Monetary Policy in a World of Cryptocurrencies*

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Abstract

Can currency competition destabilize central banks' control of interest rates and prices? Yes, it can. In a two-currency world, the growth rate of cryptocurrency sets a lower bound on the nominal interest rate and the attainable inflation rate. In a world of multiple competing currencies issued by profit-maximizing agents, the central bank completely loses control of the nominal interest rate and the inflation rate, which are both determined by structural factors, and thus not subject to manipulation, a result welcomed by the proponents of currency competition. The article also proposes some fixes for the classical problem of indeterminacy of exchange rates.

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In recent years cryptocurrencies have attracted the attention of consumers, media and policymakers.[1] Cryptocurrencies are digital currencies, not physically minted. Monetary history offers other examples of uncoined money. For centuries, since Charlemagne, an "imaginary" money existed but served only as unit of account and never as, unlike today's cryptocurrencies, medium of exchange.[2] Nor is the coexistence of multiple currencies within the borders of the same nation a recent phenomenon. Medieval Europe was characterized by the presence of multiple media of exchange of different metallic content.[3] More recently, some nations contended with dollarization or eurization.[4]

However, the landscape in which digital currencies are now emerging is quite peculiar: they have appeared within nations dominated by a single fiat currency just as central banks have succeeded in controlling the value of their currencies and taming inflation.

In this perspective, this article asks whether the presence of multiple currencies can jeopardize the primary function of central banking – controlling prices and inflation – or eventually limit their operational tools – e.g. the interest rate. The short answer is: yes it can.

The analysis posits a simple endowment perfect-foresight monetary economy along the lines of Lucas and Stokey (1987), in which currency provides liquidity services. For the benchmark single-currency model, the results are established: the central bank can control the rate of inflation by setting the nominal interest rate; the (initial) price level instead is determined by an appropriate real tax policy.[5] The combination of these two policies (interest-rate targeting and fiscal policy) determines the path of the price level in all periods, and the central bank can achieve any desired inflation rate by setting the right nominal interest rate.

First I add to this benchmark a privately issued currency that is perfectly substitutable for the government's currency in providing liquidity services. The private currency is "minted" each period according to a constant growth rate μ. The first important result is that the presence of a second currency significantly limits the central bank's maneuvering room and the achievable equilibrium allocation. For an equilibrium to exist, the gross nominal interest rate set by the central bank cannot be lower than $\beta^{-1}(1+\mu)$, β being the consumers' rate of time preference. As a consequence the equilibrium inflation rate in government currency is bounded below by the growth rate of private currency, μ. When private currency is set to grow at a

[1] See BIS (2018).

[2] See Einaudi (1936) for an analysis of the "imaginary" money from the time of Charlemagne to the French Revolution. Loyo (2002) studies optimal choice of unit account in a context of multiple units.

[3] Cipolla (1956,1982,1990) describes several cases in the monetary history of coexistence of multiple currencies.

[4] See Calvo and Vegh (1997) for an analysis of dollarization.

[5] Canzoneri, Cumby and Diba (2001), Niepelt (2004), Sims (1994, 2000, 2013) and Woodford (1995, 2001, 2003), among others, present results of the benchmark single-currency case.

positive rate, i.e. $\mu > 0$, price stability is not attainable.

A pervasive result in this environment, along the lines of Kareken and Wallace (1981), is that the exchange rate between the two currencies is indeterminate. However, if fiscal policy is set in the same way as in the benchmark single-currency framework, this indeterminacy does not bring instability to the price level (in units of government currency).[6]

Next I extend the framework to allow for multiple currencies in a market of profit-maximizing issuers where entry is endogenous and subject to a fixed cost. The results become even more striking: the central bank completely loses control of the nominal interest rate, which is entirely determined by structural factors (the intertemporal discount factor, the exit rate and the fixed cost of entry). As a consequence, the inflation rate too becomes a function of the same structural factors and is therefore out of the control of the authorities. However, the (initial) price level in units of government currency is still determinate if fiscal policy is set appropriately as in the benchmark case. Multiple currency competition, unlike the two-currency variety, can also preclude all possible instability of the path of real money balances in each currency.

I then turn back to analyze exchange-rate determination in the two-currency economy. This problem can be solved if the private currency is issued in a centralized system and the issuer has some taxation power. This approach has novel features with respect to the straightforward application of the fiscal theory of the price level to a two-currency economy: only one of the two monetary authorities can issue securities that have a solely pecuniary return.[7] In general, currencies that are not the liabilities of some agent, like cryptocurrencies, can hardly have a determined exchange rate. The latter is therefore subject to non-fundamental sunspot shocks.

This paper is related to the recent literature prompted by the increasing number of cryptocurrencies, which has revived interest in multiple-currency monetary models. Fernandez-Villaverde and Sanches (2018) evaluate the role of competing private currencies whose supply is determined by profit maximization.[8] Their results differ substantially from mine. They find that an appropriately defined price stability equilibrium can arise under certain restrictions on the cost function for private money production. But, when the marginal cost goes to zero, price stability cannot be an equilibrium.

Motivated by these results, they argue in favour of Milton Friedman's view that a purely private system of fiduciary currencies would inevitably lead to price insta-

[6] It creates indeterminacy of the equilibrium path of government money supply with no consequence for the equilibrium path of consumption except but at time zero.

[7] The fiscal theory of the price level, instead, requires that the government issue at least some securities with only pecuniary value.

[8] Klein (1974) is an early example of a model of currency competition with profit-maximizing suppliers.

bility (Friedman, 1960). Further they find that competition does not achieve the efficient allocation and is socially wasteful. If anything, my currency competition model is reminiscent of Hayek's view that unfettered competition in the currency market is beneficial for society (Hayek, 1976) insofar as I find that the efficient allocation can be reached when the fixed cost of entry goes to zero.[9] Further, unlike Fernandez-Villaverde and Sanches (2018), I find no equilibrium multiplicity or any hyperinflationary equilibria.

The difference might depend on their assumption of a fixed number of issuers without no new entry whereas I allow for entry at a fixed cost. We share the result that a competing currency can restrict the set of possible equilibria. But even in this case the results differ considerably. In their model, there cannot be an equilibrium in which the real interest rate equals the rate of time preference unless private money is driven out of the market, whereas in mine the real interest rate is always tied, in equilibrium, to the rate of time preference. The restrictions that I find are on nominal variables – interest rate and prices.

Schilling and Uhlig (2018) also analyze coexistence and competition between traditional fiat money and cryptocurrencies. But, they are more concerned with the determination of the price of cryptocurrencies, deriving interesting bounds and asset-price relationships. On policy, they assume that the government has always full control of the inflation rate. With respect to monetary policy, they emphasize the connection between the indeterminacy of the cryptocurrency, prices and government money supply. This result emerges also in my two-currency model but not in the multiple-currency profit-maximizing framework. Garratt and Wallace (2017) too are interested in the determinacy of the exchange rate between currencies and revisit the indeterminacy result of Kareken and Wallace (1981). In my analysis, exchange-rate indeterminacy, if present, is not particularly relevant to the way in which competing currencies affect the findings of traditional monetary policy analysis. I also suggest ways to overcome the indeterminacy problem in a centralized system of private currency creation.

Woodford (1995) is the benchmark reference for the analysis of price determination through interest-rate targeting and fiscal rules in a single-currency economy. He also studies a "free banking" regime in which deposits, in units of the single currency, compete with government money in providing liquidity services. In this case, he also finds that the nominal interest rate and the inflation rate are determined by structural factors. However, he does not analyze multiple-currency models.

The paper is structured as follows. Section 1 presents the two-currency model and solves for the equilibrium, and Section 2 analyzes this equilibrium. Section 3 discusses

[9] Marimon et al. (2012) also find that currency competition can achieve efficiency. However, they do not model entry, so in their framework each profit-maximizing issuer would set its price level to infinite to wipe out its existing stock of liabilities. Kovbasyuk (2018), instead, finds that private currency (tokens) can tend to inflate prices and harm agents who hold fiat currency.

what a competing currency implies for standard monetary policy analysis. Section 4 extends the model to multiple currencies with competition among profit-maximizing issuers and assesses the implications. Section 5, going back to the two-currency model, shows how it is possible to solve the problem of indeterminacy of the private-currency exchange rate. Section 6 concludes.

1 The model

We consider a two-currency economy, one issued by the government and one privately. The model follows the lines of Lucas and Stokey (1987) and Benigno and Robatto (2018) in which there are two goods: a "cash" good and a "credit" good. The "cash" good can only be purchased by money.

Consumers have preferences of the following form:

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \ln C_t + X_t \right\}, \tag{1}$$

where β is the intertemporal discount factor with $0 < \beta < 1$; C is the "cash" and X the "credit" good. C can be purchased for money

$$C_t \leq \frac{M_{t-1}}{P_t} + \frac{M^*_{t-1}}{P^*_t} \tag{2}$$

in which M_t is the government-issued and M^*_t the privately issued money. Money can be material (coins or banknotes) or digital, and in either form it carries no interest payment; P_t is the price of the consumption good in terms of the government currency, P^*_t is the price in the private currency.

The cash constraint (2) assumes that the two types of cash are perfect substitutes for the purchase of the consumption good C. This assumption simplifies the analysis at little cost in generality. At least, it enables the model to challenge the results that derive from the single-currency framework to a greater extent.[10]

In the credit market the consumption good X_t is subject to the budget constraint

$$\frac{B_t}{P_t(1+i_t)} + \frac{M_t}{P_t} + \frac{M^*_t}{P^*_t} + X_t \leq \frac{B_{t-1}}{P_t} + \left(\frac{M_{t-1}}{P_t} + \frac{M^*_{t-1}}{P^*_t} - C_t \right) + Y + \frac{T_t}{P_t} + \frac{T^*_t}{P^*_t}, \tag{3}$$

in which B is a risk-free interest-bearing security in units of the government currency; Y is the constant endowment of the two consumption goods; T_t are government transfers in units of government currency and T^*_t are the private issuer's transfers in units of its currency.

[10]Schilling and Uhlig (2018) also assume perfect substitutability between the two moneys.

In writing the budget constraint (3), we make two important assumptions: first, interest-bearing securities provide no liquidity services; second, they are only denominated in the government currency. The significance of the latter assumption is discussed later. B_t can be positive, in which case it is an asset for the household, or negative, in which case it is debt. I allow the private sector to borrow by issuing debt denominated in government currency but not in the privately-issued currency. And this debt, when issued, is paid back with certainty, being subject to an appropriate borrowing limit.

Before addressing this limit, note that the impossibility of arbitrage has two implications for equilibrium. First, it requires that

$$\frac{P_{t+1}}{P_t} = \frac{P^*_{t+1}}{P_t} \tag{4}$$

for each $t \geq t_0$ and therefore that the exchange rate S_t – the price of the privately-issued currency in government currency (i.e. $S_t \equiv P_t/P^*_t$) – is constant over time at a value S, which has to be determined. The equivalence of money returns shown in (4) follows from the assumption of perfect substitutability between the two moneys in the cash constraint.

The second implication of the absence of arbitrage is that the nominal interest rate in government currency is non-negative, $i_t \geq 0$. If it were negative, households could make infinite profits by borrowing at negative interest rates and investing the proceed in cash.

Given these two results, the flow budget constraint (3) can be written as

$$\frac{W_t}{(1+i_t)P_t} + C_t + X_t + \frac{i_t}{1+i_t}\frac{M_t}{P_t} + \frac{i_t}{1+i_t}\frac{SM^*_t}{P_t} \leq \frac{W_{t-1}}{P_t} + Y + \frac{T_t}{P_t} + \frac{ST^*_t}{P_t},$$

in which nominal wealth is defined in units of government currency, as

$$W_t \equiv B_t + M_t + SM^*_t.$$

The natural borrowing limit is then written as

$$\frac{W_{t-1}}{P_t} \geq -\sum_{T=t}^{\infty} Q_{t,T}\left(Y + \frac{T_T}{P_T} + \frac{ST^*_T}{P_T}\right) > -\infty \tag{5}$$

for each $t \geq t_0$ given an appropriate discount factor $Q_{t,T}$ with $Q_{t,t} \equiv 1$ and

$$Q_{t,T} = \frac{P_T}{P_t}\prod_{j=1}^{T}\frac{1}{1+i_{t+j}},$$

for $T > t$.

The natural borrowing limit (5) is the maximum amount of net debt that the consumer can carry in a certain period of time and repay with certainty, i.e. with current and future net income and assuming that future consumption and asset holdings are going to be equal to zero. The finite borrowing limit is a requirement for consumption to be bounded in the optimization problem. This assumption can be equivalently written as

$$\sum_{t=t_0}^{\infty} Q_{t_0,t}\left(\frac{T_t}{P_t}\right) < \infty \qquad \sum_{t=t_0}^{\infty} Q_{t_0,t}\left(\frac{ST_t^*}{P_t}\right) < \infty.$$

For a bounded consumption plan to exist, the following two infinite sums must have a finite value

$$\sum_{t=t_0}^{\infty} Q_{t_0,t}\left(\frac{i_t}{1+i_t}\frac{M_t^g}{P_t}\right) < \infty \qquad \sum_{t=t_0}^{\infty} Q_{t_0,t}\left(\frac{i_t}{1+i_t}\frac{SM_t^{*p}}{P_t}\right) < \infty.$$

We now turn to the optimality conditions. The first-order condition with respect to the consumption of the cash good is

$$C_t = \frac{1}{1+\lambda_t}$$

for each $t \geq t_0$ in which $\lambda_t \geq 0$ is the Lagrange multiplier associated with the cash constraint (2). The optimality condition with respect to holdings of the interest-bearing security B_t is

$$1 + i_t = \frac{1}{\beta}\frac{P_{t+1}}{P_t} \tag{6}$$

at each time $t \geq t_0$ while those with respect to M_t and M_t^* are:

$$1 + \lambda_{t+1} = \frac{1}{\beta}\frac{P_{t+1}}{P_t} \tag{7}$$

at each time $t \geq t_0$. Combining (6) and (7), it follows that $\lambda_{t+1} = i_t$ for each $t \geq t_0+1$. Finally, the intertemporal budget constraint holds with equality,

$$\frac{W_{t_0-1}}{P_{t_0}} + \sum_{t=t_0}^{\infty} \beta^{t-t_0}\left(Y + \frac{T_t}{P_t} + \frac{ST_t^*}{P_t}\right) = \sum_{t=t_0}^{\infty} \beta^{t-t_0}\left(C_t + X_t + \frac{i_t}{1+i_t}\frac{M_t}{P_t} + \frac{i_t}{1+i_t}\frac{SM_t^*}{P_t}\right). \tag{8}$$

Let us now specify the budget constraint of the two currency issuers. The government's monetary authority is subject to the following budget constraint

$$M_t^g - \frac{B_t^g}{1+i_t} = T_t + M_{t-1}^g - B_{t-1}^g$$

where M_t^g is the supply of cash, and B_t^g is the assets held by the government, if positive, or the debt issued, if negative. In the first case, we implicitly assume that the private sector takes on debt denominated in government currency and fully repaid because of the natural borrowing limit, and that the government, through the central bank, holds this debt. In the second case, we assume that the government issues also interest-bearing liabilities that are held by the private sector and that are fully repaid, since the government's liabilities define the unit of account in government currency.

The private issuer is instead subject to the following constraint

$$M_t^{*p} = T_t^* + M_{t-1}^{*p},$$

which can be interpreted either as the budget constraint of an agent issuing money in a centralized system or simply as an identity regulating how private money is created in a decentralized system. The latter interpretation is akin to the way in which some cryptocurrencies, such as Bitcoins, are created nowadays.

2 Equilibrium

Equilibrium in the goods market implies that the sum of the consumption of the two goods is equal to the constant endowment

$$C_t + X_t = Y.$$

Equilibrium in the market for the interest-bearing security in government currency requires that

$$B_t + B_t^g = 0,$$

while equilibrium in the cash market for the two currencies implies that supply and demand equalize for each currency

$$M_t = M_t^g,$$

$$M_t^* = M_t^{*p}.$$

We can now summarize the set of equilibrium conditions concisely.

The first equation to consider is the Euler equation (6) derived from household optimality condition with respect to the interest-bearing security B :

$$1 + i_t = \frac{1}{\beta}\frac{P_{t+1}}{P_t} \tag{9}$$

which holds for each $t \geq t_0$. The Euler equation links the nominal interest rate to the real rate and the future gross inflation rate.

The second equilibrium condition is the consumer's intertemporal budget constraint (8) which, using equilibrium in the good and asset markets, can be written as

$$\frac{M^g_{t_0-1} - B^g_{t_0-1} + SM^*_{t_0-1}}{P_{t_0}} + \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left(\frac{T_t}{P_t} + \frac{ST^*_t}{P_t} \right) = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\frac{i_t}{1+i_t} \left(\frac{M^g_t}{P_t} + \frac{SM^{*p}_t}{P_t} \right) \right]. \tag{10}$$

The mirror image of this constraint is an aggregate intertemporal budget constraint consolidating both currency issuers. In (10), the initial value of the real liabilities of both agents plus the present discounted value of real transfers should be equal to the seigniorage revenues accruing to both suppliers from issuance of non-interest bearing liabilities that have non-pecuniary benefits for the consumer.

In equilibrium, for a bounded consumption path to exist, it should be the case that all infinite sums have a finite value, i.e.

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{T_t}{P_t} < \infty \qquad \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{ST^*_t}{P_t} < \infty, \tag{11}$$

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{i_t}{1+i_t} \frac{M^g_t}{P_t} < \infty \qquad \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{i_t}{1+i_t} \frac{SM^{*p}_t}{P_t} < \infty. \tag{12}$$

Although there is an aggregate intertemporal budget constraint that pools the two issuers together, each has a different flow budget constraint on policy decisions. The following constraint applies to the government

$$M^g_t - \frac{B^g_t}{1+i_t} = T_t + M^g_{t-1} - B^g_{t-1} \tag{13}$$

given initial conditions $M^g_{t_0-1}$ and $B^g_{t_0-1}$.

Private issuance of currency instead follows the law of motion

$$M^{*p}_t = T^*_t + M^{*p}_{t-1}, \tag{14}$$

given an initial condition $M^{*p}_{t_0-1}$.

From the optimality condition of the household, consumption of good C is inversely related to the nominal interest rate

$$C_t = \frac{1}{1+i_{t-1}}, \tag{15}$$

for each $t \geq t_0 + 1$, and

$$C_{t_0} = \frac{1}{1+\lambda_{t_0}}$$

at time t_0. The cash-in-advance constraint implies that

$$\frac{M^g_{t-1}}{P_t} + \bar{S}\frac{M^{*p}_{t-1}}{P_t} \geq C_t \tag{16}$$

for each $t \geq t_0$ with equality whenever $i_{t-1} > 0$.

To close the set of equilibrium conditions, one must specify the monetary policy regime.[11] There are three degrees of freedom according to which one can specify monetary policy for the two issuers. I assume that the government authority eventually sets the path $\{i_t, T_t\}_{t=t_0}^{\infty}$ as a function of other variables. In the flow budget constraint (13), this means that given initial conditions $M^g_{t_0-1}$ and $B^g_{t_0-1}$, the government is setting the size of the balance sheet $M^g_t - B^g_t/(1+i_t)$ and leaving the private sector to allocate its wealth optimally between the two government securities. I assume that the private issuer sets the transfer $\{T^*_t\}_{t=t_0}^{\infty}$ which, given the initial condition on $M^{*p}_{t_0-1}$, directly implies the path of its money supply according to (14). This specification is perfectly in line with how such cryptocurrencies, as Bitcoins, are now being issued, namely by a proof-of-work system that determines the new units to be emitted at each point in time.

Definition 1 *An equilibrium is a set of sequences* $\{C_t, P_t, i_t, M^g_t, M^{*p}_t, B^g_t, T_t, T^*_t\}_{t=t_0}^{\infty}$ *with* $C_t, P_t, i_t, M^g_t, M^{*p}_t \geq 0$ *at each time* $t \geq t_0$ *and non-negative values* $\bar{S}$ *and* λ_{t_0} *that are consistent with the monetary-policy regime and that satisfy (9), (13), (14), (16) (with equality whenever* $i_{t-1} > 0$*) for each* $t \geq t_0$ *and (15) for each* $t \geq t_0 + 1$ *together with the intertemporal budget constraint (10), the bounds (11), (12) and the constraint* $C_{t_0} = (1+\mu_{t_0})^{-1}$*, given initial conditions* $M^g_{t_0-1}, B^g_{t_0-1}, M^{*p}_{t_0-1}$.

Analysis will be restricted to a certain class of policy rules.

Definition 2 *Assume the following specification of the monetary policy regime. The government sets a constant interest rate policy* $i_t = i$ *at each* $t \geq t_0$ *and the following transfer policy*

$$\frac{T_t}{P_t} = \frac{i}{1+i}\frac{M^g_t}{P_t} - (1-\beta)\tau \tag{17}$$

for each $t \geq t_0$ *and for some* τ *different from zero. The private currency issuer sets* $T^*_t = \mu M^{*p}_{t-1}$ *at each* $t \geq t_0$ *with* $\mu > -1$*, to achieve a constant growth rate,* $1+\mu$*, of its money supply.*

Let us focus on the implications of such a monetary policy regime. First, by the Fisher equation (9), the constant-interest-rate policy implies that inflation too is constant

$$\frac{P_{t+1}}{P_t} = \beta(1+i). \tag{18}$$

[11] Analysis of all possible regimes is beyond the scope of this work.

The inflation rate is positive whenever $(1+i) > \beta^{-1}$, negative when $(1+i) < \beta^{-1}$. A price stability policy is in place when the nominal interest rate is set equal to the inverse of the consumer's rate of time preference, i.e. $(1+i) = \beta^{-1}$.

The government also sets a transfer policy in which it rebates its entire seigniorage revenue to the consumers – the first element on the right-hand side of (17) – minus a constant proportional to a non-zero value τ. In the terminology of the literature on price-level determination, the transfer rule (17) is an "active" transfer policy.

Finally, it is assumed that the private issuer sets a constant growth rate of supply; growth can also be negative, where the issuer is able to destroy money by some electronic algorithm.

The assumptions on the monetary policy regime carry important implications for the conditions of possible equilibrium. Assuming $i > 0$, the cash constraint (16) holds with equality. After plugging in the private issuer's supply rule and $C_t = 1/(1+i)$, we obtain

$$\frac{M_t^g}{P_{t+1}} + S\frac{(1+\mu)^{t+1-t_0} M_{t_0-1}^{*p}}{P_{t+1}} = \frac{1}{1+i}$$

into which one can substitute (18) to obtain

$$\frac{M_t^g}{P_{t+1}} + S\left(\frac{1+\mu}{\beta(1+i)}\right)^{t+1-t_0} \frac{M_{t_0-1}^{*p}}{P_{t_0}} = \frac{1}{1+i}.$$

From this it follows that for an equilibrium to exist it must be the case that $1+\mu \leq \beta(1+i)$: otherwise the government money supply will be negative within some finite period of time, which is not feasible.

Note, however, that when $i = 0$, then the cash constraint (16) holds with inequality so it is possible for real money balances to grow without bound.

Moreover, under the monetary policy regime defined above, we explicitly write the following summations

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \left(\frac{ST_t^*}{P_t}\right) = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left(\frac{S\mu M_{t-1}^{*p}}{P_t}\right) = \mu\frac{SM_{t_0-1}^{*p}}{P_{t_0}} \sum_{t=t_0}^{\infty} \left(\frac{1+\mu}{1+i}\right)^{t-t_0} \tag{19}$$

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{i_t}{1+i_t} \left(\frac{SM_t^*}{P_t}\right) = \frac{i(1+\mu)}{1+i} \frac{SM_{t_0-1}^{*p}}{P_{t_0}} \sum_{t=t_0}^{\infty} \left(\frac{1+\mu}{1+i}\right)^{t-t_0} \tag{20}$$

where in (19) the transfer rule of the private issuer is substituted into the first equality and the equilibrium condition (18) into the second equality. Similarly in (20). The first sum is finite whenever $\mu < i$ for any $i \geq 0$, while (20) requires that $\mu < i$ only when $i > 0$.

The results of the above discussion are grouped in the following Lemma.

Lemma 3 *Given the monetary policy regime specified in Definition 2, a necessary condition for an equilibrium to exist is: i)* $1 + \mu \leq \beta(1 + i)$ *when* $i > 0$ *and ii)* $\mu < 0$ *when* $i = 0$.

We now analyze whether given the above specification of the policy regime the price level and the exchange rate can be determined.

3 Equilibrium implications of competing currencies

Before analyzing the consequences of a multiple-currency environment, let me first outline the results of the benchmark one-currency model. Assume that there is only the government currency and that the government sets the same policies as in Definition 2: a constant nominal-interest-rate policy and an appropriate real transfer policy. First, by setting the nominal interest rate at a target i it can also set the gross inflation rate, Π, to any desired constant number given by $\Pi = \beta(1 + i)$, which is a function of the interest rate chosen. Second, the initial price level and the entire price path are fully determined by the fiscal policy rule (17). The central bank thus has full control of the price level and its growth. Note that to determine the price level it is critical that the government issue some liabilities with only pecuniary return, such as security B. Moreover, given interest-rate targeting, the money supply path is endogenously determined by the cash-in-advance constraint (2).

Adding a competing currency alters these conclusions substantially. Given Lemma 3, the first result says that, if the government wants to fix a positive nominal interest, it should set the nominal interest rate to satisfy the inequality $1 + i \geq \beta^{-1}(1 + \mu)$, otherwise it could set the interest to zero provided $\mu < 0$. That is, the rate of growth of the private currency constrains the government's choice of its interest rate, in order for an equilibrium to exist.

There are also implications for the feasible inflation targets. Consider first the case in which the government wants to set a positive interest rate. Inserting the requirement $(1 + i) \geq \beta^{-1}(1 + \mu)$ into the Fisher equation (9), we obtain that in equilibrium the gross inflation rate Π (in government currency) must satisfy the following inequality $\Pi \geq (1 + \mu)$. So, given the exogenous growth rate of private currency, in equilibrium the inflation rate in government currency is bounded below by that number. In particular, if μ is positive, price stability cannot be an equilibrium. Consider now the case in which $\mu < 0$. For an equilibrium to exist, the government can also set $i = 0$ according to Lemma 3, which implies in (18) that prices are falling at the rate β and are therefore not necessarily bounded above by $1 + \mu$.

We now investigate whether a competing currency can create problems for the determination of the price level. The following Proposition summarizes these results.

Proposition 4 *Given the monetary policy regime of Definition 2, an equilibrium exists if and only if: (i) the government sets a positive interest rate and* $(1+i) \geqslant \beta^{-1}(1+\mu)$*; or (ii) the government sets* $i = 0$ *given* $\mu < 0$*. In either case, the path of the price level in government currency is determined, but the exchange rate* S *and consumption at time* t_0 *are not. Moreover the inflation rate in government currency is bounded below by the growth of private currency, i.e.* $P_{t+1}/P_t \geq (1+\mu)$*, when the nominal interest rate is positive.*

Proof. See Appendix A.1. ■

Lemma 3 and Proposition 4 have two interesting implications. First, the presence of a second currency causes no problem for the determinacy of the price level. The potential problem would be the non-existence of a single intertemporal budget constraint for each currency supplier. But given that $i > \mu$ in any equilibrium, an intertemporal budget constraint holds for the supplier(s) of private currency. To see this, note that given $i > \mu$ it follows that

$$\lim_{T \to \infty} \beta^{T-t_0} \frac{SM_T^{*p}}{P_{T+1}} = \frac{SM_{t_0-1}^{*p}}{\beta P_{t_0}} \left(\frac{1+\mu}{1+i} \right)^{T+1-t_0} = 0$$

which, together with the flow budget constraint (14), implies an intertemporal budget constraint for the private currency issuers. See the Appendix for details. The existence of an intertemporal budget constraint for the private currency issuer together with the aggregate intertemporal budget constraint (10) implies that the constraint holds for the government exactly in the same form as in the benchmark case of a single currency. Applying the real transfer policy (17), we get price determination. However, while this analysis precludes multiple equilibria for prices, the rate of inflation is affected by the presence of a competing currency, in line with the foregoing discussion.

The second result, where multiple equilibria arise, is the indeterminacy of the exchange rate, as in Kareken and Wallace (1981) and recently restated by Garratt and Wallace (2017) and Schilling and Uhlig (2018). In my model, this results in the indeterminacy of consumption at time t_0 and of the equilibrium path of government money starting from period $t_0 + 1$.[12]

To see the first result, consider the cash constraint at time t_0

$$\frac{M_{t_0-1}}{P_{t_0}} + \frac{SM_{t_0-1}^{*}}{P_{t_0}} \geq C_{t_0}$$

and note that since P_{t_0} is determined and M_{t_0-1}, $M_{t_0-1}^{*}$ are given, variations in S translate into variations in C_{t_0} unless consumption is at the efficient level. To see the

[12] The path of consumption starting from period $t_0 + 1$ is instead fully determined by the path of nominal interest rates set by the government.

second result, consider the cash constraint (16) when $i > 0$ and use (15) and (18) to obtain

$$\frac{M_t^g}{P_t} + S\frac{M_t^{*p}}{P_t} = \beta$$

for each $t \geq t_0$. Since P_t is determined and M_t^{*P} is also given by the process of private money creation, the indeterminacy of the exchange rate S translates directly into indeterminacy of the path of government money supply. By contrast prices are completely insulated from the indeterminacy of the exchange rate, although their rate of increase is bounded below by the rate of growth of private currency, as noted earlier.

4 Multiple-currency competition

Now let us extend the above framework to a multiple-currency environment. Each currency issuer operates in a centralized system and chooses its money supply to maximize profit. Entry into the market is subject to a fixed cost, and the number of issuers is endogenously determined by the zero-rent condition. Government currency is supplied as in the benchmark model, with a monetary policy of a constant nominal interest rate and an appropriate real transfer policy.

A concise summary of the results is that the government completely loses control of the nominal interest rate and of the inflation rate, which are now both determined by structural factors. The price level and real money balances in each currency are also determined.

Assume that at time $\mathrm{t}_0 - 1$ there is only government money; therefore at t_0 the cash-in-advance constraint is:

$$C_{t_0} \leq \frac{M_{t_0-1}}{P_{t_0}}.$$

Starting from time t_0 other issuers can enter and the liquidity constraint generalizes to

$$C_t \leq \frac{M_{t-1}}{P_t} + \sum_{j=1}^{J_t} \frac{M_{t-1}^j}{P_t^j}$$

for each $t \geq t_0 + 1$, which includes a generic number J_t of private currencies (to be determined in equilibrium) plus the government currency: M_{t-1}^j are money holdings of currency j and P_t^j is the price of goods in terms of currency j.

At each time $\bar{t}$ with $\bar{t} \geq t_0$, a number $N_{\bar{t}}$ of currency issuers enters the market, each paying a fixed cost $\Phi > 0$. In this group, a generic issuer of type j chooses the sequence $\{M_t^j\}_{t=\bar{t}}^{\infty}$ so as to maximize its liquidity rent (specified below) subject to exit rate which occurs with probability $\delta > 0$. In the case of exit at time $t + 1$, the money holdings M_t^j chosen in period t still provide liquidity services in period $t + 1$

and become useless only at the end of the period. Therefore, the first-order condition with respect to M_t^j is

$$\frac{1}{P_t^j} = \beta(1 - \delta + \lambda_{t+1})\frac{1}{P_{t+1}^j}, \tag{21}$$

since the currency has unitary payoff with probability $1 - \delta$ and zero otherwise and always provides liquidity services, receiving the premium λ_{t+1}. The above condition holds for any $t \geq \bar{t}$.

Given that first-order condition (7) still applies, comparing it with (21), we get that

$$\frac{P_{t+1}^j}{P_t^j} = \frac{P_{t+1}}{P_t} - \beta\delta \tag{22}$$

which implies that the return on private money (the inverse of its inflation rate) should be higher than on government money in order to compensate for its exogenous rate of exit. Given the latter, the number J_t of private currencies follows the law of motion $J_t = N_t + (1-\delta)J_{t-1}$ with initial condition $J_{t_0-1} = 0$.

Each issuer is subject to the flow budget constraint

$$M_{t-1}^j + T_t^j = M_t^j,$$

starting from its entry at time $\bar{t}$ with $M_{\bar{t}-1}^j = 0$. Real profits at time t are given by the liquidity premium λ_{t+1} in period $t+1$ multiplied by the real money balances, i.e. $\lambda_{t+1}M_t^j/P_{t+1}^j$ discounted by the rate of time preference β. Each supplier is assumed to maximize the discounted present value of real profits by choosing the sequence $\{M_t^j\}_{t=\bar{t}}^{\infty}$ on condition that it will still be on the market, and therefore it maximizes:

$$\beta\sum_{t=\bar{t}}^{\infty}[(1-\delta)\beta]^{t-\bar{t}}\lambda_{t+1}\frac{M_t^j}{P_{t+1}^j} \tag{23}$$

starting from its entry period $\bar{t}$.

In an equilibrium with competing currencies, then, the following results obtain.

Proposition 5 *Given a fixed entry cost $\Phi > 0$ and an exogenous exit probability $0 < \delta < 1$, in an equilibrium with currency competition and profit-maximizing issuers: (i) consumption is constant and equal to*

$$C_t = 1 - z^{\frac{1}{2}} > 0, \tag{24}$$

for each $t \geq t_0 + 1$ with $z \equiv (1-(1-\delta)\beta)\Phi/\beta$ (assuming $z < 1$); (ii) the supply of real money balances of each private issuer is constant and equal to

$$\frac{M_t^*}{P_{t+1}^*} = z^{\frac{1}{2}} - z > 0; \tag{25}$$

and (iii) *the number of private suppliers of currency is given by*

$$J_t = z^{-\frac{1}{2}} \left(1 - \frac{M_t^g}{P_{t+1}} \frac{1}{1 - z^{-\frac{1}{2}}} \right). \tag{26}$$

Proof. See Appendix A.2. ■

Some interesting implications derive from Proposition 5: (i) the nominal interest rate and (ii) the inflation rate in each currency are constant and are functions of structural parameters.[13] To see the first result, combine the constant consumption level of equation (24) with the equilibrium condition $C_t = 1/(1 + i_{t-1})$ at each time $t \geq t_0 + 1$. It follows that the nominal interest rate is constant and determined by the entry cost Φ, the exogenous exit rate δ and the intertemporal discount factor β through the parameter z

$$1 + i_t = \frac{1}{1 - z^{\frac{1}{2}}}, \tag{27}$$

for each $t \geq t_0$. One implication of currency competition is therefore that the government loses control of the nominal interest rate.[14] Moreover, to see that structural factors also determine the inflation rate in each currency, combine (27) with the Euler equation (9) to obtain

$$\frac{P_{t+1}}{P_t} = \frac{\beta}{1 - z^{\frac{1}{2}}} \qquad \frac{P_{t+1}^*}{P_t^*} = \frac{\beta}{1 - z^{\frac{1}{2}}} - \delta\beta, \tag{28}$$

where the second equality follows from (22).

We collect these results in the following Corollary.

Corollary 6 *Given a fixed entry cost* $\Phi > 0$ *and an exogenous exit probability* $0 < \delta < 1$, *in an equilibrium with currency competition and profit-maximizing issuers:* (i) *the nominal interest rate and* (ii) *the inflation rates are determined by structural factors:*

$$1 + i_t = \frac{1}{1 - z^{\frac{1}{2}}},$$

$$\frac{P_{t+1}}{P_t} = \frac{\beta}{1 - z^{\frac{1}{2}}} \qquad \frac{P_{t+1}^*}{P_t^*} = \frac{\beta}{1 - z^{\frac{1}{2}}} - \delta\beta.$$

[13] With appropriate qualifications a similar result could arise in a model of competition among securities all denominated in the same currency, see Woodford (1995).

[14] The real interest rate, however, is always tied to the rate of time preference. This is different from the finding of Fernandez-Villaverde and Sanchez (2018), namely that there is no equilibrium in which the real interest rate is equal to the rate of time preference.

Note that in the special case of zero entry cost there are an infinite number of issuers and the competition drives liquidity premia to zero. The efficient allocation is attained.[15] Therefore the nominal interest rate is at the zero lower bound, which is equivalent to have full satiation of liquidity. Prices in government currency fall at rate β and all other prices – in private currencies – at $\beta(1-\delta)$. These results are perfectly in line with Hayek (1974), who advocated private money creation way to achieve efficiency in the liquidity provision. The loss of control over the nominal interest rate and inflation too is consistent with his position that government should not have monopoly power to manipulate interest rates and prices. Rather, unfettered competition can work to keep inflation under control.

An important further result of Proposition 5 is that equilibrium real money balances for each of private currency are uniquely determined and constant.[16] They are not affected by exchange rate indeterminacy, which was instead pervasive in the two-currency model (Section 1). As a consequence, real money balances for government currency are also determined again departing from the two-currency model.

The key remaining question is whether there is some indeterminacy of the price level, in government currency, now that interest and inflation rates are determined by exogenous factors. The answer is no, the argument follows the foregoing discussion with some caveats. For one thing, we need to appropriately redefine the monetary regime.

Definition 7 *Assume the following specification of the monetary policy regime. The government sets the sequence of money supply $\{M_t^g\}_{t=t_0}^{\infty}$ and the following transfer policy*

$$\frac{T_t}{P_t} = \frac{i_t}{1+i_t}\frac{M_t^g}{P_t} - (1-\beta)\tau \tag{29}$$

for each $t \geq t_0$ and for some τ different from zero.

With respect to the benchmark monetary policy regime (Definition 2), here the interest-rate policy is replaced by a money supply policy.

Proposition 8 *Given the monetary policy regime of Definition 7 and the results of Proposition 5, in an equilibrium under currency competition the path of the price level in government currency is determined as is consumption at time t_0.*

Proof. See Appendix A.3. ■

This result contradicts Friedman (1960), who held that purely private fiduciary currencies would inevitably produce price instability. Fernandez-Villaverde and Sanchez

[15] Fernandez-Villaverde and Sanchez (2018) instead find that private money creation can be socially wasteful.

[16] This is again in contrast with Fernandez-Villaverde and Sanches (2018), who find multiple equilibria, including hyperinflationary equilibria with real money balances that converge to zero.

(2017) reach a similar conclusion: "in a monetary system with competitive issuers the supply of each brand becomes unbounded when the marginal cost goes to zero...Private entrepreneurs always have an incentive to mint just a little bit more of the currency." In my model, what eliminates this is the possibility of entry into the market, which drives all rents to zero. Moreover, an appropriate real transfer policy fully determines the path of prices.

5 Determinacy of the exchange rate

Let us return to the two-currency model to see whether it is possible to solve the problem of indeterminacy of the exchange rate of the cryptocurrency. I have already observed that this is not the main reason for monetary policymakers to be concerned over the existence of other currencies.

We need to explore alternative monetary regimes for private issuance to address the indeterminacy problem. A characteristic of some cryptocurrencies is that they are not the liability of any single agent (decentralization), while government currency is the liability of the central bank (centralization). The rule of constant money-supply growth for private currency (Definition 2) can describe the behavior of a decentralized as well as of a centralized system. On the other hand, transfer policy (17) can be more naturally instituted by an agent that has full control of its balance sheet and that can back the value of its liability by a real transfer/tax policy. To obtain determinacy, a one possibility is to posit that the private currency is issued by an agent in a centralized system that at some point can switch to an "active" transfer policy of the same kind as (17).

Definition 9 *Assume the following specification of the monetary policy regime. The government sets a constant interest rate policy $i_t = i$ at each $t \geq t_0$ and the following transfer policy*

$$\frac{T_t}{P_t} = \frac{i}{1+i}\frac{M_t^g}{P_t} - (1-\beta)\tau \tag{30}$$

for each $t \geq t_0$ and for some τ different from zero. The private currency issuer sets $T_t^* = \mu M_{t-1}^{*p}$ for each $t_0 \leq t < \tilde{t}$ with $\mu > -1$, to achieve a constant growth rate, $1+\mu$, of its money supply and switches to a real transfer policy

$$\frac{T_t^*}{P_t^*} = \frac{i}{1+i}\frac{M_t^{*p}}{P_t^*} - (1-\beta)\tau^* \tag{31}$$

for each $t \geq \tilde{t}$ and some $\tau^* > 0$.

The difference with respect to the monetary policy regime set in Definition 2 is the assumption that the private issuer switches to a real transfer policy at or after time $\tilde{t}$, which may coincide with the initial period t_0 or be postponed far into the future. This change to the monetary regime is sufficient for an equilibrium to have determinacy of all the variables, and in particular the price level (in government currency) and the nominal exchange rate. A necessary condition, in the following Proposition, is that the government-set nominal interest rate be positive.

Proposition 10 *Given the monetary policy regime of Definition 9, in an equilibrium with a positive nominal interest rate $i_t = i > 0$ for each $t \geq t_0$, the equilibrium is determinate.*

Proof. See Appendix A.4. ■

It is worth highlighting some of the key elements of the proof of Proposition 10. First is the requirement of a positive interest rate. In this case the liquidity constraint (16) holds with equality, and it follows that in equilibrium real money balances are bounded and therefore $\lim_{t\to\infty} \beta^t M_{t-1}^{*p}/P_t^* = 0$. This condition is sufficient to separate the aggregate intertemporal budget constraint into two components, one for the government and the other for the private issuer. It then follows that given the backing implied by the regime of Definition 9 through transfer rules (30) and (31), the path of prices in each currency is determined and so is the exchange rate.

The proposal of Definition 9 might seem to be a straightforward extension of the fiscal theory to a two-currency setting, but there is one key difference. While the fiscal theory applied to a single currency requires that the monetary authority issue at least some securities with only pecuniary return, like B, in the two-currency extension only one currency issuer should supply such securities. Indeed, supposing that the private issuer of currency also issues interest-bearing securities with only pecuniary return, let's say B^{*p}, the fact that $\lim_{t\to\infty} \beta^t M_{t-1}^{*p}/P_t^* = 0$ in any equilibrium with positive interest rate no longer implies the possibility of defining two separate intertemporal budget constraints. Consequently prices and the exchange rate cannot be determined separately.

Instead, the proposal of Definition 9 could be interpreted as a combination of the fiscal theory approach for the government and the proposal of Obstfeld and Rogoff (1983) for private money, since currency is exchanged for goods (a stream of goods) at a certain point in time. However, in my model, the purpose of this exchange is different. In their case, it serves to rule out multiple equilibria and in particular hyperinflationary solutions. In mine, explosive equilibria are ruled out and the purpose of exchanging currency for goods is to determine the constant exchange rate between the two currencies.

Note, further, that under the monetary regime proposed the path of private money supply is determined by the private issuer's policy, while that of the government is endogenously determined by the cash constraint (16).

Finally, note that the monetary policy regime of Definition 9, given a positive nominal interest rate, does not necessarily imply the existence of an equilibrium. But if an equilibrium does exist under that regime, it features determinate prices and exchange rate.

6 Conclusion

This paper analyzes models of coexistence between government and privately-issued currencies. Competing currencies can limit the ability of the central bank to use the interest rate as a policy instrument; they can also restrict the attainable equilibrium inflation rate. In a market with free entry for multiple currencies with profit-maximizing issuers, the central bank completely loses control of the interest rate and the inflation rate, which both come to be determined by structural factors (rate of time preference, entry cost, exit rate).

I have kept the analysis as simple as possible in order to focus on this important topic, which has recently received a good deal of attention. In particular, I assume that private and government currencies are perfect substitutes, delivering the same liquidity services but an extension to imperfect substitutability does not alter the results significantly. Most interesting would be to devise a model in which acceptability or unacceptability of currencies as medium of exchange is endogenous – a task I leave to future research. Another future avenue of research is extension to a multi-country world, as a way of studying competition among international reserve currencies and national currencies.

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A Appendix

A.1 Proof of Proposition 4.

Proof. Consider first the case in which $i > 0$. Lemma 3 implies that in an equilibrium $(1+i) \geqslant \beta^{-1}(1+\mu)$. Since $0 < \beta < 1$, then $i > \mu$. When instead $i = 0$, Lemma 3 also implies that $i > \mu$. Therefore in any equilibrium $i > \mu$. Consider the flow budget constraint (14) and iterate it forward to obtain

$$\frac{SM^{*p}_{t_0-1}}{P_{t_0}} + \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{ST^*_t}{P_t} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{i_t}{1+i_t} \frac{SM^{*p}_t}{P_t} + \lim_{T\to\infty} \beta^{T-t_0} \frac{SM^{*p}_T}{P_{T+1}}. \tag{1}$$

Note that we can write

$$\lim_{T\to\infty} \beta^{T-t_0} \frac{SM^{*p}_T}{P_{T+1}} = \frac{SM^{*p}_{t_0-1}}{\beta P_{t_0}} \left(\frac{1+\mu}{1+i}\right)^{T+1-t_0} = 0$$

using in the first equality $P_{T+1} = [(1+i)\beta]^{T+1-t_0} P_{t_0}$ and $M^{*p}_T = (1+\mu)^{T+1-t_0} M^{*p}_{t_0-1}$. The second equality follows since $i > \mu$ in any equilibrium. Moreover, under this inequality the two sums in (1) are also finite. We can therefore write

$$\frac{SM^{*p}_{t_0-1}}{P_{t_0}} + \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{ST^*_t}{P_t} = \frac{i}{1+i} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{SM^{*p}_t}{P_t}. \tag{2}$$

Substituting (2) into (10) we obtain:

$$\frac{M^g_{t_0-1} - B^g_{t_0-1}}{P_{t_0}} + \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{T_t}{P_t} = \frac{i}{1+i} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{M^g_t}{P_t}. \tag{3}$$

Plugging in the active monetary policy (17), (3) simplifies to

$$\frac{M^g_{t_0-1} - B^g_{t_0-1}}{P_{t_0}} = \tau, \tag{4}$$

which determines the price level P_{t_0}. Note that in the case $M^g_{t_0-1} - B^g_{t_0-1} > 0$, i.e. when government is a net debtor, then it should be that $\tau > 0$. Conversely, when $M^g_{t_0-1} - B^g_{t_0-1} < 0$ it should be that $\tau < 0$. Given a constant interest rate policy i, it follows from (9) that the entire path of prices $\{P_t\}_{t=t_0}^{\infty}$ is determined; further, using (15) the path of consumption starting from period $t_0 + 1$ is also determined.

Consider first the case $i > 0$; this implies

$$\frac{M^g_t}{P_t} + S\frac{M^{*p}_t}{P_t} = \beta$$

for each $t \geq t_0$ and in particular at time $t_0 + 1$ in which case

$$\frac{M^g_{t_0}}{P_{t_0}} + S\frac{M^{*p}_{t_0-1}(1+\mu)}{P_{t_0}} = \beta. \tag{5}$$

Moreover, substituting (17) into the government budget constraint we get

$$M^g_{t_0-1} - B^g_{t_0-1} = (1-\beta)P_{t_0}\tau + \frac{M^g_{t_0}}{1+i} - \frac{B^g_{t_0}}{1+i}. \tag{6}$$

Since P_{t_0} is determined in (4), the two restrictions (5) and (6) are insufficient to determine the three variables S, $B^g_{t_0}$ and $M^g_{t_0}$. Note also that the liquidity constraint at time t_0 cannot determine the variables of interest. Indeed,

$$\frac{M^g_{t_0-1}}{P_{t_0}} + S\frac{M^{*p}_{t_0-1}}{P_{t_0}} \geq C_{t_0} \tag{7}$$

cannot disentangle S and C_{t_0}.

Consider now the case in which the government sets $i = 0$ provided $\mu < 0$. Equation (6) also applies when $i = 0$, but now (5) is replaced by

$$\frac{M^g_{t_0}}{P_{t_0}} + S\frac{M^{*p}_{t_0-1}(1+\mu)}{P_{t_0}} > 1$$

which cannot determine S and $M^g_{t_0}$. Note instead that (7) still applies, so that C_{t_0} too is not determined. To see the result of indeterminacy from a different angle, suppose instead that S is determined, so that (5) determines $M^g_{t_0}$, (6) determines $B^g_{t_0}$, and so forth using the foregoing two equations in the subsequent periods. Further, given initial conditions $M^g_{t_0-1}$ and $M^{*p}_{t_0-1}$, the values of S and P_{t_0} determine the amount of initial real money balances. If the left-hand side of (7) is greater than the unitary value, then $C_{t_0} = 1$ and $\lambda_{t_0} = 0$. If it is less, then C_{t_0} is constrained by this value and $\lambda_{t_0} = C^{-1}_{t_0} - 1$. ■

A.2 Proof of Proposition 5

Proof. Consider the optimization problem of a generic supplier of private currency. Each supplier internalizes that it has some power on the rents given by λ_t. If liquidity rents are zero, i.e. $\lambda_t = 0$, the supply of currency will be infinitely elastic. But if $\lambda_t > 0$, rents are inversely related to the overall supply of liquidity, as shown by the following equation

$$C_t = \frac{1}{1+\lambda_t} = \frac{M_{t-1}}{P_t} + \sum_{j=1}^{J_t} \frac{M^j_{t-1}}{P^j_t}. \tag{8}$$

Inserting the above constraint into the objective function (23) and deriving it with respect to the sequence $\{M_t^j\}_{t=\bar{t}}^{\infty}$, one gets

$$C_{t+1}\left(1 - C_{t+1}\right) = \frac{M_t^j}{P_{t+1}^j}, \tag{9}$$

for each $t \geq \bar{t}$ in which $\bar{t}$ is the entry time of a generic issuer j of private currency. Therefore, the above equality holds for every $\bar{t} \geq t_0$. Since the left-hand side is independent of j, so too is the right-hand side, and the equilibrium is therefore symmetric. Defining $M_t^j/P_{t+1}^j = M_t^*/P_{t+1}^*$ for each j, we can write (9) as

$$C_{t+1} = \left(\frac{M_t}{P_{t+1}} + J_{t+1}\frac{M_t^*}{P_{t+1}^*}\right) = 1 - \frac{\frac{M_t^*}{P_{t+1}^*}}{\left(\frac{M_t}{P_{t+1}} + J_{t+1}\frac{M_t^*}{P_{t+1}^*}\right)}, \tag{10}$$

which holds for every $t \geq t_0$. Free entry proceeds until there are no rents left in the market, i.e. at each point in time the discounted present value of the profit of each issuer entering the market is equal to the fixed cost

$$\beta\sum_{t=\bar{t}}^{\infty}[(1-\delta)\beta]^{t-\bar{t}}\lambda_{t+1}\frac{M_t^j}{P_{t+1}^j} = \Phi, \tag{11}$$

for every $\bar{t} \geq t_0$. We can further write (11), using (8) and (9), as

$$\sum_{t=\bar{t}}^{\infty}[(1-\delta)\beta]^{t-\bar{t}}(1 - C_{t+1})^2 = \frac{\Phi}{\beta}. \tag{12}$$

Since (12) applies at each $\bar{t} \geq t_0$, this implies that

$$C_t = 1 - z^{\frac{1}{2}}, \tag{13}$$

for each $t \geq t_0 + 1$. Moreover, using (9) and (13), we get

$$\frac{M_t^*}{P_{t+1}^*} = z^{\frac{1}{2}} - z,$$

for each $t \geq t_0$. Combining this with (10) and (13), the number of supplier in the market, at each point in time, is given by

$$J_t = z^{-\frac{1}{2}}\left(1 - \frac{M_t}{P_{t+1}}\frac{1}{1 - z^{-\frac{1}{2}}}\right).$$

■

A.3 Proof of Proposition 8

Proof. To determine the price level in government currency, consider first the consumer's flow budget constraint, which in this more general case can be written as

$$\begin{aligned}\frac{B_t}{P_t(1+i_t)} + \frac{M_t}{P_t} + J_t\frac{M_t^*}{P_t^*} + X_t \quad \leq \quad & \frac{B_{t-1}}{P_t} + \left(\frac{M_{t-1}}{P_t} + (1-\delta)J_{t-1}\frac{M_{t-1}^*}{P_t^*} - C_t\right) + \\ & + Y + \frac{T_t}{P_t} + J_t\frac{T_t^*}{P_t^*}.\end{aligned}$$

The consumer's intertemporal budget constraint together with goods market equilibrium implies the following aggregate resource constraint:

$$\frac{M_{t_0-1}^g - B_{t_0-1}^g}{P_{t_0}} + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\left(\frac{T_t}{P_t} + J_t\frac{T_t^*}{P_t^*}\right) = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\left[\frac{i_t}{1+i_t}\frac{M_t^g}{P_t} + J_t\beta\frac{M_t^*}{P_{t+1}^*}\lambda_{t+1}\right].$$

Using the law of motion $J_t = (1-\delta)J_{t-1} + N_t$, this can be written as

$$\begin{aligned}& \frac{M_{t_0-1}^g - B_{t_0-1}^g}{P_{t_0}} + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\frac{T_t}{P_t} + \sum_{\bar{t}=t_0}^{\infty}\beta^{\bar{t}-t_0}N_{\bar{t}}\sum_{t=\bar{t}}^{\infty}[(1-\delta)\beta]^{t-\bar{t}}\frac{T_t^*}{P_t^*} \\ = \quad & \sum_{t=t_0}^{\infty}\beta^{t-t_0}\frac{i_t}{1+i_t}\frac{M_t^g}{P_t} + \sum_{\bar{t}=t_0}^{\infty}\beta^{\bar{t}-t_0}N_{\bar{t}}\sum_{t=\bar{t}}^{\infty}[(1-\delta)\beta]^{t-\bar{t}}\beta\lambda_{t+1}\frac{M_t^*}{P_{t+1}^*}. \qquad (14)\end{aligned}$$

Consider now the flow budget constraint of a generic private currency issuer, expressed in real terms,

$$\frac{M_{t-1}^*}{P_t^*} + \frac{T_t^*}{P_t^*} = \frac{M_t^*}{P_t^*} = \beta(1-\delta+\lambda_{t+1})\frac{M_{t-1}^*}{P_t^*}$$

in which in order to obtain the second equality we have used (21). The above budget constraint can be reiterated forward starting from the entry period $\bar{t}$ to obtain

$$\sum_{t=\bar{t}}^{\infty}[(1-\delta)\beta]^{t-\bar{t}}\frac{T_t^*}{P_t^*} = \beta\sum_{t=\bar{t}}^{\infty}[(1-\delta)\beta]^{t-\bar{t}}\frac{M_t^*}{P_{t+1}^*}\lambda_{t+1} \qquad (15)$$

using the fact that M_t^*/P_{t+1}^* is bounded in all periods given (25) and therefore $\lim_{t\to\infty}(1-\delta)^t\beta^t M_t^*/P_{t+1}^* = 0$. Note, further, that the right-hand side is also finite and equal to Φ given the zero-rent condition. Profit maximization under currency competition and free market entry therefore implies that an intertemporal budget constraint holds for each private issuer. We can then plug (15) into (14) to obtain

$$\frac{M_{t_0-1}^g - B_{t_0-1}^g}{P_{t_0}} + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\frac{T_t}{P_t} = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\left[\frac{i_t}{1+i_t}\frac{M_t^g}{P_t}\right],$$

which reduces to

$$\frac{M^g_{t_0-1} - B^g_{t_0-1}}{P_{t_0}} = \tau,$$

having used (29). Therefore the price level at time t_0 is determined by the above equation. Given the cash constraint $C_{t_0} \leq M^g_{t_0-1}/P_{t_0}$ consumption at time t_0 is determined by $C_{t_0} = \min(1, M^g_{t_0-1}/P_{t_0})$. Given that the interest rate is constant at (27), the path of the price level is determined by $P_{t+1} = \beta(1+i)P_t$ for each $t \geq t_0$. Note from Proposition 5 that since the government monetary authority sets $\{M^g_t\}^{\infty}_{t=t_0}$ and since $\{P_t\}^{\infty}_{t=t_0}$ is determined, it follows that J_t too is determined. Moreover, using the flow budget constraint of the government and the real transfer policy (29), we obtain

$$\frac{M^g_{t-1} - B^g_{t-1}}{P_t} = (1-\beta)\tau + \beta\left(\frac{M^g_t - B^g_t}{P_{t+1}}\right),$$

which determines the sequence $\{B^g_t\}^{\infty}_{t=t_0}$ given that all other variables are determined. ■

A.4 Proof of Proposition 10

Proof. When the nominal interest rate is positive, (16) holds with equality and

$$\frac{M^g_{t-1}}{P_t} + \frac{M^{*p}_{t-1}}{P^*_t} = \frac{1}{1+i}$$

holds in equilibrium, which implies that the real money balances are non-negative and bounded above at every point in time. Therefore in equilibrium $\lim_{t\to\infty} \beta^t M^{*p}_{t-1}/P^*_t = 0$, which implies $\lim_{t\to\infty}\beta^t(M^g_{t-1} - B^g_{t-1})/P_t$ given the transversality condition of the consumer's problem.[17] Using $\lim_{t\to\infty}\beta^t(M^g_{t-1} - B^g_{t-1})/P_t$, the flow budget constraint of the government (13) and $P_{t+1} = \beta P_t(1+i)$, we obtain

$$\frac{M^g_{t_0-1} - B^g_{t_0-1}}{P_{t_0}} + \sum_{t=t_0}^{\infty} \beta^{t-t_0}\frac{T_t}{P_t} = \frac{i}{1+i}\sum_{t=t_0}^{\infty} \beta^{t-t_0}\frac{M^g_t}{P_t},$$

in which (30) can be used to get

$$\frac{M^g_{t_0-1} - B^g_{t_0-1}}{P_{t_0}} = \tau$$

and to determine the price level P_{t_0} at time t_0. Using $P_{t+1} = \beta P_t(1+i)$ starting from $t = t_0$, one can determine the sequence $\{P_t\}^{\infty}_{t=t_0}$. Using $\lim_{t\to\infty} \beta^t M^{*p}_{t-1}/P^*_t = 0$,

[17] When $i = 0$, it is not necessarily the case that $\lim_{t\to\infty} \beta^t M^{*p}_{t-1}/P^*_t = 0$.

$P_{t+1} = \beta P_t(1+i)$ and (14) starting from period $t = \tilde{t}$, we obtain

$$\frac{M^{*p}_{\tilde{t}-1}}{P^*_{\tilde{t}}} + \sum_{t=\tilde{t}}^{\infty} \beta^{t-\tilde{t}} \frac{T^*_t}{P^*_t} = \frac{i}{1+i} \sum_{t=\tilde{t}}^{\infty} \beta^{t-\tilde{t}} \frac{M^{*p}_t}{P^*_t}$$

into which (31) and $M^{*p}_{\tilde{t}-1} = (1+\mu)^{\tilde{t}-t_0} M^{*p}_{t_0-1}$ can be substituted to obtain

$$\frac{(1+\mu)^{\tilde{t}-t_0} M^{*p}_{t_0-1}}{P^*_{\tilde{t}}} = \tau^*, \tag{16}$$

which determines $P^*_{\tilde{t}}$. Using (4) the entire sequence $\{P^*_t\}_{t=t_0}^{\infty}$ can be determined, and therefore also the nominal exchange rate S from $S = P_t/P^*_t$. Using $P_{t+1} = \beta P_t(1+i)$, (14) for time $t \geq \tilde{t}$ and (31) we obtain

$$\frac{M^{*p}_{t-1}}{P^*_t} = (1-\beta)\tau^* + \beta \frac{M^{*p}_t}{P^*_{t+1}}, \tag{17}$$

which can determine the path of private money supply from period $\tilde{t}$ onward, whereas in previous periods the path of private money issuance is determined by the transfer rule $T^*_t = \mu M^{*p}_{t-1}$ for each $t_0 \leq t < \tilde{t}$ given initial condition $M^{*p}_{t_0-1}$. Consumption is determined by (15) for each $t \geq t_0 + 1$, whereas at time t_0 it is determined by

$$C_{t_0} = \min\left\{ \frac{M^g_{t_0-1}}{P_{t_0}} + \frac{M^{*p}_{t_0-1}}{P^*_{t_0}}, 1 \right\}.$$

The sequence of government money issuance is determined by

$$\frac{M^g_t}{P_{t+1}} + \frac{M^{*p}_t}{P^*_{t+1}} = \frac{1}{1+i}$$

for each $t \geq t_0$ given the path of all the other variables. Using $P_{t+1} = \beta P_t(1+i)$, equation (13) for time $t \geq t_0$ and the policy rule (30) we obtain

$$\frac{M^g_{t-1} - B^g_{t-1}}{P_t} = (1-\beta)\tau + \beta \left(\frac{M^g_t - B^g_t}{P_{t+1}} \right),$$

which determines the sequence $\{B^g_t\}_{t=t_0}^{\infty}$ given that all the other variables are already determined. Note that the existence of an equilibrium imposes restrictions on the parameter μ and the time $\tilde{t}$. Indeed, it should be the case that

$$\left(\frac{1+\mu}{\beta(1+i)} \right)^{\tilde{t}+1-t_0} \frac{M^{*p}_{t_0-1}}{P_{t_0}} \leq \frac{1}{1+i}$$

given the value P_{t_0} determined in equilibrium. ■