is $c_r(1,\omega) = \frac{1}{2}\left[1+\sqrt{\omega}\right]^2$. Everyone has the same one-period utility $u(x) = \ln(x)$, with discount factor $\beta \in (0,1)$.

To track equilibrium paths, notice that at any date, robot prices must satisfy

$$p_r(t) \leq c_r(1,\omega_r(t)) = \frac{1}{2}\left[1+\sqrt{\omega(t)}\right]^2. \tag{18}$$

with equality if the robot sector is active.

Case 1: $\nu \leq 1/2$. Then automation cannot ever occur. For if it did at any date t, then $\omega(t) = \nu^{-1}p_r(t)$. Substituting this into (18) which now holds with equality, we see that

$$p_r(t) = \frac{1}{2}\left[1+\sqrt{\nu^{-1}p_r(t)}\right]^2 > \frac{1}{2}\nu^{-1}p_r(t),$$

which contradicts $\nu \leq 1/2$. So at every date the robot sector shuts down. The economy effectively consists of a single consumption and capital good with aggregate Cobb-Douglas production and a 50% share of labor in national income at every date.

Case 2: $\nu > 1/2$. Then, if the economy exhibits sustained growth of per-capita income — as it indeed will if some household types are patient enough[11] — all sectors j that grow must be "asymptotically fully automated": $a_j(t) = r_j(\tau)/h_j(\tau) \to \infty$ as $t \to \infty$. For suppose this assertion is false. Then $a_j(\tau)$ must be bounded in at least one growing sector j along a subsequence $\{\tau\}$ of dates. Since the total amount of human labor in the economy is bounded, so must be the *overall* labor input in that sector. Then sustained growth implies that machine capital used in j — and hence the capital-labor ratio — grows without bound, implying $w(\tau) \to \infty$. In the absence of full automation, unit labor cost $\omega_j(\tau)$ will equal the

[11] We provide precise conditions below in Theorem 1.

human wage $w(\tau)$, and also converge to ∞. By (18),

$$p_r(\tau) \leq \frac{1}{2}\left[1+\sqrt{\omega(\tau)}\right]^2 = \frac{1}{2}\left[1+\sqrt{w(\tau)}\right]^2,$$

so that along the same subsequence,

$$\frac{\nu^{-1}p_r(\tau)}{w(\tau)} \leq \frac{1}{2\nu}\left[\frac{1}{\sqrt{w(\tau)}}+1\right]^2 \to \frac{1}{2\nu} < 1 \text{ as } t \to \infty,$$

but that would imply $\omega_j(\tau) \leq \nu^{-1}p_r(\tau) < w(\tau)$ for large τ, a contradiction.

Intuitively, the absence of automation implies an ever-growing scarcity of labor which causes the human wage to grow (without bound, this latter qualification implicitly presumed from now on). But that triggers automation when the human wage becomes large. If $\nu > 1/2$, it is possible to dispense with humans altogether, and still produce robots at a finite unit cost (using machine capital and robots). Specifically, there exists $p_r^* < \infty$ satisfying $p_r^* = \frac{1}{2}\left[1+\sqrt{\nu^{-1}p_r^*}\right]^2$, if and only if $\nu > 1/2$. Then p_r^* is an upper bound to the price of robots, making automation inevitable in all growing sectors. That bounds the human wage above, and hence the aggregate income earned by human workers. It follows that the share of labor in national income must converge to 0 in the long run.

We now provide a condition which explains the key distinction between the two cases above, and drives automation and income distribution in the general model.

4.2. Self-Replication. Recall the "no-protection" assumption $\ell_j^o(0,r) > 0$ for some $r > 0$, for every j and o. By linear homogeneity, $\ell_j^o(0,r)/r$ is independent of r for $r > 0$; call this ratio ν_j^o. In what follows, we ask you to temporarily forget that the capital rental rate is the numeraire. Consider the unit cost minimization problem in the robot sector, when each type o of effective labor in that sector is priced at ν_j^{o-1} per unit, and the capital rental rate is equal to η. Look at the limit of this unit cost as $\eta \to 0$:

$$\lim_{\eta\to 0} c_r\left(\eta, \{\nu_r^{o-1}\}\right).$$

It turns out that this limit bears on the possible automation of the robot sector itself.

PROPOSITION **1.** *Suppose the robot sector satisfies the following "self-replication" condition:*

(19) $$\lim_{\eta\to 0} c_r\left(\eta, \{\nu_j^{o-1}\}\right) < 1.$$

Then there is a nonempty compact set P^* of strictly positive solutions to the equation

$$p_r = c_r\left(1, \{\nu_r^{o-1} p_r\}\right), \tag{20}$$

and in equilibrium, $p_r(t) \leq \sup P^* < \infty$ for all t: the robot price is bounded relative to the rental on capital. If at any t, the robot sector is automated, then $p_r(t) \in P^*$.

We prove Proposition 1 in the main text as it is simple and intuitive. Because ν_j^o units of effective labor in occupation o can be produced by a single robot unit, it must be that $\omega_j^o \leq \nu_r^{o-1} p_r$. This option imposes an upper bound to the price of robot services:

$$p_r = c_r\left(1, \{\omega_r^o\}\right) \leq c_r\left(1, \{\nu_r^{o-1} p_r\}\right). \tag{21}$$

Figure 1 depicts $c_r(1, \{\nu_r^{o-1} p_r\})$ as a function of p_r. Because machines are indispensable for producing robot services, this function initially lies above the 45^0 line. Self-replication coupled with linear homogeneity guarantees that it ultimately dips below — and stays below — the 45^0 line; see Panel A of Figure 1. Then P^* is the set of intersections with the 45^0 line, as described by (20). It is nonempty and compact,[12] and (21) is equivalent to the assertion of the Proposition that $p_r(t) \leq \sup P^*$ for all t in any equilibrium. So the price of robot services (relative to machine capital) is bounded above if self-replication holds. Indeed, if the robot sector *is* automated, then that price must be one of the solutions in P^*. This pin on the robot price (relative to capital rental), $p_r \in P^*$, can be viewed as a variant of the Nonsubstitution Theorem (Arrow 1951, Samuelson 1951). Of course, automation may never be full but only asymptotic. In that case — though not stated formally in the Proposition — the robot price will lie very close to some element of P^*; see, for instance, the proof of Proposition 2.

Conversely if self-replication fails, a finite positive solution to (20) could fail to exist, as shown in Panel B, and it will *necessarily* fail to exist if f_r is quasi-concave. In this case, the robot producing sector can never be automated, and the price of robot prices is unbounded. The resulting implications are spelt out in Proposition 3.

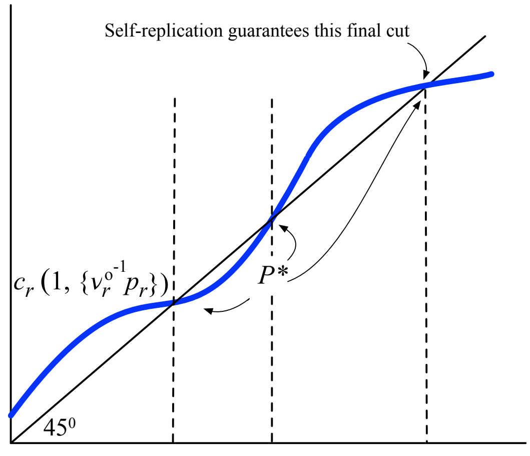
Let's examine the self-replication condition (19) in the special CES class, again with just one occupation in the robot sector. We have:

$$f_r(k, \ell) = \left[\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\ell^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

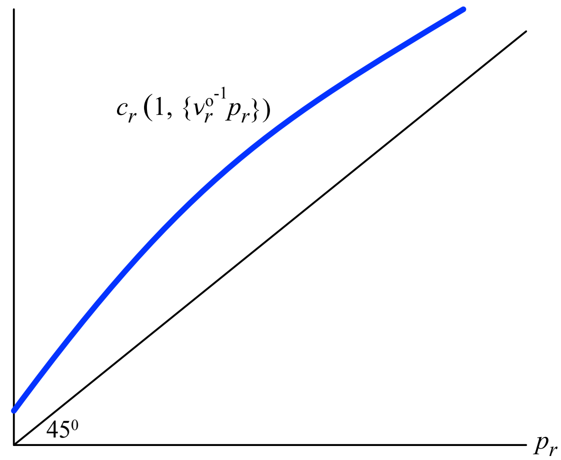
with $\alpha \in (0, 1)$ and the elasticity of substitution $\sigma \geq 0$. The unit cost function is

$$c_r(\eta, \nu_r^{-1}) = \left[\alpha^\sigma \eta^{1-\sigma} + (1-\alpha)^\sigma \nu_r^{\sigma-1}\right]^{1/(1-\sigma)}.$$

[12] If f_r is quasi-concave, then P^* is a singleton — there is a unique positive solution to (20).



(a) Self-Replication Holds



(b) Self-Replication Fails

Figure 1. Self-Replication and the Bound on Robot Price

So our limit equals zero when $\sigma \geq 1$, which includes the Cobb-Douglas case ("enough" substitution is available). But it is positive when $\sigma < 1$. For instance, if the production function is "almost" Leontief, labor costs will matter for unit cost no matter how cheap machines are. In this latter case, (19) does restrict the value of ν_r. Specifically,

(22) $$\text{Either } \sigma \geq 1, \text{ or } \sigma \in (0,1) \text{ and } \nu_r > (1-\alpha)^{\sigma/1-\sigma}.$$

4.3. Automation and the Declining Labor Share Under Long Run Growth. In conjunction with sufficient consumer patience, the self-replication condition in the robot sector has strong implications for long run growth, automation and income distribution.

THEOREM **1.** *Assume the self-replication condition* (19) *holds, and that for some m,*

(23) $$\beta_m \left[(1-\delta) + \frac{1}{c_k\left(1, \{\nu_k^{o-1} \sup P^*\}\right)} \right] > 1.$$

where P^* is defined as in Proposition 1. Then:

(i) *Per-capita national income grows: $Y(t) \to \infty$.*

(ii) *Each sector j that grows exhibits asymptotic full automation:*

(24) $$\frac{\sum_{o \in O_j} r_j^o(t)}{\sum_{o \in O_j} h_j^o(t)} \to \infty \text{ as } t \to \infty.$$

(iii) *If preferences are asymptotically homothetic, the national income share of human labor converges to zero as $t \to \infty$, and that of physical capital converges to 1.*

We describe the underlying argument; a formal proof is in the Appendix. Part (i) states that per capita income grows (without bound) if (23) holds. This is a condition placed on the primitives of the model. When the robot production function is quasiconcave, P^* is an easy-to-compute singleton. Otherwise P^* is multivalued but still based on primitives. We show that (23) is sufficient for an m-type to accumulate unbounded wealth.

The first difficulty is to account for moving capital prices. While bounds can be placed on these prices, there will in general be capital gains (or losses). To sidestep the spikes of accumulation and decumulation that could might arise from these anticipated gains and losses, we cumulate the relevant Euler equations for financial wealth. Recalling the indirect utilities v_m and $\gamma(t)$, the equilibrium rate of return on financial assets, we have:

$$v'_m(z_m(t), \mathbf{p}(t)) \geq \beta_m \gamma(t) v'_m(z_m(t+1), \mathbf{p}(t+1)), \tag{25}$$

with equality holding if financial wealth is actively accumulated. From (9),

$$\gamma(t) = \frac{1 + (1-\delta)p_k(t+1)}{p_k(t)} = \left[\frac{p_k(t+1)}{p_k(t)}\right]\left[(1-\delta) + \frac{1}{p_k(t+1)}\right],$$

where the second equality decomposes the return into the product of capital gains and the rental income (augmented by any undepreciated capital) on a unit of wealth. If we compound the Euler inequality in (25) over dates $0, \ldots, t$, where $t \geq 2$, then we have

$$v'_m(z_m(0), \mathbf{p}(0)) \geq \beta_m^{t-1} \frac{(1-\delta)p_k(t) + 1}{p_k(0)} \left\{\prod_{\tau=1}^{t-1}\left[(1-\delta) + \frac{1}{p_k(\tau)}\right]\right\} v'_m(z_m(t), \mathbf{p}(t)),$$

which eliminates temporary spikes and dips in capital gains. Because $p_k(\tau)$ is bounded by the option to automate: $p_k(\tau) \leq c_k(1, \nu_k^{-1} p_r(\tau)) \leq c_k(1, \nu_k^{-1} \sup P^*)$, we have:

$$v'_m(z_m(0), \mathbf{p}(0)) \geq \frac{\beta_m^{t-1}\left[(1-\delta) + \frac{1}{c_k\left(1, \{\nu_k^{o-1} \sup P^*\}\right)}\right]^{t-2}}{c_k\left(1, \{\nu_k^{o-1} \sup P^*\}\right)} v'_m(z_m(t), \mathbf{p}(t)). \tag{26}$$

Applying (23), (26) implies that $v'_m(z_m(t), \mathbf{p}(t)) \to 0$ as $t \to \infty$. Further bounds on equilibrium prices $\{\mathbf{p}(t)\}$ (see Appendix) then allow us to conclude that the consumption expenditure of type m households must grow. With bounded debt, the same is true *a fortiori* for per capita consumption and income in the economy. Because returns to capital are bounded away from zero under self-replication, the option to automate liberates owners of capital from labor scarcity as the economy grows.

Part (ii) asserts that any such sector that grows exhibits asymptotic full automation in the sense of (24). The proof is easily obtained by contradiction: if a sector grows, at least one of its effective labor inputs must grow with it — this is a consequence of unbounded steepness with respect to at least one effective labor input, and the fact that self-replication holds (so the price of effective labor is bounded relative to capital). But the total available supply of (raw) human labor is bounded. Therefore each such growing occupation within the sector must have $a_j^o(t) \to \infty$. Because all bounded occupations become insignificant relative to the growing ones, the result follows.

Part (iii) is the heart of our thesis, and needs a more subtle argument. We want to allow for the case in which there is no *uniform* threshold for automation — the idea being that at any human wage, there are always productive sectors where humans continue to be a desirable presence. In fact, humans may well be persistently present in every occupation, asymptotically automated or not,[13] but with asymptotic automation their income share cannot be preserved. However, the presence of an unbounded number of occupations opens the possibility of "human shelters" that provide opportunities for humans to stay ahead of automation waves. To do so, they must perennially accumulate human capital, moving across occupations. Indeed, in these relatively protected sectors, the human wage could be very high. In Proposition 2 below, we provide conditions under which in any equilibrium with growth, the highest human wage across all sectors grows *unboundedly* over time. If humans acquire the skills to enter these yet-to-be-automated sectors, their wages might conceivably grow in step with per capita income.

At this point, the endogeneity of prices and wages that we've emphasized throughout takes center stage. The willing absorption of humans into sectors requires that there be adequate demand for those *outputs*. When all labor is clumped into an efficiency-unit aggregate, this demand question is eliminated by definition, because relative wages cannot change over the sectors that are thus aggregated with brute force. With an endogenous wage structure, this is no longer the case. Part (iii) shows that if demand is asymptotically homothetic, then the economy runs out of steam in its ability to shelter labor. For the human wage share to stay positive in the long run, household expenditures shares on yet-to-be automated sectors must remain sizable. Under asymptotic homotheticity, this cannot happen, so wage incentives do not climb at the required pace.

Hometheticity is sufficient for Theorem 1, but not necessary. Without formalizing what follows, consider a version of our model in which there is no homotheticity, so that demand can keep shifting persistently with rising income. Then Theorem 1 would still remain true as long as those shifting preferences favor subsets of goods in a way that is neutral, on average,

[13] To see why, consider the example of an ℓ_j function provided just after equation (2).

to their susceptibility to automation. Under self-replication, there appears to be only one way to break the stranglehold imposed by the theorem, and that is if preferences are *both* non-homothetic *and* move progressively in favor of goods that aford human protection; it is easy to construct examples of this.[14]

4.4. Persistent Growth in Human Wages. The discussion in Section 4.3 suggests that a vanishing share of labor income could co-exist with unbounded growth in human wages. Several combinations of sufficient conditions can deliver such a result; we provide one. Suppose that there is just one occupation per sector (nothing hinges on this). Consider the marginal rate of substitution between humans and robots in producing effective labor in sector j, that is, the marginal human input that needs to be added to keep effective labor constant as robot services are decreased by a constant infinitesimal amount. Let θ_j denote the limiting marginal rate of substitution as the ratio of human to robot labor in sector j converges to zero. This serves as a measure of the "local relative efficiency" of robots relative to humans in sector j in a neighborhood of the automation corner.

Given asymptotically homothetic preferences, we make the following "full-support" assumption on limit demand: for each m and each price $\mathbf{p} \gg 0$, $d_{mi}(\mathbf{p}) > 0$ for $i \in I$.

PROPOSITION **2.** *Suppose that the conditions of Theorem 1 hold, including asymptotic homotheticity of preferences, and suppose that the full-support condition holds. Then:*

(a) *The highest wage earned by humans across sectors grows without bound as $t \to \infty$, if there is a sequence of goods $i = 1, 2, \ldots$ with $\theta_i \to 0$.*

(b) *The wage earned by every human grows without bound if education costs incurred in order to move across any pair of occupations is bounded.*

As a special case, all the conditions of the proposition hold when human labor is freely mobile, when preferences are CES and Dixit-Stiglitz, and if the aggregator for effective labor in each sector is as described in the example after equation (2). Theorem 1 tells us that any growing sector must eventually be automated, so robot output must grow to meet that rising demand. Given the limited endowment of human labor, this is only possible if the robot sector *itself* is asymptotically automated. Hence the tail sequence of robot prices has limit points contained in the set P^*. (It will additionally converge if the robot technology

[14]Comin, Danieli and Mestieri (2019) describe non-homotheticities in demand which raise the share of services and lower that of agriculture, and are associated with rising wage polarization. They do not investigate the implications for the decline in overall labor income share. Karabarbounis and Neiman (2014) argue that this decline is mainly intrasectoral, and not driven by changing intersectoral composition.

is quasiconcave.) Because any sector i can be contested by humans, the corresponding tail sequence of human wages is asymptotically bounded below by $p_r(t)/\theta_i$. Given that $p_r(t)$ enters into a strictly positive compact set, the highest human wages diverge to infinity along the double sequence $t \to \infty$ and $\{i\}$ such that $\theta_i \to 0$.

When education costs are bounded, (b) states that the wage earned by *every* human must grow. This is because wage differences across sectors are bounded. Hence if wages in any sector grow, so must they also grow in every sector in which some humans choose to work. However, if education costs are unbounded, some humans may remain trapped in certain occupations with a finite limiting wage.

4.5. The Failure of Self-Replication. A failure of self-replication means that the price of robots cannot be severed from human wages. Human workers are indispensable in the production of robot services, causing robot prices to climb with wages as labor scarcity grows. The scope for automation in all sectors is therefore restricted, so inducing wages to grow faster. The example in Section 4.1 already makes this clear, but more can be said. Under broad conditions, self-replication is formally *necessary* for Theorem 1.

To develop this argument, we place some restrictions on our general environment. Once again, without any real loss of generality, we assume just one occupation per sector. The first restriction is a general version of the condition that the production function f_j defined on capital and effective labor has an elasticity of substitution smaller than 1 in every sector j. For any sector j, and any effective price of labor ω, consider the set $\Xi_j(\omega)$ of labor-capital ratios $\xi = \ell/k$ that minimize unit cost of production, and let

$$\Lambda_j(\omega) \equiv \min_{\xi \in \Xi_j(\omega)} \frac{\omega\xi(\omega)}{1 + \omega\xi(\omega)}$$

be the lowest ratio of labor income to total income in that sector thus generated. Just for a moment, think of the production function f_j in this sector as CES with elasticity of substitution lower than 1. Then we know that $\Lambda_j(\omega)$ is increasing in ω, with $\Lambda_j(0) = 0$ and $\Lambda_j(\infty) = 1$. In particular, given any lower bound $\underline{\omega} > 0$, we have

$$\inf_{\omega \geq \underline{\omega}} \Lambda_j(\omega) > 0.$$

In our more general setting without constant elasticity (or indeed concavity), we impose the above condition, and uniformly so across sectors:

$$\inf_j \inf_{\omega \geq \underline{\omega}} \Lambda_j(\omega) > 0. \tag{27}$$

Next, we make additional assumptions on the production function for robots. We assume that it is strictly quasiconcave, in addition to being linearly homogeneous. We assume further that $\ell_r(h, 0) > 0$ for some $h > 0$, so that the labor aggregate in the robot sector can be produced by humans alone. This restriction is analogous to the feasibility of automation, as already assumed. Call such a technology *regular*.

PROPOSITION **3.** *Suppose that (27) holds and the robot production function is regular. Then, if the self-replication condition fails, in any equilibrium the share of human labor in national income is bounded away from zero.*

Proposition 3 makes it clear that *both* the asymmetry of human and physical capital accumulation *and* the self-replication condition on the robot sector are needed for our results. Indeed, the self-replication condition is logically necessary in a broad class of environments. Without it, robot prices cannot be divorced from the wages of human labor. As labor becomes more expensive, so do robots, and the forces of automation are attenuated — sufficiently attenuated, as it turns out, under the conditions of Proposition 3 so that the share of human labor does not decline in a sustained way over time.

4.6. Within-Occupation Human Capital. We now extend the theory to incorporate the acquisition of intra-occupational skill. First note that the device of several occupations within a sector can be interpreted to mean that these are different skill levels within the same job. As long as there is a finite (or even compact) set of such skill levels, the theory already accommodates such cases, by redefining different levels of skill as different occupations. However, that is still in contrast to the unbounded scope for accumulation of physical capital within any sector. Could our model be extended to similarly accommodate the unbounded accumulation of skill within a sector?

We already know that the answer cannot be an unqualified yes: there are macroeconomic models which generate balanced labor income shares once human capital can be accumulated to an unbounded degree in efficiency units, with no changes in relative prices. So studying this extension will help identify the precise nature of the asymmetry needed between physical and human capital accumulation in our model.

For expositional clarity, we revert to the common-sense notion of an occupation, and do not interpret varying levels of skill as constituting distinct occupations. We extend our model to allow workers to acquire varying levels of skill within any given occupation, and place no upper bound on the amount of such skill that can be accumulated. We model skill in the conventional manner, as a certain number of efficiency units. Let the production function for effective labor services in occupation o in some sector be $\ell_j^o(\mu_j^o h^o, r^o)$, where μ^o is the productivity of a human in that occupation. (We've dropped sector subscripts for ease in writing.) Wages are paid per unit of productivity, just as in the standard model based on efficiency units, so the income of a person with productivity μ^o is $w^o\mu^o$, where w^o is the occupation-specific "efficiency unit human wage."

Everything else in the model is kept unchanged, but we must specify the technology of productivity acquisition. To this end, we extend the education function as follows: let $e(\mu, \mu', o, o')$ denote the units of education needed to move from "starting productivity" μ in occupation o to "destination productivity" μ' in sector o', *where o could be equal to o'*. In particular, one can both invest within an occupation and across occupations, generally with heterogeneous cost implications. Moreover, continued on-the-job education can depend on baseline levels of productivity already acquired in that sector.

We take e to be smooth in its first two arguments with partial derivatives denoted by e_1 (with respect to starting productivity) and e_2 (with respect to destination productivity), typically negative in the first case and positive in the second. We place the following restrictions on the education function:

(H.1) For any o and $S > 0$, there is $M < \infty$ such that $e_2(\mu, \mu, o, o) \geq S$ for all $\mu \geq M$.

(H.2) For any o, there is $L^o \geq 0$ with $e_1(\mu, \mu', o, o') \in [-L^o, 0]$ for all (μ, μ') and o'.

(H.3) For each occupation o', there is a bound $\hat{\mu}^{o'}$ such that for every starting $o \neq o'$ and productivity μ, $e(\mu, \mu', o, o') = \infty$ for $\mu' \geq \hat{\mu}^{o'}$.

(H.1) states that *within* any occupation, the marginal cost of skill acquisition ultimately becomes very high, as baseline productivity increases. (H.2) states that while a higher starting productivity may bring down the cost of achieving any destination productivity in the same or different occupation, the marginal savings are bounded. (H.3) states if an individual is switching occupations, there is some upper bound to the productivity with which she can immediately start in the new occupation. Of these three, the one that matters the most is (H.1). This condition does not automatically seal off unbounded skill accumulation, because the equilibrium price and wage structure matters as well: the returns to skill may grow fast enough to outpace the rising marginal cost. But as shall now see, the self-replication condition suffices to prevent such an outcome.

PROPOSITION **4.** *Suppose that within- and cross-sector human capital are accumulated via an education function satisfying H.1–H.3. Suppose, moreover, that the self-replication condition (19) is satisfied, and preferences are asymptotically homothetic. Then, if (23) holds, there is sustained per-capita income growth, and the income share of labor goes to zero.*

The Appendix contains a detailed proof; we describe the main step here. Under self-replication, each sectoral price is bounded below and above over time by strictly positive, finite numbers, just as before; see Lemma 2. In general, wages will not be bounded, but the argument that follows separates two cases.

In the first, some occupational wage grows; see formal proof for precision regarding subsequences, etc. But then, the feasibility of automation allows us to prove that the share of human labor income in total effective-labor revenue in that occupation must converge to zero; see part (ii) of Lemma 3. The second possibility is that some occupational wage is bounded. Then (H.1) chokes off the incentive to acquire within-occupation productivity, given that the price of education is bounded below. The gains from such acquisition include direct wage benefits from the occupation, as well as cost savings on *future* investments, but these are all bounded, by our conditions on the education function. At the same time, the cost of incremental productivity climbs without bound. These observations ensure that when the occupational wage is bounded, so is productivity per person. With this boundedness result in hand, we can essentially follow the existing line of proof in Theorem 1. The two possibilities taken together replicate our previous result.

4.7. Sectors With Full Human Protection. We've assumed that full automation is technically feasible, but what of "protected" occupations in which production is impossible without humans: $\ell(0, r) = 0$? Examples might include "live music" or "hand-made pottery," with a human element in production by the very nature of the good. Of course, it is still possible that the ratio of human to robot could become vanishingly small over time. In the live-music example, it might be possible to increase the size of the audience without bound for any live concert, and "hand-made pottery" could be judiciously redefined to include minimal human intervention. The debate is philosophical and possibly endless, as anyone who's seen *Blade Runner* or heard of the Turing test will know.

For expositional simplicity, assume one occupation per sector. Say that sector (or occupation) j is *unprotected* if $\ell_j(0, r) > 0$ for $r > 0$, as assumed so far, and *protected* if $\ell_j(0, r) = 0$ for all $r \geq 0$. When preferences are asymptotically homothetic, say that the asymptotic demand system $\mathbf{d}_m(\mathbf{p})$ is *elastic* if $\sum_{i \in Q} p_i^n d_i(\mathbf{p}^n) \to 0$ along any sequence prices $\{\mathbf{p}^n\}$ in which $\mathbf{p}_Q^n \to \infty$ while all other prices are bounded above.

PROPOSITION **5.** *Suppose that all intermediate goods sectors and some final goods sectors are unprotected, and that the self-replication condition (19) holds. Then:*

(i) *Under (23), there is sustained per-capita income growth.*

(ii) *For every unprotected sector on which expenditure grows, there is asymptotic automation and the output price is bounded.*

(iii) *For every protected sector on which expenditure grows, there is asymptotic automation and the output price is unbounded.*

(iv) *Suppose that preferences are asymptotically homothetic, and that the expenditure shares of all sectors converge to a limit expenditure share vector. Then the limit share of human labor in national income is bounded above by the asymptotic share of expenditure on protected sectors. Moreover, if the demand system for every type is elastic, then once again the share of human labor in national income converges to zero.*

We omit a formal proof; much of it follows already covered ground. Part (i) is proved by exactly the same argument as in part (i) of Theorem 1. The proof of part (ii) is a special case of (ii) of Theorem 1, noting that the growth of output *value* is the same as the growth of *physical* output — prices must be bounded. Part (iii) is new. There are two cases: either the price of the protected good grows, or its physical output does. Under the former, the price of the labor aggregate must grow in that sector — and so too must human wages, given that robot prices are bounded, by self-replication. Asymptotic automation then follows from Lemma 3(iii) in the Appendix.

In the latter case, with output growing, there are two possibilities: (a) The labor aggregate *also* grows in that sector, but then we have asymptotic automation, given that the stock of raw labor is bounded. Moreover, since the sector is protected, the price of the labor aggregate must grow, and so must the price of final output. (b) The labor aggregate is bounded, but then capital must grow, implying an ever-increasing price for the labor aggregate. That in turn can only happen if the human wage grows, and once again we obtain asymptotic automation. Moreover, the price of the final output must grow.

Exactly the same argument as in Theorem 1(iii) shows that the labor share as a ratio of expenditure on all unprotected sectors converges to zero. Therefore the overall labor share in national income must be asymptotically bounded above by the asymptotic share of expenditure on protected sectors. Finally, observe that all unprotected prices are bounded (Lemma 2(ii)) and all protected prices in growing sectors are unbounded. Moreover, by assumption, all protected goods are final goods. Then, with an elastic demand system, the expenditure share on all protected goods must fall to zero, and by the upper bound just established, so must the share of human labor in national income.

5. Technical Progress

We extend the theory to incorporate directed technical progress. “Directedness" means that technical progress is geared to input scarcity. The key assumption we make is that the opportunities for such progress are symmetric over all inputs, and across all sectors. This is not to deny the possibility that the nature of science and technology might generate biases in certain directions. But studying the effect of such exogenous biases would not need a theory. If they were to favor unbridled automation, our earlier results would be a foregone conclusion. If they favored the augmentation of human quality over robots, that would raise the share of humans in national income instead.

The two possibilities combined point to a long run “balanced-growth" view of technical progress; see Acemoglu and Restrepo (2018, 2019), with antecedents that include Hicks (1932), Salter (1966), Galor and Maov (2000), and Acemoglu (1998, 2002), among many others. Acemoglu and Restrepo (2018) generate balanced growth by assuming that newly developed tasks lie in the human domain, enjoying temporary protection from the robot invasion. But the robots are also hard at work, automating existing tasks and perennially chasing the moving human frontier. Balance is achieved by equilibrium across these two forces. This is a potentially fruitful approach, but one that is — perhaps unavoidably — laden with questions. Why can’t new tasks that favor *robots* also appear on the frontier? Or (the flip side): why cannot technical progress allow humans to recover their edge in old tasks? What if there is technical progress in machines?

In this section, we adopt a more neutral view. There is a fixed number of sectors and tasks, with technical progress taking place on the intensive rather than extensive margin. There is no *a priori* restriction on sectors in which humans and robots can be active. In contrast, Acemoglu and Restrepo study the extensive margin, where the task space is extended in the asymmetric way already described. Within each sector/occupation, we allow for technical progress in machine capital, in human labor, and in robot services. Technical progress in each of these sectors plays an important role in our analysis.[15]

The sole asymmetry we retain — one already developed — is the difference between the scaling-up of physical and human capital. It will turn out that this asymmetry is augmented in the presence of symmetric (though directed) technical progress. Moving away from this symmetric benchmark, if there are stronger opportunities for technical progress that heighten the productivity of machine capital or robots relative to humans, our results will *a fortiori* be reinforced. And in the opposite case they will be moderated.

We presume that productivity growth in any factor in any given sector spills over to the same factor in *other* sectors.[16] There is empirical evidence of such spillovers (see, e.g., Bernstein and Nadiri 1988 and Johnson 2018). Such spillovers will ensure that the rate of productivity growth of any factor cannot diverge across sectors in the long run.[17]

5.1. Framework. To simplify the exposition we narrow the baseline model. First, we consider only finitely many sectors and one occupation per sector. Second, we assume that the ℓ-aggregator in each sector exhibits perfect, linear substitution between robots and humans. To allow for changes in the productivity of every input, we attach coefficients to all three inputs as follows:

$$y_j = f_j(\theta_j k_j, \mu_j h_j + \nu_j r_j),$$

where the same assumptions are made on f_j as before, and $(\theta_j, \mu_j, \nu_j) \gg 0$ are intra-sectoral factor productivities that can be changed by deliberate technical progress.

We assume that self-replication holds throughout. Because productivities will change over time, the condition is stated relative to *starting* robot productivity $\nu_r(0)$ at date 0:

$$\nu_r(0) > \lim_{\eta \to 0} c_r(\eta, 1). \tag{28}$$

Robot productivity in subsequent periods will be endogenous but no lower. So we impose no restrictions involving these, but it is easy to see that all our results hold for any competitive equilibrium for which (28) holds at some date t along the equilibrium path, and *a fortiori* at all dates thereafter.

[15] Acemoglu and Restrepo abstract from non-robot machine capital in their model.

[16] Acemoglu and Restrepo (2019) assume a single sector, avoiding the complications from multiple sectors and inter-sectoral spillovers. But this is not a fundamental source of difference in the results.

[17] In the *complete* absence of spillovers, Theorem 2 may not hold; details are available on request.

5.2. Endogenous Technical Progress. Each factor-sector pair is serviced by a short-lived inventor whose activities and returns are external to the economy in question.[18] She can increase the sectoral productivity of that factor. Assume that generic productivity $\pi_j(t)$ of a factor in sector j at t can be increased at some rate $\rho_j(t)$ through R&D investment. A fixed proportion $\gamma > 0$ of this growth spills over to the same factor in other sectors, subject to a maximum rate $\bar{\rho}$ of productivity growth in any sector. So for any sector j:

$$\pi_j(t+1) - \pi_j(t) = \max\left\{\rho_j(t)\pi_j(t) + \gamma\sum_{j'\neq j}\rho_{j'}(t)\pi_{j'}(t), \quad \bar{\rho}\pi_j(t)\right\}. \tag{29}$$

The associated R&D cost of the inventor is $\kappa(\rho_j(t))$, where $\kappa'(0) = 0$, $\kappa'(\rho) > 0$ and $\kappa''(\rho) > 0$ for all ρ. View R&D investment as a game played across sectors and factors by inventors. Spillovers are in the public domain, utilized freely by all firms. The spillover into sector j from other sectors depends on R&D investments by inventors in those sectors. As these are correctly anticipated in a Nash equilibrium, the sector j inventor will invest in R&D only if the spillovers into j from other sectors are smaller than $\bar{\rho}\pi_j(t)$. This provides scope for the inventor to license the proprietary advance $\rho_j(t)\pi_j(t)$ to firms operating in sector j in period $t+1$. The license fee is levied per (natural) unit of the factor employed by the firm at $t+1$.

Each inventor takes prices as given, as in the competitive innovation models of Grossman and Hart (1979) and Makowski (1980). The maximum unit license fee $L_j(t+1)$ that can be charged by a date-t inventor is

$$L_j(t+1) = q_j(t+1)\rho_j(t) \tag{30}$$

where $q_j(t)$ is the factor price at t.[19] Intuitively, the "effective factor price" for licensees must rise by exactly the same proportion as the proprietary productivity advance.

Therefore, if $x_j(t)$ is the use of that factor in sector j at date t, the total return earned by the inventor equals $L_j(t+1)x_j(t+1) = \rho_j(t)E_j(t+1)$, where $E_j(t+1) \equiv q_j(t+1)x_j(t+1)$

[18]We can fully integrate the inventor into the economy by providing her with a technology that depends on machine capital and human/robot labor. We avoid that recursive extension here. However, one difference that will arise is that the R&D sector will not be perfectly competitive, and will earn profits that typically constitute a positive fraction of national income. The extent to which humans can be replaced by robots in the R&D sector is then an additional determinant of the labor income share. This will be driven by the logic of cost minimization in the production of R&D, in a manner similar to that in other sectors.

[19]One efficiency unit of the factor costs $q_j(t+1)/\pi_j(t)$ for someone without access to the technical advance, and $q_j(t+1)/(1+\rho_j(t))\pi_j(t)$ for someone with access. The difference in unit cost is $q_j(t+1)\rho_j(t)/\pi_j(t)(1+\rho_j(t))$, so this can be sucked out as a license fee per efficiency unit. Multiplying by the number of efficiency units $\pi_j(t)(1+\rho_j(t))$ made possible by the advance, we obtain expression (30).

is the total bill for that factor in sector j. Given the R&D choices $\{\rho_j'(t)\}_{j'\neq j}$ by inventors in other sectors, and given that our inventor is specific to the sector at hand, she will invest until $\kappa'(\rho_j(t)) = E_j(t+1)$, provided $\kappa'^{-1}(E_j(t+1))\pi_j(t) + \gamma \sum_{j'\neq j} \rho_{j'}(t)\pi_{j'}(t) < \bar{\rho}$. Otherwise, she will invest $\max\{0, \bar{\rho} - \gamma \frac{1}{\pi_j(t)} \sum_{j'\neq i} \rho_{j'}(t)\pi_{j'}(t)\}$. So in Nash equilibrium, the rate of productivity increase must satisfy:

$$\frac{\pi_j(t+1) - \pi_j(t)}{\pi_j(t)} = \max\left\{\kappa'^{-1}(E_j(t+1)) + \gamma \frac{1}{\pi_j(t)} \sum_{j'\neq j} \rho_{j'}(t)\pi_{j'}(t), \quad \bar{\rho}\right\} \tag{31}$$

5.3. Equilibrium. An equilibrium looks in part like the competitive equilibrium in Section 3.6. Because licensees transfer all their surplus to the inventor, their current production decisions are the same as they would have been in the absence of license purchases. With the rental rate on capital as numeraire, an equilibrium is a sequence of wages $\{\boldsymbol{w}(t), w_r(t), w_e(t), w_k(t)\}$, prices $\{\mathbf{p}(t), p_r(t), p_e(t), p_k(t)\}$, quantities $\{F_m(t), z_m(t), e_m(t), j_m(t), k_j(t), r_j(t$ for every person and every sector, and productivity coefficients $\{\theta_j(t), \mu_j(t), \nu_j(t)\}$ for every input, sector, and date, such that:

(a) Given the sequence of productivities, the remaining sequence of outcomes constitutes a competitive equilibrium; and

(b) At every date, given equilibrium prices, all productivity changes and fees are the outcome of a Nash equilibrium as described earlier in this section.

5.4. Automation and the Vanishing Labor Share with Technical Progress. To state the main result of this section, we place two additional restrictions. First, we assume that household expenditure *shares* on each final good are bounded away from zero:

[E] For any individual type m and good i, $\inf_{(\mathbf{p},z)\gg 0} p_i x_{mi}(\mathbf{p}, z)/z > 0$.

As there are a finite number of goods, this is innocuous. Next, we make an additional assumption on the production technology:

[F] There is some final good i for which $\lim_{\omega\to 0} c_i(1,\omega) > 0$.

It is easy to verify that Condition F holds as long as there is at least one final good sector with constant capital-labor elasticity of substitution strictly smaller than 1.[20]

[20] We believe that both conditions E and F are technical. Condition E is possibly needed for the result, but is mild. As for F, we do know that it can be replaced by other conditions. For instance, for any sector j, define the intensive-form function $g_j(e) \equiv f_j(e, 1)$. Then the following condition substitutes for F: [F′] There is some

THEOREM **2.** *Assume the self-replication condition* (28)*, and Conditions E and F. Then in any equilibrium which exhibits unbounded accumulation of machine capital, the income share of human labor in the economy as a whole must converge to zero as* $t \to \infty$*.*

Theorem 2 resurrects our earlier prediction, and continues to highlight the effects of asymmetry across human and physical capital accumulation. The theorem now makes a stronger assumption on growth, asking that capital be accumulated in equilibrium. It is possible to provide sufficient conditions for this, along the lines of Theorem 1. Technical progress induces a downward drift on prices (relative to incomes), which is an “automatic" — albeit endogenous — source of real income growth. For machine capital to be willingly accumulated despite this drift, the degree of patience must clear a higher threshold (which depends on the maximal rate $\bar{\rho}$ of technical progress).

We outline the main steps to prove Theorem 2.

OBSERVATION **1.** (a) *For each factor and sector* j*, there is* $M < \infty$ *such that factor productivity in* j *grows at rate* $\bar{\rho}$ *if the factor bill* $q_j x_j$ *for that factor exceeds* M*.*

(b) *If the spillover rate* γ *is positive, the relative productivity* $\pi_j(t)/\pi_{j'}(t)$ *of any factor between any pair of sectors* j *and* j' *is bounded.*

Part (a) follows from (31). If the factor bill $E_j(t+1)$ is large enough, the right-hand-side of (31) will equal the maximal rate $\bar{\rho}$ of technical progress. This reflects the force of the intensive margin in motivating technical progress in sector j. Part (b) ensures that the benefits of technical progress in any sector spills over to other sectors, thereby preventing the unbounded inter-sectoral divergence of factor productivity.

Next, the self-replication condition (28) places a bound on robot prices $p_r(t)/\nu_r(t)$ in efficiency units, relative to the capital rental rate $1/\theta_r(t)$ (in effective units) in the robot sector. Spillovers across sectors then imply that the same is true of *every* sector in the long run, a consequence of Observation 1(b). This generates the following sector-specific bound on efficiency-adjusted price of robot services relative to machine rentals:

---

sector j such that the infimum elasticity of the intensive form is positive:

$$\inf_{e>0} \frac{g_j'(e)e}{g_j(e)} > 0.$$

Conditions F and F$'$ are not nested. A CES production function with elasticity of substitution smaller than 1 satisfies F but not F$'$. A Cobb-Douglas production function satisfies F$'$ but not F.

LEMMA **1.** *Assume* (28)*. For any j, there is $B_j < \infty$ such that*

(32) $$\theta_j(t)\frac{p_r(t)}{\nu_r(t)} \leq B_j \textit{ for all } t$$

in any equilibrium.

OBSERVATION **2.** *If Conditions E and F hold, then in any equilibrium with $K(t) \to \infty$, it must be the case that $k_i(t) \to \infty$ for some sector i.*

This observation rules out the arcane possibility that unbounded accumulation of machine capital will be accompanied by cross-sectoral cycles of ever-increasing amplitude, so that capital in any given sector is bounded along some subsequence of dates. Condition E ensures that sectoral expenditure shares cannot fluctuate “too much" across dates, so the output of each good must track overall growth. Condition F implies a bounded ratio of output value to machine capital for at least some sectors. In these sectors, the amount of machine capital allocated must therefore grow.

Observation 2 implies that the rate of technical progress in machines must eventually occur at the maximal rate $\bar{\rho}$ in some sector. These advances spill over to every other sector. Hence, *the rate of technical progress in machine capital in any sector cannot fall behind the rate of technical progress in human labor in the long run.*

Combined, these three observations have the following implication. The asymmetric growth in endowments in natural units between machine capital and human labor generates a bias (at least weakly) in technical progress in favor of capital. That compounds the distributional shift in favor of physical capital, so our preceding results concerning the long run income share of human labor continue to hold in the presence of directed technical progress. Specifically, in any sector j that employs human labor at any date t, humans must be cost-effective relative to robots, which is the first inequality below:

$$\frac{w_j(t)}{\mu_j(t)} \leq \frac{p_r(t)}{\nu_j(t)} \leq \frac{B_j}{\theta_j(t)\nu_j(t)},$$

but the second inequality also applies, by Observation 1. Combining the two,

$$w_j(t) \leq \frac{\mu_j(t)}{\theta_j(t)}\frac{B_j}{\nu_j(t)}.$$

Now, $\frac{\mu_j(t)}{\theta_j(t)}$ is bounded: as already noted, machine capital productivity grows at least as fast as human productivity in every sector in the long run, while $\nu_j(t)$ is non-decreasing in t.

So human wages are bounded, implying that the share of human labor income in national income must converge to 0 in the long run, as $K_t \to \infty$.

We remark on the very last line in the previous paragraph. It is true that human wages are bounded, but only relative to our chosen numeraire, which is the rental rate on machine capital in natural units. Because technical progress occurs in all sectors, machine capital becomes highly productive over time, which leads to a progressive decline in the prices of final goods, relative to the same numeraire. Human wages are bounded below in that numeraire, and so by any measure of the cost-of-living — that is, relative to any index number defined on the basket of final goods — real incomes must diverge to infinity. The fact that the share of human labor in national income nevertheless converges to zero again reveals the contrast between absolute and relative behavior in human incomes, already emphasized at different points above.

As before, the argument for Theorem 2 rests on the underlying asymmetry between the growth of physical capital and human labor in natural units. Unbounded capital accumulation relative to human labor implies a corresponding asymmetry in factor bills, and thereby in technical progress on the intensive margin. That induces a parallel asymmetry in rates of technical progress, precisely unveiled by our insistence on the level-ground assumption that the *opportunities* for technical progress are symmetric.

6. Concluding Remarks

We study long-term automation and decline in the labor share, driven by capital accumulation rather than biased technical progress or rising markups. Our argument relies on a fundamental asymmetry across physical and human capital in modern economies. While physical capital can be scaled up for the same activity and accumulates in natural units, human capital accumulates via education and training that alters choice into higher-skilled occupations, but — from the vantage point of a household or individual — cannot scale up the quantity of labor for a *given* occupation to an unlimited degree. Under a self-replication condition on the technology of the robot-producing sector, we show that the share of capital in national income approaches 100%.

The self-replication condition is central to our findings. So the condition itself merits greater scrutiny. There is greater recognition that the “production of robots by means of robots" is not merely a hypothetical possibility:

> “They are a dream of researchers but perhaps a nightmare for highly skilled computer programmers: artificially intelligent machines that can build other artificially intelligent machines . . . Jeff Dean, one of Google’s leading engineers, spotlighted a Google project called AutoML . . . [which] is a machine-learning algorithm that learns to build other machine-learning algorithms. With it, Google may soon find a way to create A.I. technology that can partly take the humans out of building the A.I. systems that many believe are the future of the technology industry." (*The New York Times*, November 5, 2017.)

The self-replication condition is placed on the robot sector *alone*, but it has economy-wide ramifications. If this condition fails, humans are generally guaranteed a positive share of national income. For either some sectors of the economy are automated, in which case the robot sector becomes active, and humans must obtain a positive share of value added in the robot sector. Or there is no automation anywhere, in which case humans are not displaced in any sector, and the standard theory applies. Therefore, in the absence of self-replication, there is a route by which the benefits of physical capital accumulation could persistently flow to humans.

Our paper also takes note of other escape routes from the possibility of an ever-falling labor share: non-homothetic demand that progressively favors sectors that protect humans, growing sectors where humans are protected from full automation by the nature of technology, or technical progress biased in favor of humans. However, while any of these scenarios is possible, we do not see any reason why they should be inevitable.

Our emphasis throughout has been on the *functional* distribution of income. Whether a household’s income remains on par with the economy — the question of the personal distribution of income — will depend on whether they invest in financial wealth or human capital (or neither, or both). This is a question we have not yet addressed, though our model provides the means to study it, and is something we plan to undertake. It will become necessary to take closer account of both the heterogeneity of the population in their preference parameters, as well as to incorporate a detailed description of credit market constraints. Both these features are currently present in the model, but play no more than a background role. Finally, we note that despite its generality, the theory presented here is simple and tractable, which may also allow it to be useful in analyzing effects of fiscal policies such as capital taxes, education subsidies, universal basic income or other policy interventions to address the distributional consequences of automation.

References

D. Acemoglu and J. Robinson (2015), “The Rise and Decline of General Laws of Capitalism," Journal of Economic Perspectives, 29(1), 3-28.

D. Acemoglu and P. Restrepo (2018), “The Race Between Man and Machine: Implications of Technology for Growth, Factor Shares and Employment," American Economic Review, 108(6): 1488-1542.

———————————— (2019), “Automation and New Tasks: How Technology Displaces and Reinstates Labor,” Journal of Economic Perspectives, 33(2): 3-30.

P. Aghion, B. Jones, and C. Jones (2019), “Artificial Intelligence and Economic Growth," in Ajay Agrawal, Joshua Gans and Avi Goldfarb (eds), *The Economics of Artificial Intelligence: An Agenda*, National Bureau of Economic Research.

K.J. Arrow (1951), “Alternative Proof of the Substitution Theorem for Leontief Models in the General Case". In Activity Analysis of Production and Allocation, ed. T.C. Koopmans. New York: John Wiley.

D. Autor, D. Dorn, L.F. Katz, C. Patterson and J. van Reenen (2017), “Concentrating on the Fall of the Labor Share," American Economic Review Papers and Proceedings, 107(5), 180-185.

J. Azar and X. Vives (2018), “Oligopoly, Macroeconomic Performance, and Competition Policy," mimeo., IISE Business School, Barcelona.

A. Banerjee and A. Newman (1993), “Occupational Choice and the Process of Development." *Journal of Political Economy* **101**, 274–298.

J.I. Bernstein and M.I. Nadiri (1988), “Interindustry R&D Spillovers, Rates of Return, and Production In High-Tech Industries," *American Economic Review - AEA Papers and Proceedings*, 78, No. 2, pp. 429-434, May 1988.

R. Chirinko and D. Mallick (2014), “The Substitution Elasticity, Factor Shares, Long Run Growth and the Low Frequency Panel Model," Working Paper, University of Illinois at Chicago.

D. Comin, A. Danieli and M. Mestieri (2019), “Income-Driven Labor Market Polarization," Working Paper.

G. Eggertsson, J. Robbins, and E.G. Wold (2018), "Kaldor and Piketty's Facts: The Rise of Monopoly Power in the United States," mimeo.

O. Galor and J. Zeira (1993), "Income Distribution and Macroeconomics." *Review of Economic Studies* **60**, 35–52.

G. Grossman, E. Helpman, E. Oberfield and T. Sampson (2020), "Endogenous Education and Long-Run Factor Shares," working paper, Princeton University.

S. Grossman and O. Hart (1979), "A Theory of Competitive Equilibrium in Stock Market Economies," *Econometrica* **47**, 293–330.

G. Gutiérrez and T. Philippon (2017), ""Declining Competition and Investment in the U.S," mimeo., NYU Stern.

W. Johnson (2018), "Economic Growth and the Evolution of Comparative Advantage in an Occupation-Based Network of Industries," PhD dissertation, Department of Economics, Boston University.

G. Kaplan and P. Zoch (2020), "Markups, Labor Market Inequality and the Nature of Work," working paper, University of Chicago.

L. Karabarbounis and B. Neiman (2014), "The Global Decline of the Labor Share," Quarterly Journal of Economics, 129(1): 61-103.

L. Ljungqvist (1993), "Economic Underdevelopment: The Case of Missing Market for Human Capital," *Journal of Development Economics* **40**, 219–239.

L. Makowski (1980), "Perfect Competition, the Profit Criterion, and the Organization of Economic Activity," *Journal of Economic Theory* **22**, 222–42.

G. Mankiw (2015), "Yes $r > g$. So What?" *American Economic Review* (Papers and Proceedings) **105**, 43–47.

D. Mookherjee and D. Ray (2002), "Is Equality Stable?," *American Economic Review* **92** (Papers and Proceedings), 253–259.

D. Mookherjee and D. Ray (2003), "Persistent Inequality," *Review of Economic Studies* **70**, 369–394.

D. Mookherjee and D. Ray (2010), "Inequality and Markets: Some Implications of Occupational Diversity," *American Economic Journal: Microeconomics* **2**, 38–76.

J. P. Neary (2003), “Globalization and Market Structure," *Journal of the European Economic Association* **1**, 245–271.

T. Piketty (2014). *Capital in the Twenty First Century*. Belknap Press.

D. Ray (2015), “Nit-Piketty: A Comment on Thomas Piketty’s Capital in the Twenty First Century," *CESifo Forum* **16** (1), 19–25.

P. A. Samuelson (1951) “Abstract of a Theorem concerning Substitutability in Open Leontief Models". in *Activity Analysis of Production and Allocation*, ed. T.C. Koopmans. New York: John Wiley.

S. Santens (2016), “Robots Will Take Your Job," *The Boston Globe*, February 24.

H. Uzawa (1961-2), “On a Two-Sector Model of Economic Growth," *Review of Economic Studies* **29**, 40–47.

J. Zeira (1998), “Workers, Machines and Economic Growth," *Quarterly Journal of Economics*, 113(4): 1091-117.

Appendix: Proofs

Proof of Theorem 1. We begin with some preliminary observations.

LEMMA **2.** *For each j, there is $\underline{p}_j > 0$ such that in any equilibrium and at any date t,*

$$p_j(t) \geq \underline{p}_j > 0 \tag{33}$$

whenever $y_j(t) > 0$. If in addition, self-replication holds, then

$$p_j(t) \leq c_j\left(1, \{\nu_j^{o-1} \sup P^*\}\right) < \infty \tag{34}$$

for all j and at every date t, where P^* is defined in Proposition 1.

Proof. See Supplementary Appendix. ∎

Consider unit costs for the effective labor aggregator ℓ (in a generic occupation o, subscript removed), with human-robot price ratio ζ:

$$c^o(\zeta, 1) = \min \zeta h + r, \text{ subject to } \ell(h, r) \geq 1. \tag{35}$$

LEMMA **3.** (i) *Suppose that $\zeta > \zeta'$. Then $h \leq h'$ and $r \geq r'$.*

(ii) *For any optimal selection $h(\zeta)$, $\zeta h(\zeta) \to 0$ as $\zeta \to \infty$.*

(iii) *For any optimal selection* $h(\zeta)$ *and* $r(\zeta)$*,* $a(\zeta) = r(\zeta)/h(\zeta) \to \infty$ *as* $\zeta \to \infty$*.*

Proof. See Supplementary Appendix. ∎

Now we prove part (i). We first show that in an equilibrium and for any group m that satisfies (23), we have $z_m(t) \to \infty$. Consider the indirect utility functions $v_m(z(t), \mathbf{p}(t))$ for individual expenditure $z(t)$ at any date t. In any equilibrium, an individual in this group has $F_0 > 0$ units of a financial asset at date 0, and thereafter makes educational and financial asset choices (and consumption choices), under fully anticipated prices, which includes a sequence of return factors $\{\gamma(t)\}$ on financial holdings. She has several necessary conditions that describe her behavior, but one set of these has to do with her choice of financial assets. Because her initial income can be strictly positive if she so pleases (there is a positive subsistence wage), her current expenditure $z_m(T)$ must be strictly positive at some date T, but then $z_m(t) > 0$ for all $t \geq T$, by the unbounded steepness of v_m in z at 0. For ease in writing set $T = 0$. It follows that the Euler equation on financial assets must hold with a particular inequality at every date $t \geq 0$:

$$v'_m(z(t), \mathbf{p}(t)) \geq \beta_m \gamma(t) v'_m(z(t+1), \mathbf{p}(t+1)). \tag{36}$$

If (36) fails, she could always transfer resources one period into the future and increase lifetime utility. (Equality may not hold because human capital could have a higher rate of return than financial assets, and the individual may not be able to marginally pull back funds from future to present, because of credit constraints.) Now we compound this Euler inequality just as in the main text to arrive at (26), reproduced here as:

$$v'_m(z_m(0), \mathbf{p}(0)) \geq \frac{\beta_m^{t-1}\left[(1-\delta) + \frac{1}{c_k\left(1, \{\nu_k^{o-1} \sup P^*\}\right)}\right]^{t-2}}{c_k\left(1, \{\nu_k^{o-1} \sup P^*\}\right)} v'_m(z_m(t), \mathbf{p}(t)). \tag{37}$$

It follows from condition (23) that $v'_m(z_m(t), \mathbf{p}(t)) \to 0$ as $t \to \infty$. But v_m is strictly increasing and concave for every $\mathbf{p}$. Moreover, every active final goods price is bounded above and below by (34) of Lemma 2.[21] Therefore (37) can *only* hold if $z_m(t) \to \infty$ as $t \to \infty$. With a bounded credit limit on every other individual, we must conclude that per-capita income $Y(t)$ as defined in (17) must go to infinity.

For part (ii), we prove that any sector j must be asymptotically fully automated along any subsequence in which its output grows. To show this, we argue first that effective labor in some occupation $o \in O_j$ in that sector must also grow. If this were false for every occupation

[21] If a final good is inactive it has no effect on v_m anyway, as it is not consumed.

in O_j, then by the unbounded steepness condition,

$$\omega_j^o(\tau) = p_j(\tau)\frac{\partial}{\partial \ell_j^o} f_j(k_j(\tau), \boldsymbol{\ell}_j(\tau)) \to \infty \text{ as } \tau \to \infty \tag{38}$$

for *some* occupation $o \in O_j$. But we know that

$$\omega_j^o(\tau) = c_j^o(w_j^o(\tau), p_r(\tau)) \le \nu_j^{o-1} p_r(\tau) \le \nu_j^{o-1} \sup P^* < \infty,$$

where the first inequality comes from the fact that automation is feasible and the second from self-replication and Proposition 1. But this boundedness of $\omega_j^o(\tau)$ contradicts (38). So effective labor $\ell_j^o(\tau)$ in some occupation $o \in O_j$ grows. But

$$\ell_j^o(\tau) = \ell_j^o(h_j^o(\tau), r_j^o(\tau)) = \ell_j^o(h_j^o(\tau), a_j^o(\tau)h_j^o(\tau)) \le \ell_j^o(1, a_j^o(\tau)),$$

where “1" is the total human labor endowment. It follows that $a_j^o(\tau) \to \infty$ for each such growing occupation.

Part (iii). For this part, we need the following

LEMMA **4.** *Let S be the set of all infinite-dimensional nonnegative vectors $\mathrm{s} \equiv (s_1, s_2, \ldots)$, with components in $[0,1]$ and $\sum_{j=1}^{\infty} s_j = 1$. Let $\mathrm{s}(t)$ be a sequence in S, and suppose that there is $\hat{\mathrm{s}} \in S$ such that $\mathrm{s}(t)$ converges pointwise to $\hat{\mathrm{s}} = (\hat{s}_j)$. Let $\boldsymbol{\Psi}(t)$ be a corresponding convergent sequence with components $(\Psi_1(t), \Psi_2(t), \ldots)$, where $\Psi_j(t) \in [0,1]$ for every j and t, with $\Psi_j(t) \to 0$ as $t \to \infty$ for every j with $\hat{s}_j > 0$. Then $\lim_{t\to\infty} \sum_{j=1}^{\infty} \Psi_j(t)\hat{s}_j(t) = 0$.*

Proof. See Supplementary Appendix. ■

For any active sector j and date t, define

$$\Psi_j(t) = \frac{\sum_{o \in O_j} w_j^o(t) h_j^o(t)}{p_j(t) y_j(t)} \in [0,1],$$

and set $\Psi_j(t) = 0$ if $y_j(t) = 0$. This is well-defined: $p_j(t) > 0$ whenever $y_j(t) > 0$ (Lemma 2). We claim that if $y_j(t) \to \infty$ along a subsequence of dates, $\Psi_j(t) \to 0$. To see this, pick any limit point of $\Psi_j(t)$ along the subsequence in question. Choose further subsequences such that for every occupation $o \in O_j$, $w_j^o(t)$ is either bounded or diverges to infinity; retain the original index t. Now, if $w_j^o(t)$ is bounded for some $o \in O_j$, then certainly

$$\frac{w_j^o(t) h_j^o(t)}{p_j(t) y_j(t)} \to 0$$

as $t \to \infty$. (Because $p_j(t)$ is bounded below, $p_j(t)y_j(t) \to \infty$.) Otherwise, if $w_j^o(t) \to \infty$ for some $o \in O_j$, $\zeta_j^o(t) = w_j^o(t)/p_r(t) \to \infty$, given that $p_r(t)$ is bounded above (Proposition 1). By linear homogeneity of ℓ_j^o and Lemma 3(ii),

$$\frac{w_j^o(t)h_j^o(t)}{p_j(t)y_j(t)} \leq \frac{w_j^o(t)h_j^o(t)}{w_j^o(t)h_j^o(t) + p_r(t)r_j^o(t)} = \frac{\zeta_j^o(t)h_j^o(t)}{\zeta_j^o(t)h_j^o(t) + r_j^o(t)} \to 0.$$

Aggregating these observations over all the occupations proves the claim.

If $\sigma(t)$ denotes the share of human labor in national income, it follows that

$$\begin{aligned} \sigma(t) &= \frac{\sum_{o\in O_j} w_j^o(t)h_j^o(t)}{Y(t)} = \frac{\sum_j \Psi_j(t)p_j(t)y_j(t)}{Y(t)} \\ &= \left[\frac{\sum_{i=1}^{\infty} \Psi_i(t)p_i(t)y_i(t)}{Y(t)}\right] + \frac{\sum_{j=e,r,k} \Psi_j(t)p_j(t)y_j(t)}{Y(t)} \end{aligned} \tag{39}$$

at every date t, where it is understood that any sector inactive at any date has an entry of 0 in the sum above. Write for every final good i active at date t:

$$\frac{p_i(t)y_i(t)}{Y(t)} = \sum_m \phi_m(t)s_{mi}(t), \tag{40}$$

where $\phi_m(t) \equiv Z_m(t)/Y(t)$ is the ratio of current aggregate *expenditure* of type m to total *income*, and $s_{mi}(t)$ is the corresponding expenditure share on good i by type m. Combining (39) and (40),

$$\sigma(t) = \sum_{i=1}^{\infty} \Psi_i(t)\left[\sum_m \phi_m(t)s_{mi}(t)\right] + \frac{\sum_{j=e,r,k} \Psi_j(t)p_j(t)y_j(t)}{Y(t)} \tag{41}$$

for all t. We will show that the right hand side of (41) converges to 0 as $t \to \infty$. To this end, pick any subsequence of dates (but retain original notation) so that $\sigma(t)$ converges. Exploiting the fact that the number of sectors is countable, use a diagonal argument to extract a further subsequence (again retain notation) so that *each* of the bounded sequences $\Psi_j(t)$, $\phi_m(t)$, $s_{mi}(t)$, $p_j(t)$, and $[p_j(t)y_j(t)]/Y(t)$ also converge.[22] The last finite sum in (41) pertains only to three sectors: e, r and k. For any of these sectors, call it j, $\Psi_j(t) \to 0$ along any subsequence for which j is consequential, and on any other subsequence $p_j(t)y_j(t)$ must be bounded, while $\Psi_j(t) \in [0,1]$. Putting these observations together with $Y(t) \to \infty$, we must conclude that this last finite term in (41) converges to 0. The rest of the argument concerns the first set of terms in (41).

[22]In particular, the ratio $\phi_m(t) = Z_m(t)/Y(t)$ is also bounded because of finite credit limits.

Let M be the set of all indices for which $\lim_t \phi_m(t) > 0$ for the subsequence under consideration. If M is empty, we are done, so assume it is nonempty. Then, using the fact that the interchange of a finite and infinite sum is always valid, we have

$$\sum_{i=1}^{\infty} \Psi_i(t) \left[\sum_m \phi_m(t) s_{mi}(t) \right] = \sum_m \phi_m(t) \left[\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \right]$$

$$= \sum_{m \in M} \phi_m(t) \left[\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \right] + \sum_{m \notin M} \phi_m(t) \left[\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \right]. \tag{42}$$

Because $\phi_m(t) \to 0$ for all $m \notin M$, the second term on the right hand side of this equation converges to 0. It remains to show that same is true of the first term. It will suffice to show that for each $m \in M$,

$$\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \to 0 \tag{43}$$

as $t \to \infty$ along our chosen subsequence. Because $\lim_t \phi_m(t) > 0$ for $m \in M$ and $Y(t) \to \infty$, it follows that expenditures diverge to infinity for a positive measure of individuals of each type m. Let $Z_m(t)$ be the aggregate expenditure of type m and $x_{mi}(t)$ the aggregate demand for good i by this type. By asymptotic homotheticity,

$$\hat{s}_{mi} \equiv \lim_t s_{mi}(t) = \lim_t \frac{p_i(t) x_{mi}(t)}{Z_m(t)} = \lim_t p_i(t) d_i^m(\mathbf{p}(t)),$$

We claim that each $p_i(t)$ is bounded above and below by strictly positive numbers. The upper bound is given by Lemma 2. For the lower bound, suppose by contradiction that I, the set of indices such that $p_j(t) \to 0$, is nonempty. Then, by assumption (ii) on the function d_m, we have $\liminf_t d_{mi}(\mathbf{p}(t)) > 0$ for some $i \in I$. But then that sector is active at all large dates, which means that its price is bounded below (see (33) of Lemma 2), a contradiction. Therefore the claim is true, and given assumption (i) on $\mathbf{d}_m$, it follows that $\hat{s}_{mi}$ forms a "bonafide share vector" with $\sum_i \hat{s}_{mi} = 1$. So the conditions in Lemma 4 are satisfied (ignore index m). Therefore this Lemma implies (43), and the income share of human labor must converge to zero. Recall (17) to write out income:

$$Y = \sum_i p_i y_i + p_e y_e + p_k y_k,$$

and express it as the sum of (machine) capital and human income:

$$\begin{aligned}Y &= \sum_i p_i y_i + p_e y_e + p_k y_k = \sum_{j\neq r}[k_j + p_r r_j + w_j h_j] \\ &= \sum_{j\neq r}[k_j + w_j h_j] + p_r[y_r - r_r] = \sum_{j\neq r}[k_j + w_j h_j] + [k_r + w_r h_r] \\ &= \sum_j [k_j + w_j h_j] = K + \sum_j w_j h_j.\end{aligned}$$

That means that the income share of capital converges to 1.

Proof of Proposition 2. (a) Under the conditions of Theorem 1, including homotheticity, there are sectors that grow to infinity, and are asymptotically automated. It follows that the robot sector is active after some date T, so that for all $t \geq T$,

$$p_r(t) = c_r(1, \omega_r(t)) \leq c_r(1, \nu_r^{-1} p_r(t)), \tag{44}$$

and moreover, $y_r(t) \to \infty$ as $t \to \infty$. We claim that every limit point of $p_r(t)$ must lie in the set P^* defined by the self-replication condition, as in Proposition 1. If this is false, then using (44), there is $\epsilon > 0$ and some subsequence of dates (retain notation) such that

$$c_r(1, \omega_r(t)) \leq c_r(1, \nu_r^{-1} p_r(t)) - \epsilon,$$

which in turn implies the existence of $\epsilon' > 0$ such that

$$\omega_r(t) \leq \nu_r^{-1} p_r(t) - \epsilon'$$

for all t large. It follows that $a_r(t) \equiv r_r(t)/h_r(t)$ must be bounded above. But then, because $y_r(t) \to \infty$, we must have $h_r(t) \to \infty$, which contradicts the finite labor endowment of the economy. So the claim is true.

By asymptotic homotheticity, the full support restriction on d_m for every m, and the positive, finite bounds on prices above and below, we see that every sector must grow. So every sector $i \in I$ is asymptotically automated. A necessary condition for this to happen is that

$$\liminf_t \theta_i w_i(t) \geq \inf P^*. \tag{45}$$

To see why, note from the previous claim that $\liminf_t p_r(t) \geq \inf P^*$. If (45) is false, then there is $\epsilon" > 0$ such that $\theta_i w_i(t) \leq p_r(t) - \epsilon"$ along some subsequence of dates, implying that $\liminf_t h_i(t)/r_i(t) > 0$ on that subsequence and contradicting asymptotic full automation. Recalling the assumption that $\theta_i \to 0$ for some sequence of goods in I, it follows immediately from (45) that $\sup_i w_i(t) \to \infty$ as $t \to \infty$.

(b) We claim that for any number W, however large, there exists a time T such that for all dates $t \geq T$, $w_j(t) \geq W$ for every sector j for which $h_j(t) > 0$. If this claim is false, there exists some sector q and a subsequence of dates (retain original notation t) such that $\sup_t w_q(t) \equiv W_q < \infty$, but $h_q(t) > 0$ for all t. Next, pick a sector $i \in I$ that nearly attains the supremum in (a). Because that supremum goes to infinity, and because educational costs are bounded, we have for all large t,

$$w_i(t) > W_q + \bar{p}_e \sup_j |E_{ji} - E_{jq}|, \tag{46}$$

where $\bar{p}_e$ is some finite upper bound on education prices given by (34). It follows that for all large t,

$$w_i(t) - p_e(t)E_{ji} > w_q(t) - p_e(t)E_{jq}.$$

This shows that no individual can ever want to enter (or remain in) sector q from any sector j (including $j = q$) for large enough t; i.e., $h_q(t) = 0$ for all large enough t along the subsequence, a contradiction. ■

Proof of Proposition 3. By the minimum subsistence bound on wages and (33) of Lemma 2, there is $\underline{\omega} > 0$ such that in any equilibrium, $\omega(t) \geq \underline{\omega}$ for all t. Recalling the definition of $\Lambda_j(\omega)$, we can easily use the linear homogeneity of f_j and invoke (27) to see that there is $\epsilon > 0$ such that the income share of effective labor in sector j satisfies:

$$\frac{\omega_j(t)\ell_j(t)}{p_j(t)y_j(t)} = \Lambda_j(\omega(t)) \geq \epsilon > 0,$$

for every t and every active sector j. Therefore, if $\Lambda(t)$ denotes the overall income share of effective labor at date t, then, because it is simply a convex combination of all the sector-specific shares,

$$\Lambda_j(t) \geq \epsilon > 0 \tag{47}$$

as well, for every date t.

Now consider any sequence of dates (retain original index t) along which the overall income share of human labor converges. Using a diagonal argument, extract a subsequence such that in every sector j, $\Lambda_j(t)$ converges — to a strictly positive limit, by (47), and the overall shares of human labor income and robot income in effective labor income converges as well. If the share of robot income in effective labor income converges to a number strictly smaller than one, then the proof is complete. Otherwise, the share of robot income in effective labor income converges to 1, and given that the latter has a positive limit share in national income, it follows that $\lim_j r_j(t) > 0$. In particular, for large dates, the robot sector is active, so that:

$$p_r(t) = c_r(1, \omega_r(t)) \leq c_r(1, \nu_r^{-1} p_r(t)). \tag{48}$$

where the latter inequality comes from the feasibility of automation in the robot sector.

Now, self-replication fails by assumption, so $\lim_{\eta \to 0} c_r(\eta, 1) \geq \nu_r$. Multiplying through by $p_r \nu_r^{-1}$, and using the concavity of the robot cost function (the first part of our regularity condition on f_r), $p_r \leq c(1, \nu_r^{-1} p_r)$ for every $p_r > 0$. Indeed, using (33) of Lemma 2 and the unbounded steepness of c at $p_r = 0$ (inherited in turn from the unbounded steepness of f_r), we make a stronger claim: there is $\epsilon > 0$ such that

$$p_r(t) \leq c_r(1, \nu_r^{-1} p_r(t)) - \epsilon. \tag{49}$$

at every conceivable *equilibrum* price $p_r(t)$ at any date.[23] Combining (48) and (49),

$$c_r(1, \omega_r(t)) \leq c_r(1, \nu_r^{-1} p_r(t)) - \epsilon,$$

which in turn implies the existence of $\epsilon' > 0$ such that

$$\omega_r(t) \leq \nu_r^{-1} p_r(t) - \epsilon'$$

for all t large. So, because the effective labor price is bounded away from what it would have been with full automation, it follows that $a_r(t) = r_r(t)/h_r(t)$ must be bounded above. But then, because effective labor in the robot sector can be produced by humans alone (the second part of our regularity condition on f_r), it must be that the share of human labor income in the total value of robot production (equal to robot income) is bounded away from 0. Therefore in this case, too, the share of human income in total effective labor income is bounded away from zero, and the proof of the proposition is complete. ∎

Proof of Proposition 4. In any equilibrium, all prices are bounded below (pointwise) by strictly positive numbers, just as before; see (33) of Lemma 2. Under self-replication, Proposition 1 additionally applies and robot prices are also bounded above exactly as before, and *independent* of human productivity. In turn, this provides pointwise upper bounds on prices in all sectors, see (34) of Lemma 2. That includes the same bound on price of capital, so part (i) of Theorem 1 holds under the same conditions and following exactly the same proof.

The remainder of the proof consists in applying the following argument at more than one point:

---

[23] Note first that $p_r(t)$ is bounded below (Lemma 2. Now consult Panel B, Figure 1. Because $c_r(1, p_r)$ is concave and initially lies strictly above the diagonal, it cannot converge back to the diagonal without actually crossing it. So it must remain separated from the diagonal by some strictly positive number.

Claim. Suppose that for some occupation $o \in \cup_j O_j$, the human wage per unit of productivity, $w^o(t)$, is bounded on the equilibrium path by some $\bar{w}^o < \infty$. Then human labor in efficiency units is also bounded along that same path.

To establish the Claim, pick some $S > 0$ such that

$$\underline{p}_e S > \frac{\bar{w}^o}{1-\beta} + \bar{p}_e L^o \tag{50}$$

where β is the largest discount factor among all types. Next, using (H.1), pick $M < \infty$, *larger than initial productivity endowment and the cross-occupation bound*, such that $e(\mu, \mu+\Delta, o, o) \geq S\Delta$ for all $\mu \geq M$. Suppose an individual contemplates a move beyond a productivity of M without changing occupation; i.e., there exists t such that she moves from $\mu^o(t-1) \geq M$ to $\mu^o(t) > \mu^o(t-1)$. Let $\Delta \equiv \mu^o(t) - \mu^o(t-1)$. Then the lifetime wage gain as a result of this move is bounded above by $\bar{w}^o\Delta/(1-\beta)$. Also, the higher productivity can lower the marginal cost of subsequent actions. By (H.2), these gains are bounded above by $\bar{p}_e L_j \Delta$, where $\bar{p}_e$ is an upper bound on the price of education. So total gains are bounded above by

$$\frac{\bar{w}^o\Delta}{1-\beta} + \bar{p}_e L_j \Delta \tag{51}$$

On the other hand, the cost of this move is given by

$$p_e(t)e(\mu^o(t-1), \mu^o(t), o, o) = p_e(t)e(\mu^o(t-1), \mu^o(t-1)+\Delta, o, o) \geq \underline{p}_e S\Delta.$$

Combining this expression with (50) and (51), we must conclude that the cost of the proposed move exceeds its benefits, so it will never be made. That proves the Claim.

For parts (ii) and (iii), minor adjustments are needed. In (ii), we prove that any sector j must be asymptotically fully automated along any subsequence in which its output grows. Just as in the proof of Theorem 1, we can first show that effective labor in some occupation $o \in O_j$ in that sector must also grow. Now we consider two possibilities. If $w_j^o(t)$ grows along some further subsequence, then the share of human labor income in total revenue accruing to effective labor in that occupation must converge to zero along that subsequence; see part (ii) of Lemma 3. The second possibility is that $w_j^o(t)$ is bounded. Then by the Claim, individual productivity is also bounded, and — given that this occupation grows — it must become asymptotically automated.

For part (iii), we need to show again that

$$\Psi_j(t) = \frac{\sum_{o \in O_j} w_j^o(t) h_j^o(t)}{p_j(t) y_j(t)} \in [0, 1],$$

converges to zero, as in the proof of Theorem 1. Very similar (and minor) changes need to be made as in the preceding paragraph, using the Claim. We omit the details. With this established, there is no change in the rest of the argument to establish Theorem 1. ■

Proof of Theorem 2: See Supplementary Appendix.