Designing Stress Scenarios*

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Abstract

We develop a tractable framework to study the optimal design stress scenarios. A risk-averse principal (e.g, a manager, a regulator) seeks to learn about the exposures of a group of agents (e.g., traders, banks) to a set of risk factors. The principal asks the agents to report their outcomes (e.g., credit losses) under a variety of scenarios that she designs. She can then take remedial actions (e.g., mandate reductions in risk exposures). The principal's program has of two parts. For a given set of scenarios, we show how to apply a Kalman filter to solve the learning problem. The optimal design is then a function of what she wants to learn and how she intends to intervene if she uncovers excessive exposures. The choice of optimal scenarios depends on the principal's prior's about risk exposures, the cost of ex-post interventions, and the potential correlation of exposures across agents.

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1 Introduction

Stress tests are ubiquitous in risk management and financial supervision. Risk officers use stress tests to set and monitor risk limits within their organizations, and financial regulators around the world use stress tests to assess the health of financial institutions. To give just a few examples: financial firms use stress tests to complement their statistical risk management tools (e.g., Value at Risk); asset managers stress test their portfolios; trading venues stress tests their counter-party exposures; regulators mandate large scale stress tests for banks and insurance companies and use the results to enforce capital requirements and validate dividend policies.[1]

Despite the growing importance of stress testing and the amount of resources devoted to them, there is little theoretical guidance on exactly how one should design the stress scenarios. A theoretical literature has focused on the trade-offs involved in the disclosure of supervisory information (see Goldstein and Sapra, 2014 for a review), which range from concerns about the reputation of the regulator (Shapiro and Skeie, 2015) to the importance of having a fiscal backstop (Faria-e-Castro et al., 2017). Though these papers provide insights as to what to do with the results of asset quality reviews, they do not model *stress testing*. They are silent about the design of forward-looking scenarios, which are the hallmarks of stress testing.

The goal of our paper is to start filling this void. There are two ways to think about stress tests: as learning mechanisms, or as tests of capital adequacy. We focus on the learning aspect of stress testing because it is the richer and more complex aspect of the exercise. Our results shed light on risk exposures and capital ratios, but we do not specify the direct link between passing the test and having a particular level of equity. To do so one would need to take a stand on many features of financial regulations that are not central to our analysis.[2]

[1]Central banks in the United States, Europe, England, Brazil, Chile, Singapore, China, Australia, and New Zealand, as well as the International Monetary Fund in Japan, have recently used stress tests to evaluate the banking sector's solvency and guide banking regulation.

[2]For instance, imagine that a bank needs the same level of ex-ante equity to satisfy a 9% capital requirement after scenario 1 or a 7% requirement after scenario 2 (presumably because scenario 2 embodies a higher degree of stress). As far as ex-ante capital adequacy is concerned, these two regulations are equivalent. The law sometimes mandates one of these numbers, in which case our model can shed light on the other, but in general the "level" of the ratio and the "degree" of the stress are not independent and a model is likely to pin down a combination as opposed to a particular value for each. A scenario used for "pass/fail" also needs to be plausible in a way that a learning scenario does not have to be. When stress test results are mechanically linked to capital requirements the choice of scenarios can be used to increase the effective requirement (if one assumes that, for some reason,

We model stress testing as a learning mechanism. We consider a principal and a potentially large number of agents. The agents can be traders within a financial firm, or they can be financial firms within a financial system. The principal can be a regulator designing supervisory tests, or a risk officer running an internal stress test. For simplicity we will use the supervisory stress testing example in much of the paper. Banks are exposed to a set of risk factors, but their exposures are unknown to the regulator. The regulator is risk averse and worries about the financial system experiencing large losses in some states of the world. The regulator then designs a set of scenarios and asks the banks to report what their losses would be under these scenarios. From their responses, the regulator learns about the underlying exposures of the banks. Based on this information, the regulator can decide to intervene, i.e., she can ask a set of banks to reduce their exposures to some factors.

Our main insight comes from writing the learning problem as a Kalman filter. The filter gives us a mapping from prior beliefs and test results into posterior beliefs. The precision of the mapping depends on the scenarios. We can then formulate the regulator's problem as an information acquisition problem in which the regulator chooses the precision of her signals. By explicitly considering the structure of the signals generated by the stress tests, we can map the feasible set of precision choices to the primitive parameters of the model, such as the priors of the regulator regarding the banks' exposures. If, for instance, the regulator is worried about a particular risk factor, we can derive the stress test that maximizes learning about the exposures to this risk factor.

Will the regulator focus a particular risk factor or will she try and learn about several factors at the same time? We show that the answer depends on her prior beliefs about correlation between the banks' risk exposures and on the marginal cost of intervention cost. The regulator can choose to mandate a broad risk reduction. This is likely to involve a lot of unnecessary changes and disruptions, but it does not require much information. If the cost of intervention is high, this strategy is not efficient and the regulator will want to learn in order to avoid unnecessary interventions. She can learn by choosing a more extreme scenario along a particular risk dimension, but extreme scenarios lead to less precise answers. The reduction in overall information quality

the baseline requirement is too weak) or to implement counter-cyclical ratios (keeping the level of stress constant as the economy improves leads to larger assumed shocks). Finally, passing the stress test means that regulators deem the institution safe and sound even in the stress scenario, and thus that lender-of-last-resort policies would be appropriate.

depends on the correlation of the risk exposures. Whether or not there is specialization in learning depends therefore on the intervention cost function and on the prior correlation matrix.

The costs of intervention and the prior beliefs of the regulator are central in determining the optimal scenario design. When the intervention costs are heterogenous, the regulator chooses to stress the factor where interventions are more costly, since this allows her to limit wasteful interventions.

When the regulator expects a high exposure to a risk factor, she stresses that factor more because the benefit from learning about it is larger. When this expected exposure is high enough, the regulator may choose to stress only that factor and concentrate all her efforts in reducing the ex-ante uncertainty along that factor's dimension.

Our model also sheds light on the role of systematic factors, within or across banks. When the exposures to two factors within a bank are correlated, learning about the exposure to one factor also provides information about the exposure to the other factor. Hence, the regulator stresses more the factors with correlated exposures and may focus on these factors if the correlation is high enough. However, due to the convexity of the information acquisition cost, the regulator's specialization is usually incomplete and she tends to put weight on all factors.

Similarly, the regulator optimally puts more weight on systemic factors in stress scenarios. When different banks have correlated exposures to a (systemic) factor, learning about one bank's exposure to this factor provides information about the other banks' exposures. The higher the expected correlation between the exposures to the systemic factor, the more precise the information about the banks' exposure to it and the less costly it becomes for the regulator to learn about it. Hence, the regulator optimally stresses systemic factors more and may choose to specialize and stress only these factors when they are systemic enough (the banks' exposures to it are sufficiently correlated).

The main advantage of our framework is that it can be easily applied to design stress tests in practice. The only inputs needed are the regulator's preferences and beliefs. Though for expositional purposes we develop our insights in the context of a small number of banks and few independent macroeconomic factors, our framework can accommodate correlated macro factors, including non-linear combinations of factors, arbitrary and general correlation structures for risk exposures across and within banks, and different preferences for the regulator. Moreover, the linear structure allows it to be scalable and very easy to implement, even with a large number of banks, scenarios and risk factors.

Literature Review

Most of the literature on stress tests focuses on banking. Several recent papers study specifically the trade-offs involved in disclosing stress test results. Goldstein and Leitner (2018) focus on the Hirshleifer (1971) effect: revealing too much information destroys risk-sharing opportunities between risk neutral investors and (effectively) risk averse bankers. These risk-sharing arrangements also play an important role in Allen and Gale (2000). Shapiro and Skeie (2015) study the reputation concerns of a regulator when there is a trade-off between moral hazard and runs. Faria-e-Castro et al. (2017) study a model of optimal disclosure where the government trades off Lemon market costs with bank run costs, and show that a fiscal backstop allows government to run more informative stress tests. Schuermann (2012) analyzes the design and governance (scenario design, models and projection, and disclosure) for more effective stress test exercises. Schuermann (2016) particularly determines how stress testing in crisis times can be adapted to normal times in order to insure adequate lending capacity and other key financial services. Orlov et al. (2017) look at the optimal disclosure policy when it is jointly determined with capital requirements, while Gick and Pausch (2014) and Inostroza and Pavan (2017) do so in the context of Bayesian persuasion.

While most of the existing literature on stress testing, theoretical and empirical, analyzes the disclosure of stress test results, Leitner and Williams (2018) focus on the disclosure of the regulator's risk modeling. They examine the trade-offs involved in disclosing the model the regulator uses to perform the stress test to banks. However, none of these papers consider the optimal scenario design, which is the focus of our paper.

Most empirical papers on stress tests focus on the information content at the time of disclosure, using an event study methodology to determine whether stress tests provide valuable information to investors. Petrella and Resti (2013) assess the impact of the 2011 European stress test exercise. For the 51 banks with publicly traded equity, they find that the publication of the detailed results provided valuable information to market participants. Similarly, Donald et al. (2014) evaluate the 2009 U.S. stress test conducted on 19 bank holding companies and find significant abnormal stock returns for banks with capital shortfalls. Candelon and Sy (2015), Bird et al. (2015), and Fernandes et al. (2015) also find significant average cumulative abnormal returns for stress tested BHCs around many of the stress test disclosure dates. Flannery et al. (2016) find that U.S. stress tests contain significant new information about assessed BHCs. Using a sample of large banks with publicly traded equity, the authors find significant average abnormal returns around many of the stress test disclosures dates. They also find that stress tests provide relatively more information about riskier and more highly leveraged bank holding companies. Glasserman and Tangirala (2016) evaluate one aspect of the relevance of scenario choices. They show that the results of U.S. stress tests are somewhat predictable, in the sense that rankings according to projected stress losses in 2013 and 2014 are correlated. Similarly, the rankings across scenarios in a given year are also correlated. They argue that regulators should experiment with more diverse scenarios, so that it is not always the same banks that project the higher losses. Acharya et al. (2014) compare the capital shortfalls from the stress tests with the capital shortfalls predicted using the systemic risk model of Acharya et al. (2016) based on equity market data. Camara et al. (2016) study the quality of the 2014 EBA stress tests using the actual micro data from the tests.

Finally, our paper is related to the large theoretical literature on information acquisition following Verrecchia (1982), Kyle (1989) and especially Van Nieuwerburgh and Veldkamp (2010). In this class of models, the cost of acquiring information pins down the set of feasible precisions and determines whether there are signals are complement or substitutes. Vives (2008) and Veldkamp (2009) provide a comprehensive review of this literature. These papers take the information processing constraint on the signal precisions as given. In contrast, our paper focuses on the design of the signals that the regulator receives and endogenizes the information processing constraint.

The rest of the paper is organized as follows. Section 2 describes the environment. Section 3 introduces the notion of stress test. Sections 4 and 5 respectively provide an application of the general environment to linear quadratic preferences for the regulator and characterize the optimal stress scenarios for this case. Section 6 concludes.

2 Environment

We consider the problem of a regulator who needs to learn about the risk exposures of a set of banks in order to take remedial actions. The regulator elicits information from the banks in the form of stress tests. In our model, a stress test is a technology used by regulators to ask questions. The banks cannot evade the questions and have to answer to the best of their abilities. Banks in our model can only lie by omission.

2.1 Banks and Risks

There is one regulator overseeing N banks indexed by $i \in [1, .., N]$ exposed to systematic and idiosyncratic risks. The state of the macro-economy is the vector $s = [s_1, .., s_J]$ where J is the number of systematic factors. The vector $x_i = [x_{i,1}, .., x_{i,J}]$ represents the exposures of bank i, with $x_{i,j}$ the exposure of bank i to factor j. The excess losses of bank i in state s are given by

$$y_i(s) = x_i \cdot s + \eta_i = \sum_{j=1}^{J} x_{i,j} s_j + \eta_i, \tag{1}$$

where η_i is a random idiosyncratic shock. We normalize the baseline scenario to $s \equiv 0$, so that our scenarios should be interpreted as deviations from the baseline.

The net worth of bank i is given by

$$w_i(s) = \bar{w}_i - y_i(s), \tag{2}$$

where $\bar{w}_i$ is the mean level of profits net of debt. For instance, we could have $\bar{w}_i = r_i - d_i$ where r_i would be average revenues and d_i outstanding debt. Given Eq. (1) and Eq. (2), the aggregate net worth of the banking system is

$$W(s) = \sum_{i=1}^{N} w_i = \bar{w} - \overline{x} \cdot s,$$

where $\overline{w}$ and $\overline{x}$ are the sum of the corresponding variables across the N banks in the economy.

We do not restrict the factors to be independent, and we can allow for a non-linear mapping by setting $s_{j+1} = s_j^2$. Our factors can be thought of as fundamental shocks in a macroeconomic model or as traditional macroeconomic variables such as GDP, unemployment, or house prices as functions of the factors. In a linear interpretation, factors can be negative (good state) or positive (bad case), and $x_{i,j}$ are positive numbers for most banks, although a particular bank could, in principle, have a short exposure. We consider several special cases below.

2.2 Regulatory Interventions and Preferences

As in Acharya et al. (2016), we assume that the regulator has preferences $U(W)$ over the total net worth of the banking system W.[3] The regulator can intervene to force the banks to reduce

[3] More generally, we could have $U([w_i]_{1..N})$. This would capture the case where the idiosyncratic failure of bank i matters regardless of the health of the banking sector as a whole. As in the systemic risk literature, we impose

their exposures. If the regulator takes action $a_i = [a_{i,1}, .., a_{i,J}]$ on bank i, the exposure of bank i to factor j becomes $(1 - a_{i,j})\, x_{i,j}$.

Intervention are costly. There are direct costs born by the regulators and the banks, as well as indirect costs from the disruption of valuable activities. We assume that the costs are convex in the size of the intervention. For simplicity, we make it quadratic and equal to $\frac{1}{2}\sum_{i,j}\phi_j a_{ij}^2$. To shorten the notation, we define the cell-by-cell multiplication operator $\circ$ as

$$(\mathbf{1} - a_i) \circ x_i \equiv [(1 - a_{i,1})\, x_{i,1}, \ldots, (1 - a_{i,J})\, x_{i,J}] \,.$$

Let $\mathcal{I}$ denote the information set of the regulator at the time when she chooses her intervention policy. The regulator's problem is to choose an intervention policy to maximize her expected utility given by

$$\mathbb{E}\left[U\left(\bar{w} - \overline{\eta} - \left(\sum_{i=1}^{N} (\mathbf{1} - a_i) \circ x_i \right) \cdot s \right) - \frac{1}{2}\sum_{i=1}^{N} \|\boldsymbol{\phi} \circ a_{ij}\|^2 \middle| \mathcal{I} \right],$$

where $\boldsymbol{\phi} = \left(\sqrt{\phi_1}, ..., \sqrt{\phi_J}\right)$ is an $N \times 1$ vector that modulates the marginal cost of intervening to reduce the risk exposure of banks along each dimension $j = 1, \ldots, J$

Example. Suppose that there is only one bank.[4] Then, the regulator chooses her intervention policy to maximize

$$\mathbb{E}\left[U\left(\bar{w} - \overline{\eta} - \sum_{j=1}^{J} (1 - a_j)\, x_j s_j \right) - \frac{1}{2}\sum_{j=1}^{J} \phi_j a_j^2 \middle| \mathcal{I} \right].$$

The first order condition (FOC) for this problem is

$$a_j = \frac{1}{\phi_j}\mathbb{E}\left[x_j s_j U'(W) \middle| \mathcal{I} \right]. \tag{3}$$

The FOC equates the marginal cost of an action to its expected marginal benefit. The expected marginal benefit of reducing risk exposure to factor j depends on the covariance between the marginal social utility $U'(W)$ and the marginal increase in the bank's net worth $x_j s_j$. Risk reduction is valuable when we expect that U' to be large when s is positive.

the restriction that only W matters. As discussed in Acharya et al. (2016), this specification naturally arises when healthy banks can efficiently take over of failed ones. As a result, a financial crisis only happens when the financial system as a whole is under-capitalized.

[4] Equivalently, all banks are identical and have the same risk exposures.

The regulator could mandate a reduction is all risky activities: this would not require much information, but it would be costly. On the other hand, if the regulator had access to adequate information, she could mandate reductions only for the activities that create significant systematic risk. We think of stress tests precisely as a way of figuring out what these activities are.

3 Stress Tests

In this section, we study how the regulator can learn about banks' risk exposures $\{x_j\}_{j=1}^J$ by running a stress test.

3.1 Definitions

The regulator has a prior belief over the distribution of exposures, both within banks and across banks. We stack the banks' exposures in one NJ vector as follows

$$\mathbf{x} \equiv \begin{bmatrix} [x_1'] \\ .. \\ [x_i'] \\ .. \\ [x_N'] \end{bmatrix}$$

and we summarize the regulator's prior over the vector of exposures $\mathbf{x}$ as

$$\mathbf{x} \sim N\left(\overline{\mathbf{x}}, \Sigma_x\right),$$

where

$$\overline{\mathbf{x}} = \begin{pmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_N \end{pmatrix} \quad \text{and } \Sigma_x = \begin{bmatrix} \Sigma_x^1 & \Sigma_x^{12} & \cdots & \Sigma_x^{1N} \\ \Sigma_x^{12} & \Sigma_x^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Sigma_x^{(N-1)N} \\ \Sigma_x^{1N} & \cdots & \Sigma_x^{(N-1)N} & \Sigma_x^N \end{bmatrix}$$

with $\Sigma_x^i = \mathbb{V}\left(x_i\right)$ for all i and $\Sigma_x^{ij} = \mathbb{C}\text{ov}\left(x_i, x_j\right)$ for all $i \neq j$. If Σ_x^i is diagonal the regulator expects the exposures of bank i to the different factors to be independent of each other. If $\Sigma_x^{ij} = 0$, the regulator's prior is that the risk exposures of bank i and j are independent.

To learn about the banks' risk exposures, the regulator can ask the banks to estimate their losses, and report their estimates, under a particular realization of the macroeconomic state. This choice of macroeconomic state is a scenario $\hat{s}$.

Definition 1. A scenario $\hat{s}$ is a realization of the row-vector of states s.

A scenario $\hat{s}$ is a row-vector of size J that represents an aggregate state of the economy. Given our normalization of the baseline scenario to $s = 0$, a scenario close to 0 is a scenario close to the baseline of the economy. A scenario $\hat{s}$ in which element $\hat{s}_j$ is large, represents a large deviation from the baseline along the dimension of factor j. The larger the distance between $\hat{s}$ and 0, the more extreme the scenario. When designing a stress test, the regulator specifies a set of scenarios for which the banks need to report their losses.

Definition 2. A stress test is a collection of scenarios $\left\{\hat{s}^{(m)}\right\}_{m=1}^{M}$ and reported losses $\{\hat{y}_i\}_{i=1}^{N}$ for each bank under each scenario.

For each scenario $\hat{s}^{(m)}$ bank i reports its net losses $\hat{y}_i^{(m)}$, or equivalently, its net worth $\hat{w}_i^{(m)}$. For simplicity, we will express all of our analysis in terms of the net losses reported by the banks $\hat{y}_i = \left[\hat{y}_i^{(1)}, ..., \hat{y}_i^{(M)}\right]$.

3.2 Stress test results

Banks use imperfect models to predict what their losses would be under the stress test scenario. More precisely, we assume bank i estimates its losses under scenario $\hat{s}_m$ as

$$\hat{y}_i^{(m)} = \hat{s}^{(m)} \cdot x_i' + e_i^{(m)} \tag{4}$$

where $e_i^{(m)} \sim N\left(0, \Sigma_{e,i}^{(m)}\right)$. We think of $e_i^{(m)}$ as capturing genuine model uncertainty because the banks themselves do not know their true exposures and because macro-economic scenarios are partly ambiguous and misspecified. $\Sigma_{e,i}^{(m)}$ determines the precision of bank i's report under scenario m. We specify the following functional form for the precision of a particular scenario $\hat{s}^{(m)}$ when there are M scenarios in the stress test

$$\Sigma_{e,i}^{(m)} \equiv \left(1 + \lambda_i(M) \left\|\alpha_0^i \cdot \hat{s}^{(m)}\right\|^{\alpha_1^i}\right) \sigma_\varepsilon^2, \tag{5}$$

where α_0^i is a positive vector of size J, $\alpha_1^i > 2$ is a scalar, and $\lambda_i(M)$ is a positive and increasing function of M. The term $\left\|\alpha_0^i \cdot \hat{s}^{(m)}\right\|$ increases with the deviation of the scenario from the baseline.

This specification captures the idea that extreme scenarios are harder to estimate, and that increasing the number of scenarios in a stress test reduces the precision of the bank's responses. The vector of weights α_0^i captures the idea that risk exposures to some factors might be easier to learn than others. The curvature α_1^i determines the sensitivity of the precision of bank i's report to extreme scenarios. The larger α_1^i, the harder it is for bank i to provide a precise estimate of its losses for scenarios away from the baseline. We assume that $\alpha_1^i > 2$ to allow for a non-trivial trade-off in the regulator's design problem. Finally, $\lambda_i(M)$ represents the increasing cost of computing expected losses for a bank with limited information processing capacity.

The structure of the bank's internal models are the outcome of learning about risk exposures from historical data. Therefore, the mistakes a bank makes in computing its expected revenues for each scenario may be correlated. The variance-covariance matrix of the errors made by bank i in computing its stress test results is given by

$$\mathbf{\Sigma}_e^i = \begin{bmatrix} \Sigma_{e,i}^{(1)} & \Sigma_{e,i}^{(1,2)} & \cdots & \Sigma_{e,i}^{(1,M)} \\ \Sigma_{e,i}^{(1,2)} & \Sigma_{e,i}^{(2)} & \ddots & \vdots \\ \vdots & \cdots & \ddots & \Sigma_{e,i}^{(M-1,M)} \\ \Sigma_{e,i}^{(1,M)} & \cdots & \Sigma_{e,i}^{(M-1,M)} & \Sigma_{e,i}^{(M)} \end{bmatrix},$$

where

$$\Sigma_{e,i}^{(m,\ell)} = \mathbb{C}ov\left[e_i^{(m)}, e_i^{(\ell)}\right] \quad \forall m, \ell = 1, ..., M.$$

Differences in $\mathbf{\Sigma}_e^i$ across banks reflect differences in information (priors), in the amount or quality of data available to each bank, or in the bank's information processing capacity. We assume that x_i and $e_i = \left[e_i^{(1)}, ..., e_i^{(M)}\right]$ are independent, but we allow banks to make correlated mistakes. The correlation among the mistakes made by bank i and bank j is given by $\mathbf{\Sigma}_e^{ij}$. The results of the stress test can be summarized in the NM vector

$$\hat{\mathbf{y}} \equiv \begin{bmatrix} [\hat{y}_1'] \\ .. \\ [\hat{y}_i'] \\ .. \\ [\hat{y}_N'] \end{bmatrix},$$

and similarly for $\mathbf{e}$.

3.3 Learning from stress tests

By implementing stress tests, the regulator elicits information from the banks which she uses to learn about the banks' risk exposures to the aggregate factors. In fact, we can interpret stress test results as signals about the banks' risk exposures.

The information available to the regulator after seeing the results of the stress test can be summarized in the following state space representation

$$\hat{\mathbf{y}} = \left(\mathbf{I}_N \otimes \hat{S}\right)\mathbf{x} + \mathbf{e}, \tag{6}$$

where $\mathbf{I}_N$ is the identity (diagonal) matrix of size N and

$$\hat{S} \equiv \begin{bmatrix} \hat{s}^{(1)} \\ .. \\ \hat{s}^{(m)} \\ .. \\ \hat{s}^{(M)} \end{bmatrix}$$

is the $M \times J$ matrix of scenarios. The covariance matrix of the error terms $\mathbf{e}$ is

$$\mathbf{\Sigma_e}\left(\hat{S}\right) \equiv \begin{bmatrix} \mathbf{\Sigma}_e^1 & \mathbf{\Sigma}_e^{12} & \cdots & \mathbf{\Sigma}_e^{1N} \\ \mathbf{\Sigma}_e^{12} & \mathbf{\Sigma}_e^2 & \ddots & \vdots \\ \vdots & \cdots & \ddots & \mathbf{\Sigma}_e^{(N-1)N} \\ \mathbf{\Sigma}_e^{1N} & \cdots & \mathbf{\Sigma}_e^{(N-1)N} & \mathbf{\Sigma}_e^N \end{bmatrix}$$

which is of size $NM \times NM$ and depends on the stress test scenarios.

Remember that the regulator observes $\hat{\mathbf{y}}$ and want to learn about $\mathbf{x}$. Expressing the stress test as in equation (6) allows us to apply the Kalman filter and to fully characterize the posteriors beliefs of the regulator after observing the results.

Proposition 1. *After observing the results $\hat{\mathbf{y}}$ of the stress test, the posterior beliefs of the regulator regarding the banks' risk exposures are*

$$\mathbf{x}|\,\hat{\mathbf{y}} \sim N\left(\hat{\mathbf{x}}, \hat{\Sigma}_{\mathbf{x}}\right) \;,$$

where the posterior mean $\hat{\mathbf{x}}$, the Kalman gain K, and the posterior covariance matrix $\hat{\Sigma}_{\mathbf{x}}$ are given *by*

$$\hat{\mathbf{x}} = \left(\mathbf{I}_{NJ} - K\left(\mathbf{I}_N \otimes \hat{S}\right)\right)\bar{\mathbf{x}} + K\hat{\mathbf{y}} \ , \tag{7}$$

$$K = \Sigma_{\mathbf{x}}\left(\mathbf{I}_N \otimes \hat{S}\right)'\left(\left(\mathbf{I}_N \otimes \hat{S}\right)\Sigma_{\mathbf{x}}\left(\mathbf{I}_N \otimes \hat{S}\right)' + \mathbf{\Sigma_e}\left(\hat{S}\right)\right)^{-1} \ , \tag{8}$$

$$\hat{\Sigma}_{\mathbf{x}} = \left(\mathbf{I}_{NJ} - K\left(\mathbf{I}_N \otimes \hat{S}\right)\right)\Sigma_{\mathbf{x}} \ . \tag{9}$$

The Kalman gain K is an $NJ \times MN$ matrix. The weight K_j is a measure of the amount of information about the exposure to risk factor j contained in the stress test result.

The posterior covariance matrix $\hat{\Sigma}_{\mathbf{x}}$ plays a critical role in our analysis. $\hat{\Sigma}_{\mathbf{x}}$ measures the residual uncertainty that persists after observing the results of the stress test. The goal of the stress test is to reduce this residual uncertainty as much as possible, along the dimensions that depend on the objective function and on the priors of the regulator.

3.4 Scenario choice as signal design

In the standard state-space representation in Equation (6), the reported losses by the banks are signals about linear combinations of the banks' exposure to the risk factors. The stress test scenarios determine the structure of the signals observed by the regulator. The scenarios also determine the precision of the banks' reported losses in Eq. (5). This defines the tradeoff at the heart of the design problem. Increasing s_j in a scenario makes the results more informative about exposures to factor j, but extreme scenarios reduce the precision of the banks' report.

When designing the scenarios, the regulator anticipates how she will interpret the results of the test. The extent to which learning will take place is captured by the *expected distribution of the posterior mean*, given by

$$\hat{\mathbf{x}} \sim N\left(\bar{\mathbf{x}}, \Sigma_{\hat{\mathbf{x}}}\right) , \tag{10}$$

where the variance of posterior beliefs is given by

$$\Sigma_{\hat{\mathbf{x}}} \equiv \Sigma_{\mathbf{x}} - \hat{\Sigma}_{\mathbf{x}} = K\left(\mathbf{I}_N \otimes \hat{S}\right)\Sigma_{\mathbf{x}}.$$

The matrix $\Sigma_{\hat{\mathbf{x}}}$ represents the expected amount of learning from stress test $\hat{S}$. If the stress test is pure noise, then $K = 0$ and $\hat{\Sigma}_{\mathbf{x}} = \Sigma_{\mathbf{x}}$. The posterior distribution is the same as the prior distribution, and there is no uncertainty about the posterior means: they are simply the priors and $\Sigma_{\hat{\mathbf{x}}} = 0$. If the test is fully informative, then $\hat{\Sigma}_{\mathbf{x}} = 0$. The regulator expect to change her mind a lot in response to the tests, and $\Sigma_{\hat{\mathbf{x}}} = \Sigma_{\mathbf{x}}$. The goal of the regulator is to maximize her learning $\Sigma_{\hat{\mathbf{x}}}$ by minimizing the residual uncertainty $\hat{\Sigma}_{\mathbf{x}}$, taking into account that the Kalman gain K is a function of the scenario, given by equation (8).

Example. Let us consider the simple case of one bank, two factors, and one scenario: $N = 1$, $J = 2$, $M = 1$ (we drop the superscript m since $M = 1$). Increasing $\hat{s}_j$ in the scenario affects learning according to

$$\frac{d\Sigma_{\hat{x}}}{d\hat{s}_j} = -\left[\frac{dK_l}{d\hat{s}_j}\hat{s}_h + K_l \mathbb{I}\left\{j = h\right\}\right]_{lh} \Sigma_x, \tag{11}$$

where $\mathbb{I}\left\{j = h\right\}$ is 1 if $j = h$ and 0 otherwise. The change in the information gain in the dimension of the risk exposure to factor j is given by

$$\frac{dK_l}{d\hat{s}_j} = \frac{\partial K_l}{\partial \hat{s}_j} + \frac{\partial K_l}{\partial \boldsymbol{\Sigma}_{\mathbf{e}}\left(\hat{S}\right)} \frac{\partial \boldsymbol{\Sigma}_{\mathbf{e}}\left(\hat{S}\right)}{\partial \hat{s}_j}. \tag{12}$$

Equation (12) shows the two effects of changing the stress scenario along dimension j. The direct effect is to change the information content about the exposure to factor j. The second, indirect effect is to change the precision of the bank's report.

These two effects depend on the prior of the regulator. Consider the case where the regulator thinks that the risk factors are uncorrelated, i.e., if $\Sigma_{x,12} = 0$. In that case we can sign the direction of the direct effect of $\hat{s}_j$ on $\Sigma_{\hat{x},jj}$ given by

$$\operatorname{sgn}\left(\left(\frac{\partial K_j}{\partial \hat{s}_j}\hat{s}_j + K_j\right)\Sigma_{x,jj}\right) = \operatorname{sgn}\left(\hat{s}_j\right). \tag{13}$$

Equation (13) shows that making the scenario more extreme by increasing the absolute size of $\hat{s}_j$ increases the amount of information the stress test results contain about the bank's risk exposure to risk factor j. Similarly, the direct effect of changes in $\hat{s}_j$ on $\Sigma_{\hat{x},hh}$ depends on the how the weight K_h that the regulator puts on the stress test result to learn about x_h changes with $\hat{s}_j$. When $\Sigma_{x,12} = 0$ we have

$$\operatorname{sgn}\left(\frac{\partial K_h}{\partial \hat{s}_j}\hat{s}_h \Sigma_{x,hh}\right) = -\operatorname{sgn}\left(\hat{s}_j\right). \tag{14}$$

As Equation (14) shows, making the scenario more extreme in the dimension of one risk factor reduces the amount of information that the stress test result contains about the bank's exposure to the other risk factor $h \neq j$. When $\Sigma_{x,12} = 0$, learning about the risk exposure of the bank to one risk factor contains no information about the bank's exposure to the other risk factor. Therefore, increasing the amount that can be learned in one dimension by making the scenario more extreme in this dimension decreases the informational gain in the other dimension.

The indirect effect of changing $\hat{s}_j$ on the amount that can be learned from the stress test depends on how the scenario affects the precision of the bank's report. A lower precision of the bank's report, i.e. a higher $\mathbf{\Sigma_e}\left(\hat{S}\right)$, always decreases the amount of information that the stress test contains about the bank's risk exposures. Formally,

$$\frac{\partial K_l}{\partial \mathbf{\Sigma_e}\left(\hat{S}\right)} \leq 0. \tag{15}$$

Moreover, given our assumption in Equation (5),

$$\frac{\partial \mathbf{\Sigma_e}\left(\hat{S}\right)}{\partial \left|\hat{s}_j\right|} > 0. \tag{16}$$

The expression for $\mathbf{\Sigma_e}\left(\hat{S}\right)$ in Equation (5) captures the idea that more extreme scenarios are harder for the bank to predict accurately. Hence, though increasing $|\hat{s}_j|$ makes the stress test results relatively more informative about the bank's exposure to factor j, it also decreases the total amount of information available by reducing the overall precision of the stress test result.

Using Equations (14), (15), and (16), the total effect of $\hat{s}_j$ on the amount amount of information about the exposure to risk factor i contained in the stress test result is

$$\operatorname{sgn}\left(\frac{d\Sigma_{\hat{x},hh}}{d\left|\hat{s}_j\right|}\right) < 0.$$

When the regulator has a prior belief that the risk exposures of the bank are uncorrelated, she faces a trade-off between learning about one factor or the other. However, when the regulator's prior is such that $\Sigma_{x,12} \neq 0$, this result may be overturned and putting more weight on $\hat{s}_j$ in the scenario can directly increase the weights K_1 and K_2 the regulator puts on the stress test result to update her beliefs on both risk exposures.

3.5 Optimal stress scenario design

Running stress tests and processing information is costly for the banks and for the regulator. The cost for the banks is captured by the precision of the bank's reports being decreasing in the number of scenarios. We can also incorporate a cost of creating additional scenarios for the regulator: choosing M scenarios for the stress test has a cost $\mathscr{C}(M)$, where $\mathscr{C}'(\cdot) > 0$ and $\mathscr{C}''(\cdot) \geq 0$. When designing a stress test, the regulator has to choose how many and which scenarios to include. The regulator's problem can be written as

$$\max_{M \in \mathbb{N} \cup \{0\}} V(M) - \mathscr{C}(M),$$

where

$$V(M) = \max_{\{s_m\}_{m=1}^{M}} \mathbb{E}\left[U\left(W\left(a^\star\left(\mathcal{I}_M\right)\right)\right) - \frac{1}{2}\left\|\boldsymbol{\phi} \circ a^\star\left(\mathcal{I}_M\right)\right\|^2\right] \tag{17}$$

and

$$V(0) = \mathbb{E}\left[U\left(W\left(a^\star\left(\mathcal{I}_0\right)\right)\right) - \frac{1}{2}\left\|\boldsymbol{\phi} \circ a^\star\left(\mathcal{I}_0\right)\right\|^2\right],$$

where $a^\star(\mathcal{I})$ is the optimal intervention policy of the regulator given her information set $\mathcal{I}$ given by Eq. (3), $\mathcal{I}_0$ is the information set of the regulator if she does not run a stress test, and

$$W(a) = \sum_{i=1}^{N}\left(\omega_i - \eta_i - \sum_{j=1}^{J}\left(1 - a_{ij}\right) x_{ij} \cdot s_j\right). \tag{18}$$

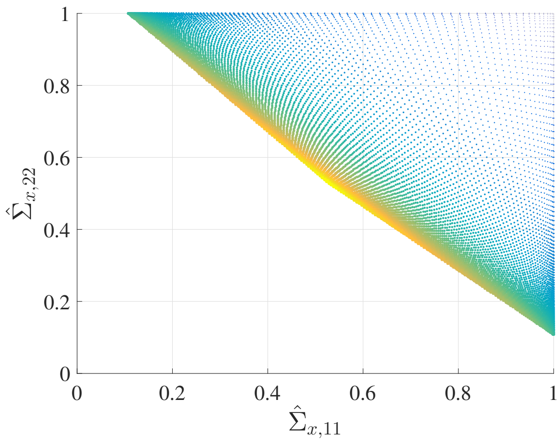
The information set on which the regulator conditions her intervention policy depends on her scenario choice. The choice of scenario affects the actual choice of policy $a^\star$ and the ex-ante distribution of this policy. However, since the regulator is choosing $a^\star$ optimally, the envelope theorem implies we can ignore the direct effect the scenario choice has on $a^\star$ and focus on the effect on its ex-ante distribution. To fix ideas, we explore this dependence in an application in the next section.

Scenario choice as information precision choice The regulator's problem in Equation (17) only depends on the scenario $\hat{s}$ indirectly through the distribution of $\hat{y}$. Given the Gaussian structure of the random variables and the linearity of the signals, this distribution only depends on the posterior precision of the exposures, $\hat{\Sigma}_x$. Therefore, choosing a scenario $\hat{s}$ is equivalent to choosing a posterior covariance matrix $\hat{\Sigma}_x$. Each scenario $\hat{s}$ determines the structure of the signals about the banks' risk exposures contained in the stress test results $\hat{y}$. This signal structure determines the amount of information that can be extracted from the stress test results and, hence, the amount of residual uncertainty that the regulator faces after observing $\hat{y}$, which is measured by $\hat{\Sigma}_x$.

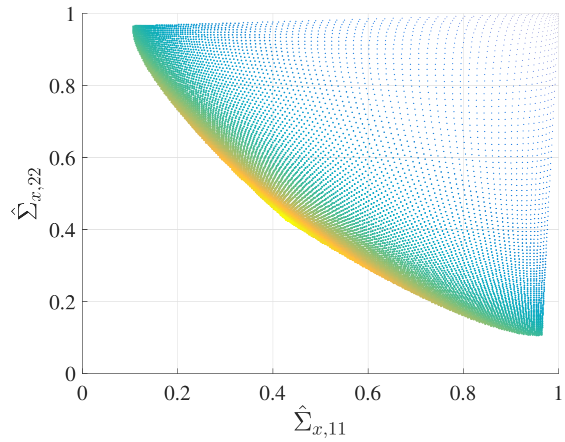
Therefore, the regulator's problem in Equation (17) can be rewritten in terms of choosing a posterior covariance matrix $\hat{\Sigma}_x \in \mathbf{\Sigma}$. The set $\mathbf{\Sigma}$ from which the regulator can choose maps a set of posterior precisions $\hat{\Sigma}_x$ for each feasible set of stress test scenarios $\hat{s}$ and it is given by Equation (9). Note that $\mathbf{\Sigma}$ is a closed and convex set. Moreover, the set $\mathbf{\Sigma}$ is the set of feasible posterior variances and it plays the role of a budget set in models in which agents choose the precision of their signals under information processing constraints. Figures (1) shows the feasible set of posterior precisions $\left\{\hat{\Sigma}_{x,jj}\right\}_{j=1,2}$ for the case in which there is one representative bank and two risk factors for different values of prior correlations among risk factors.

Choosing a more extreme scenario has two effects on the amount of information that the regulator can acquire. On one hand, a higher value of $\hat{s}_i$ increases the weight the bank's stress test results put on the bank's exposure to factor i. On the other hand, more extreme scenarios are harder to predict and the stress test results become more noisy, i.e., the variance of the error term $\mathbf{\Sigma}_{\mathbf{e}}\left(\hat{S}\right)$ is larger. For scenarios that are close to the benchmark, small values of $(\hat{s}_1, \hat{s}_2)$, the first effect dominates and making the scenario more extreme translates into lower posterior variances (acquiring more information). For more extreme scenarios, the second effect dominates and moving away form the benchmark reduces the amount of information the regulator can get from the stress test. These countervailing effects limit how much the regulator can learn form the stress test, as can be seen in Figures 1.

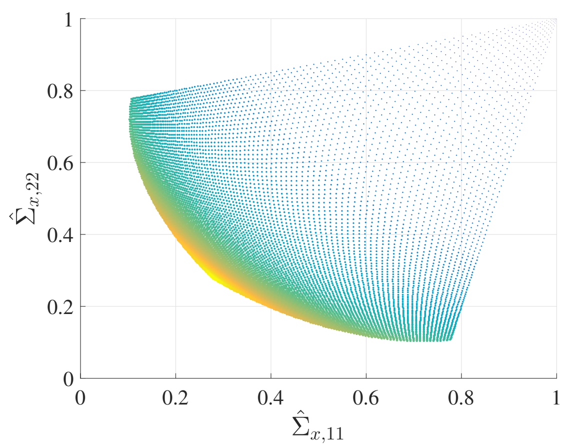
The prior correlation between the bank's risks exposures also determines the amount of information that the regulator can acquire from the stress test. When the regulator's prior is such that risk exposures are correlated, the cost of increasing the value of $\hat{s}_i$ in terms of decreasing the information about risk exposure j is lower. Hence, the regulator can learn more from the stress tests and reduce the posterior variances $\left\{\hat{\Sigma}_{x,jj}\right\}_j$. At the same time, the regulator cannot learn about the bank's exposure to factor 1 without learning about the bank's exposure to factor 2. Hence, as it can be seen from panels a), b), c) and d) Figures 1, the boundary of set of feasible posterior precisions, $\mathbf{\Sigma}$, becomes more convex.



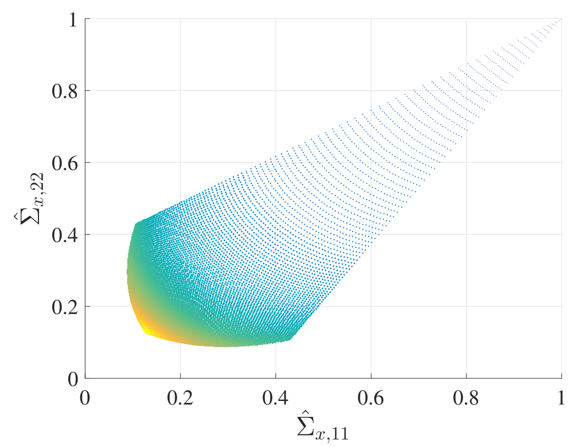
(a) Prior correlation in exposures $\Sigma_{x,12} = 0$



(b) Prior correlation in exposures $\Sigma_{x,12} = 0.2$



(c) Prior correlation in exposures $\Sigma_{x,12} = 0.5$



(d) Prior correlation in exposures $\Sigma_{x,12} = 0.8$

Figure 1: Feasible set of posterior variances, $\left(\hat{\Sigma}_{x,11}, \hat{\Sigma}_{x,22}\right)$ when there are two factors and one representative bank for different values of the regulator's prior correlation among the bank's risk exposures to factors 1 and 2.

Note: Figures 1 illustrates the set of feasible posterior variances, $\mathbf{\Sigma}$ for different values of prior correlations among risk exposures. The parameters used are $M = 1$, $N = 1$, $J = 2$, $\lambda(M) = 0.05M$, $\alpha_0 = [1, 1]$, $\alpha_1 = 5$,$\sigma_\varepsilon^2 = 1$, $\Sigma_{x,11} = \Sigma_{x,22} = 1$.

4 Linear quadratic preferences

We consider the case in which the regulator has linear quadratic preferences over the aggregate net worth of the banking sector. In this case, the regulator's utility is given by

$$U\left(W\right)=W-\frac{\gamma}{2}W^{2}.$$

We normalize our model in such a way that the factors are orthogonal, mean zero and unit variance: $\mathbb{E}\left[s\right]=0$ and $\mathbb{V}\left[s\right]=I_J$, where $\mathbb{V}$ denotes the variance-covariance matrix and I_J is the identity matrix of size J. We start by looking at the regulator's optimal intervention choice. Then, we analyze the optimal scenario choice. To do this, we first focus on how the scenario choice affects the structure of the information available to the regulator and then proceed to illustrate how the optimal scenario choice depends on the primitives of the model. To simplify the analysis, we assume that the cost function $\mathscr{C}\left(\cdot\right)$ is such that it is too costly for the regulator to choose more than one scenario.

4.1 Intervention policy

When there are N banks and J factors the banking sector's wealth is given by $W=\bar{\omega}-\bar{\eta}-\sum_{i=1}^{N}\sum_{j=1}^{J}\left(1-a_{i,j}^{\star}\left(\hat{y}\right)\right)x_{i,j}s_j$. With linear quadratic preferences, the first order condition that characterizes the optimal action choice in Equation (3) becomes

$$\phi_j a_{i,j}^{\star}\left(\hat{y}\right)=\mathbb{E}\left[x_{i,j}s_j\left(1+\gamma\sum_{n=1}^{N}\sum_{h=1}^{J}\left(1-a_{n,h}^{\star}\left(\hat{y}\right)\right)x_{n,h}s_h\right)\middle|\,\hat{y}\right]\quad\forall i=1,...,N,\forall j=1,...,J. \tag{19}$$

Moreover, given our normalization of the risk factors, the intervention policy problem is separable in the dimension of the risk factor. More specifically, the regulator chooses her intervention along dimension j for all banks, $a_j^{\star}\equiv\left(a_{1,j}^{\star}\left(\hat{y}\right),a_{2,j}^{\star}\left(\hat{y}\right),...,a_{N,j}^{\star}\left(\hat{y}\right)\right)$, independently of her intervention on all other dimensions. In this case, the optimal intervention policy in Equation (19) can be written as

$$a_j^{\star}=\left(\phi_j\mathbb{I}_N+\gamma\mathbb{E}\left[x_jx_j'\middle|\,\hat{y}\right]\mathbb{E}\left[s_j^2\right]\right)^{-1}\gamma\mathbb{E}\left[x_jx_j'\middle|\,\hat{y}\right]\mathbf{1}_{N\times1}\mathbb{E}\left[s_j^2\right]\quad\forall j=1,\ldots,J. \tag{20}$$

Equation (20) shows that the optimal intervention policy of the regulator depends on the stress test only through the stress test's informational content. The stress test results affect the regulator's behavior by changing her perceived risk in the economy, which is measured by

$\left\{\gamma\mathbb{E}\left[x_j x_j' \middle| \hat{y}\right]\mathbb{E}\left[s_j^2\right]\right\}_j$. Note that $\mathbb{E}\left[x_j x_j' \middle| \hat{y}\right] = \hat{\Sigma}_{x,j} + \hat{x}_j\hat{x}_j'$. Hence, the optimal intervention along dimension j depends on the amount of information revealed by the stress test about the exposure of the banks to risk factor j, measured by $\hat{\Sigma}_{x,j}$, and on the stress test results through the posterior mean of this risk exposures $\hat{x}_j$. This is consistent with the regulator designing the stress test to extract information from the banks. Moreover, since the regulator's intervention problem is separable in the dimension of the risk factor, the regulator only cares about $\hat{\Sigma}_{x,jj} \equiv \left\{\hat{\Sigma}_{x,jj}^{in}\right\}_{i,n} \forall j \in J$ when she chooses her intervention policy.

Proposition 2. *The regulator intervenes more in bank i along dimension j when the perceived risks coming from bank i being exposed to risk factor j, $\left\{\gamma\mathbb{E}\left[x_{i,j}x_{n,j}\middle| \hat{y}\right]\mathbb{E}\left[s_j^2\right]\right\}_n$, is higher.*

As proposition 2 shows, a risk averse regulator will intervene more to reduce risk along dimension j for bank i the higher her perceived risks associated with bank i being exposed to risk factor j. This risks have four components: risk aversion γ; the variance of risk factor j, $\mathbb{E}\left[s_j^2\right]$; the expected exposures of the banks to risk factor j, $\{\hat{x}_{n,j}\}_n$; and the precision with which the regulator estimates these exposures, $\hat{\Sigma}_{x,jj}$. Intuitively, a higher γ implies that the regulator dislikes risk more and, therefore, prefers the banking sector to be exposed to less risk overall. Therefore, a more risk averse regulator intervenes more in all dimensions, in particular she intervenes more in bank i along dimension j. Similarly, a higher uncertainty about the net worth of the banking sector coming from bank i being exposed to factor j, measured by $\sum_{n=1}^{N}\mathbb{E}\left[x_{i,j}x_{n,j}\middle| \hat{y}\right]\mathbb{E}\left[s_j^2\right]$, also induces the regulator to intervene more. This uncertainty can be driven by the volatility of the macroeconomic risk factor j, $\mathbb{E}\left[s_j^2\right]$, by imprecise estimates of the banks' risk exposures,$\hat{\Sigma}_{x,jj}$, or by high expected exposure to the risk factor j, high $\hat{x}_{i,j}$.

4.2 Scenario choice

Suppose that the regulator optimally chooses one scenario for the stress test. In this case, the regulator chooses a scenario $\hat{s} = [\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_J]'$ to solve

$$\max_{\hat{s}\in\bar{\mathbb{R}}^J} \mathbb{E}_{\hat{y},\eta}\left[W - \frac{\gamma}{2}W^2 - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{J}\phi_j\left(a_{i,j}^{\star}(\hat{y})\right)^2\right], \tag{21}$$

which is the same as

$$\min_{\hat{s}\in\bar{\mathbb{R}}^J} \mathbb{E}_{\hat{y}}\left[\gamma\left(\sum_{j=1}^{J}\left(\mathbf{1}_{1\times N} - a_j^{\star}(\hat{y})'\right)\mathbb{E}\left[x_j x_j'\middle| \hat{y}\right]\left(\mathbf{1}_{1\times N} - a_j^{\star}(\hat{y})'\right)'\mathbb{E}\left[s_j^2\right]\right)^2 + \sum_{j=1}^{J}\phi_j a_j^{\star}(\hat{y})' a_j^{\star}(\hat{y})\right],$$

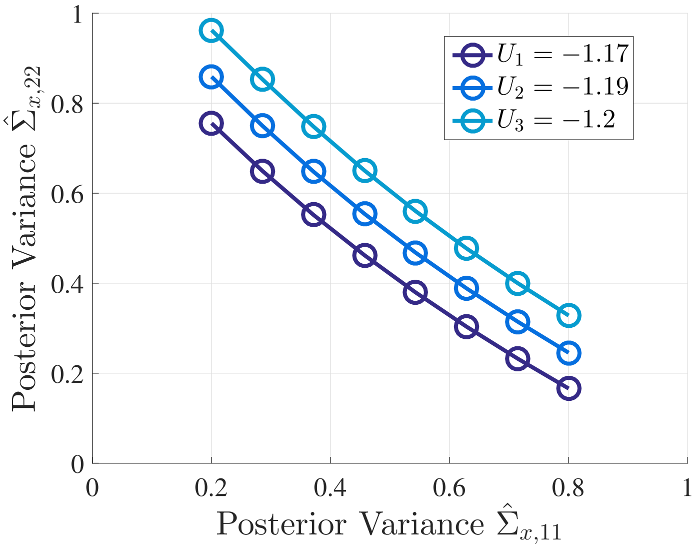


Figure 2: Indifference curves of the regulator with linear quadratic preferences

Note: Figures 2 illustrates the regulator's indifference curves as a function of the posterior variances, $\left\{\Sigma_{x,jj}\right\}_j$. The parameters used are $N = 1$, $J = 2$, $\gamma = 1$, and $\phi_1 = \phi_2 = 1$.

where $a_j^\star(\hat{y})$ is given by Equation (20). Although $\hat{s}$ does not appear explicitly in Equation (21) it determines the distribution of $\hat{x}$ and, as can be seen from Equation (19), the optimal intervention policy choice $a_j^\star(\hat{y})$. Given our assumption about the orthogonality of the risk factors, the problem is separable in the risk exposures to factors 1 and 2 and the problem can be rewritten in terms of the precisions with which the exposures to these risk factors are estimated. The feasible set of posterior precisions over which the regulator can choose, $\mathbf{\Sigma}$, is given by the Kalman filter formulas in Equations (8) and (9). The regulator's utility can be written as a function the posterior precisions $\left\{\hat{\Sigma}_{x,jj}\right\}_j$. Figure (2) illustrates the indifference curves of a regulator when there is one representative bank and two factors. Intuitively, a risk-averse regulator has higher utility the lower the posterior uncertainty about the bank's risk exposures.

Learning and ex-post interventions There are two ways in which the regulator can reduce the risk she faces: learning about the risk exposures and intervening to force the banks to reduce their exposures to the risk factors. Whether these tools are compliments or substitutes depends on the cost of intervention relative to the regulators risk aversion. When the cost of intervening along dimension j is too convex relative to the regulators risk aversion, the regulator prefers to learn about all risk exposures and chooses small interventions along all dimensions. In this case, learning and intervening are complements and the optimal scenario stresses all risk factors. When the cost of intervention is less convex, it is relatively cheap to have large interventions. In this case, learning and intervening ex-post are substitutes. The regulator chooses to intervene more in those dimensions about which the stress test provides less information. As the next section illustrates, whether learning and intervening ex-post are complements or substitutes depends on the parameters of the model.

5 Optimal scenario design

The regulator cares about the stress test result because it contains information about the banks' exposures to the risk factors. As we discussed in the previous section, the regulator's choice of scenario determines the signals that she has available to choose her intervention policy. Hence, it is natural that the regulator's problem can be cast in terms of choosing the precision of the information available to her. Formulating the regulator's problem in this way unfolds the problem in two. First, we can focus on the information acquisition aspect embedded in the stress scenario design. Once one understands the choice of information, we can use the insights of Section 3.4 to understand the optimal scenario choice.

The regulator can choose scenarios that stress only one factor or scenarios that stress multiple factors. She can choose extreme scenarios or scenarios closer to the benchmark. The optimal choice of scenario for the regulator, which determines the amount and type of learning from the stress test that can be attained by the regulator, depends on the intervention costs to reduce the banks' exposure to the different factors, on the regulator's prior distribution over the banks' risk exposures, and on the distribution of the errors in the banks' reported losses. In these Section, we explore how these parameters affect the optimal scenario choice by providing numerical examples that illustrate the economic forces at play. We start by analyzing the case in which there is only one bank to focus on how the intervention costs, the prior mean of the risk exposures and the correlation among them within a bank determine the scenario choice. Then, we consider the case with two banks and concentrate on how having systemic factors impacts the stress scenario design.

5.1 One bank

When there is only one bank in the economy, the net worth of the banking system is given by $W = \bar{w} - \eta - \sum_{j=1}^{J} \left(1 - a_j^\star(\hat{y})\right) x_j s_j$. Then, the optimal intervention policy in Equation (20) becomes

$$a_j^\star(\hat{y}) = \frac{\gamma \left(\hat{\Sigma}_{x,jj} + \hat{x}_j^2\right) \mathbb{E}\left[s_j^2\right]}{\phi_j + \gamma \left(\hat{\Sigma}_{x,jj} + \hat{x}_j^2\right) \mathbb{E}\left[s_j^2\right]}. \tag{22}$$

As in the general case in Equation (20), conditional on an expected risk exposure, the regulator wants to intervene more the more uncertain she is about that risk exposure and the lower the cost of intervening along that factor's dimension. Intuitively, the regulator is willing to bear a higher exposure to factor j when the exposure to factor j is more precisely estimated or when it is too expensive to reduce the exposure to that factor es-post.

Using the optimal intervention policy, the regulator's information choice problem can be rewritten as minimizing sum of marginal intervention costs as follows

$$\min_{\left\{\hat{\Sigma}_{x,jj}\right\}_j \in \mathbf{\Sigma}} \sum_{j=1}^{J} \phi_j \mathbb{E}_{\hat{y}} \left[a_j^\star(\hat{y})\right].$$

Since ex-post interventions are costly for the regulator, she chooses posterior variances to minimize the total amount of expected interventions weighted by the average marginal cost of intervening. Even though the objective function is separable in the dimension of the factors, the precision choices along the factor dimensions are not independent of each other. The precision budget set $\mathbf{\Sigma}$ restricts the amount of information about the exposures to different factors that the regulator can choose. For example, when the regulator's prior is that risk exposures to different factors are uncorrelated, increasing the precision of the stress test results as a signal of the exposure to one factor comes at the expense of receiving less precise information about all other factors to which the bank is exposed.

From Equation (22) we know that the optimal intervention to reduce the exposure to factor j, $a_j^\star(\hat{y})$, is increasing in the uncertainty about the bank's exposure to factor j,$\hat{\Sigma}_{x,jj}$, and in the bank's expected exposure to that factor,$\hat{x}_j$, and that

$$\hat{x}_j = \left(\Sigma_{x,jj} - \hat{\Sigma}_{x,jj}\right)^{\frac{1}{2}} z + \bar{x}_j$$

where $z \sim N(0,1)$. Then, the derivative of the expected intervention in dimension j with respect to $\hat{\Sigma}_{x,jj}$ is

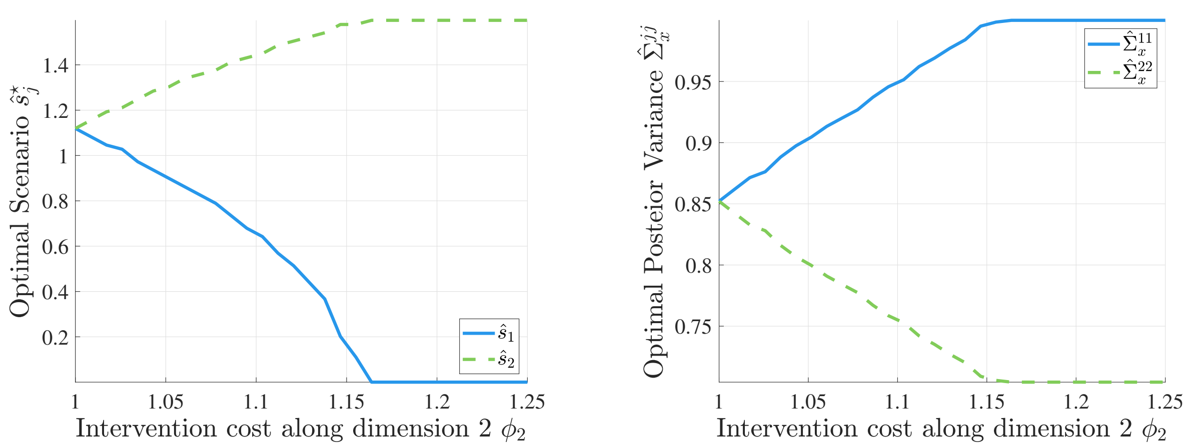
$$\frac{d\mathbb{E}\left[a_j^{\star}\left(\hat{y}\right)\right]}{d\hat{\Sigma}_{x,jj}}=\mathbb{E}_{\hat{y}}\left[\frac{\partial a_j^{\star}\left(\hat{y}\right)}{\partial\left(\hat{\Sigma}_{x,jj}+\hat{x}_j^2\right)\mathbb{E}\left[s_j\right]^2}\frac{\partial\left(\hat{\Sigma}_{x,jj}+\hat{x}_j^2\right)}{\partial\hat{\Sigma}_{x,jj}}\mathbb{E}\left[s_j\right]^2\right]. \tag{23}$$

Equation (23) represents the change in the expected intervention by the regulator along dimension j as the posterior precision changes. When the uncertainty about the bank's net worth coming from factor j, measured by $\left(\hat{\Sigma}_{x,jj}+\hat{x}_j^2\right)\mathbb{E}\left[s_j\right]^2$, increases, the regulator intervenes more along dimension j. The second term in Equation (23) represents the change in this uncertainty as the residual uncertainty about the bank's exposure to risk factor j changes. Since these two terms are positive, learning about the exposure to factor j and intervening the reduce the bank's exposure to that factor are substitute instruments to reduce the overall exposure to risk. Intuitively, when the cost to intervene along dimension j increases, learning becomes relatively cheaper and the regulator stresses that dimension more. The increases precision of the regulator's information along dimension j implies that she can intervene more accurately when it is more expensive to do so and it reduces the overall expected costs of intervention.

The regulator does not only need to decide whether to learn or intervene along dimension j, but she also needs to allocate her resources across the different factor dimensions. Should the regulator learn only about the bank's exposure to one factor? Should she learn about multiple factors? How much weight should the regulator put in each factor when designing the stress scenario? In the remainder of this section we provide numerical examples to illustrate how the relative intervention cost and the regulator's prior beliefs about the bank's risk exposures determine her optimal scenario choice. For most exercises we focus on the two factor case and use the following baseline parameters $\gamma=1$, $\phi_1=\phi_2=1$, $\bar{x}=[1,1]$, $\lambda\left(M\right)=M$, $M=1$, $\alpha_0=[1,1]$, $\alpha_1=3$, $\sigma_\varepsilon^2=1$, and $\Sigma_x=I_J$.

Intervention costs

Everything else equal, the regulator will choose to intervene less along dimensions with higher intervention costs. Since learning and intervening are substitute instruments to reduce risk, this implies that a higher intervention costs to reduce the bank's exposure to factor j will lead to more learning about the bank's exposure to this factor (lower posterior variance). This translates into an optimal stress scenario that stresses the factor with the higher intervention cost more. When the intervention cost along dimension j becomes high enough, the regulator finds it optimal to specialize and learn only about factor j from the stress test to minimize the expected costs of



(a) Optimal scenario choice

(b) Optimal posterior

Figure 3: Optimal scenario and information choice as a function the intervention cost to reduce the exposure to factor 2, ϕ_2.

Note: Figure 3 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of the intervention cost along the dimension of factor 2. The parameters used are $N = 1$, $J = 2$, $\gamma = 1$, $\phi_1 = 1$, $\bar{x}$=[1, 1], $\lambda(M) = M$, $M = 1$, $\alpha_0 = [1, 1]$, $\alpha_1 = 3$,$\sigma_\varepsilon^2 = 1$, and $\Sigma_x = \boldsymbol{I}_J$.

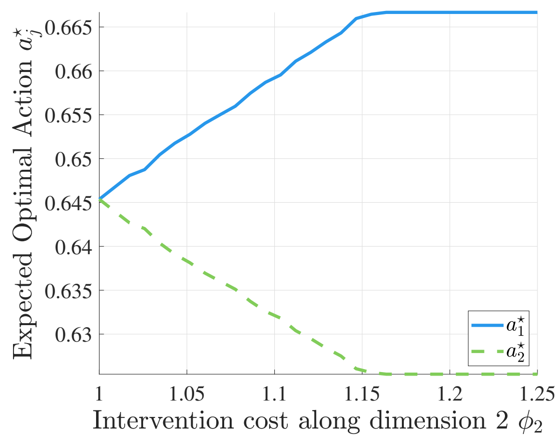


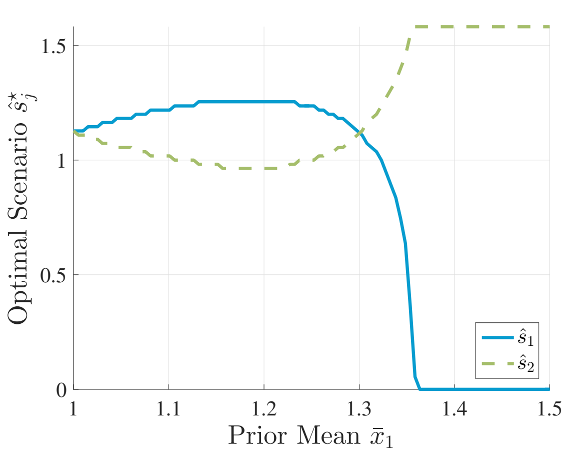
Figure 4: Optimal expected ex-post interventions as a function the intervention cost to reduce the exposure to factor 2, ϕ_2.

Note: Figure 4 illustrates the regulator's optimal expected ex-post interventions to reduce the bank's exposure to factors 1 and 2 as a function of the intervention cost along the dimension of factor 2. The parameters used are $N = 1$, $J = 2$, $\gamma = 1$, $\phi_1 = 1$, $\bar{x}$=[1, 1], $\lambda(M) = M$, $M = 1$, $\alpha_0 = [1, 1]$, $\alpha_1 = 3$,$\sigma_\varepsilon^2 = 1$, and $\Sigma_x = \boldsymbol{I}_J$.

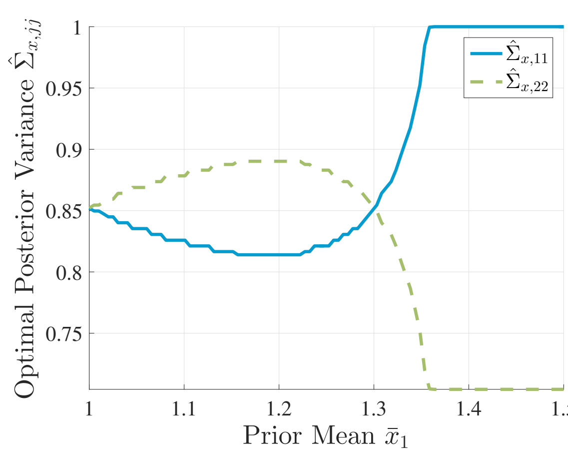
intervention. Figure (3) illustrates the optimal scenario choice and the implied posterior precision as the cost of intervening to reduce the bank's exposure to factor 2 increases.

Figure (4) shows the expected optimal intervention as a function of the intervention cost to reduce the exposure to factor 2. The expected intervention cost is lower in the dimension of factor 2 for two reasons. First, it is more costly to intervene to reduce the exposure to factor 2. Second, the regulator has more precise information about the bank's exposure to factor 2, which allows her to intervene more accurately. As the cost of intervention ϕ_2 increases, the expected intervention along dimension 2 decreases since it becomes more expensive to intervene and the regulator stresses factor 2 more in the optimal stress scenario.

Prior mean risk exposure The prior mean of the exposure of the bank to each factors affects the how much the regulator wants to learn about each risk exposure. When the prior expected risk exposure to a factor is large, the risk along dimension 1 becomes more important and the regulator wants to learn more about the bank's exposure which he expects to be bigger. When risk exposures are uncorrelated, this is achieved by increasing factor 1 in the stress test scenario and decreasing factor 2. When the expected risk exposure is high enough, the regulator finds it optimal to set factor 2 to zero in the stress test and learn only about factor 1. This does not imply



(a) Optimal scenario choice



(b) Optimal posterior variances

Figure 5: Optimal scenario and information choice as a function of the prior mean $\bar{x}_1$.

Note: Figure 5 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of the prior mean of the risk exposure to factor 1. The parameters used are $N = 1$, $J = 2$, $\gamma = 1$, $\phi_1 = \phi_2 = 1$, $\bar{x}$=[1, 1], $\lambda(M) = M$, $M = 1$, $\alpha_0 = [1, 1]$, $\alpha_1 = 3$,$\sigma_\varepsilon^2 = 1$, and $\Sigma_x = \boldsymbol{I}_N$.

that the regulator stops caring about the bank's exposure to risk factor 2. However, the regulator will choose to reduce the risk generated by the bank's exposure to factor 2 by intervening more heavily ex-post and not learning about it at all. Panel (a) in Figure 5 shows the weights of risk factors 1 and 2 in the optimal scenario as $\bar{x}_1$ changes. Panel (b) in the same figure shows the posterior precisions for the bank's risk exposures as a function of the expected risk exposure to factor 1.

Prior correlation in risk exposures within banks The prior correlation among risk exposures plays a crucial role in determining how much the regulator can learn from a stress test, as it can be seen from the feasible set of posterior precisions, $\boldsymbol{\Sigma}$, in Figures (1). When the prior correlation among risk exposures is zero, the regulator faces a trade-off between learning about one risk exposure or the other. Scenarios that provide a lot of information about the bank's risk exposure to factor 1 contain very little information about the bank's risk exposure to factors 2 and 3. For example, an extreme scenario that puts all the weight on risk factor 1 and no weight on the other risk factors contains no information about the bank's risk exposure to factors 2 and

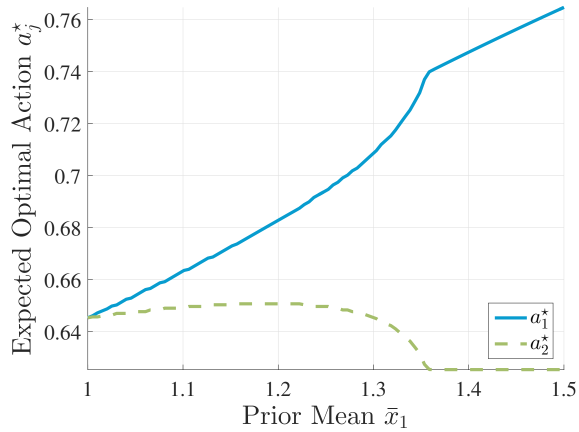


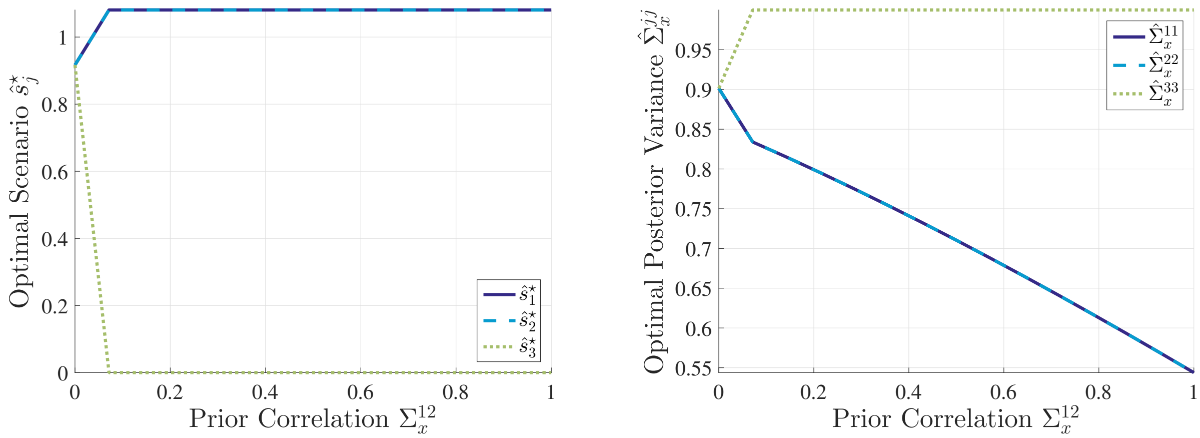
Figure 6: Optimal expected ex-post interventions as a function of the prior mean $\bar{x}_1$.

Note: Figure 5 illustrates the regulator's the regulator's optimal expected ex-post interventions to reduce the bank's exposure to factors 1 and 2 as a function of the prior mean exposure to factor 1.The parameters used are $N = 1$, $J = 2$, $\gamma = 1$, $\phi_1 = \phi_2 = 1$, $\bar{x}$=$[1,1]$, $\lambda(M) = M$, $M = 1$, $\alpha_0 = [1,1]$, $\alpha_1 = 3$,$\sigma_\varepsilon^2 = 1$, and $\Sigma_x = \boldsymbol{I}_J$.

3 at all. When the correlation between risk exposures is non-zero, this trade-off is attenuated as signals about one risk exposure always contain some information about the other.

Panel (a) in Figure 7 plots the weight on risk factor 1 in an optimal scenario as a function of the regulator's prior correlation among risk exposures. As one can see in this figure, the correlation in risk exposures does not affect the scenario choice. However, as panel (b) shows, the amount of information that the regulator can get from the same scenario changes as the prior correlation in risk exposures varies. As bank's risk exposures become more correlated in the bank's prior, the more informative the stress test.

When there are more than two factors, the scenario choice will be affected by the prior correlation among the bank's risk exposures to factors 1 and 2. As this correlation increases, the regulator will put more weight on the factors to which the bank's exposures are correlated and decrease the weight of the remaining factor. When the exposures of the bank to factors 1 and 2 are very correlated, the regulator finds it optimal to learn only about these two risk exposures and chooses not to deviate from the benchmark for factor 3.



(a) Optimal scenario choice

(b) Optimal posterior variance

Figure 7: Optimal scenario and information choice as a function of the prior correlation in risk exposures, $\Sigma_{x,12}$.

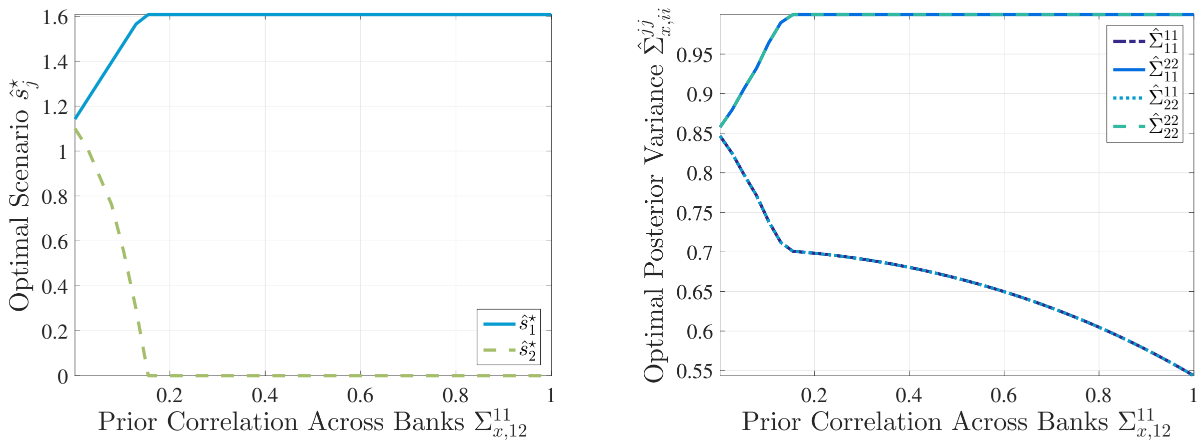
Note: Figure 7 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of prior correlation in risk exposures. The parameters used are $N = 1$, $J = 3$, $\gamma = 1$, $\phi_1 = \phi_2 = \phi_3 = 1$,$\bar{x}$=$[1, 1]$, $\lambda(M) = M$, $M = 1$, $\alpha_0 = [1, 1]$, $\alpha_1 = 3$,$\sigma_\varepsilon^2 = 1$, and $\Sigma_x = \boldsymbol{I}_J$.

5.2 Two banks

When there are multiple (types of) banks in the economy, the regulator's beliefs about the risk exposures within and across banks determine the optimal choice of stress scenario. Consider the case in which there are two banks and two risk factors. In this case, the banking sector's wealth is given by $W = \bar{\omega} - \bar{\eta} - \sum_{i=1}^2 \left(\sum_{j=1}^2 \left(1 - a_{i,j}^\star(\hat{y})\right) x_{i,j} s_j\right)$. With linear quadratic preferences, the system that characterizes optimal action choices in Equation (20) becomes

$$a_{i,j}^\star(\hat{y}) = \frac{\gamma \mathbb{E}\left[s_j^2\right]\left(\phi_j \sum_{n=1}^2 \mathbb{E}\left[x_{i,j} x_{n,j} \middle| \hat{y}\right] + \gamma \mathbb{E}\left[s_j^2\right]\left(\mathbb{E}\left[x_{i,j}^2 \middle| \hat{y}\right] \mathbb{E}\left[x_{z,j}^2 \middle| \hat{y}\right] - \left(\mathbb{E}\left[x_{i,j} x_{z,j} \middle| \hat{y}\right]\right)^2\right)\right)}{\phi_j^2 + \gamma \mathbb{E}\left[s_j^2\right]\left(\phi_j \sum_{n=1}^2 \mathbb{E}\left[x_{n,j}^2 \middle| \hat{y}\right] + \gamma \mathbb{E}\left[s_j^2\right]\left(\mathbb{E}\left[x_{i,j}^2 \middle| \hat{y}\right] \mathbb{E}\left[x_{z,j}^2 \middle| \hat{y}\right] - \left(\mathbb{E}\left[x_{i,j} x_{z,j} \middle| \hat{y}\right]\right)^2\right)\right)} \quad \forall i, j = 1, 2. \tag{24}$$

As in the one bank case, the optimal ex-post intervention for bank i along the dimension of factor j depends on the intervention cost ϕ_j and on the uncertainty about bank i's exposure to factor j. However, when there are multiple banks, the optimal intervention $a_{i,j}^\star$ also depends on how the posterior covariance of exposures to factor j across banks, as it can be seen from Equation (24). This posterior covariance in turn depends on the regulator's prior beliefs about the banks' risk exposures and, in turn, on the amount of scenario choice. In the remainder of this section, we provide numerical examples to understand how the regulator's beliefs affect the optimal stress scenario design when there are multiple banks. We use the following baseline parameters. $N = 2$,



(a) Optimal scenario choice

(b) Optimal posterior variance

Figure 8: Optimal scenario and information choice as a function of the prior correlation between the bank's risk exposures to factor 1, $\Sigma^{11}_{x,12}$.

Note: Figure 8 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of prior correlation in risk exposures. The parameters used are $N = 2$, $J = 2$, $\gamma = 1$, $\phi_1 = \phi_2 = 1$, $\bar{x}$=[1, 1], $\lambda(M) = M$, $M = 1$, $\alpha_0 = [1, 1]$, $\alpha_1 = 3$,$\sigma^2_\varepsilon = 1$, and $\Sigma_x = \boldsymbol{I}_{NJ}$.

$J = 2$, $\gamma = 1$, $\phi_1 = \phi_2 = 1$, $\bar{x} = [1, 1]$, $\lambda(M) = M$, $M = 1$, $\alpha_0 = [1, 1]$, $\alpha_1 = 3$, $\sigma^2_\varepsilon = 1$, and $\Sigma_x = I_{NJ}$.

Correlated risk exposures across banks When there are multiple banks, the exposures to the risk factors are likely to be correlated among banks, even when the factors are orthogonal. Figure 8 shows the optimal scenario weights as a function of the correlation between the exposure to risk factor 1 for banks 1 and 2, when there are two banks, two risk factors and one scenario.

When the exposures to factor 1 are correlated across banks, the stress test results of bank i contain information about bank i's exposures to both risk factors and about bank j's exposure to factor 1, as long as the weight of factor 1 in the stress test's scenario is non-zero. Figure 8 shows that, in this case, learning about factor 1 becomes more valuable for the regulator to the point that the stress scenario only puts weight on factor 1 when the risk exposures of the banks to factor 1 are very correlated. In this case, the regulator does not learn about the risk exposures to factor 2 and only chooses to intervene ex-post to reduce the losses associated with this factor. The posterior variance of the risk exposures to factor 2 remain constant and equal to the prior. On the other hand, the posterior variance of the exposure of the banks to factor 1 decreases mechanically with $\Sigma^{12}_{x,11}$ since the the stress test of bank j becomes more informative of bank i's exposure to factor 1 as this correlation increases.

6 Conclusion

Despite the grow-in importance of stress testing for financial regulation, economists still lack a theory of the design of stress scenarios. We model stress testing as a learning mechanism and we derive optimal scenarios. We show how the design of these optimal scenarios depends on the cost of interventions, the prior beliefs of the regulator, the precision of regulatory information, and the presence of systemic risk factors.

Our approach is consistent with the general principles of current policies implement in various jurisdiction, but it has the advantage that our optimal scenarios are not arbitrary. For example, the current policy on stress scenario design in the U.S. allows for the stress scenarios to "follow either a recession approach, a probabilistic approach, or an approach based on historical experiences."[5] These concepts are somewhat vague and have generated much discussion among banks and regulators. Some commentators argue that scenarios should be predictable while others advocate a flexible design to accommodate emerging risks and changing exposures. Our learning approach shows how to incorporate this goals in the design of the stress scenarios.

[5] See 12 CFR Part 252 Appendix A.

References

Acharya, V., R. Engle, and D. Pierret (2014). Testing macroprudential stress tests: The risk of regulatory risk weights. *Journal of Monetary Economics 65*, 36 – 53. Carnegie-Rochester-NYU Conference Series on Public Policy "A Century of Money, Banking, and Financial Instability" held at the Tepper School of Business, Carnegie Mellon University on November 15-16, 2013.

Acharya, V. V., L. H. Pedersen, T. Philippon, and M. Richardson (2016). Measuring systemic risk. *The Review of Financial Studies*.

Allen, F. and D. Gale (2000). Financial contagion. *Journal of Political Economy 108*(1), 1–33.

Bird, A., S. Karolyi, T. Ruchti, and A. Sudbury (2015). Bank regulator bias and the efficacy of stress test disclosures. Working Paper.

Camara, B., P. Pessarossi, and T. Philippon (2016). Backtesting european stress tests. NBER Working Paper.

Candelon, B. and A. Sy (2015). How do markets react to stress tests? IMF Working Paper.

Donald, M., S. Peristiani, and V. Savino (2014). The information value of the stress test. *Journal of Money Credit and Banking 46*(7), 1479–1500.

Faria-e-Castro, M., J. Martinez, and T. Philippon (2017). Runs versus lemons: Information disclosure and fiscal capacity. *Review of Economic Studies 84(4)*, 1683–1707.

Fernandes, M., D. Igan, and M. Pinheiro (2015). March madness in wall street: (what) does the market learn from stress tests? IMF Working Paper.

Flannery, M., B. Hirtle, and A. Kovner (2016, March). Evaluating the information in the federal reserve stress tests. FRBNY Staff Report.

Gick, W. and T. Pausch (2014). Bayesian persuasion by stress test disclosure. Technical report, Deutsche Bundesbank.

Glasserman, P. and G. Tangirala (2016). Are the federal reserve's stress test results predictable? *Journal of Alternative Investments 18*(4).

Goldstein, I. and Y. Leitner (2018). Stress tests and information disclosure. *Journal of Economc Theory 177*(13-26), 34–69.

Goldstein, I. and H. Sapra (2014, March). Should banks' stress test results be disclosed? an analysis of the costs and benefits. *Foundations and Trends(R) in Finance 8*(1), 1–54.

Hirshleifer, J. (1971). The private and social value of information and the reward to inventive activity. *The American Economic Review 61*(4), pp. 561–574.

Inostroza, N. and A. Pavan (2017). Persuation in global games with an application to stess testing. Technical report, Nortwestern University working paper.

Kyle, A. S. (1989). Informed speculation with imperfect competition. *Review of Economic Studies 56*, 317–356.

Leitner, Y. and B. Williams (2018). Model secrecy and stress tests. *working paper*.

Orlov, D., P. Zryumov, and A. Skrzypacz (2017). Design of macro-prudential stress tests. *working paper*.

Petrella, G. and A. Resti (2013). Supervisors as information producers: Do stress tests reduce bank opaqueness? *Journal of Banking & Finance 37*(12), 5406–5420.

Schuermann, T. (2012). Stress testing banks. Wharton Working Paper.

Schuermann, T. (2016). Stress testing in wartime and in peacetime. Working Paper.

Shapiro, J. and D. Skeie (2015). Information management in banking crises. *Review of Financial Studies 28*(8), 2322–2363.

Van Nieuwerburgh, S. and L. Veldkamp (2010). Information acquisition and under-diversificatio. *The Review of Economic Studies 77(2)*, 779–805.

Veldkamp, L. B. A. B. A. (2009). Information Choice in Macroeconomics and Finance. *Manuscript, New York University*.

Verrecchia, R. E. (1982). Information acquisition in a noisy rational expectations economy. *Econometrica: Journal of the Econometric Society*, 1415–1430.

Vives, X. (2008). *Information and Learning in Markets: the Impact of Market Microstructure*. Princeton University Press.

Appendix

The Appendix contains some auxiliary calculation for formulas in the text. It needs to be completed.

A Learning

A.1 Proof of Proposition 1

The following state space representation holds

$$\mathbf{x} = \mathbf{x}$$
$$\hat{\mathbf{y}} = \left(\mathbf{I}_N \otimes \hat{S}\right)\mathbf{x} + \mathbf{e}$$

where $\mathbf{e} \perp \mathbf{x}$. Therefore, the standard Kalman filter formulas applied to the context of our model gives the result in the proposition.

A.2 Example

If $M = 1$ and $J = 2$, we have that for each bank i

$$\hat{x}_i = (I_2 - K_i S)\,\overline{x}_i + K_i \hat{y}_i \tag{A.1}$$

$$K_i = \begin{bmatrix} \Sigma^i_{x,1} & \Sigma^i_{x,12} \\ \Sigma^i_{x,12} & \Sigma^i_{x,2} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \left(\begin{bmatrix} s_1 & s_2 \end{bmatrix} \left(\begin{bmatrix} \Sigma^i_{x,11} & \Sigma^i_{x,12} \\ \Sigma^i_{x,12} & \Sigma^i_{x,22} \end{bmatrix} \right) \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \Sigma^i_e \right)^{-1} = \begin{bmatrix} \frac{\Sigma^i_{x,11}s_1 + \Sigma^i_{x,12}s_2}{\Sigma^i_{x,11}(s_1)^2 + 2\Sigma^i_{x,12}s_1 s_2 + \Sigma^i_{x,22}s_2^2 + \Sigma^i_e} \\ \frac{\Sigma^i_{x,12}s_1 + \Sigma^i_{x,22}s_2}{\Sigma^i_{x,11}(s_1)^2 + 2\Sigma^i_{x,12}s_1 s_2 + \Sigma^i_{x,22}s_2^2 + \Sigma^i_e} \end{bmatrix}$$

$$\hat{\Sigma}^i_x = \left(I_2 - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \end{bmatrix} \right) \begin{bmatrix} \Sigma^i_{x,11} & \Sigma^i_{x,12} \\ \Sigma^i_{x,12} & \Sigma^i_{x,22} \end{bmatrix} = \begin{bmatrix} (1 - K_1 s_1)\,\Sigma^i_{x,11} + K_1 s_2 \Sigma^i_{x,12} & (1 - K_1 s_1)\,\Sigma^i_{x,12} + K_1 s_2 \Sigma^i_{x,22} \\ K_2 s_2 \Sigma^i_{x,11} + (1 - K_2 s_2)\,\Sigma^i_{x,12} & K_2 s_2 \Sigma^i_{x,12} + (1 - K_2 s_2)\,\Sigma^i_{x,22} \end{bmatrix}.$$

Also,

$$\hat{y}_i \sim N\left(\overline{x}_1 s_1 + \overline{x}_2 s_2, \sigma^2_{\hat{y}_i}\right),$$

where $\sigma^2_{\hat{y}i} = s_1^2\left(\Sigma^i_{x,11} + \Sigma^i_{e,11}\right) + 2 s_1 s_2 \left(\Sigma^i_{x,12} + \Sigma^i_{e,12}\right) + s_2^2\left(\Sigma^i_{x,22} + \Sigma^i_{e,22}\right)$. Then, unconditionally of the realization of $\hat{y}_i$ the distribution of $\hat{x}_i$ is

$$\hat{x}_i \sim N\left(\overline{x}_i, K_i \sigma^2_{\hat{y}i} K_i^T\right).$$

Using the expressions for K_i we have that

$$\begin{aligned} \frac{\partial K_1}{\partial s_1} &= \frac{\Sigma^i_{x,11}\left(\Sigma^i_{x,11}(\hat{s}_1)^2 + 2\Sigma^i_{x,12}\hat{s}_1\hat{s}_2 + \Sigma^i_{x,22}(\hat{s}_2)^2 + \Sigma^i_e\right) - \left(\Sigma^i_{x,11}\hat{s}_1 + \Sigma^i_{x,12}\hat{s}_2\right)\left(2\Sigma^i_{x,11}\hat{s}_1 + 2\Sigma^i_{x,12}\hat{s}_2\right)}{\left(\Sigma^i_{x,11}(\hat{s}_1)^2 + 2\Sigma^i_{x,12}\hat{s}_1\hat{s}_2 + \Sigma^i_{x,22}(\hat{s}_2)^2 + \Sigma^i_e\right)^2} \\ &= \frac{-\left(\Sigma^i_{x,11}\hat{s}_1\right)^2 + \Sigma^i_{x,11}\Sigma^i_{x,22}(\hat{s}_2)^2 - 2\Sigma^i_{x,12}\hat{s}_2\left(\Sigma^i_{x,11}s_1 + \Sigma^i_{x,12}s_2\right) + \Sigma^i_{x,11}\Sigma^i_e}{\left(\Sigma^i_{x,11}(\hat{s}_1)^2 + 2\Sigma^i_{x,12}\hat{s}_1\hat{s}_2 + \Sigma^i_{x,22}(\hat{s}_2)^2 + \Sigma^i_e\right)^2} \end{aligned}$$

$$\frac{\partial K_2}{\partial s_2} = \frac{-\left(\Sigma^i_{x,22}\hat{s}_2\right)^2 + \Sigma^i_{x,22}\Sigma^i_{x,11}(\hat{s}_1)^2 - 2\Sigma^i_{x,12}\hat{s}_1\left(\Sigma^i_{x,12}\hat{s}_1 + \Sigma^i_{x,22}\hat{s}_2\right) + \Sigma^i_{x,22}\Sigma^i_e}{\left(\Sigma^i_{x,11}(\hat{s}_1)^2 + 2\Sigma^i_{x,12}\hat{s}_1\hat{s}_2 + \Sigma^i_{x,22}(\hat{s}_2)^2 + \Sigma^i_e\right)^2}$$

$$\frac{\partial K_2}{\partial s_1}=\frac{\Sigma^i_{x,12}\left(\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+2\Sigma^i_{x,12}\hat{s}_1\hat{s}_2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e\right)-\left(\Sigma^i_{x,12}\hat{s}_1+\Sigma^i_{x,22}\hat{s}_2\right)\left(2\Sigma^i_{x,11}\hat{s}_1+2\Sigma^i_{x,12}\hat{s}_2\right)}{\left(\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+2\Sigma^i_{x,12}\hat{s}_1\hat{s}_2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e\right)^2}$$

$$\frac{\partial K_1}{\partial s_2}=\frac{\Sigma^i_{x,12}\left(\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+2\Sigma^i_{x,12}\hat{s}_1\hat{s}_2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e\right)-\left(\Sigma^i_{x,11}\hat{s}_1+\Sigma^i_{x,12}\hat{s}_2\right)\left(2\Sigma^i_{x,12}\hat{s}_1+2\Sigma^i_{x,22}\hat{s}_2\right)}{\left(\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+2\Sigma^i_{x,12}\hat{s}_1\hat{s}_2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e\right)^2}$$

If the regulator has a prior such that the bank's exposures to the risk factors are uncorrelated, i.e., if $\Sigma_{x,12}=0$, we have

$$\frac{\partial K_1}{\partial s_1}=\Sigma^i_{x,11}\frac{-\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e}{\left(\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e\right)^2}$$

$$\frac{\partial K_2}{\partial s_2}=\frac{-\left(\Sigma^i_{x,22}\hat{s}_2\right)^2+\Sigma^i_{x,22}\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+\Sigma^i_{x,22}\Sigma^i_e}{\left(\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e\right)^2}$$

$$\frac{\partial K_2}{\partial s_1}=-\frac{2\Sigma^i_{x,22}\hat{s}_2\Sigma^i_{x,11}\hat{s}_1}{\left(\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e\right)^2}$$

$$\frac{\partial K_1}{\partial s_2}=-\frac{2\Sigma^i_{x,11}\hat{s}_1\Sigma^i_{x,22}\hat{s}_2}{\left(\Sigma^i_{x,11}\left(\hat{s}_1\right)^2+\Sigma^i_{x,22}\left(\hat{s}_2\right)^2+\Sigma^i_e\right)^2}$$

B Optimal Intervention policy

Under linear quadratic preferences, the first order condition that characterizes the regulator's optimal intervention policy is

$$\phi_j a^\star_{i,j}\left(\hat{y}\right)=\mathbb{E}\left[x_{i,j}s_j\left(1-\gamma W\right)\middle|\,\hat{y}\right]\quad \forall i=1,...,N,\quad \forall j=1,...,J,$$

where $W=\sum_{i=1}^N\left(\sum_{j=1}^J\left(1-a^\star_{i,j}\left(\hat{y}\right)\right)x_{i,j}s_j+\eta_i-d_i\right)$. This is the same as

$$\phi_j a^\star_{i,j}\left(\hat{y}\right)=\mathbb{E}\left[x_{i,j}s_j\left(1+\gamma\sum_{n=1}^N\sum_{h=1}^J\left(1-a^\star_{n,h}\left(\hat{y}\right)\right)x_{n,h}s_h+\gamma\sum_{i=1}^N d_i\right)\middle|\,\hat{y}\right].$$

Since $\mathbb{E}\left[s_j\right]=0$ for all j, we have

$$\phi_j a^\star_{i,j}\left(\hat{y}\right)=\gamma\sum_{n=1}^N\left(1-a^\star_{n,j}\left(\hat{y}\right)\right)\mathbb{E}\left[x_{i,j}x_{n,j}\middle|\,\hat{y}\right]\mathbb{E}\left[s_j^2\right].$$

Let $a^\star_j=\left(a^\star_{1,j},\ldots,a^\star_{N,j}\right)'$. Then, the FOCs can be written as

$$\phi_j a^\star_j=-\gamma\begin{bmatrix}\mathbb{E}\left[x_{1,j}x_{1,j}\middle|\,\hat{y}\right] & \mathbb{E}\left[x_{1,j}x_{2,j}\middle|\,\hat{y}\right] & \ldots & \mathbb{E}\left[x_{1,j}x_{N,j}\middle|\,\hat{y}\right]\\ \mathbb{E}\left[x_{2,j}x_{1,j}\middle|\,\hat{y}\right] & \ddots & \ldots & \vdots\\ \vdots & \ldots & \ldots & \vdots\\ \mathbb{E}\left[x_{N,j}x_{1,j}\middle|\,\hat{y}\right] & \ldots & \ldots & \mathbb{E}\left[x_{N,j}x_{N,j}\middle|\,\hat{y}\right]\end{bmatrix}\mathbb{E}\left[s_j^2\right]a^\star_j+\begin{bmatrix}\gamma\sum_{n=1}^N\mathbb{E}\left[x_{1,j}x_{n,j}\middle|\,\hat{y}\right]\mathbb{E}\left[s_j^2\right]\\ \gamma\sum_{n=1}^N\mathbb{E}\left[x_{2,j}x_{n,j}\middle|\,\hat{y}\right]\mathbb{E}\left[s_j^2\right]\\ \vdots\\ \gamma\sum_{n=1}^N\mathbb{E}\left[x_{N,j}x_{n,j}\middle|\,\hat{y}\right]\mathbb{E}\left[s_j^2\right]\end{bmatrix},$$

which is the same as

$$\phi_j a_j^\star = -\left(\hat{\Sigma}_{x,j} + \hat{x}_j\hat{x}_j'\right)\gamma\mathbb{E}\left[s_j^2\right]a_j^\star + \left(\hat{\Sigma}_{x,j} + \hat{x}_j\hat{x}_j'\right)\mathbf{1}_{N\times 1}\gamma\mathbb{E}\left[s_j^2\right],$$

where $\hat{x}_j = (\hat{x}_{i,j}, \ldots, \hat{x}_{N,j})'$. Then,

$$a_j^\star = \left(\phi_j\mathbb{I}_N + \gamma\left(\hat{\Sigma}_{x,j} + \hat{x}_j\hat{x}_j'\right)\mathbb{E}\left[s_j^2\right]\right)^{-1}\gamma\left(\hat{\Sigma}_{x,j} + \hat{x}_j\hat{x}_j'\right)\mathbf{1}_{N\times 1}\mathbb{E}\left[s_j^2\right] \quad \forall j = 1, \ldots, J \tag{A.2}$$

B.1 Proof of Proposition 2

It follows directly from the system in Equation (A.2).

B.2 One bank

When there is only one bank, the optimal intervention policy in Equation (20) becomes

$$\begin{bmatrix} \phi_1 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0\phi_J \end{bmatrix} a^\star(\hat{y}) = -\gamma \begin{bmatrix} \mathbb{E}\left[x_1^2\middle|\hat{y}\right]\mathbb{E}\left[s_1^2\right] & 0 & \cdots & 0 & 0 \\ 0 & \mathbb{E}\left[x_2^2\middle|\hat{y}\right]\mathbb{E}\left[s_2^2\right] & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mathbb{E}\left[x_J^2\middle|\hat{y}\right]\mathbb{E}\left[s_J^2\right] \end{bmatrix} a^\star(\hat{y})$$
$$+ \begin{bmatrix} \gamma\mathbb{E}\left[x_1^2\middle|\hat{y}\right]\mathbb{E}\left[s_1^2\right] \\ \gamma\mathbb{E}\left[x_2^2\middle|\hat{y}\right]\mathbb{E}\left[s_2^2\right] \\ \vdots \\ \gamma\mathbb{E}\left[x_{J-1}^2\middle|\hat{y}\right]\mathbb{E}\left[s_{J-1}^2\right] \\ \gamma\mathbb{E}\left[x_J^2\middle|\hat{y}\right]\mathbb{E}\left[s_J^2\right] \end{bmatrix}$$

This is the same as

$$a_j^\star(\hat{y}) = \frac{\gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]}{\phi_j + \gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]} = \frac{\gamma\left(\hat{\Sigma}_{x,jj} + \hat{x}_j^2\right)\mathbb{E}\left[s_j^2\right]}{\phi_j + \gamma\left(\hat{\Sigma}_{x,jj} + \hat{x}_j^2\right)\mathbb{E}\left[s_j^2\right]} \quad \forall j. \tag{A.3}$$

B.3 Two banks

When there are two banks and two factors, the regulator's optimal intervention policy is characterized by the following systems

$$a_j^\star = \left(\phi_j\mathbb{I}_2 + \gamma\begin{bmatrix} \mathbb{E}\left[x_{1,j}x_{1,j}\middle|\hat{y}\right] & \mathbb{E}\left[x_{1,j}x_{2,j}\middle|\hat{y}\right] \\ \mathbb{E}\left[x_{2,j}x_{1,j}\middle|\hat{y}\right] & \mathbb{E}\left[x_{2,j}x_{2,j}\middle|\hat{y}\right] \end{bmatrix}\mathbb{E}\left[s_j^2\right]\right)^{-1}\begin{bmatrix} \gamma\sum_{n=1}^N\mathbb{E}\left[x_{1,j}x_{n,j}\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right] \\ \gamma\sum_{n=1}^N\mathbb{E}\left[x_{2,j}x_{n,j}\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right] \end{bmatrix},$$

or, equivalently,

$$a_{ij}^\star = \frac{\gamma\mathbb{E}\left[s_j^2\right]\left(\phi_j\sum_{n=1}^2\mathbb{E}\left[x_{i,j}x_{n,j}\middle|\hat{y}\right] + \gamma\mathbb{E}\left[s_j^2\right]\left(\mathbb{E}\left[x_{i,j}^2\middle|\hat{y}\right]\mathbb{E}\left[x_{z,j}^2\middle|\hat{y}\right] - \left(\mathbb{E}\left[x_{i,j}x_{z,j}\middle|\hat{y}\right]\right)^2\right)\right)}{\phi_j^2 + \gamma\mathbb{E}\left[s_j^2\right]\left(\phi_j\sum_{n=1}^2\mathbb{E}\left[x_{n,j}^2\middle|\hat{y}\right] + \gamma\mathbb{E}\left[s_j^2\right]\left(\mathbb{E}\left[x_{i,j}^2\middle|\hat{y}\right]\mathbb{E}\left[x_{z,j}^2\middle|\hat{y}\right] - \left(\mathbb{E}\left[x_{i,j}x_{z,j}\middle|\hat{y}\right]\right)^2\right)\right)} \quad \forall i, j.$$

C Scenario choice

Since $\mathbb{E}[s_j] = \mathbb{E}[s_j s_k] = 0$ for all $j \neq k$, the objective function of the regulator can be written as

$$\begin{aligned}
O \equiv & \mathbb{E}_{\hat{x},s,\eta}\left[W - \frac{\gamma}{2}W^2 - \frac{1}{2}\sum_{j=1}^{J}\phi_j a_j^\star(\hat{y})' a_j^\star(\hat{y})\right] \\
= & \mathbb{E}_{x,e,\eta,s}\left[\bar{\omega} - \bar{\eta} - \sum_{i=1}^{J}\left(\mathbf{1}_{1\times N} - a_j^\star(\hat{y})'\right) x_{,j} s_j\right] \\
& - \mathbb{E}_{x,e,\eta,s}\left[\frac{\gamma}{2}\left(\bar{\omega} - \bar{\eta} - \sum_{j=1}^{J}\left(\mathbf{1}_{1\times N} - a_j^\star(\hat{y})'\right) x_{,j} s_j\right)^2 + \frac{1}{2}\sum_{j=1}^{J}\phi_j a_j^\star(\hat{y})' a_j^\star(\hat{y})\right] \\
= & \bar{\omega} - \frac{\gamma}{2}\left(\bar{\omega}^2 + \sigma_\eta^2\right) \\
& - \mathbb{E}_{x,e,s}\left[\sum_{i=1}^{J}\left(\mathbf{1}_{1\times N} - a_j^\star(\hat{y})'\right) x_{,j} s_j + \frac{\gamma}{2}\left(\sum_{j=1}^{J}\left(\mathbf{1}_{1\times N} - a_j^\star(\hat{y})'\right) x_{,j} s_j\right)^2 - \frac{1}{2}\sum_{j=1}^{J}\phi_j a_j^\star(\hat{y})' a_j^\star(\hat{y})\right] \\
= & \bar{\omega} - \frac{\gamma}{2}\left(\bar{\omega}^2 + \sigma_\eta^2\right) \\
& - \mathbb{E}_{\hat{y}}\left[\frac{\gamma}{2}\left(\sum_{j=1}^{J}\left(\mathbf{1}_{1\times N} - a_j^\star(\hat{y})'\right)\mathbb{E}\left[x_j x_j' \middle| \hat{y}\right]\left(\mathbf{1}_{1\times N} - a_j^\star(\hat{y})'\right)'\mathbb{E}\left[s_j^2\right]\right)^2 - \frac{1}{2}\sum_{j=1}^{J}\phi_j a_j^\star(\hat{y})' a_j^\star(\hat{y})\right] \\
= & \bar{\omega} - \frac{\gamma}{2}\left(\bar{\omega}^2 + \sigma_\eta^2\right) \qquad \text{(A.4)} \\
& - \frac{1}{2}\sum_{j=1}^{J}\mathbb{E}_{\hat{y}}\left[\gamma\left(\mathbf{1}_{1\times N} - a_j^\star(\hat{y})'\right)\mathbb{E}\left[x_j x_j' \middle| \hat{y}\right]\left(\mathbf{1}_{1\times N} - a_j^\star(\hat{y})'\right)'\mathbb{E}\left[s_j^2\right] + \sum_{j=1}^{J}\phi_j a_j^\star(\hat{y})' a_j^\star(\hat{y})\right],
\end{aligned}$$

where $a_j^\star = \left(a_{1,j}^\star, \ldots, a_{N,j}^\star\right)'$ and $\hat{x}_j = (\hat{x}_{i,j}, \ldots, \hat{x}_{N,j})'$.

C.1 One bank

When there is only one bank, the objective function of the regulator in Equation (A.4) becomes

$$O = \omega - \frac{\gamma}{2}\left(\omega^2 + \sigma_\eta^2\right) - \frac{1}{2}\sum_{j=1}^{J}\mathbb{E}_{\hat{y}}\left[\gamma\left(1 - a_j^\star(\hat{y})\right)^2\mathbb{E}\left[x_j^2 \middle| \hat{y}\right]\mathbb{E}\left[s_j^2\right] + \phi_j\left(a_j^\star(\hat{y})\right)^2\right].$$

Using the he optimal intervention policy in Equation (A.3) we have

$$
\begin{aligned}
O &= \omega - \frac{\gamma}{2}\left(\omega^2+\sigma_\eta^2\right) - \frac{1}{2}\sum_{j=1}^{J}\phi_j \mathbb{E}_{\hat{y}}\left[\gamma\left(\frac{1}{1+\gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]}\right)^2 \mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right] + \left(\frac{\gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]}{\phi_j+\gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]}\right)^2\right] \\
&= \omega - \frac{\gamma}{2}\left(\omega^2+\sigma_\eta^2\right) - \frac{1}{2}\sum_{j=1}^{J}\phi_j \mathbb{E}_{\hat{y}}\left[\left(\frac{1}{\phi_j+\gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]}\right)^2 \gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]\left(\phi_j+\gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]\right)\right] \\
&= \omega - \frac{\gamma}{2}\left(\omega^2+\sigma_\eta^2\right) - \frac{1}{2}\sum_{j=1}^{J}\phi_j \mathbb{E}_{\hat{y}}\left[\frac{\gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]}{\phi_j+\gamma\mathbb{E}\left[x_j^2\middle|\hat{y}\right]\mathbb{E}\left[s_j^2\right]}\right] \\
&= \omega - \frac{\gamma}{2}\left(\omega^2+\sigma_\eta^2\right) - \frac{1}{2}\sum_{j=1}^{J}\phi_j \mathbb{E}_{\hat{y}}\left[\frac{\gamma\left(\hat{\Sigma}_{x,jj}+\hat{x}_j^2\right)\mathbb{E}\left[s_j^2\right]}{\phi_j+\gamma\left(\hat{\Sigma}_{x,jj}+\hat{x}_j^2\right)\mathbb{E}\left[s_j^2\right]}\right] \\
&= \omega - \frac{\gamma}{2}\left(\omega^2+\sigma_\eta^2\right) - \frac{1}{2}\sum_{j=1}^{J}\phi_j \mathbb{E}_{\hat{y}}\left[a_j^\star\left(\hat{y}\right)\right]
\end{aligned}
$$

Since $\hat{x}_j \sim N\left(\bar{x}_j, \Sigma_{x,jj}-\hat{\Sigma}_{x,jj}\right)$ we have $\hat{x}_j = \left(\Sigma_{x,jj}-\hat{\Sigma}_{x,jj}\right)^{\frac{1}{2}} z + \bar{x}_j$ where $z \sim N(0,1)$. In this case, the objective function is separable in $\hat{\Sigma}_{x,jj}$.

Let,

$$
\boldsymbol{\Sigma} = \left\{\left\{\hat{\Sigma}_{x,jj}\right\} : \hat{\Sigma}_{x,jj} = \Sigma_{x,jj}\left(1-K_j\left(\hat{s}\right)\hat{s}_j\right) - \Sigma_{x,hj}K_j\left(\hat{s}\right)\hat{s}_h \quad \forall h,j=1,..,J, h\neq j \quad \text{for some} \quad \hat{s}\in\Omega\right\}, \tag{A.5}
$$

where Ω is the set of scenarios. Then, the regulator's problem can be written in terms of choosing posterior variances as

$$
\min_{\left\{\hat{\Sigma}_{x,jj}\right\}\in\boldsymbol{\Sigma}} \frac{1}{2}\sum_{i=1}^{J}\phi_j \mathbb{E}_{\hat{x},e,\eta,s}\left[a_j^\star\left(\hat{y}\right)\right].
$$