

Forecasting and Stress Testing with Quantile Vector Autoregression*

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Abstract

We introduce and estimate a quantile vector autoregressive model. Unlike standard VAR model which model only the average interaction, quantile VAR allows one to model the interaction between any quantile of the endogenous variables. The methodology illustrates how to estimate and forecast multivariate quantiles within a structural model. The model is estimated using real and financial variables for the euro area. The results show that the dynamic properties of the model change significantly when the economy is hit by abnormal financial and real shocks, with respect to tranquil times. The econometric framework is used to perform multi period ahead stress testing exercises, where the euro area economy is hit with a series of financial and real economic shocks which mimic those that occurred during the recent crises.

Keywords: Regression quantiles; Structural VAR; Growth at Risk.

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1 Introduction

Vector autoregressive (VAR) models are the empirical workhorse of macroeconomists. These models often rely on constant coefficients and i.i.d. Gaussian innovations. There is, however, substantial empirical evidence that macroeconomic variables are characterised by nonlinearities and asymmetries which cannot be captured by simple linear Gaussian models (Perez-Quiros and Timmermann 2000, Hubrich and Tetlow 2015, Kilian and Vigfusson 2017, Adrian et al. 2019b). This paper shows how structural VAR models can be estimated with quantile regression methods, thus offering a robust alternative to study asymmetric dynamics in time series econometrics.

Quantile regression has a long and illustrious history in econometrics. It was introduced by Koenker and Bassett (1978) and has found many applications in economics (Koenker 2005, 2017). Early applications to univariate time series include Engle and Manganelli (2004) and Koenker and Xiao (2006). White, Kim and Manganelli (2010, 2015) develop a framework to model multivariate quantiles. Schueler (2014) introduces a Bayesian quantile structural vector autoregressive model.

Two long standing challenges of the regression quantile literature are how to deal with multiple variables and how to forecast in a time series context. We provide a solution to both problems and show that they are closely connected. The VAR for VaR model of White et al. (2015) represents the starting point of our model, as it provides the general framework for inference. Casting the problem in a multivariate framework such as a VAR model immediately raises the issue of the definition of structural shocks and identification. We show that structural identification and quantile modelling of multiple variables are different sides of the same coin. We identify the quantile VAR by estimating a recursive model, where the first variables of the system are allowed to contemporaneously affect the remaining variables. This corresponds to performing a Cholesky decomposition in a standard VAR model and falls within the recursive conditioning framework of Chesher (2003). The

quantile at any time of the second variable, say, becomes a random variable as it depends on the contemporaneous value taken by the first variable, and as such it is characterised by a certain distribution. By taking any quantile of this quantile distribution we can estimate the quantile value taken by the second random variable when the first random variable is equal to its own quantile. This reasoning can be repeated recursively for all the cross section of random variables, therefore giving the quantile of the quantile of the cross section at any given point in time.

This intuition holds also for forecasting future quantiles. Since future quantiles depend on future shocks, they are themselves random variables. By taking specific quantiles of these random variables, we can characterise their future distribution. Any quantile forecast at any point in time is therefore the quantile of future quantiles. The logic is similar to the one used to factor any likelihood into a product of marginal and conditional densities. We refer to this relationship as to the law of iterated quantiles. A key difference with the law of iterated expectations is that while expectations are additive, quantiles are not, so that the quantile of the quantile of a sum of random variables is not equal to the quantile of the sum.

Our econometric framework is general enough to cover the modelling of multiple quantiles of multiple random variables. It is this multivariate approach that gives the flexibility to assess the impact of any desired scenario. Stress testing can be thought of as an estimate of the reaction of the endogenous random variables when the system is hit by a sequence of quantile shocks. Stress scenarios are nothing else than an arbitrary series (to be chosen by the policy maker or calibrated to past crises) of quantile shocks hitting the environment.

We estimate a quantile VAR model on euro area data for industrial production and an indicator of financial distress. We find that severe financial shocks – defined as a tail quantile realization – are transmitted to the real economy only when the economy is simultaneously hit by a real negative

shock. Modelling the mean dynamics with a standard VAR misses most of this action. Furthermore, shutting down the financial channel of transmission in the system significantly changes the dynamics of the real economy when hit by negative shocks, but leaves the dynamics largely unaffected in normal conditions.

These results are broadly in line with those found by Adrian et al. (2019b) for the U.S. economy. The empirical model estimated by Adrian et al. (2019b) is equivalent to estimating only one equation of our quantile VAR model. The advantage of quantile VAR is that it allows us to perform impulse response analyses and to forecast the quantiles of the endogenous variables. We find that by hitting the system with a financial shock there is a strong and persistent asymmetric impact on the distribution of industrial production, which takes about two years to be absorbed.

Quantile VAR provides also the natural environment to perform stress testing exercises. At its core, stress testing is a forecast of what happens to the system when it is hit by an arbitrary sequence of negative shocks. If the euro area is hit by a sequence of six monthly consecutive financial and real tail shocks, its industrial production contracts by a cumulated amount of more than 10% over the same period. This contrasts with a median increase of industrial production of around 2%, a forecast which would hold under normal circumstances.

The paper is organized as follows. Section 2 develops the general quantile structural vector autoregressive framework. It provides the links with standard OLS structural VAR, derives the asymptotic distributions, and shows how to do forecasting with quantile structural VAR. Section 3 estimates the quantile VAR model for the euro area and performs a stress testing exercise. Section 4 concludes.

2 Quantile Vector Autoregression

This section defines the concept of structural quantile impulse response function, shows how to compute quantile VAR forecasts and provides the asymptotic properties of the model.

2.1 The Law of Iterated Quantiles and Quantile Impulse Response Functions

Consider a sequence of random variables $\{\tilde{Y}_t : t = 1, \dots, T\}$, where \tilde{Y}_t is an $n \times 1$ vector with i^{th} element denoted by \tilde{Y}_{it} for $i \in \{1, \dots, n\}$.

Consider the following structural vector autoregressive model, written in recursive and reduced form:

$$\tilde{Y}_{t+1} = \omega + A_0 \tilde{Y}_{t+1} + A_1 \tilde{Y}_t + \epsilon_{t+1} \quad \epsilon_{t+1} \sim i.i.d.(0, \Sigma) \quad (1)$$

$$= \mu_t + (I_n - A_0)^{-1} \epsilon_{t+1} \quad (2)$$

where $\mu_t \equiv (I_n - A_0)^{-1} \omega + (I_n - A_0)^{-1} A_1 \tilde{Y}_t$, A_0 and A_1 are a $n \times n$ coefficient matrices, ω is a $n \times 1$ vector of constants, ϵ_{t+1} is a $n \times 1$ vector of i.i.d. structural shocks with Σ a diagonal matrix, and I_n is a n -dimensional identity matrix. Imposing that A_0 has a lower triangular structure, the identification of this system is equivalent to assuming a Choleski decomposition of the errors from a standard reduced form vector autoregressive model (see, for instance, chapter 2 of Lutkepohl 2005).

The expected value of the process (1) at time $t + H$, given Ω_{t+H} , the information available at time $t + H$, is:

$$\begin{aligned} E_{t+H}(\tilde{Y}_{t+H+1}) &\equiv E(\tilde{Y}_{t+H+1} | \Omega_{t+H}) \\ &= \mu_{t+H} \\ &= \nu + B \tilde{Y}_{t+H} \end{aligned}$$

where $\nu \equiv (I_n - A_0)^{-1}\omega$ and $B \equiv (I_n - A_0)^{-1}A_1$, which together with (2) can be solved backwards in terms of the structural shocks $\{\epsilon_{t+h}\}_{h=1}^H$, for $H \geq 1$:

$$\mu_{t+H} = \sum_{h=0}^H B^h \nu + B^{H+1} \tilde{Y}_t + \sum_{h=1}^H B^{H-h+1} (I_n - A_0)^{-1} \epsilon_{t+h}$$

Since μ_{t+H} depends on future shocks, it is a random variable. The standard way to characterise the properties of this random variable is to compute the expectation of its future expectations:

$$E_t(\cdots E_{t+H-1}(\mu_{t+H})) = \sum_{h=0}^H B^h \nu + B^{H+1} \tilde{Y}_t$$

This is convenient because the expectation of future expectations depends only on the estimated parameters and \tilde{Y}_t . In principle, one could choose to characterise the properties of μ_{t+H} by looking at any other part of its distribution, at the cost, however, of estimating additional parameters.

The impulse-response function is defined by the marginal impact that a structural shock has on the expected value of future expectations, via the impact it has on \tilde{Y}_t :

$$\partial E_t(\cdots E_{t+H-1}(\mu_{t+H}))/\partial \epsilon'_t = B^{H+1} (I_n - A_0)^{-1} \quad \text{for } H \geq 1 \quad (3)$$

This framework motivates our definition of a quantile structural vector autoregressive model. Since we want to consider the possibility of jointly modelling multiple quantiles, we need additional notation. For our purposes, it is important to define a recursive information set, which allows us to work with structural models. Define $\Omega_{it} \equiv \{\tilde{Y}_{1t}, \dots, \tilde{Y}_{i-1,t}, \tilde{Y}_{t-1}, \tilde{Y}_{t-2}, \dots\}$ for $i \in \{2, \dots, n\}$ and $\Omega_{1t} \equiv \{\tilde{Y}_{t-1}, \tilde{Y}_{t-2}, \dots\}$, so that the information set Ω_{2t} , say, contains all the lagged values of \tilde{Y}_t as well as the contemporaneous value of \tilde{Y}_{1t} . We consider also p distinct quantiles, $0 < \theta_1 < \theta_2 < \dots < \theta_p < 1$.¹ The

¹The model can be generalised to the case where quantile indices are different for

quantile structural vector autoregressive model is defined as follows, written again in recursive and reduced form:

$$Y_{t+1} = \omega^\theta + A_0^\theta Y_{t+1} + A_1^\theta Y_t + \epsilon_{t+1}^\theta, \quad P(\epsilon_{i,t+1}^{\theta_j} < 0 | \Omega_{it}) = \theta_j, \quad (4)$$

$$i = 1, \dots, n, \quad j = 1, \dots, p$$

$$= q_t^\theta + (I_{np} - A_0^\theta)^{-1} \epsilon_{t+1}^\theta \quad (5)$$

where $q_t^\theta = (I_{np} - A_0^\theta)^{-1} \omega^\theta + (I_{np} - A_0^\theta)^{-1} A_1^\theta Y_t$. The dependent variable Y_t is now an np -vector, which is obtained as $Y_t = \iota_p \otimes \tilde{Y}_t$, where ι_p is a p -vector of ones, and $\epsilon_t^\theta \equiv [\epsilon_{1t}^{\theta_1}, \dots, \epsilon_{nt}^{\theta_1}, \dots, \epsilon_{1t}^{\theta_p}, \dots, \epsilon_{nt}^{\theta_p}]'$. The matrices A_0^θ and A_1^θ are block diagonal, to avoid trivial multicollinearity problems. We further impose that the diagonal blocks of A_0^θ are lower triangular matrices with zeros along their main diagonal, reflecting the recursive identification assumption of the system. The probability relationship defining the regression quantile follows the recursive structure of the identification assumption.

An explicit example may help. Consider a model with two endogenous random variables and two quantiles, say 50% and 90%. System (4) can be written explicitly as:

$$\begin{bmatrix} \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \\ \hline \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \end{bmatrix} = \begin{bmatrix} \omega_1^5 \\ \omega_2^5 \\ \hline \omega_1^9 \\ \omega_2^9 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \vdots & 0 & 0 \\ a_{021}^5 & 0 & \vdots & 0 & 0 \\ \hline 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & a_{021}^9 & 0 \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \\ \hline \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \end{bmatrix} + \\ + \begin{bmatrix} a_{11}^5 & a_{12}^5 & \vdots & 0 & 0 \\ a_{21}^5 & a_{22}^5 & \vdots & 0 & 0 \\ \hline 0 & 0 & \vdots & a_{11}^9 & a_{12}^9 \\ 0 & 0 & \vdots & a_{21}^9 & a_{22}^9 \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1,t} \\ \tilde{Y}_{2,t} \\ \hline \tilde{Y}_{1,t} \\ \tilde{Y}_{2,t} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t+1}^5 \\ \epsilon_{2,t+1}^5 \\ \hline \epsilon_{1,t+1}^9 \\ \epsilon_{2,t+1}^9 \end{bmatrix}$$

Given the recursive structure of (4), the quantiles of $\tilde{Y}_{2,t+1}, \dots, \tilde{Y}_{n,t+1}$ are different elements of Y_t . See White et al. (2015) for details.

random vectors at time t , as they depend on the vector of contemporaneous structural shocks via the term $A_0^\theta Y_{t+1}$. Consider the θ_j quantile of $\tilde{Y}_{n,t+1}$ and write the random part of $\tilde{Y}_{n,t+1}$ as $a_1 \epsilon_{1,t+1}^{\theta_j} + \dots + a_{n-1} \epsilon_{n-1,t+1}^{\theta_j} + \epsilon_{n,t+1}^{\theta_j}$, for suitably chosen scalars a_1, \dots, a_{n-1} . The θ_j quantile of this term, given the information set Ω_{nt} , is $a_1 \epsilon_{1,t+1}^{\theta_j} + \dots + a_{n-1} \epsilon_{n-1,t+1}^{\theta_j}$, by the quantile property of $\epsilon_{n,t+1}^{\theta_j}$ in (4). In turn, the θ_j quantile of this θ_j quantile conditional on the information set $\Omega_{n-1,t}$ is $a_1 \epsilon_{1,t+1}^{\theta_j} + \dots + a_{n-2} \epsilon_{n-2,t+1}^{\theta_j}$ ². Repeating this argument for all the cross section of variables in Y_{t+1} we obtain that the θ_j quantile of all the cross section of θ_j quantiles of the shocks $(I_{np} - A_0^\theta)^{-1} \epsilon_{t+1}^\theta$ conditional on all the lagged and the recursive contemporaneous dependent variables is zero. We write for brevity $Q_t^\theta((I_{np} - A_0^\theta)^{-1} \epsilon_{t+1}^\theta) = 0$, meaning that this contemporaneous recursive iteration has been applied to each element of the vector. That is, for any np -vector x , we define $Q_t^\theta(x) \equiv [Q_t^{\theta_1}(x_1^1), \dots, Q_t^{\theta_1}(\dots Q_t^{\theta_1}(x_n^1)), \dots, Q_t^{\theta_p}(x_1^p), \dots, Q_t^{\theta_p}(\dots Q_t^{\theta_p}(x_n^p))]'$ and $Q_t^{\theta_j}(x_i^j)$ is implicitly defined by $P(x_i^j < Q_t^{\theta_j}(x_i^j) | \Omega_{it}) = \theta_j$. This reasoning implies also that $Q_t^\theta(Y_{t+1}) = q_t^\theta$.

If system (1) is the data generating process, then $\omega^\theta = \iota_p \otimes \omega + \kappa^\theta$, where κ^θ is the np -vector containing the θ quantiles of ϵ_{t+1} , $A_0^\theta = I_p \otimes A_0$ and $A_1^\theta = I_p \otimes A_1$. Under this assumption, the VAR and quantile VAR are characterized by identical dynamics. In general, however, this need not be the case. In homoskedastic linear regression models, the conditioning variables shift the location of the conditional density of Y_t , but they have no effect on conditional dispersion or shape. Quantile regression is a semiparametric technique which allows different covariates to affect different parts of the distribution. If and how this happens is an empirical question. In our empirical applications, we find that estimates of quantile regression slopes and quantile impulse response functions vary across quantiles. This may happen either because of unmodelled time varying higher order moments, and/or because

²Here we have a slight abuse of notation, as $P(a_{n-1} \epsilon_{n-1,t+1}^{\theta_j} < 0) = 1 - \theta_j$ when $a_{n-1} < 0$, and so it becomes the $(1 - \theta_j)$ quantile.

the conditioning variables affect the conditional distribution of the dependent variables in a nonlinear way. These effects cannot be detected with standard OLS VAR estimates.

The θ quantile of process (4) at time $t+H$, given the information available at time $t+H$, is:

$$\begin{aligned} Q_{t+H}^\theta(Y_{t+H+1}) &= q_{t+H}^\theta \\ &= \nu^\theta + B^\theta Y_{t+H} \\ &= \nu^\theta + B^\theta q_{t+H-1}^\theta + B^\theta (I_{np} - A_0^\theta)^{-1} \epsilon_{t+H}^\theta \end{aligned} \quad (6)$$

where $\nu^\theta \equiv (I_{np} - A_0^\theta)^{-1} \omega^\theta$ and $B^\theta \equiv (I_{np} - A_0^\theta)^{-1} A_1^\theta$. Recursive substitution gives:

$$q_{t+H}^\theta = \sum_{h=0}^H (B^\theta)^h \nu^\theta + (B^\theta)^{H+1} Y_t + \sum_{h=1}^H (B^\theta)^{H-h+1} (I_{np} - A_0^\theta)^{-1} \epsilon_{t+h}^\theta \quad (7)$$

Notice again that like μ_{t+H} also q_{t+H}^θ is a random vector at time t , as it depends on the vector of future structural shocks ϵ_{t+h}^θ . Applying recursions over time similar to those outlined above gives the θ quantile of future θ quantiles:

$$Q_t^\theta(\cdots Q_{t+H-1}^\theta(q_{t+H}^\theta)) = \sum_{h=0}^H (B^\theta)^h \nu^\theta + (B^\theta)^{H+1} Y_t \quad (8)$$

because by the previous reasoning $Q_{t+h-1}^\theta((B^\theta)^{H-h+1} (I_{np} - A_0^\theta)^{-1} \epsilon_{t+h}^\theta) = 0$ for all h . We refer to equation (8) as to the *Law of Iterated Quantiles* (LIQ). Notice the difference with respect to the Law of Iterated Expectations (LIE). For LIE, given any generic random variable X_t with finite expectation, it holds that:

$$E_t(X_{t+1} + X_{t+2}) = E_t(E_{t+1}(X_{t+1} + X_{t+2}))$$

For the LIQ, instead, this is generally not the case:

$$Q_t^\theta(X_{t+1} + X_{t+2}) \neq Q_t^\theta(Q_{t+1}^\theta(X_{t+1} + X_{t+2}))$$

From equation (7) or (8), it is possible to define the quantile impulse response function as the marginal impact that a structural shock has on the quantile of future quantiles:

$$\partial Q_t^\theta(\dots Q_{t+H-1}^\theta(q_{t+H}^\theta))/\partial(\epsilon_t^\theta)' = (B^\theta)^{H+1}(I_{np} - A_0^\theta)^{-1} \quad \text{for } h \geq 1 \quad (9)$$

Standard OLS impulse response functions measure the impact of a structural shock on the expectation of expectations of future values of the endogenous variables. The law of iterated quantiles, instead, implies that quantile impulse response functions measure the impact of a structural shock on the quantile of the quantiles of future values of the endogenous variables. In other words, future quantiles are random variables themselves and will therefore be characterized by a distribution. The quantile impulse response function traces the impact of shocks on the quantiles of the distribution of future quantiles.

2.2 Forecasting and stress testing

Forecasts are future values taken by parts of the distribution of the dependent variables of interest, and are obtained by giving specific values to the error terms. In the case of the OLS, forecasts are future values taken by the mean of the distribution obtained by setting future mean shocks to zero. In the case of quantile regression models, forecasts are the values taken by specific quantiles of the distribution obtained by setting the corresponding future quantile shocks to zero.

To formalize, define $S_{j_{t+1}}$ the $n \times np$ matrix selecting specific quantile shocks from the vector ϵ_{t+1}^θ . That is, $S_{j_{t+1}}\epsilon_{t+1}^\theta = [\epsilon_{1,t+1}^{\theta_{j_{t+1}}^1}, \dots, \epsilon_{n,t+1}^{\theta_{j_{t+1}}^n}]'$ for $j_{t+1}^i \in$

$\{1, \dots, p\}$ and $i \in \{1, \dots, n\}$. Then by (4), the forecast of \tilde{Y}_{t+1} , conditional on setting the shocks identified by the matrix $S_{j_{t+1}}$ to zero, is:

$$\begin{aligned} \hat{Y}_{t+1}|S_{j_{t+1}} &= S_{j_{t+1}}Y_{t+1} \\ &= S_{j_{t+1}}(\omega^\theta + A_0^\theta Y_{t+1} + A_1^\theta Y_t) \\ &= \bar{\omega}_{t+1}^\theta + \bar{B}_{t+1}^\theta Y_t \end{aligned}$$

where $\bar{\omega}_{t+1}^\theta \equiv (I_n - S_{j_{t+1}}A_0^\theta\bar{S})^{-1}S_{j_{t+1}}\omega^\theta$, $\bar{B}_{t+1}^\theta \equiv (I_n - S_{j_{t+1}}A_0^\theta\bar{S})^{-1}S_{j_{t+1}}A_1^\theta$, and \bar{S} is the $pn \times n$ duplication matrix such that $Y_{t+1} = \bar{S}S_{j_{t+1}}Y_{t+1}$.

Solving this equation forward, for any given sequence $\{S_{j_{t+h}}\}_{h=1}^H$, we obtain the forecast of the dependent variables at any future point in time H :

$$\begin{aligned} \hat{Y}_{t+H}|\{S_{j_{t+h}}\}_{h=1}^H &= \bar{\omega}_{t+H}^\theta + \bar{B}_{t+H}^\theta\bar{\omega}_{t+H-1}^\theta + \dots + \\ &+ (\bar{B}_{t+H}^\theta\bar{B}_{t+H-1}^\theta \dots \bar{B}_{t+2}^\theta)\bar{\omega}_{t+1}^\theta + \\ &+ (\bar{B}_{t+H}^\theta\bar{B}_{t+H-1}^\theta \dots \bar{B}_{t+1}^\theta)Y_t \end{aligned} \quad (10)$$

For instance, the forecast of Y_{t+H} conditional on future shocks taking their median values can be obtained by choosing the $\{S_{j_{t+h}}\}_{h=1}^H$ matrices such that they select the median quantile and setting the corresponding median shocks to zero.

Equation (10) is a generalization of (8). Relationship (8) implicitly assumes a specific sequence of shocks and does not take into account the cross restrictions which bind the different quantile shocks of the same random variable together. For instance, the first element of (8) is the θ_1 quantile associated with the first dependent variable of all the future and cross-sectional θ_1 quantiles of the dependent variables. This corresponds to the first element of (10) when the sequence $\{S_{j_{t+h}}\}_{h=1}^H$ selects the following shocks $\{\epsilon_{1,t+1}^{\theta_1}, \dots, \epsilon_{n,t+1}^{\theta_1}, \dots, \epsilon_{1,t+H}^{\theta_1}, \dots, \epsilon_{n,t+H}^{\theta_1}\}$ to be set to zero. Equation (10) allows one to forecast any quantile of *any* future and cross-sectional quantile.

It is also possible to rewrite the impulse response function in terms of (10).

Suppose that we are interested in the median forecast of all future medians and suppose that the median corresponds to the θ_j quantile. Let S_j be the matrix selecting the median elements of system (4). The median forecast is then given by $S_j Q_t^\theta(\cdots Q_{t+H-1}^\theta(q_{t+H}^\theta))$, where the quantile of the quantile function is specified in (8). This is equivalent to (10) with $\hat{Y}_{t+H}|\{S_{j_{t+h}} = S_j\}_{h=1}^H$. We are now interested in how this median forecast would change, had we observed the shock $\epsilon_{it}^{\theta_{j'}} = 0$, for some $j' \in \{1, \dots, p\}$. Denoting with \ddot{Y}_t the shocked vector, the change in forecast is given by:

$$\begin{aligned} \hat{Y}_{t+H}|\{\ddot{Y}_t, \{S_{j_{t+h}} = S_j\}_{h=1}^H\} - \hat{Y}_{t+H}|\{S_{j_{t+h}} = S_j\}_{h=1}^H \\ = \bar{B}_{t+H}^\theta \bar{B}_{t+H-1}^\theta \cdots \bar{B}_{t+1}^\theta (\ddot{Y}_t - Y_t) \end{aligned} \quad (11)$$

which is proportional to the corresponding element of (9). The generic impulse response function for any quantile of any future quantile is given by $\hat{Y}_{t+H}|\{\ddot{Y}_t, \{S_{j_{t+h}}\}_{h=1}^H\} - \hat{Y}_{t+H}|\{S_{j_{t+h}}\}_{h=1}^H$.

The greater generality and flexibility of (10) provides the natural environment to perform stress testing exercises. A policy maker interested in how the endogenous variables react to a given stressful scenario can first define the scenario by setting a series of future tail (say, 10% or 1%) quantile shocks to zero, and then obtain the forecast of the endogenous variables conditional on the chosen scenario.

Finally, it is straightforward to compute average step ahead forecasts from the QVAR model. Suppose that at time T the interest lies in the average H -step ahead values of the dependent variables, that is:

$$Y_{T,H} \equiv H^{-1} \sum_{h=1}^H Y_{T+h} \quad (12)$$

Then, the forecast, conditional on the sequence of shocks $\{S_{j_{T+h}}\}_{h=1}^H$, is:

$$\hat{Y}_{T,H}|\{S_{j_{T+h}}\}_{h=1}^H \equiv H^{-1} \sum_{h=1}^H \hat{Y}_{T+h}|\{S_{j_{T+i}}\}_{i=1}^h$$

where $\hat{Y}_{T+h}|\{S_{j_{T+i}}\}_{i=1}^h$ is defined in (10).

2.3 General quantile VAR(q) model

Model (4) can be easily generalized to any VAR(q) model using its companion form. Define the npq vectors $\bar{\omega} \equiv [(\omega^\theta)', 0', \dots, 0']'$, $\bar{Y}_{t+1} \equiv [Y'_{t+1}, Y'_t, \dots, Y'_{t-q+2}]'$, $\varepsilon_{t+1} \equiv [(\varepsilon_{t+1}^\theta)', 0', \dots, 0']'$, and the $(npq \times npq)$ matrices

$$A^0 = \begin{bmatrix} A_0^\theta & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \text{ and } A^1 = \begin{bmatrix} A_1^\theta & A_2^\theta & \dots & A_q^\theta \\ I_{np} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \dots & I_{np} & \mathbf{0} \end{bmatrix}.$$

Then the companion form of the VAR(q) model is:

$$\bar{Y}_{t+1} = \bar{\omega} + A^0 \bar{Y}_{t+1} + A^1 \bar{Y}_t + \varepsilon_{t+1} \quad (13)$$

All the results of the previous sections extend to model (13).

2.4 Estimation and Asymptotics

The recursive QVAR model (4) can be estimated using the framework developed by White, Kim and Manganelli (2015). Let $q_t^\theta(\beta) \equiv \omega^\theta + A_0^\theta Y_t + A_1^\theta Y_{t-1}$ and $q_{it}^{\theta j}(\beta)$ the j^{th} quantile of the i^{th} variable of the vector $q_t^\theta(\beta)$, where we have made explicit the dependence on β , the vector containing all the unknown parameters in ω^θ , A_0^θ , and A_1^θ . Define the quasi-maximum likelihood

estimator $\hat{\beta}$ as the solution of the optimization problem:

$$\hat{\beta} = \arg \min_{\beta} T^{-1} \sum_{t=1}^T \left\{ \sum_{i=1}^n \sum_{j=1}^p \rho_{\theta} \left(\tilde{Y}_{it} - q_{it}^{\theta_j}(\beta) \right) \right\}, \quad (14)$$

where $\rho_{\theta}(u) \equiv u(\theta - I(u < 0))$ is the standard check function of quantile regressions.

Under the assumptions of theorems 1 and 2 of White et al. (2015), $\hat{\beta}$ is consistent and asymptotically normally distributed. The asymptotic distribution is:

$$\sqrt{T}(\hat{\beta} - \beta^*) \xrightarrow{d} N(0, Q^{-1}VQ^{-1}) \quad (15)$$

where

$$\begin{aligned} Q &\equiv \sum_{i=1}^n \sum_{j=1}^p E[f_{it}^{\theta_j}(0) \nabla q_{it}^{\theta_j}(\beta^*) \nabla' q_{it}^{\theta_j}(\beta^*)] \\ V &\equiv E[\eta_t \eta_t'] \\ \eta_t &\equiv \sum_{i=1}^n \sum_{j=1}^p \nabla q_{it}^{\theta_j}(\beta^*) \psi^{\theta_j}(\epsilon_{it}^{\theta_j}) \\ \psi^{\theta_j}(\epsilon_{it}^{\theta_j}) &\equiv \theta_j - I(\epsilon_{it}^{\theta_j} \leq 0) \\ \epsilon_{it}^{\theta_j} &\equiv \tilde{Y}_{it} - q_{it}^{\theta_j}(\beta^*) \end{aligned}$$

and $f_{it}^{\theta_j}(0)$ is the conditional density function of $\epsilon_{it}^{\theta_j}$ evaluated at 0. The asymptotic variance-covariance matrix can be consistently estimated as suggested in theorems 3 and 4 of White et al. (2015), or using bootstrap based methods in the spirit of Buchinsky (1995).³

³Modern statistical softwares contain packages for regression quantile estimation and inference. This paper uses the interior point algorithm discussed by Koenker and Park (1996).

To obtain the standard errors of the forecasts in (10), let

$$\bar{B} \equiv [\bar{\omega}, A^0, A^1]$$

where $\bar{\omega}$, A^0 and A^1 are defined in (13). Define

$$vec(\bar{B}) = R\beta + \gamma \quad (16)$$

where R is a $(npq(1 + 2npq) \times b)$ matrix of restrictions with b the size of β and γ is the corresponding vector of 0 and 1 constraints (see chapter 5 of Lütkepohl, 2005). The matrix R can be easily constructed in a software by creating a matrix of 0s and an index ϕ which identifies the position of the elements of $vec(\bar{B})$ different from 0 and 1, and then setting $R(\phi(i), i) = 1$, for $i = 1, \dots, b$. Letting $\bar{\omega} = \bar{B}K_\omega$, $A^0 = \bar{B}K_0$ and $A^1 = \bar{B}K_1$, for suitable K_ω , K_0 and K_1 matrices, the standard error of the forecast can be obtained from a Taylor expansion:

$$\begin{aligned} Y_{T,H}(\hat{\beta}) &\equiv \hat{Y}_{T,H} | \{S_{j_{t+h}}\}_{h=1}^H \\ &\approx Y_{T,H}(\beta^*) + \Phi(\hat{\beta})(\hat{\beta} - \beta^*) \end{aligned}$$

where the term $\Phi(\hat{\beta}) \equiv \partial Y_{T,H}(\hat{\beta}) / \partial \beta'$ can be computed numerically or applying the rules of matrix differentiation (see, for instance, Lütkepohl, 2005). From the asymptotic properties of $\hat{\beta}$, it follows that:

$$\sqrt{T}(Y_{T,H}(\hat{\beta}) - Y_{T,H}(\beta^*)) \xrightarrow{d} N(0, \Phi(\beta^*)Q^{-1}VQ^{-1}\Phi'(\beta^*)) \quad (17)$$

The standard errors associated with the impulse response function (9) can be computed in a similar fashion.

3 Is growth in Europe vulnerable to financial distress?

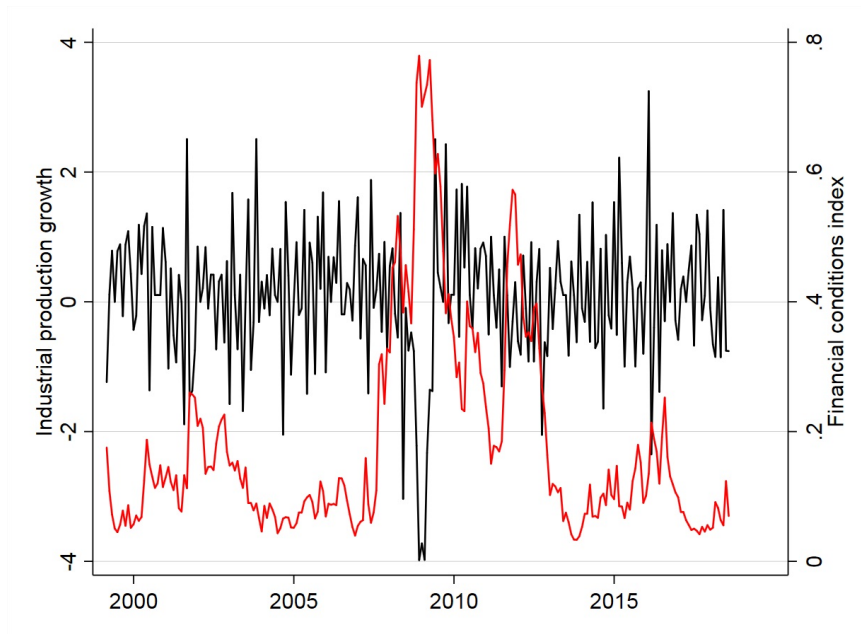
We apply the methodology developed in the previous section to model the interaction between real and financial variables in Europe. We study the interrelationship between the euro area industrial production (\tilde{Y}_{1t}) and the composite indicator of systemic stress in the financial system (CISS, \tilde{Y}_{2t}) of Hollo, Kremer and Lo Duca (2012). Adrian et al. (2019b) have shown that there are substantial asymmetries in the relationship between the US real GDP growth and financial conditions. In particular, they find that the estimated lower quantiles of the distribution of future GDP growth are significantly affected by financial conditions, while the upper quantiles appear to be more stable over time. The quantile model specification of Adrian et al. (2019b) is the following:

$$\tilde{Y}_{1,t+1} = \omega_1^\theta + a_{11}^\theta \tilde{Y}_{1,t} + a_{12}^\theta \tilde{Y}_{2t} + \epsilon_{t+1}^\theta \quad (18)$$

They estimate this model for $\theta \in \{0.05, 0.25, 0.75, .95\}$. This corresponds to the first line of model (4). An obvious drawback of neglecting to model the second line of the quantile VAR model is that forecasting becomes impossible. In fact, for the four quarters ahead analysis, they have to resort to direct estimation, whereby they quantile regress the four quarter ahead GDP directly on current GDP and financial conditions. Our framework, instead, allows us to estimate the model at the highest possible frequency and still to study the forecasting properties of the system as well as to test the presence of any feedback effect.

We start by reporting in figure 1 the monthly time series of industrial production and CISS in the euro area from January 1999 until July 2018. The data is downloaded from the Statistical Data Warehouse database of the

Figure 1: Real and financial variables



Note: Time series evolution of euro area industrial production (black line) and CISS (red line). Monthly data, source: ECB.

ECB.⁴ A cursory view at the plot reveals a clear negative correlation between the two time series, especially during the Great Financial Crisis.

Next, we estimate the quantile VAR model (4):

$$\begin{aligned}\tilde{Y}_{1,t+1} &= \omega_1^\theta + a_{11}^\theta \tilde{Y}_{1t} + a_{12}^\theta \tilde{Y}_{2t} + \epsilon_{1,t+1}^\theta \\ \tilde{Y}_{2,t+1} &= \omega_2^\theta + a_{01}^\theta \tilde{Y}_{1,t+1} + a_{21}^\theta \tilde{Y}_{1t} + a_{22}^\theta \tilde{Y}_{2t} + \epsilon_{2,t+1}^\theta\end{aligned}$$

By ordering CISS after industrial production, we impose the structural identification assumption that financial variables can react contemporaneously to real variables, but real variables react to financial developments only with a lag. This corresponds to a Choleski identification where shocks to real economic variables have an immediate impact on financial variables,

⁴Available at <https://sdw.ecb.de/home.do>.

while shocks to financial variables are allowed to affect real variables only with a lag. Given the speed at which financial markets react to news, this seems like a reasonable assumption.

As pointed out by Adrian et al. (2019a), the interaction between real and financial variables can be tested by checking whether the off-diagonal coefficients of the matrices A_0^θ and A_1^θ are statistically different from zero:

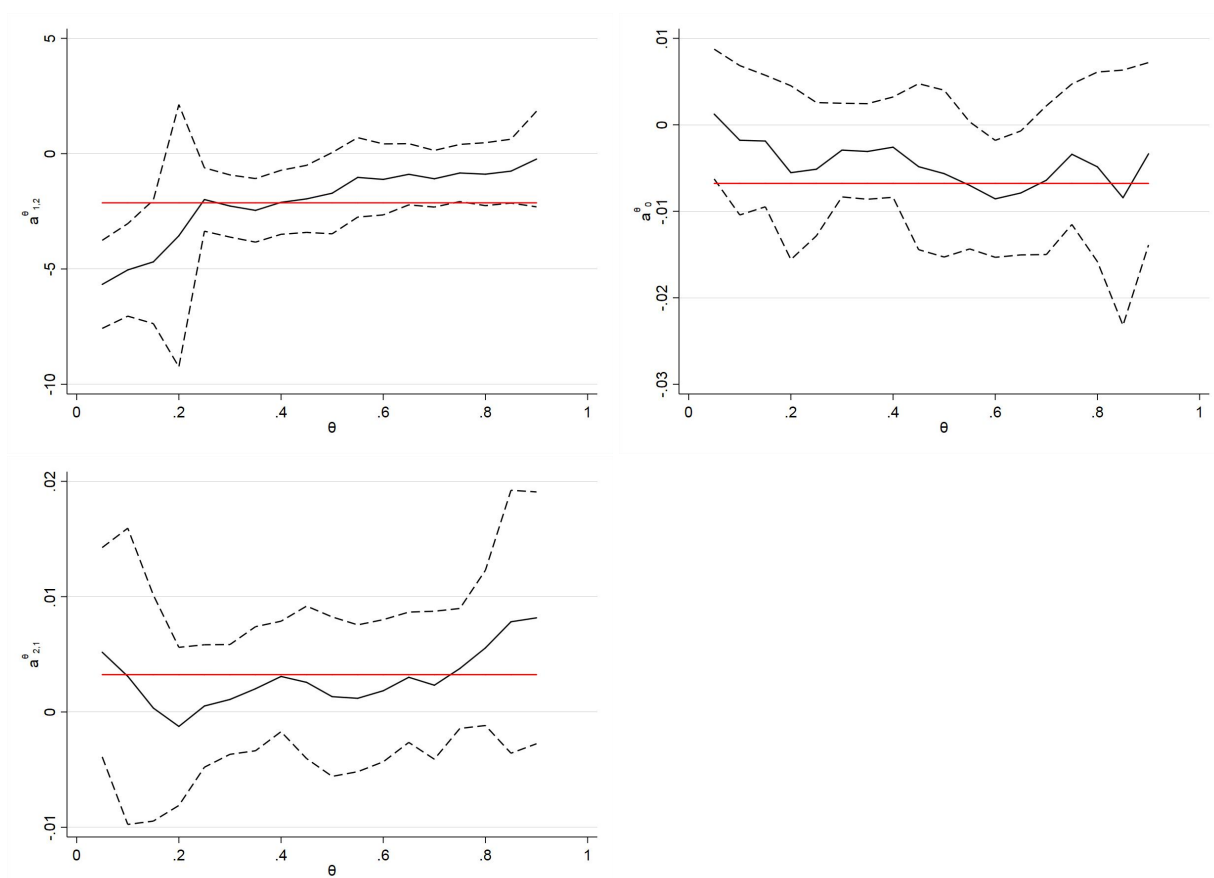
$$H_0 : a_{12}^\theta = a_0^\theta = a_{21}^\theta = 0 \quad (19)$$

Figure 2 reports the estimated quantile coefficients $a_{12}^\theta, a_0^\theta, a_{21}^\theta$ for $\theta \in \{0.05, 0, 10, 0, 15, \dots, 0.95\}$, together with the OLS estimate. We observe the presence of substantial asymmetries, especially in the a_{12}^θ coefficient, which cannot be detected with standard OLS models. The coefficient estimates of a_{12}^θ are consistent with the findings of Adrian et al. (2019b), whereby financial conditions significantly affect the left tail of the distribution of industrial production, but not the right tail.

In the top panel of figure 3, we show that the impact of financial conditions is not only statistically significant, but also economically relevant. The figure reports the 10% and 90% quantiles of industrial production. It reveals that worsening of financial conditions impacts the left tail by about two percentage points. The middle line represents the estimated expected value of industrial production according to a standard OLS VAR model. Notice that the impact of the financial crisis is much more muted relative to the one obtained with the 10% quantile. For comparison, in the bottom panel of figure 3 we report the same time series quantile estimates of industrial production where the off-diagonal coefficient a_{12}^θ has been set to zero.

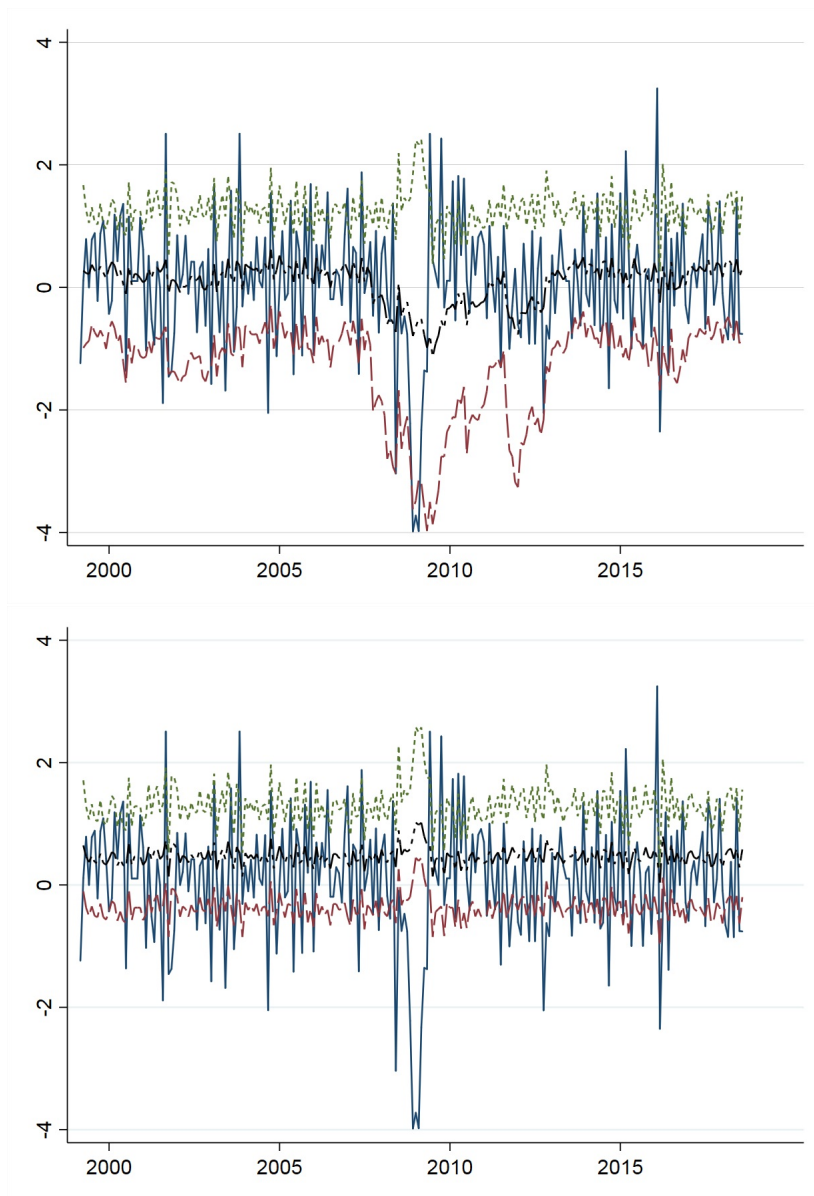
In figure 4 we compute a three dimensional quantile impulse response function corresponding to (9), which studies how different quantiles of industrial production react to a shock to CISS. The thought experiment is the following: How different the various quantiles would have been if we had observed a different realization in the financial conditions of the euro area

Figure 2: Testing interactions between real and financial variables



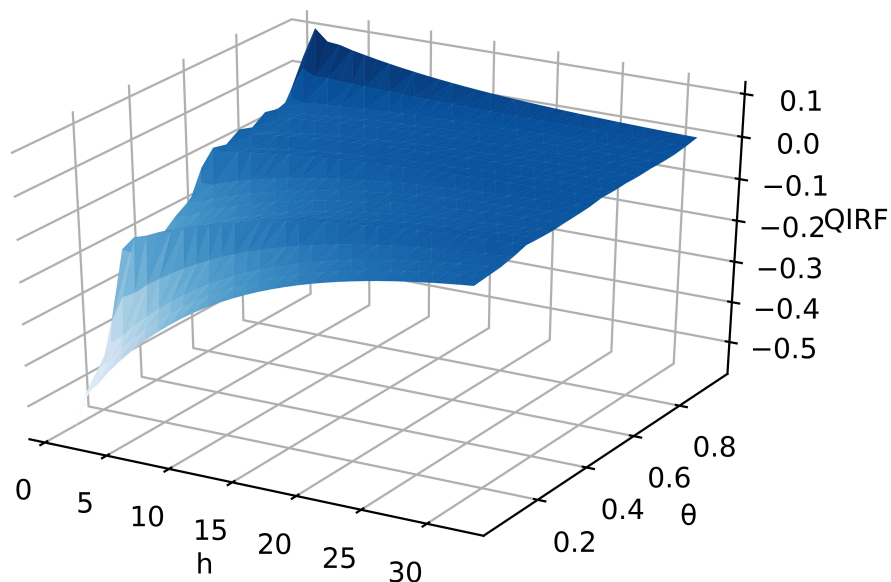
Note: Estimated coefficients of the off diagonal elements at different θ quantiles, with 90% confidence intervals. The flat line represent the OLS estimate.

Figure 3: Euro area growth at risk



Note: Time series estimates of the 10% and 90% quantiles of euro area industrial production, together with the mean estimate according to a standard OLS VAR. The top panel represents the unrestricted estimates, the bottom panel restricts the off-diagonal coefficients to be zero.

Figure 4: Quantile impulse response function for the euro area industrial production



Note: The figure reports how a shock to the financial variable would affect the estimates of the different quantiles of euro area industrial production at different time horizons.

economy? The change in quantile forecasts is measured along the vertical axis (QIRF), while the horizontal plane contains the different quantiles (θ) and time horizons (h). We continue to notice substantial asymmetric impacts in different parts of the distribution, but the chart now reveals that these asymmetries disappear after around 24 periods, which corresponds to two years. This analysis highlights the advantage of our framework. It is an internally consistent fully dynamic model of the real and financial variables of the euro area economy, which allows us to study the propagation of shocks across the different parts of the distribution and through time.

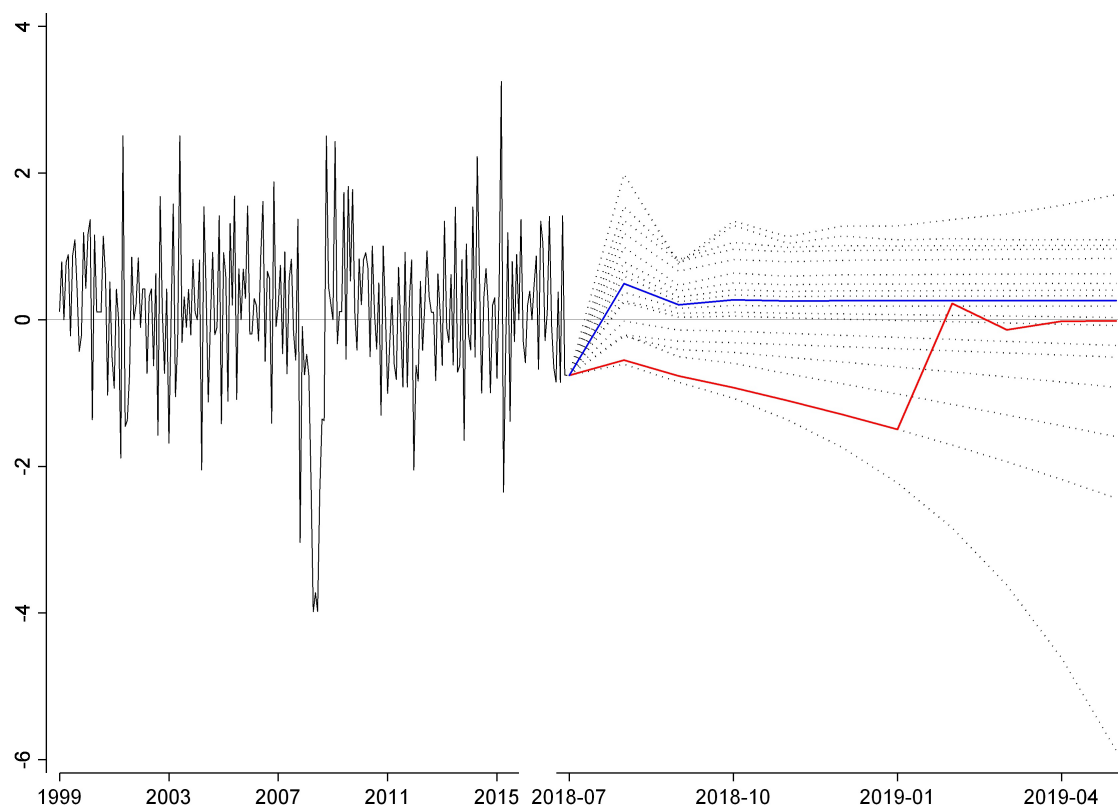
We conclude our empirical illustration of the quantile VAR model with a forecasting and stress testing exercise.

In figure 5, we report the distribution forecast of industrial production several months ahead, conditional on the future endogenous variables being hit by different quantile shocks. Each dotted line corresponds to alternative specifications for the sequence of $\{S_{j_{T+h}}\}_{h=1}^{10}$ matrices in (10). Unlike the forecast within an OLS VAR, which can only set the future OLS shocks to zero, within the quantile VAR we are free to set to zero any future series of shocks. The various dots at each point in time can be thought as possible realizations from the distribution of the future random variables.

We have highlighted two specific scenarios. The one in blue corresponds to a situation where the sequence of future random variables are set to their median values. This roughly corresponds to the results that one would obtain from a standard OLS VAR analysis. Our framework, however, allows us also to create arbitrary stress scenarios and to assess their impact. In the same figure, we have highlighted in red the forecast of the system associated with the following stress testing exercise. We assume that the euro area economy will be hit by a series of six consecutive 90% quantile shocks to its financial system and 10% quantile real economy shocks. This can be seen by the fact that the red line initially follows the trajectory of the second from the bottom dotted line, which traces the forecasts associated with consecutive 90% and 10% quantile shocks. After that, we assume that the system is hit by a series of median shocks, reverting to normal functioning. The number of consecutive tail financial shocks is calibrated to mimic the situation of euro area sovereign debt crisis. We see that industrial production contracts by a maximum of around 2%.

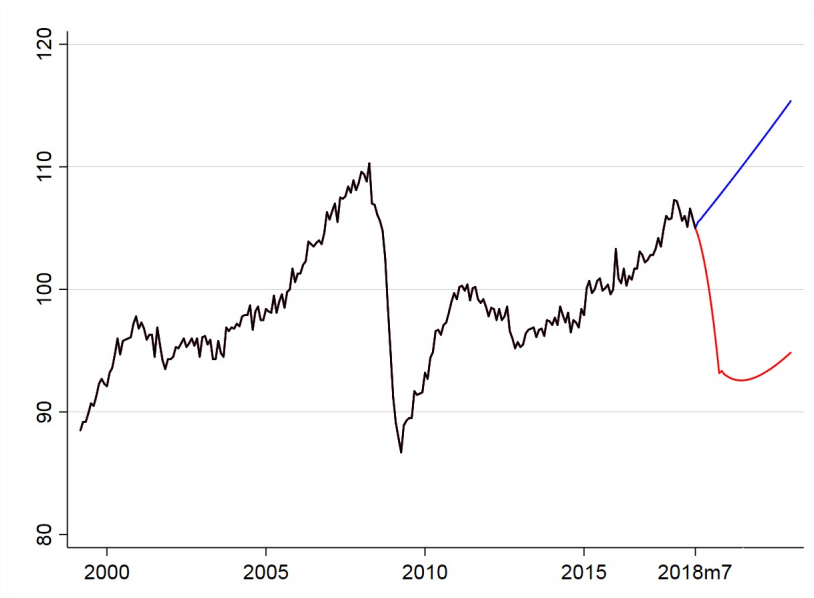
Figure 6 reports the implication of the scenarios of figure 5 in levels of industrial production. Notice that the chosen stress scenario implies an overall contraction in industrial production of more than 10% over 6 months, a contraction falling somewhere in between the one experienced during the

Figure 5: Forecasting and stress testing the real and financial variables in the euro area



Note: The figure reports the time series of industrial production for the euro area together with the forecasts associated with different scenarios. The path highlighted in blue corresponds to a scenario where both the real and financial variables are hit by a sequence of median shocks. The path highlighted in red corresponds to the stress scenario where the financial variable is hit by a 90% shock and the real variable by a 10% shock for six consecutive months, followed by median shocks.

Figure 6: Evolution of industrial production under alternative scenarios



Note: The figure reports the historical time series of industrial production together with its projected levels as of July 2018 under the stress scenario (red line) and median scenario (blue line). The stress scenario is defined as in figure 5 as a sequence of six monthly 90% financial and 10% real shocks, followed by a sequence of median shocks.

financial crisis in 2008-2009 and that of the euro area sovereign debt crisis in 2012. Charts of this type can be used by policy makers to calibrate the severity of the stress test according to their own preferences.

4 Conclusion

We have developed a quantile VAR model and used it to forecast and stress test the interaction between real and financial variables in the euro area. Unlike OLS VAR, quantile VAR models each quantile of the distribution. This provides the natural modelling environment to design particular stress scenarios and test the impact that they have on the economy. A stress scenario is just a sequence of tail quantile shocks, which can be chosen arbitrarily by the policy maker or calibrated to mimic previous crisis episodes. We find the presence of strong asymmetries in the transmission of financial shocks in the euro area, with negative financial shocks being particularly harmful when coupled with negative real shocks. By modelling the average interaction between the random variables, OLS VAR models miss most of these detrimental interactions.

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