Sentiment and speculation in a market with heterogeneous beliefs

Ian Martin    Dimitris Papadimitriou

July, 2019
Introduction

- Agents disagree about the probabilities of good/bad news
- Optimists go long; pessimists go short
- If the market rallies, optimists get rich; if the market sells off, pessimists get rich
- In either case, ex post winners’ beliefs become overrepresented in prices
- This *sentiment* effect boosts volatility, and hence risk premia
- Sentiment induces *speculation*: agents trade at prices that they think are not warranted by fundamentals, in anticipation of adjusting their positions in future
Setup

- Agents indexed by $h \in (0, 1)$ are initially endowed with one unit of a risky asset.
- The asset evolves on a binomial tree with exogenous terminal payoffs.
- The interest rate is normalized to zero.
- Agent $h$ thinks the probability of an up-move is $h$.
- Agents have log utility over terminal wealth.
The mass of agents with belief $h$ follows a beta distribution, pdf

$$f(h) \propto h^{\alpha-1}(1-h)^{\beta-1} \quad \text{where } \alpha, \beta > 0$$
Equilibrium (1): individual optimization

- With log utility, agents behave myopically
- Solve backwards: the price of the risky asset is $p_d$ or $p_u$ next period
- Agent $h$ chooses number of units of risky asset held:

$$x_h = w_h \left( \frac{h}{p - p_d} - \frac{1 - h}{p_u - p} \right)$$

- If $p_d < p_u$ then pessimistic agents, $h \approx 0$, are short, and optimists, $h \approx 1$, are long
After $m$ up and $n$ down steps, agent $h$’s share of aggregate wealth is

$$\frac{w_h}{p} = \frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} h^m (1 - h)^n$$

The richest agent is $h = m/(m + n)$; this agent looks right in hindsight.
Equilibrium (2): market clearing

- Price \( p \) clears the market. At time \( t = m + n \),

\[
p = \frac{p_u p_d}{H_{m,t} p_d + (1 - H_{m,t}) p_u}
\]

where

\[
H_{m,t} = \frac{m + \alpha}{t + \alpha + \beta} = \int_0^1 h \frac{w_h f(h)}{p} dh
\]

is wealth-weighted average belief

- Write \( p^* \) for risk-neutral probability of an up-move, defined via

\[
p = p^* p_u + (1 - p^*) p_d
\]
Two special investors

- Share of wealth agent $h$ invests in the risky asset is $\frac{h - p^*}{H_{m,t} - p^*}$.
- Representative agent—“Mr. Market”—with $h = H_{m,t}$ invests fully in the risky asset.
- The agent with $h = p^*$ invests fully in the bond.
Example 1: Geometric payoffs, uniform belief distribution

\[ p = 0.96 \]
\[ \bar{p} = 1.00 \]
\[ H_{0,0} = 0.50 \]
\[ p^* = 0.29 \]
\[ p = 1.69 \]
\[ \bar{p} = 1.50 \]
\[ H_{1,1} = 0.67 \]
\[ p^* = 0.50 \]
\[ p = 0.56 \]
\[ \bar{p} = 0.56 \]
\[ p^* = 0.20 \]
\[ p = 1.13 \]
\[ p = 2.25 \]

\( p \): price. \( \bar{p} \): price in homog. economy. \( H_{m,t} \): identity of rep agent. \( p^* \): risk-neutral prob.
Mr. Market perceives a higher Sharpe ratio in the up state than the down state.

This is the opposite of what any individual thinks.
Example 2: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than $10^{-15}$
  - $h = 0.25$ thinks default prob is less than $10^{-6}$
  - $h = 0.10$ thinks default prob is less than 0.6%
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is just over 60%

- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at?
Example 2: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than $10^{-15}$
  - $h = 0.25$ thinks default prob is less than $10^{-6}$
  - $h = 0.10$ thinks default prob is less than 0.6%
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is just over 60%

- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at? at $95.63$
Example 2: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than $10^{-15}$
  - $h = 0.25$ thinks default prob is less than $10^{-6}$
  - $h = 0.10$ thinks default prob is less than 0.6%
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is just over 60%
- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at? at $95.63$
- Who would go short, at this price?
Example 2: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than $10^{-15}$
  - $h = 0.25$ thinks default prob is less than $10^{-6}$
  - $h = 0.10$ thinks default prob is less than 0.6%
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is just over 60%
- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at? at $95.63$
- Who would go short, at this price? everyone below $h = 0.48$!
Example 2: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than $10^{-15}$
  - $h = 0.25$ thinks default prob is less than $10^{-6}$
  - $h = 0.10$ thinks default prob is less than 0.6%
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is just over 60%

- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at? at $95.63$
- Who would go short, at this price? everyone below $h = 0.48!$
- Who will stay short?
Example 2: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than $10^{-15}$
  - $h = 0.25$ thinks default prob is less than $10^{-6}$
  - $h = 0.10$ thinks default prob is less than 0.6%
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is just over 60%
- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at? at $95.63$
- Who would go short, at this price? everyone below $h = 0.48$!
- Who will stay short? marginal agent $p^*$ at time 0, 1, 2, ... is $h = 0.48, 0.31, 0.22, ...$
Example 2: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than $10^{-15}$
  - $h = 0.25$ thinks default prob is less than $10^{-6}$
  - $h = 0.10$ thinks default prob is less than 0.6%
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is just over 60%

- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at? at $95.63$
- Who would go short, at this price? everyone below $h = 0.48!$
- Who will stay short? marginal agent $p^*$ at time 0, 1, 2, ... is $h = 0.48, 0.31, 0.22, ...$; only $h < 0.006$ stay short to the bitter end
Figure: The risky bond’s price over time following consistently bad news.
Figure: The number of units of the risky bond held by different agents, $x_{h,t}$, plotted against time.
Example 3: A diffusion limit

- Slice the period from 0 to $T$ into $2N$ short periods
- Geometrically increasing terminal payoffs with volatility $\sigma$ (Cox–Ross–Rubinstein)
- Tune down per-period disagreement by parametrizing $\alpha = \beta = \theta N$
- Low $\theta$: lots of disagreement. $\theta \to \infty$: homogeneous economy
- The belief distribution is very spiky, so we write $h = \tilde{E}[h] + z\sqrt{\tilde{\text{var}}[h]}$
- Now let $N \to \infty$. All agents perceive returns as lognormal and agree on ($\mathbb{P}$) volatility

$$\text{annualized return vol}_{0 \to t} = \left( \frac{\theta + 1}{\theta + \frac{t}{T}} \right) \sigma$$

- But they disagree on risk premia...
Result (Subjective expectations)

Agent $z$’s annualized expected return is

$$\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} = \frac{\theta + 1}{\theta + \frac{t}{T}} \left[ \frac{z \sigma}{\sqrt{\theta T}} + \frac{\theta + 1}{\theta + \frac{t}{T}} \sigma^2 \right]$$

In particular, the cross-sectional average expected return is

$$\bar{\mathbb{E}} \frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} = \frac{(\theta + 1)^2 \left( \theta + \frac{t}{2T} \right)}{\theta \left( \theta + \frac{t}{T} \right)^2} \sigma^2$$

Disagreement is the cross-sectional standard deviation of expected returns:

$$\text{disagreement} = \frac{\theta + 1}{\theta + \frac{t}{T} \sqrt{\theta T}} \sigma$$
Result (Option pricing and the volatility term structure)

The time-0 price of a call option with maturity $t$ and strike price $K$ is

$$p_0 \Phi(d_1) - K \Phi(d_1 - \tilde{\sigma}_t \sqrt{t})$$

where

$$d_1 = \frac{\log(p_0/K) + \frac{1}{2} \tilde{\sigma}_t^2 t}{\tilde{\sigma}_t \sqrt{t}}$$

and

$$\tilde{\sigma}_t = \frac{\theta + 1}{\sqrt{\theta(\theta + t/\theta)}} \sigma$$

In particular, short-dated options have $\tilde{\sigma}_0 = \frac{\theta + 1}{\theta} \sigma$ and long-dated options have $\tilde{\sigma}_T = \sqrt{\frac{\theta + 1}{\theta}} \sigma$
Figure: The term structures of implied and physical volatility.

- Variance risk premium \( \frac{1}{T} (\text{var}^* \log R_{0 \rightarrow T} - \text{var} \log R_{0 \rightarrow T}) = \frac{\sigma^2}{\theta} \)
An illustrative calibration

<table>
<thead>
<tr>
<th></th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1mo implied vol</td>
<td>18.6%</td>
<td>18.6%</td>
</tr>
<tr>
<td>1yr implied vol</td>
<td>18.2%</td>
<td>18.1%</td>
</tr>
<tr>
<td>2yr implied vol</td>
<td>17.7%</td>
<td>17.9%</td>
</tr>
<tr>
<td>1yr disagreement</td>
<td>4.4%</td>
<td>4.8%</td>
</tr>
<tr>
<td>10yr disagreement</td>
<td>2.9%</td>
<td>2.9%</td>
</tr>
<tr>
<td>1yr mean risk premium</td>
<td>3.3%</td>
<td>3.8%</td>
</tr>
<tr>
<td>10yr mean risk premium</td>
<td>1.9%</td>
<td>3.6%</td>
</tr>
</tbody>
</table>

- We set $T = 10$ and $\sigma = 12\%$
- We set belief heterogeneity parameter $\theta$ to 1.8
Figure: Volatility term structures in the baseline calibration with $\theta = 1.8$. 

Martin & Papadimitriou (LSE) 

Sentiment and speculation 

July, 2019 19 / 24
Figure: Volatility term structures in a “crisis” calibration with $\theta = 0.2$. 
Speculation and Sharpe ratios

- As agents have different beliefs but agree on market prices, they have different SDFs.
- Results so far supply the static Sharpe ratio of the risky asset.

Result

The maximum Sharpe ratio (as perceived by investor z) is finite for $\theta > 1$ and is equal to

$$MSR_{0 \to T}^{(z)} = \frac{\theta}{\sqrt{\theta^2 - 1}} \exp \left\{ \frac{\left[ z\sqrt{\theta} + (\theta + 1) \sigma \sqrt{T} \right]^2}{\theta (\theta - 1)} \right\} - 1$$
Figure: Max Sharpe ratio (annualized) as perceived by investor $z$. Baseline calibration.
The gloomy investor

- The **gloomy investor** who perceives the smallest MSR has $z = z_g$,
  \[ z_g = -\frac{\theta + 1}{\sqrt{\theta}} \sigma \sqrt{T} \]

- This investor perceives zero instantaneous Sharpe ratio, but a positive MSR associated with a contrarian strategy: buy if the market sells off, sell if the market rallies
  \[ \text{MSR}_{0\to T}^{(z_g)} = \sqrt{\frac{\theta}{\sqrt{\theta^2 - 1}}} - 1 \]
Agents have target prices

- Terminal wealth of agent $z$ is

$$W^{(z)}(p_T) = p_0 \sqrt{\frac{\theta + 1}{\theta}} \exp \left\{ \frac{1}{2} (z - z_g)^2 - \frac{1}{2(1 + \theta)\sigma^2 T} \left[ \log \left( \frac{p_T}{K^{(z)}} \right) \right]^2 \right\}$$

- Target price $K^{(z)}$ for investor $z$—their ideal outcome—satisfies

$$\log K^{(z)} = \mathbb{E}^{(z)} \log p_T + (z - z_g)\sigma \sqrt{\theta T}$$

- Gloomy investor $z = z_g$ wants to be proved right: ideal outcome equals expected outcome (in logs)

- But extremists are happiest if the market moves even more than they expect
Figure: MSR strategies (solid) are far from optimal strategies (dashed). Log scale on x-axis.
Conclusions

- Sentiment generates volatility, speculation, and volume
- Sentiment can push prices either up or down
- Extreme outcomes are far more important than in a homogeneous economy
- Downward-sloping vol term structure in a diffusion limit, and a variance risk premium
- Agents have target prices
- Moderate investors are contrarian, “short vol”, supply liquidity to extremists