Estimating the Consequences of Climate Change from Variation in Weather*

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I formally relate the consequences of climate change to the time series variation in weather extensively explored by recent empirical literature. I show that whether conventional fixed effects regressions underestimate or overestimate the effect of climate on actions (such as adaptation investments) depends primarily on whether actions are intertemporal substitutes (owing to resource constraints) or intertemporal complements (owing to adjustment costs). I also derive the conditions under which fixed effects regressions can recover the marginal effect of climate change on payoffs. I show that these conditions become less restrictive when regressions control for lags of weather and for forecasts of future weather.

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1 Introduction

A pressing empirical agenda seeks to estimate the economic costs of climate change. Ignorance of these costs has severely hampered economists' ability to give concrete policy recommendations (Pindyck, 2013). However, while climate primarily varies over space, so too do many unobserved variables that are potentially correlated with climate.[1] Seeking credible identification, an explosively growing empirical literature instead uses variation in a location's weather over time to estimate the consequences of transient weather shocks.[2] The hope is that transient weather shocks identify—or at worst bound—the effects of a change in climate.

Identifying the consequences of climate change from responses to transient weather shocks combines two challenges: (i) empirical researchers must credibly identify the consequences of transient weather shocks, and (ii) the consequences of transient weather shocks must be informative about the consequences of climate change. Challenge (i) is the challenge central to empirical work throughout economics, seeking as-good-as-random assignment of the weather treatment. Empirical researchers have addressed challenge (i) by including time and unit fixed effects, usually taking for granted that the remaining idiosyncratic variation in weather is exogenous.[3] Challenge (ii) is less standard. The recent empirical literature seeks to approximate the effect of one treatment (a change in climate) that is never observed from the estimated effect of a different treatment (a transient change in weather). Whether this mapping between treatments succeeds has been the subject of much discussion but little formal analysis.[4]

I here undertake the first formal analysis that precisely delineates what and how we can learn about the climate from the weather. A change in climate differs from a weather shock in being repeated period after period and in affecting expectations of weather far out into the future. Linking weather to climate therefore requires analyzing a dynamic model that can capture the distinction between transient and permanent changes in weather. I study an agent (equivalently, firm) who is exposed to stochastic weather outcomes. The agent chooses actions (equivalently, investments) that suit the weather. The actions chosen in different

[1] For many years, empirical analyses did rely on cross-sectional variation in climate to identify the economic consequences of climate change (e.g., Mendelsohn et al., 1994; Schlenker et al., 2005; Nordhaus, 2006). However, cross-sectional analyses fell out of favor due to concerns about omitted variables bias. See Dell et al. (2014) and Auffhammer (2018b) for expositions and Massetti and Mendelsohn (2018) for a review.

[2] For recent reviews, see Dell et al. (2014), Carleton and Hsiang (2016), and Heal and Park (2016). Blanc and Schlenker (2017) discuss the strengths and weaknesses of relying on panel variation in weather.

[3] For instance, Dell et al. (2014, 741) write that "the primary advantage of the new literature is identification", and Blanc and Schlenker (2017, 262) describe "weather anomalies" as "ideal right-hand side variables" because "they are random and exogenous". We will see that the existence of forecasts and the likelihood of serial correlation in weather in fact complicate identification.

[4] For instance, Dell et al. (2014, 771–772) emphasize that "short-run changes over annual or other relatively brief periods are not necessarily analogous to the long-run changes in average weather patterns that may occur with climate change."

periods may be complements or substitutes: when actions are intertemporal complements, choosing a high action in the previous period reduces the cost of choosing a high action today, but when actions are intertemporal substitutes, choosing a high action in the previous period increases the cost of choosing a high action today. The first case is one of adjustment costs, and the second case describes actions that draw from a finite reservoir of resources or time.[5] When choosing actions, the agent knows the current weather, has access to forecasts of future weather, and relies on knowledge of the climate to generate forecasts of weather at longer horizons. A change in the climate alters both the distribution of potential weather outcomes and the agent's expectation of future weather outcomes.

I show several novel results. First, I describe when fixed effects estimators of the effects of weather on actions understate or overstate the long-run effect of climate on actions. Much empirical research has sought to estimate the consequences of climate change for decision variables or functions of decision variables, including productivity (Heal and Park, 2013; Zhang et al., 2018), health (Deschenes, 2014), crime (Ranson, 2014), time allocation (Graff Zivin and Neidell, 2014), and energy use (Auffhammer and Aroonruengsawat, 2011; Deschênes and Greenstone, 2011; Auffhammer, 2018a). I show that if actions are neither intertemporal complements nor substitutes, then empirical researchers can approximate the effects of climate by combining the effects of current weather, lagged weather, and forecasts. However, the standard practice fails to control for forecasts. I show that failing to control for forecasts can recover the effects of climate change in only a narrower range of cases. Further, standard approaches use only the coefficient on contemporary weather when estimating the effects of climate change, but I show that this calculation can recover the effects of climate change in only the most special of cases.

Under more general relationships between current and past actions, I show that researchers can only bound the effects of climate change. Many economists have intuited that short-run adaptation responses to weather are likely to be smaller than long-run adaptation responses to climate (e.g., Deschênes and Greenstone, 2007). I show that two forces can favor this result. First, when actions are durable, a forward-looking agent will undertake more actions in response to a climate shock that also changes the next period's weather than in response to a transient weather shock. I show that estimating responses to forecasts can capture this channel. Second, when actions are intertemporal complements (as in adjustment cost models), the actions an agent takes in response to a transient weather shock are constrained by the agent's desire to not change actions too much from period to period, but when the same weather shock is repeated period after period, even a myopic agent eventually achieves a larger change in activity through a sequence of incremental adjustments.

[5] Both types of stories exist in the literature. For instance, in studies of the agricultural impacts of climate change, Deschênes and Greenstone (2007) conjecture that long-run adjustments to changes in climate should be greater than short-run adjustments to weather shocks because there may be costs to adjusting crops, whereas Fisher et al. (2012) and Blanc and Schlenker (2017) emphasize that constraints on storage and groundwater pumping, respectively, could reverse that conclusion.

The latter effect reverses if actions are intertemporal substitutes because agents will then undertake actions in response to transient weather shocks that they would not sustain in the face of a permanent change in climate. In that case, responses to transient weather shocks can overstate responses to permanent changes in climate. This last result is consistent with conjectures in the literature (e.g., Fisher et al., 2012; Auffhammer and Schlenker, 2014; Blanc and Schlenker, 2017; Auffhammer, 2018b). I show that controlling for lagged actions can indicate whether actions are intertemporal complements or substitutes and thus whether researchers obtain an upper or a lower bound on the effect of climate.

I also describe the conditions under which fixed effects estimators can recover the marginal effect of climate on payoffs. Much empirical research has sought to estimate the consequences of climate change for profits (e.g., Deschênes and Greenstone, 2007) and for variables such as gross output or income that are potentially related to aggregate payoffs (e.g., Dell et al., 2012; Burke et al., 2015; Deryugina and Hsiang, 2017). I show that the effect of climate change is in general hard to estimate because it depends on how actions change with the climate, which we have seen is often imperfectly represented by responses to transient weather shocks. However, I also identify several cases in which a combination of fixed effects estimators can exactly recover the effects of marginal climate change from time series variation in weather. These cases include the benchmark quadratic adjustment cost model, a model with linear interactions between current and past actions, a simple model of resource-dependent costs, and a model in which current and past actions are neither intertemporal substitutes nor complements. In all cases, it is important to control for forecasts: some special cases hold only if the coefficient on forecasts is small, and other special cases require this coefficient in order to recover the effect of climate change.

Figure 1 depicts the intuition underlying one special case in which time series variation in weather can identify the marginal effect of climate. Consider estimating the effect of temperature on agricultural profits, as in Deschênes and Greenstone (2007). Each solid curve in the left panel plots profits as a function of current inputs (such as labor and irrigation), conditional on growing season temperature. As we move to the right, the solid curves condition on increasingly warm growing seasons. In static environments, agents maximize profits by choosing inputs at the peaks of these curves, such as points a and b. The dotted line gives the effect on time t profits of time t temperature. Small changes in temperature do not have first-order effects on profits through input choices. This is the content of the envelope theorem, as applied by Deschênes and Greenstone (2007) and subsequent literature. One prominent argument extrapolates this envelope theorem reasoning to conclude that the effects of climate on profits are exactly identified by transient weather shocks (Hsiang, 2016; Deryugina and Hsiang, 2017): if climate differs from weather only through beliefs that affect input choices, then the first-order effects of climate are equivalent to the first-order effects of weather because these input choices do not have first-order consequences for profits.

However, this envelope theorem argument misses the dynamics that distinguish climate from weather. A change in climate affects past and future weather, not just current weather.

First consider the consequences of affecting past weather. Imagine that changing inputs imposes adjustment costs, so that time t profits also depend on time $t-1$ inputs. If last year was hot, then last year's input choices reflect that outcome and it becomes less costly to choose high inputs this year. The dashed curve in the left panel of Figure 1 plots profits in a current hot year conditional on having already adjusted last year's input choices in response to last year's being hot. Profits increase at input levels around point b because adjustment costs are reduced. Profits also increase because the optimal input level increases to point c, reflecting that current choices are less constrained by previous choices. Last year's input decisions can therefore have first-order effects on time t profits by changing the adjustment costs faced at time t. Because a transient weather shock will not capture how climate affects the trajectory of previous input decisions, we may expect a transient change in weather to fail to identify the effects of climate.[6]

Now consider the implications of climate affecting future weather. A change in climate leads agents to expect the subsequent year $t+1$ to once again be hot and thus to expect to choose a high input level in year $t+1$. Applying more inputs at time t now carries the dynamic benefit of reducing time $t+1$ adjustment costs. As a result, the dynamically optimal input choice is point d, where the marginal effect on this year's profit is negative but the marginal effect on expected intertemporal profits is zero (see equation (1) below).[7] Envelope theorem arguments assume that profit-maximizing inputs always occur where the marginal effect on time t profit is zero. These arguments fail when agents choose inputs with an eye to their implications for future years, whether because current inputs affect future years' adjustment costs, because current inputs affect the availability of resources in future years, or because current inputs are forward-looking adaptation decisions that directly protect against future weather.[8]

How can we estimate the difference between points a and d? The right panel of Figure 1 again plots profits as a function of current inputs, but it now holds current weather fixed between curves and instead varies only the previous year's input choices. The curve labeled "ss" depicts profits when the typical temperature has occurred many years in a row, so that previous inputs reached a steady state. The other two curves depict this year's profits under the typical temperature outcome but with higher ("H") and lower ("L") choices of inputs in the previous year. The adjustment costs imposed by these past choices constrain this year's choice of inputs and thereby reduce profits.

The dotted curve gives the effect on myopically optimized profits of changing last year's input choices. For any given previous input choice, the myopically profit-maximizing input

[6] Time $t-1$ actions can have first-order consequences on payoffs because those actions are predetermined at time t. The envelope theorem rules out first-order effects only via choices made at time t.

[7] As will be discussed, the dynamically optimal input level could be to the left or the right of point c, but in either case, expecting the subsequent year to be hot would shift the dynamically optimal input level to the right in order to reduce the adjustment costs faced in that subsequent year.

[8] Of course, the envelope theorem does hold in a dynamic setting: the envelope theorem now applies to the intertemporal value function, not to the flow payoffs typically studied by empirical researchers.

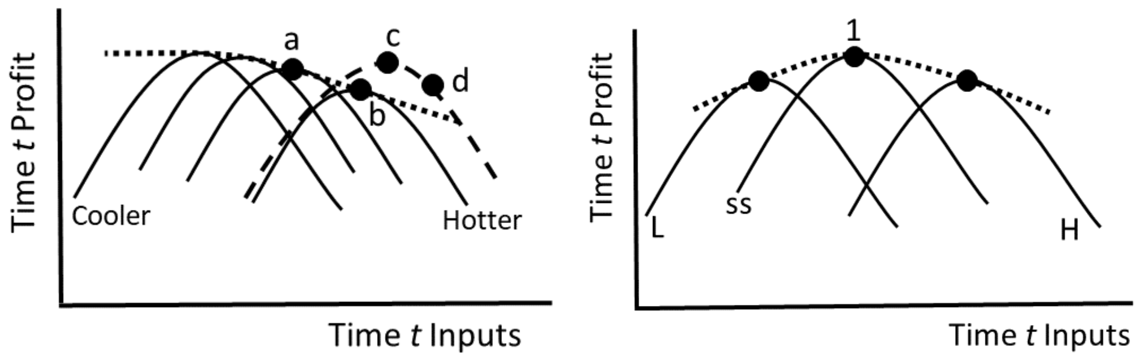


Figure 1: Left: Profits against inputs, conditional on temperature. Temperature is higher for curves farther to the right. The dotted curve through points a and b gives the effect on profits of increasing temperature in the absence of long-run adaptation. Point c accounts for adaptation to previous hot years, and point d accounts for expecting next year to again be hot. Right: Profits against inputs, conditional on past input choices. The curve labeled "ss" sets previous inputs to the steady state that would result if the current temperature had been repeated indefinitely.

choice finds the peak of the curve. The dotted curve has a peak at the myopically optimal labor input implied by curve "ss" because adjustment costs vanish in that case. Around this point (labeled 1), a change in past weather does not have first-order effects through input choices. So the left panel's point c converges to point b. Now imagine that the agent expects the typical temperature to also occur next year. Because this year's input choices do not have first-order effects on next year's profits around point 1, the myopically optimal input choice is also dynamically optimal. So the left panel's point d converges to point c. Combining these results, the left panel's point d converges to point b around point 1, so that the treatment effect of a transient weather shock indeed recovers the effect of permanently changing the weather. The key assumptions are, first, that agents tend to be near their steady-state actions and, second, that small changes in past actions do not have first-order effects on payoffs around a steady state (i.e., that point 1 occurs at a flat point of the dotted line in the right panel). The formal analysis identifies the class of profit functions for which this second assumption holds. Consistent with the intuition given here, it shows that a benchmark adjustment cost model is indeed a member of that class.

Despite the importance of empirically estimating the costs of climate change and the sharpness of informal debates around the relevance of the recent empirical literature to climate change, there has been remarkably little formal analysis of the economic link between weather and climate. Previous analysis has consisted in heuristic appeals to the envelope theorem in static environments (Hsiang, 2016; Deryugina and Hsiang, 2017), but as described above, a static environment misses the distinction between transient and permanent weather shocks that is critical to distinguishing weather and climate. A few other papers are also related. First, in an initial expositional analysis, I showed how envelope theorem arguments can fail in a three-period model (Lemoine, 2017). The present work precisely analyzes the consequences of climate change in an infinite-horizon model and constructively shows which types of empirical estimates can be informative about the climate. Second, Kelly et al. (2005) and Kala (2017) study learning about the climate from observed weather. I here abstract from learning in order to focus on mechanisms more relevant to the recent empirical literature.[9] Third, calibrated simulations have shown that dynamic responses are critical to the effects of climate on timber markets (Sohngen and Mendelsohn, 1998; Guo and Costello, 2013) and to the cost of increased cyclone risk (Bakkensen and Barrage, 2018). I develop a general analytic setting that precisely disentangles several types of dynamic responses and relates them to widely used fixed effects estimators. Finally, some empirical papers have demonstrated that actions do respond to forecasts of future weather (e.g., Neidell, 2009; Rosenzweig and Udry, 2013, 2014; Wood et al., 2014).[10] In particular, Shrader (2017) and Taraz (2017) use variation in forecasts and past weather outcomes, respectively, to estimate ex-ante adaptation to weather events. I will formally demonstrate that it is critical to estimate responses both to forecasts and to lagged weather when seeking to learn about the consequences of climate change.

The next section describes the setting. Sections 3 and 4 analyze the effects of climate on agents' chosen actions and payoffs, respectively, and connect these consequences to conventional fixed effects estimators. Section 5 examines recent long difference estimators, showing that they capture the same variation in transient weather shocks as standard fixed effects estimators. Section 6 discusses the implications for empirical work of the treatment effect of climate change varying with locations' current climates. The final section describes caveats and potential extensions. The appendix contains proofs.

[9] Kelly et al. (2005) frame the cost of learning as an adjustment cost. Quiggin and Horowitz (1999, 2003) discuss broader costs of adjusting to a change in climate. These papers' adjustment costs are conceptually distinct from the adjustment costs studied here. I follow the empirical literature in studying the long-run cost of changing the climate without modeling the transition from one climate to another. The present use of "adjustment costs" follows much other economics literature in referring to the cost of changing decisions from their previous level, where decisions here respond to variations in weather. The present paper studies how these adjustment costs hinder estimation of the consequences of climate change from weather, not how they affect the cost of transitioning from one climate to another. I return to this point in the conclusion.

[10] Severen et al. (2016) show that land markets capitalize expectations of future climate change and correct cross-sectional analyses in the tradition of Mendelsohn et al. (1994) for this effect. In contrast, I here study responses to widely available, shorter-run forecasts in a time series context and show how to use them to improve panel analyses in the tradition of Deschênes and Greenstone (2007).

2 General Setting

An agent is repeatedly exposed to stochastic weather outcomes and takes actions based on realized weather and on information about future weather. The realized weather in period t is w_t and the agent's chosen action is A_t.[11] This action may be interpreted as a level of activity (e.g., time spent outdoors, energy used for heating or cooling, irrigation applied to a field) or as a stock of capital (e.g., outdoor gear, size or efficiency of furnace, number or efficiency of irrigation lines). The agent's time t payoffs are $\pi(A_t, A_{t-1}, w_t, w_{t-1})$. Letting subscripts indicate partial derivatives with respect to the indicated argument, I assume declining marginal benefits of current and past actions ($\pi_{11} < 0, \pi_{22} \leq 0$).

I interpret actions as adaptations that become more valuable with high weather outcomes ($\pi_{13}, \pi_{23} \geq 0$). Relating to the literature on climate adaptation (e.g., Fankhauser et al., 1999; Mendelsohn, 2000), a case with $\pi_{13} > 0$ reflects adaptation that can occur after weather is realized ("reactive" or "ex-post" adaptation), whereas a case with $\pi_{23} > 0$ reflects adaptation that can occur before weather is realized ("anticipatory" or "ex-ante" adaptation). I allow for adaptation to play both roles at once. The possibility that $\pi_4 \neq 0$ reflects potential delayed impacts from the previous period's weather, with π_{14} and π_{24} capturing the potential for ex-post adaptation to alter these delayed impacts. Consistent with the normalizations above, I assume $\pi_{14}, \pi_{24} \geq 0$

I allow π_{12} to be positive or negative, with $(\pi_{12})^2 < \pi_{11}\pi_{22}$. When $\pi_{12} < 0$, actions are "intertemporal substitutes", so that choosing a higher level of past actions increases the cost of choosing higher actions today. I identify this case with resource constraint stories.[12] For instance, pumping groundwater today raises the cost of pumping groundwater tomorrow, or calling in sick today increases the cost of calling in sick tomorrow. When $\pi_{12} > 0$, actions are "intertemporal complements", so that choosing a higher level of past actions increases the benefit from choosing higher actions today. I identify this case with adjustment cost stories.[13] For instance, small changes to cropping practices or work schedules may be easier to implement than large changes. The magnitude of π_{12} affects the agent's preferred timing of adaptation. As $|\pi_{12}|$ becomes large, the agent prefers to begin adapting before the weather event arrives, but when $|\pi_{12}|$ is small, the agent may wait to undertake most adaptation only once the weather event has arrived.[14]

[11] For expositional purposes, I treat actions and the weather index as being one-dimensional. Generalizing to vector-valued actions and weather is straightforward but increases notation without further insight.

[12] Relating to the literature on resource extraction, the case with $\pi_{12} < 0$ can be seen as reflecting stock-dependent extraction costs (Heal, 1976).

[13] The benchmark quadratic adjustment cost model has $\pi_{12} = k$ for some $k > 0$.

[14] The magnitude of π_{12} is related to the distinction between ex-post and ex-ante adaptation insofar as it affects the agent's preferred timing of adaptation actions. However, π_{12} incentivizes early adaptation only to reduce the costs of later adaptation, not because early adaptation provides protection from weather events. I reserve the terms ex-ante and ex-post adaptation to refer to the effects of actions on the marginal benefit of weather captured by π_{13}, π_{23}, π_{14}, and π_{24}.

The agent observes time t weather before selecting her time t action. The agent also knows the background climate C, which controls long-run average weather. We can interpret climate and weather as temperature. At all times before $t-1$, the agent's only information about time t weather consists in knowledge of the climate. However, at time $t-1$ the agent receives a signal about time t weather, which generates an updated forecast f_{t-1}. We have $f_{t-1} = C + \zeta \nu_{t-1}$, where ν_{t-1} is a mean-zero, serially uncorrelated random variable with variance $\tau^2 > 0$. The forecast is an unbiased predictor of time t weather: $w_t = f_{t-1} + \zeta \epsilon_t$, where ϵ_t is a mean-zero, serially uncorrelated random variable with variance $\sigma^2 > 0$.[15] The parameter $\zeta \geq 0$ is a perturbation parameter useful for analysis (see Judd, 1996). The covariance between ϵ_t and ν_t is ρ, which implies that the covariance between w_t and w_{t-1} is $\zeta^2 \rho$.

The agent maximizes the present value of payoffs over an infinite horizon:

$$\max_{\{A_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t E_0 \left[\pi(A_t, A_{t-1}, w_t, w_{t-1}) \right],$$

where $\beta \in [0, 1)$ is the per-period discount factor, A_{-1} is given, and E_0 denotes expectations at the time 0 information set. The solution satisfies the following Bellman equation:

$$\begin{aligned} V(Z_t, w_t, f_t, y_t; \zeta) = &\max_{A_t} \left\{ \pi(A_t, Z_t, w_t, y_t) + \beta E_t \left[V(Z_{t+1}, w_{t+1}, f_{t+1}, y_{t+1}; \zeta) \right] \right\} \\ \text{s.t.} \quad Z_{t+1} = &A_t \\ w_{t+1} = &f_t + \zeta \epsilon_{t+1} \\ f_{t+1} = &C + \zeta \nu_{t+1} \\ y_{t+1} = &w_t. \end{aligned}$$

The state variables Z_t and y_t capture the previous period's actions and weather, respectively.

The setting is sufficiently general to describe many applications of interest. For instance, much empirical literature has studied the effects of weather on energy use. The agent could then be choosing indoor temperature in each period, where payoffs depend on current actions through energy use and depend on weather through thermal comfort. Empirical literature has also studied the effect of weather on agricultural profits. The decision variable could then be irrigation, labor, fertilizer, or crop varieties, the dependence of payoffs on these choices reflects the cost of purchasing these in each year, adjustment costs reflect the cost of

[15] Consistent with much previous literature, climate here controls average weather. One might wonder about the dependence of higher moments of the weather distribution on climate. In fact, the effects of climate change on the variance of the weather are poorly understood and spatially heterogeneous (e.g., Huntingford et al., 2013; Lemoine and Kapnick, 2016). Further, we need to know not just how climate change affects the variance of realized weather but how it affects the forecastability of weather: the variance of the weather more than one period ahead is $\zeta^2(\sigma^2 + \tau^2)$, so we need to apportion any change in variance between σ^2 and τ^2. I leave such an extension to future work.

changing equipment and plans from year to year, and weather costs reflect the deviation in crop yields from their maximum possible value. Finally, much empirical work has studied the effect of weather on labor productivity. The decision variable could be effort, the dependence of payoffs on weather can reflect current thermal stress as well as the effects of the previous day's weather via sleep and physiological functioning, the resource constraint is one of tasks needing to be done, and forecasts allow the agent to plan tasks and vacation time around weather outcomes.

I will often impose one of the following two assumptions:

Assumption 1. ζ^2 *is small.*

Assumption 2. π *is quadratic.*

Either assumption will limit the consequences of stochasticity for optimal policy, whether by limiting the variance of weather outcomes (Assumption 1) or by making the policy function independent of that variance (Assumption 2).[16]

3 Estimating the Effect of Climate on Actions

I now consider how to estimate the effect of climate on actions from time series variation in weather. Much empirical research has sought to estimate the consequences of climate change for decision variables or functions of decision variables, including productivity (Heal and Park, 2013; Zhang et al., 2018), health (Deschenes, 2014), crime (Ranson, 2014), time allocation (Graff Zivin and Neidell, 2014), and energy use (Auffhammer and Aroonruengsawat, 2011; Deschênes and Greenstone, 2011; Auffhammer, 2018a). Further, we will see that the effects of climate on payoffs are closely related to its effects on actions. I first analyze the evolution of actions over time, then derive the effect of climate change, and finally compare that effect to the estimates from fixed effects regressions of actions on weather.

Consider a deterministic system, with $\zeta = 0$. The first-order condition is:

$$0 = \pi_1(A_t, Z_t, C, C) + \beta V_1(Z_{t+1}, C, C, C; 0).$$

The envelope theorem yields:

$$V_1(Z_t, C, C, C; 0) = \pi_2(A_t, Z_t, C, C).$$

Advancing this forward by one timestep and substituting into the first-order condition, we have the Euler equation:

$$0 = \pi_1(A_t, A_{t-1}, C, C) + \beta \pi_2(A_{t+1}, A_t, C, C). \tag{1}$$

[16] Applying Assumption 2 will make the policy function independent of the variance of weather, but the chosen policy will still be affected by the variance of weather because that chosen policy depends on the realized weather.

If $\pi_2 = 0$, then we have a static optimization problem and the agent maximizes payoffs by setting the marginal time t benefit of time t actions to zero. In Figure 1, points a, b, and c would in fact be optimal. However, matters are different if π_2 is nonzero. A case with $\pi_2 > 0$ is a case of deferred benefits: actions taken in period t provide benefits in the future, as with capital investments. In the presence of these benefits, the optimal time t action must have a negative marginal flow benefit in time t, as with point d in Figure 1. A case with $\pi_2 < 0$ is a case of deferred costs: actions taken in period t impose costs in the future, as with the use of scarce resources or as when taking out a loan. In the presence of deferred costs, the optimal time t action must have a positive marginal flow benefit in time t.[17]

A steady state $\bar{A}$ of the deterministic system is implicitly defined by

$$0 = \pi_1(\bar{A}, \bar{A}, C, C) + \beta\pi_2(\bar{A}, \bar{A}, C, C). \tag{2}$$

Define $\bar{\pi} \triangleq \pi(\bar{A}, \bar{A}, C, C)$. The following lemma describes the uniqueness and stability of the steady state.

Lemma 1. *$\bar{A}$ is locally saddle-path stable if and only if $(1+\beta)|\bar{\pi}_{12}| < -\bar{\pi}_{11} - \beta\bar{\pi}_{22}$, in which case $\bar{A}$ is unique.*

Proof. See appendix. □

I henceforth assume that $(1+\beta)|\bar{\pi}_{12}| < -\bar{\pi}_{11} - \beta\bar{\pi}_{22}$, so that the deterministic steady state is unique and saddle-path stable.

Now consider optimal policy in the stochastic system. The first-order condition is:

$$0 = \pi_1(A_t, Z_t, w_t, y_t) + \beta E_t[V_1(Z_{t+1}, w_{t+1}, f_{t+1}, y_{t+1}; \zeta)].$$

The envelope theorem yields:

$$V_1(Z_t, w_t, f_t, y_t; \zeta) = \pi_2(A_t, Z_t, w_t, y_t).$$

Advancing this forward by one timestep and substituting into the first-order condition, we have the stochastic Euler equation:

$$0 = \pi_1(A_t, A_{t-1}, w_t, w_{t-1}) + \beta E_t[\pi_2(A_{t+1}, A_t, w_{t+1}, w_t)].$$

I analyze the stochastic system by approximating around the steady state and $\zeta = 0$ (Judd, 1996).

The following lemma expresses A_t in terms of deviations in predetermined variables from their means.

[17] In Figure 1, the dynamically optimal actions would now be to the left of points a, b, and c. The effect of expecting the subsequent year to be hot would still shift that action to the right, so point d would still be to the right of the corrected point c.

Lemma 2. *Let either Assumption 1 or 2 hold, and let $(A_{t-1} - \bar{A})^2$ be small. Then:*

$$\begin{aligned} A_t =& \bar{A} + \overbrace{\frac{\bar{\pi}_{14}}{\chi}(w_{t-1} - C) + \frac{\bar{\pi}_{12}}{\chi}(A_{t-1} - \bar{A})}^{\textit{effects of past weather}} + \overbrace{\frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}(w_t - C)}^{\textit{effects of current weather}} \\ &+ \underbrace{\frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}(f_t - C)}_{\textit{effects of future weather}}, \end{aligned} \tag{3}$$

where

$$\chi \triangleq -\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda} > |\bar{\pi}_{12}|$$

and $|\lambda| < 1$.

Proof. See appendix. □

We see time t actions determined by past, current, and future weather. Actions depend on past weather in two ways. First, past weather affects the marginal payoffs from current actions directly when $\bar{\pi}_{14} \neq 0$. This is a form of ex-post adaptation. Second, past weather affects past actions, and these past actions affect current actions when $\bar{\pi}_{12} \neq 0$.[18] When actions are intertemporal complements ($\bar{\pi}_{12} > 0$), high values of past actions justify higher actions today, but when actions are intertemporal substitutes ($\bar{\pi}_{12} < 0$), high values of past actions justify lower actions today. In the former case, maintaining the high action over time reduces adjustment costs, but in the latter case, the past high action depletes the resources needed to maintain a high action today.

Actions also depend on current weather, in three ways. First, actions respond to current weather as a means of mitigating its immediate harm or amplifying its immediate benefits. This channel is controlled by $\bar{\pi}_{13}$. Second, actions respond to current weather when current actions can mitigate the harm or amplify the benefits incurred by current weather in future periods. This channel is controlled by $\bar{\pi}_{24}$ and arises only for forward-looking agents. As an example of the distinction between the two channels, an agent may avoid going outside on a cold day both to minimize discomfort from the current temperature and to avoid getting sick in the near future. Both of these channels are forms of ex-post adaptation. Third, when $\bar{\pi}_{14} \neq 0$, the current weather will directly affect the agent's chosen action in the next period. A forward-looking agent anticipates this incentive and adjusts her current action in preparation for that choice. This channel vanishes when $\bar{\pi}_{12} = 0$ because today's actions then do not directly interact with subsequent actions.

Finally, actions also depend on future weather, both directly and indirectly. The direct channel reflects the possibility of ex-ante adaptation, controlled by $\bar{\pi}_{23}$. When today's actions

[18] Because past actions are also affected by expectations of current weather, it is more precise to say that current actions depend on past weather and past forecasts, not just past weather.

are durable investments that can control the effects of future weather, the agent chooses today's actions based on expectations of that future weather. The indirect channel reflects how the agent begins adjusting actions today in anticipation of the actions she will want to take in the subsequent period. When the agent receives a higher forecast, she expects to take a higher action in the subsequent period, controlled by $\bar{\pi}_{13}$ and $\beta\bar{\pi}_{24}$. When $\bar{\pi}_{12} > 0$, the indirect channel leads the agent to choose high actions today as a means of reducing adjustment costs, but when $\bar{\pi}_{12} < 0$, the indirect channel leads the agent to choose low actions today as a means of conserving resources.

We want to know how actions change, on average, with the climate index C. If either Assumption 1 or 2 holds and $E_0[(A_1 - \bar{A})^2]$ is small, then

$$E_0[A_2] = \bar{A} + \frac{\bar{\pi}_{12}}{\chi}(E_0[A_1] - \bar{A}).$$

$E_0[(A_2 - \bar{A})^2]$ must be small because $|\bar{\pi}_{12}|/\chi < 1$. Iterating forward, we find, for $t > 1$,

$$E_0[A_t] = \bar{A} + \left(\frac{\bar{\pi}_{12}}{\chi}\right)^{t-1}(E_0[A_1] - \bar{A}).$$

As $t \to \infty$, we have:

$$E_0[A_t] \to \bar{A}.$$

Differentiating equation (2) via the implicit function theorem, we have established the following lemma.

Lemma 3. *Let either Assumption 1 or 2 hold, and let $E_0[(A_1 - \bar{A})^2]$ be small. Then, as $t \to \infty$,*

$$\frac{\mathrm{d}E_0[A_t]}{\mathrm{d}C} \to \frac{\mathrm{d}\bar{A}}{\mathrm{d}C} = \frac{\overbrace{\bar{\pi}_{13} + \bar{\pi}_{14} + \beta\bar{\pi}_{24}}^{\textit{ex-post}} + \overbrace{\beta\bar{\pi}_{23}}^{\textit{ex-ante}}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \geq 0. \tag{4}$$

Expected future actions increase in the climate index because I normalized high actions to be more beneficial when the weather index is large. Equation (4) captures how climate change alters weather in all periods: the past, the present, and the future. We see the various forms of ex-post adaptation captured by $\bar{\pi}_{13}$, $\bar{\pi}_{14}$, and $\beta\bar{\pi}_{24}$. We also see the possibility of ex-ante adaptation, controlled by $\bar{\pi}_{23}$ and arising because the agent understands that the altered climate affects weather in subsequent periods. Finally, observe that $\bar{\pi}_{12}$ enters through the denominator in (4). When actions are intertemporal substitutes ($\bar{\pi}_{12} < 0$), this term reduces the magnitude of the response to climate change, as when resource constraints make long-run responses smaller than short-run responses. However, when actions are intertemporal complements ($\bar{\pi}_{12} > 0$), this term increases the magnitude of the response to climate change, as when adjustment costs allow long-run responses to exceed short-run responses.

Now consider attempting to estimate (4) from time series variation in weather. Label agents by j and imagine that they are in the same climate C with the same payoff function π and the same stochastic process driving forecasts and weather, though each agent may draw a different sequence of weather and forecasts. $\mathrm{d}\bar{A}/\,\mathrm{d}C$ is therefore identical for all agents. Consider the following fixed effects regression:

$$A_{jt} = \alpha_j + \psi_t + \Gamma_1 w_{jt} + \Gamma_2 w_{j(t-1)} + \Gamma_3 f_{jt} + \Gamma_4 A_{j(t-1)} + \eta_{jt}, \tag{5}$$

where w_{jt} and f_{jt} are the weather and forecasts relevant to agent j, α_j is a fixed effect for agent j, ψ_t is a time fixed effect, and η_{jt} is an error term, which I assume to be uncorrelated with the covariates.[19] I use a hat to denote the probability limit of each estimator.

The following lemma relates the estimated coefficients to the effect of climate change.

Lemma 4. *Let either Assumption 1 or 2 hold, and let $(A_{j(t-1)} - \bar{A})^2$ be small for all observations. Then:*

$$\hat{\Gamma}_1 + \hat{\Gamma}_2 + \hat{\Gamma}_3 = \omega \left(\frac{\mathrm{d}\bar{A}}{\mathrm{d}C} + \beta \frac{\bar{\pi}_{14} + \bar{\pi}_{13} + \beta \bar{\pi}_{24}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta \bar{\pi}_{22}} \Omega \right), \tag{6}$$

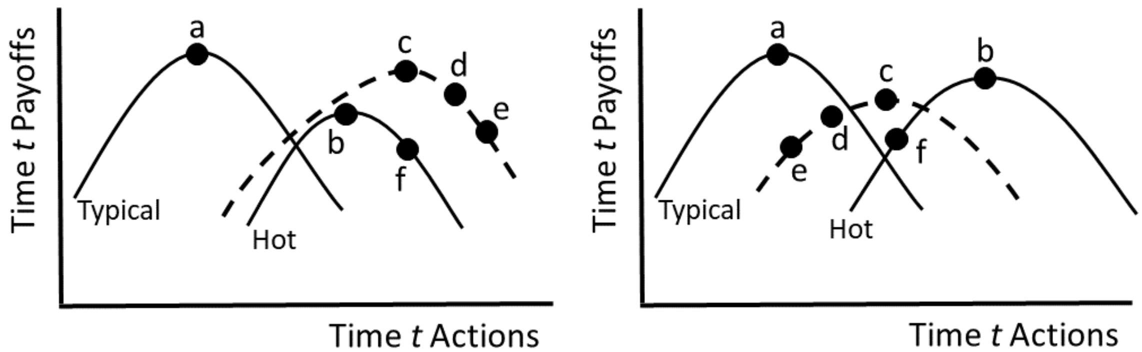
where $\bar{\pi}_{12} > 0$ implies $\omega \in (0,1)$ and $\Omega > 0$ and $\bar{\pi}_{12} < 0$ implies $\omega > 1$ and $\Omega < 0$.

Proof. See appendix. □

The three coefficients capture the three temporal relationships altered by climate change: $\hat{\Gamma}_1$ recovers consequences of altering current weather, $\hat{\Gamma}_2$ recovers consequences of altering past weather, and $\hat{\Gamma}_3$ recovers consequences of altering expectations of future weather. However, we cannot in general recover the response to a permanent change in climate from the estimated response to transient weather shocks. The reason for this failure is the possibility that $\bar{\pi}_{12} \neq 0$.

Relationships of intertemporal substitutability or complementarity drive two types of wedges between the estimator in (6) and the effect of climate change in (4). First consider adjustment cost stories, which have $\bar{\pi}_{12} > 0$. The second term in parentheses in (6) reflects how the agent adjusts today's actions in expectation of today's weather changing tomorrow's desired actions. In the presence of adjustment costs, the agent shifts today's actions towards the level that will be chosen tomorrow ($\Omega > 0$). This effect vanishes as actions become more similar between today and tomorrow, so it tends to make today's actions more responsive

[19] I do not explicitly model the unobservable characteristics that motivate the fixed effects specification because they are not central to the question of interest. These unobservables relate to challenge (i) described in the introduction. See Dell et al. (2014) and Auffhammer (2018b), among others, for standard expositions of identification in the climate-economy literature.



(a) Adjustment costs (b) Resource constraints

Figure 2: The effects of climate and of transient weather shocks on actions, for cases with adjustment costs (left) and resource constraints (right). Points a, b, c, and d are as in Figure 1. Point e depicts a case in which expected changes in weather are transient, and point f depicts a case in which previous changes in weather were only transient. The difference between points d and f reflects the difference between responses to climate and estimates from transient weather shocks, from equation (6).

to weather variation than to changes in climate. The $\omega \in (0, 1)$ captures how adjustment costs tend to diminish the magnitude of any change in actions. This effect also vanishes over time as the agent completes all of the desired adjustments, so it tends to make actions less responsive to weather variation than to changes in climate.[20] These two biases conflict, and it is difficult to sign their net effect in general.

Now consider resource constraint stories, which have $\bar{\pi}_{12} < 0$. The second term in parentheses now reflects how the agent conserves resources for tomorrow by shifting today's actions away from the level that will be chosen tomorrow ($\Omega < 0$). This effect again vanishes as time passes and actions become more similar between today and tomorrow, so it tends to make actions less responsive to weather variation than to changes in climate. The $\omega > 1$ captures how resource constraints tend to allow for more extreme actions when the actions will be maintained for only a short period of time. As a result, actions are more responsive to weather variation than to changes in climate. Once again, the two biases conflict and it is difficult in general to sign their net effect.

Figure 2 illustrates the intuition. Begin with the left panel, which is a case of adjustment costs. For exposition, ignore the possibility of ex-ante adaptation or delayed effects. As in the left panel of Figure 1, the solid curves depict time t payoffs conditional on time t actions, with point a indicating an action chosen in a typical period and point b indicating an action chosen in a hotter period. The difference between points a and b is controlled by $\bar{\pi}_{13}$ and

[20]This effect has the flavor of Le Châtelier's principle.

reflects the effects of current weather in equation (3). Also as in the left panel of Figure 1, the dashed line reflects payoffs conditional on both the current period being hot and the past period having been hot. When the previous period was hot, the agent chose higher actions than otherwise. Those past choices of high actions reduce the cost of choosing high actions in the current period. The current period's choice of action therefore increases to point c. The difference between points b and c is controlled by $\bar{\pi}_{12}$ and reflects the effects of past weather in equation (3). Finally, point d reflects the implications of expecting the subsequent period to again be hot and thus of expecting to choose a high action in the subsequent period.[21] The marginal benefit of choosing a high action in the current period increases because high current actions reduce future adjustment costs. The agent's optimal action therefore increases to point d, where the marginal effect on current-year profits is negative. The difference between points c and d is controlled by $\beta\bar{\pi}_{23}$ and $\beta\bar{\pi}_{13}\bar{\pi}_{12}$, with $\bar{\pi}_{23}$ controlling how current actions interact with the subsequent period's weather, $\bar{\pi}_{13}$ controlling how the subsequent period's action responds to its weather, $\bar{\pi}_{12}$ controlling the adjustment costs that would be incurred, and β controlling the current agent's concern for future costs. The difference between points c and d reflects the effects of future weather in equation (3).

Now consider how a transient increase in expected weather differs from a change in climate. The action that will be desired in a subsequent period is more different from the current period's action when the high forecast reflects a transient shock. In that case, the adjustment will be larger and the current period's action increases to an even larger point, represented by point e. The change to point e reflects $\Omega > 0$ and illustrates how agents may respond more strongly to transient weather shocks. In addition, past actions are lower following a transient shock to past weather than they would be if a longer history of weather had changed as a result of a shift in climate. The resulting adjustment costs bring payoffs closer to the rightmost solid curve than to the dashed curve. Those adjustment costs reduce actions to a point such as f. The change to point f reflects $\omega \in (0, 1)$ and illustrates how agents may respond less strongly to transient weather shocks. It is difficult in general to determine whether point f is to the left or to the right of point d because point f results from the combination of a rightward shift from point d to point e and a leftward shift from point e to point f.

The right panel of Figure 2 depicts a resource constraint story. The solid curves and points a and b are as before, and the dashed curve again depicts a case in which the previous period was also hot and thus saw the agent choose high actions. However, that dashed curve has now shifted down because the previous period's high actions increase the cost of the current period's actions by having used scarce resources such as groundwater or time. Point c is therefore now to the left of point b. Point d again reflects the expectation of the subsequent period being hot, but it is now shifted to the left of point c because the expectation of high

[21] For exposition, I have plotted points a, b, and c as occurring at the myopically optimal actions, where $\pi_1 = 0$. However, as described around equation (1), these points could in general have $\pi_1 \neq 0$. In either case, point d would be to the right of point c under the given story.

actions in the subsequent period increases the benefit of freeing up resources by choosing low actions in the current period.[22] If the forecasted event is transient, then the subsequent period's action will be especially different from the current period's action and the benefit of freeing up resources that much larger. Point e is therefore now to the left of point d, reflecting $\Omega < 0$ and illustrating how agents may respond less to transient weather shocks. However, past actions are also lower following transient weather shocks than following a shift in climate, so resources are not as scarce in the current period if climate change has not yet occurred. Point f therefore is now to the right of point e, reflecting $\omega > 1$ and illustrating how agents may respond more to transient weather shocks. It is difficult in general to determine whether point f is to the left or to the right of point d because point f results from the combination of a leftward shift from point d to point e and a rightward shift from point e to point f.

Nonetheless, the following proposition shows that we can make progress in special cases.

Proposition 1. *Let either Assumption 1 or 2 hold, and let $(A_{j(t-1)} - \bar{A})^2$ be small for all observations.*

1. *Testing Intertemporal Substitutes/Complements:* $\bar{\pi}_{12} < 0$ *if and only if* $\hat{\Gamma}_4 < 0$.
2. *Independence From Past Actions: If* $\bar{\pi}_{12} = 0$, *then (i)* $\hat{\Gamma}_1 + \hat{\Gamma}_2 + \hat{\Gamma}_3 = \mathrm{d}\bar{A}/\,\mathrm{d}C$ *and (ii)* $\hat{\Gamma}_4 = 0$.
3. *No Ex-Ante Adaptation or Delayed Effects: If* $\bar{\pi}_{23} = 0$, *then* $\bar{\pi}_{12} < 0$ *if and only if* $\hat{\Gamma}_3 < 0$. *If, in addition,* $\bar{\pi}_{14}, \bar{\pi}_{24} = 0$ *and* $\bar{\pi}_{13} > 0$, *then* $\hat{\Gamma}_1 > \mathrm{d}\bar{A}/\,\mathrm{d}C$ *if and only if* $\bar{\pi}_{12} < 0$.
4. *No Ex-Post Adaptation: If* $\bar{\pi}_{13}, \bar{\pi}_{14}, \bar{\pi}_{24} = 0$ *and* $\beta\bar{\pi}_{23} > 0$, *then (i)* $\hat{\Gamma}_3 > \mathrm{d}\bar{A}/\,\mathrm{d}C$ *if and only if* $\bar{\pi}_{12} < 0$ *and (ii)* $\hat{\Gamma}_1 = \hat{\Gamma}_2 = 0$.
5. *Myopic Agents: If* $\beta = 0$, *then* $\hat{\Gamma}_3 = 0$. *If, in addition, either* $\bar{\pi}_{13} > 0$ *or* $\bar{\pi}_{14} > 0$, *then* $\hat{\Gamma}_1 + \hat{\Gamma}_2 > \mathrm{d}\bar{A}/\,\mathrm{d}C$ *if and only if* $\bar{\pi}_{12} < 0$.

Proof. See appendix. □

The first result says that we can learn whether actions are intertemporal substitutes or complements by considering the coefficient on previous actions. When this coefficient is small, we may not need to worry about the biases introduced by these relationships. Further, the second result in Proposition 1 confirms that we can exactly recover the effect of climate from transient weather shocks when that coefficient is zero. The intuition should be clear

[22]In general, point d could be to the right or the left of point c because the effects of $\beta\bar{\pi}_{23}$ conflict with the channel discussed here, but I temporarily ignore the effect of $\beta\bar{\pi}_{23}$ because it is not relevant to the point of interest here.

from the foregoing discussion: both sources of bias vanish when $\bar{\pi}_{12} = 0$ because $\omega = 1$ and $\Omega = 0$.

The remaining results describe cases in which we cannot exactly recover the response to a long-run change in climate but can bound it. The third result describes a case without either ex-ante adaptation or delayed effects, so that only current weather matters and people respond to that weather only as it happens. This case may adequately describe household decisions about how to set a thermostat. If there is no ex-ante adaptation, then forecasts matter only because they shape expectations of future actions, not because they allow actions that can directly interact with future weather. And if there are no delayed effects, then we do not need to concern ourselves with the direct consequences of past weather. $\hat{\Gamma}_1$ then captures all the channels of interest and is distorted only via the ω in equation (6).

In contrast, the fourth result considers a case without ex-post adaptation. Here adaptation actions require a lead time, as with the decision to buy an air conditioning unit or the decision about which crop to plant. In this case, current weather should have no effect on observed actions and the effects of climate change arise only through altered expectations, not through effects on either current or past weather. $\hat{\Gamma}_3$ then captures all the channels of interest and is distorted only via the ω in equation (6).

Some previous literature has highlighted expectations as being the sole difference between weather and climate (e.g., Hsiang, 2016; Deryugina and Hsiang, 2017), but the final result shows that differences remain even for myopic agents, for whom expectations are irrelevant. Even though forecasts do not matter to myopic agents ($\hat{\Gamma}_3 = 0$), their responses to weather still fail to recover the effect of climate change. The reason is that their current actions do depend on past actions, even though they fail to anticipate this dependence. This dependence constrains short-run responses when actions are intertemporal complements and constrains long-run responses when actions are intertemporal substitutes. Formally, setting $\beta = 0$ in equation (6) eliminates the bias introduced by Ω but does not affect the bias introduced by ω.

We have developed intuition and results for a regression like (5), but the empirical literature has, almost without exception, not estimated that type of equation. Instead, researchers have often estimated equations of the form

$$A_{jt} = \alpha_j + \psi_t + \gamma_1 w_{jt} + \gamma_2 w_{j(t-1)} + \delta_{jt}. \tag{7}$$

Note that the error term δ_{jt} is now correlated with the covariates because it includes f_{jt} and $A_{j(t-1)}$.[23] We then have:

[23]One could in principle include $A_{j(t-1)}$ as a control on the right-hand side of‘(7), but standard advice (e.g., Angrist and Pischke, 2009, Chapter 5) recommends against controlling for both fixed effects and lagged dependent variables because of the likelihood of introducing Nickell (1981) omitted variables bias. Standard practice in the empirical climate-economy literature has emphasized fixed effects instead of lagged dependent variables, as seen in the appendix to Dell et al. (2014). Controlling for $A_{j(t-1)}$ would not change the primary results in Proposition 2.

Proposition 2. *Let either Assumption 1 or 2 hold, and let $(A_{j(t-1)} - \bar{A})^2$ be small for all observations.*

1. *Fix $\bar{\pi}_{12} = 0$ and consider $\hat{\gamma}_1$:*
   (a) $\hat{\gamma}_1 = \mathrm{d}\bar{A}/\,\mathrm{d}C$ *if* $\bar{\pi}_{14} = 0$ *and* $\beta\bar{\pi}_{23} = 0$.
   (b) $\hat{\gamma}_1 \in (\hat{\Gamma}_1, \mathrm{d}\bar{A}/\,\mathrm{d}C)$ *if* $\rho\beta\bar{\pi}_{23} > 0$.
   (c) $\hat{\gamma}_1 = \hat{\Gamma}_1$ *if* $\rho\beta\bar{\pi}_{23} = 0$.
2. *Fix $\bar{\pi}_{12} = 0$ and consider $\hat{\gamma}_1 + \hat{\gamma}_2$:*
   (a) $\hat{\gamma}_1 + \hat{\gamma}_2 = \mathrm{d}\bar{A}/\,\mathrm{d}C$ *if* $\beta\bar{\pi}_{23} = 0$.
   (b) $\hat{\gamma}_1 + \hat{\gamma}_2 \in (\hat{\Gamma}_1 + \hat{\Gamma}_2, \mathrm{d}\bar{A}/\,\mathrm{d}C)$ *if* $\rho\beta\bar{\pi}_{23} > 0$.
   (c) $\hat{\gamma}_1 + \hat{\gamma}_2 = \hat{\Gamma}_1 + \hat{\Gamma}_2$ *if* $\rho\beta\bar{\pi}_{23} = 0$.
3. *If $\beta = 0$, then $\hat{\gamma}_1 < \mathrm{d}\bar{A}/\,\mathrm{d}C$ if $\bar{\pi}_{12} > 0$. If, in addition, $\bar{\pi}_{14} = 0$, then $\hat{\gamma}_1 > \mathrm{d}\bar{A}/\,\mathrm{d}C$ if $\bar{\pi}_{12} < 0$.*

Proof. See appendix. □

Regression (7) does not control for forecasts or for past actions, so these affect the estimators $\hat{\gamma}_1$ and $\hat{\gamma}_2$ as omitted variables. The first set of results establishes what we can learn from $\hat{\gamma}_1$, which is the coefficient of interest in much previous empirical literature. Assume that the marginal benefit of current actions is independent of past actions ($\bar{\pi}_{12} = 0$). $\hat{\gamma}_1$ captures part of the effect of time t forecasts through their covariance ρ with ϵ_t; however, the proof shows that $\hat{\gamma}_1$ can never capture the total effect of forecasts. Omitted variables bias helps, but it cannot replace explicitly controlling for forecasts. Further, $\hat{\gamma}_1$ also misses the interaction between time t actions and past weather. Putting these pieces together, $\hat{\gamma}_1$ can fully recover climate impacts only if there is no ex-ante adaptation that would use forecasts ($\beta\bar{\pi}_{23} = 0$) and past weather shocks do not matter directly ($\bar{\pi}_{14} = 0$). In other cases, $\hat{\gamma}_1$ underestimates the effect of climate change by only partially capturing these channels. The performance of $\hat{\gamma}_1$ improves as weather becomes more serially correlated (i.e., as ρ increases) because omitted variables bias becomes stronger, but one could do better by estimating equation (5) and combining $\hat{\Gamma}_1$ with $\hat{\Gamma}_2$ and $\hat{\Gamma}_3$.

The second set of results shows that combining $\hat{\gamma}_1$ and $\hat{\gamma}_2$ captures the interaction between time t actions and past weather but still fails to capture the total effect of forecasts. Omitted variables bias allows $\hat{\gamma}_1 + \hat{\gamma}_2$ to potentially perform better than $\hat{\Gamma}_1 + \hat{\Gamma}_2$, but if weather is serially uncorrelated ($\rho = 0$), then omitted variables bias from forecasts vanishes and these two estimators are equivalent. Further, one could do better by estimating equation (5) and combining $\hat{\Gamma}_1 + \hat{\Gamma}_2$ with $\hat{\Gamma}_3$.

The final result again shows that expectations are not the only factor driving a wedge between weather and climate. When actors are myopic ($\beta = 0$) and current actions are independent of previous weather ($\bar{\pi}_{14} = 0$), climate directly matters for decision-making only by affecting present weather. However, the effect of climate on past weather matters indirectly even in this case by shaping past actions. When actions are intertemporal substitutes ($\bar{\pi}_{12} < 0$), these past actions constrain the long-run response to climate more than the response to transient weather events, but when actions are intertemporal complements ($\bar{\pi}_{12} > 0$), these past actions constrain the response to transient weather events more than the long-run response to climate.

4 Estimating the Effect of Climate on Payoffs

Beyond the effects of climate on decision variables, much empirical research has sought to estimate the consequences of climate change for profits (e.g., Deschênes and Greenstone, 2007) and for variables such as gross output or income that are potentially related to aggregate payoffs (e.g., Dell et al., 2012; Burke et al., 2015; Deryugina and Hsiang, 2017). I now consider how to estimate the effect of climate on long-run payoffs from time series variation in weather.

Using either Assumption 1 or Assumption 2, we have:

$$\begin{aligned}
E_0[\pi(A_t, A_{t-1}, w_t, w_{t-1})] =& \bar{\pi} + \bar{\pi}_1(E_0[A_t] - \bar{A}) + \bar{\pi}_2(E_0[A_{t-1}] - \bar{A}) \\
&+ \frac{1}{2}\bar{\pi}_{11}E_0[(A_t - \bar{A})^2] + \frac{1}{2}\bar{\pi}_{22}E_0[(A_{t-1} - \bar{A})^2] + \frac{1}{2}(\bar{\pi}_{33} + \bar{\pi}_{44})\zeta^2(\sigma^2 + \tau^2) \\
&+ \bar{\pi}_{12}E_0[(A_t - \bar{A})(A_{t-1} - \bar{A})] + \bar{\pi}_{13}Cov_0[A_t, w_t] + \bar{\pi}_{23}Cov_0[A_{t-1}, w_t] \\
&+ \bar{\pi}_{14}Cov_0[A_t, w_{t-1}] + \bar{\pi}_{24}Cov_0[A_{t-1}, w_{t-1}] + \bar{\pi}_{34}\zeta^2\rho,
\end{aligned}$$

for $t > 1$. Differentiating with respect to C, applying either assumption again, and using our earlier result that $E_0[A_t] \to \bar{A}$ for t sufficiently large, we find that, as t becomes large,

$$\frac{\mathrm{d}E_0[\pi(A_t, A_{t-1}, w_t, w_{t-1})]}{\mathrm{d}C} \to \bar{\pi}_3 + \bar{\pi}_4 + [\bar{\pi}_1 + \bar{\pi}_2]\frac{\mathrm{d}\bar{A}}{\mathrm{d}C}. \tag{8}$$

The marginal effect of climate on long-run payoffs is composed of the direct effect of a larger weather index, in both the present ($\bar{\pi}_3$) and the past ($\bar{\pi}_4$), and the effects of changing long-run actions, including both present actions ($\bar{\pi}_1$) and past actions ($\bar{\pi}_2$). From the Euler equation (1), we have $\bar{\pi}_1 = -\beta\bar{\pi}_2$, so as t becomes large,

$$\frac{\mathrm{d}E_0[\pi(A_t, A_{t-1}, w_t, w_{t-1})]}{\mathrm{d}C} \to \bar{\pi}_3 + \bar{\pi}_4 + (1 - \beta)\bar{\pi}_2\frac{\mathrm{d}\bar{A}}{\mathrm{d}C}. \tag{9}$$

Whether economic responses increase or decrease payoffs depends on the sign of $\bar{\pi}_2$. As described in Section 3, a case with $\bar{\pi}_2 > 0$ is a case in which higher actions impose costs today but provide benefits tomorrow, as when undertaking adaptation investments that take time to build. A case with $\bar{\pi}_2 < 0$ is a case in which higher actions provide benefits today but impose costs tomorrow, as when borrowing money or selling from storage. Undertaking more actions because of climate change increases payoffs if and only if actions are of the former type.

I will assess the importance of the following assumption for our ability to estimate the effect of climate on payoffs from within-unit weather variation:

Assumption 3. *$\pi_2(A_t, A_{t-1}, w_t, w_{t-1}) = K\pi_1(A_t, A_{t-1}, w_t, w_{t-1})$ if $A_{t-1} = A_t$, for $K \neq -\beta$.*

This assumption makes the marginal benefit of past actions around a steady state proportional to the marginal benefit of current actions around a steady state. Consider a few special cases. First, a quadratic adjustment cost model yields $K = 0$: π_2 is proportional to $A_t - A_{t-1}$, which is equal to 0 when $A_t = A_{t-1}$. Second, a linear interaction model yields $K = 1$: if $\pi = g(A_t A_{t-1})$, then $\pi_1 = A_{t-1} g'(A_t A_{t-1})$ and $\pi_2 = A_t g'(A_t A_{t-1})$. Third, a model in which the returns to resource extraction decline in previous extraction can yield $K = -1$: if $\pi = g(A_t/A_{t-1})$, then $\pi_1 = g'(A_t/A_{t-1})/A_{t-1}$ and $\pi_2 = -A_t g'(A_t/A_{t-1})/(A_{t-1}^2)$. Finally, a model without dynamic linkages has $\pi_2(\cdot,\cdot,\cdot,\cdot) = 0$ and thus $K = 0$.

Most empirical researchers will not observe the full set of actions available to agents or firms. As a result, empirical researchers may estimate the following regression:

$$\pi_{jt} = \alpha_j + \psi_t + X_{jt}\theta + \eta_{jt}, \tag{10}$$

where we again label firms by j and η_{jt} is again an error term. The vector of covariates X_{jt} is

$$\begin{bmatrix} w_{jt} & f_{jt} & w_{j(t-1)} & f_{j(t-1)} & \dots & w_{j(t-K)} & f_{j(t-K)} \end{bmatrix}.$$

As before, I assume that η_{jt} would be uncorrelated with the covariates if they included lags of actions.[24] However, because actions are unobserved by the econometrician, they can introduce omitted variables bias. We are interested in the vector of coefficients θ. I denote each element with a subscript corresponding to the covariate it multiplies, and I again use a hat to denote the probability limit of each element.

Proposition 3. *Let Assumption 1 hold, or let Assumption 2 hold with the ϵ and ν normally distributed. Also let $(A_{j(t-1)} - \bar{A})^2$ and $(A_{jt} - \bar{A})^2$ be small for all observations and let each agent's average actions be $\bar{A}$. Then, for t large and $K > 2$,*

1. *The Estimators: $\hat{\theta}_{w_t} = \bar{\pi}_3 + \bar{\pi}_1\hat{\Gamma}_1$ and $\hat{\theta}_{f_t} = \bar{\pi}_1\hat{\Gamma}_3$. If $\bar{\pi}_{12} = 0$, then $\hat{\theta}_{w_{t-1}} = \bar{\pi}_4 + \bar{\pi}_2\hat{\Gamma}_1 + \bar{\pi}_1\hat{\Gamma}_2$, $\hat{\theta}_{f_{t-1}} = \bar{\pi}_2\hat{\Gamma}_3$, and $\hat{\theta}_{w_{t-2}} = \bar{\pi}_2\hat{\Gamma}_2$.*

[24] I do not explicitly model the unobservable characteristics that motivate the fixed effects specification because they are not central to the question of interest. See footnote 19 above.

2. *When Actions Vanish: If Assumption 3 holds, then* $\mathrm{d}E_0[\pi_t]/\,\mathrm{d}C = \bar{\pi}_3 + \bar{\pi}_4 = \hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}}$ *and* $\hat{\theta}_{f_t} = \hat{\theta}_{f_{t-1}} = \hat{\theta}_{w_{t-2}} = 0$.

3. *Independence From Past Actions: If* $\bar{\pi}_{12} = 0$, *then* $\mathrm{d}E_0[\pi_t]/\,\mathrm{d}C = \hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{w_{t-2}} + \hat{\theta}_{f_t} + \hat{\theta}_{f_{t-1}}$.

4. *No Ex-Ante Adaptation or Delayed Effects: Let* $\bar{\pi}_{23}, \bar{\pi}_{14}, \bar{\pi}_{24} = 0$ *and* $\bar{\pi}_{13} > 0$. *Then* $\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} < \mathrm{d}E_0[\pi_t]/\,\mathrm{d}C$ *if and only if* $\bar{\pi}_2\bar{\pi}_{12} > 0$.

5. *No Ex-Post Adaptation: Let* $\bar{\pi}_{13}, \bar{\pi}_{14}, \bar{\pi}_{24} = 0$ *and* $\beta\bar{\pi}_{23} > 0$. *Then* $\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{f_t} + \hat{\theta}_{f_{t-1}} < \mathrm{d}E_0[\pi_t]/\,\mathrm{d}C$ *if and only if* $\bar{\pi}_2\bar{\pi}_{12} > 0$.

6. *Myopic Agents: If* $\beta = 0$, *then* $\hat{\theta}_{f_t} = \hat{\theta}_{f_{t-1}} = 0$. *If, in addition, either* $\bar{\pi}_{13} > 0$ *or* $\bar{\pi}_{14} > 0$, *then* $\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{w_{t-2}} < \mathrm{d}E_0[\pi_t]/\,\mathrm{d}C$ *if and only if* $\bar{\pi}_2\bar{\pi}_{12} > 0$.

Proof. See appendix. □

From equation (8), estimating the effects of climate on payoffs requires estimating four terms: $\bar{\pi}_3$, $\bar{\pi}_4$, $\bar{\pi}_1[\,\mathrm{d}\bar{A}/\,\mathrm{d}C]$, and $\bar{\pi}_2[\,\mathrm{d}\bar{A}/\,\mathrm{d}C]$. The first result of the proposition shows that the coefficients from (10) are closely related to these terms. In particular, the direct effects of weather ($\bar{\pi}_3$ and $\bar{\pi}_4$) are captured by $\hat{\theta}_{w_t}$ and $\hat{\theta}_{w_{t-1}}$. The second part of the proposition shows that when Assumption 3 holds, these direct effects on weather suffice to describe the effect of climate on payoffs. Assumption 3 and the Euler equation (1) imply that $\bar{\pi}_2 = \bar{\pi}_1 = 0$: an optimizing agent sets the marginal benefit of actions to zero around a steady state, as with point 1 in Figure 1. In this case, the consequences of marginal climate change are independent of changes in actions.[25] Therefore, summing $\hat{\theta}_{w_t}$ and $\hat{\theta}_{w_{t-1}}$ fully captures the effects of climate on payoffs. Further, we can test whether Assumption 3 holds by examining the magnitudes of the coefficients on forecasts and on sufficiently long lags of weather: because these variables matter for payoffs only through their effects on actions, they cannot affect payoffs if Assumption 3 indeed holds.

The third part of the proposition describes an additional special case in which we can recover the effect of climate on payoffs from (10). From Proposition 1, we know that $\hat{\Gamma}_1 + \hat{\Gamma}_2 + \hat{\Gamma}_3$ recovers the effect of climate on actions when $\bar{\pi}_{12} = 0$. The first part of Proposition 3 showed that $\hat{\theta}_{w_t}$ and $\hat{\theta}_{w_{t-1}}$ recover $\hat{\Gamma}_1$ (the consequences of altering current weather), $\hat{\theta}_{w_{t-1}}$ and $\hat{\theta}_{w_{t-2}}$ recover $\hat{\Gamma}_2$ (the consequences of altering past weather), and $\hat{\theta}_{f_t}$ and $\hat{\theta}_{f_{t-1}}$ recover

[25] As described earlier, one of the special cases of Assumption 3 is a model with no dynamic linkages ($\pi_2(\cdot,\cdot,\cdot,\cdot) = 0$), in which case the agent solves a series of independent, static decision problems. We have therefore recovered the result obtained by previous appeals to the envelope theorem in static settings (Hsiang, 2016; Deryugina and Hsiang, 2017), but we now see that those settings were a rather special case: envelope theorem arguments do not suffice to make actions irrelevant in a model with dynamic linkages. See also footnote 27.

$\hat{\Gamma}_3$ (the consequences of altering expectations of future weather). The coefficient in each pair with the larger time index captures the effect on payoffs via current actions and the other coefficient captures the effect on payoffs via past actions. Therefore, summing $\hat{\theta}_{w_t}$, $\hat{\theta}_{w_{t-1}}$, $\hat{\theta}_{w_{t-2}}$, $\hat{\theta}_{f_t}$, and $\hat{\theta}_{f_{t-1}}$ recovers $(\bar{\pi}_1 + \bar{\pi}_2)[\,\mathrm{d}\bar{A}/\,\mathrm{d}C]$ when $\bar{\pi}_{12} = 0$. When neither adjustment cost nor resource constraint stories apply, we can recover the effect of climate on payoffs from a regression with sufficiently long lags of weather and forecasts.[26]

The fourth part of the proposition signs the bias in our estimate of the effect of climate on payoffs when $\bar{\pi}_{12} \neq 0$ but there is no ex-ante adaptation or delayed effects. In this case, only current weather matters for actions. This channel is captured by $\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}}$. From Proposition 1, we know that adjustment costs and resource constraints lead us to misestimate effects on actions: $\hat{\Gamma}_1 > \,\mathrm{d}\bar{A}/\,\mathrm{d}C$ if and only if $\bar{\pi}_{12} < 0$. Equation (9) showed that undertaking more actions because of climate change provides benefits if and only if $\bar{\pi}_2 > 0$. Therefore, $\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}}$ is an overly pessimistic estimate of the effects of climate on payoffs in two cases: when actions are beneficial and variation in weather underestimates how actions respond to climate change ($\bar{\pi}_2, \bar{\pi}_{12} > 0$), and when actions impose costs and actions respond less to climate change than to short-lived changes in weather ($\bar{\pi}_2, \bar{\pi}_{12} < 0$). Intuitively, a case with $\bar{\pi}_2, \bar{\pi}_{12} > 0$ is one in which actions are costly to adjust and are undertaken for future benefits, as with diverting crops to storage, and a case with $\bar{\pi}_2, \bar{\pi}_{12} < 0$ is one in which actions impose long-run costs but will not be maintained for long, as may be true of groundwater withdrawals. In either case, extrapolating from responses to weather overstates the cost of climate change; in other cases, the estimator $\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}}$ is an overly optimistic estimate of the effect of climate change.

The fifth part of the proposition signs the bias in our estimate of the effect of climate on payoffs when $\bar{\pi}_{12} \neq 0$ but there is no ex-post adaptation. In this case, only expectations of future weather matter for actions. This channel is is captured by $\hat{\theta}_{f_t} + \hat{\theta}_{f_{t-1}}$. From Proposition 1, we know that adjustment costs and resource constraints lead us to misestimate effects on actions: $\hat{\Gamma}_3 > \,\mathrm{d}\bar{A}/\,\mathrm{d}C$ if and only if $\bar{\pi}_{12} < 0$. From here, the intuition for the result follows the case without ex-ante adaptation or delayed effects.

The final result considers a case with myopic agents. When agents are myopic, the Euler equation (1) implies that $\bar{\pi}_1 = 0$. The effects of climate on current actions no longer matter for payoffs, but the effects of climate on past actions can still matter for payoffs.[27]

[26]The requirement that $K > 2$ ensures that omitted variables bias does not affect the needed coefficients. If we allowed for longer lags in the consequences of weather and/or in the consequences of forward-looking investments, then the estimator of climate consequences would include longer lags of forecasts and weather than the estimator described in part 3 of Proposition 3. We would then require that K be strictly larger than the longest lag.

[27]Because myopic agents solve a static decision problem, the envelope theorem now makes current actions independent of payoffs. But a static decision problem is not equivalent to a static decision-making environment: because past actions are predetermined variables, the envelope theorem has no bearing on how the decision problems are linked through the history of weather. In Figures 1 and 2, even myopic agents find point c instead of remaining at point b following a weather shock.

Proposition 1 already established that we tend to underestimate changes in actions when $\bar{\pi}_{12} > 0$. In this case, we estimate an overly pessimistic effect of climate on payoffs if and only if past actions provide current benefits ($\bar{\pi}_2 > 0$). On the other hand, we tend to overestimate the changes in actions when $\bar{\pi}_{12} < 0$. In this case, we estimate an overly optimistic effect of climate on payoffs if and only if past actions impose current costs ($\bar{\pi}_2 < 0$). In sum, stripping away expectations eliminates several channels through which climate affects payoffs, but stripping away expectations does not eliminate all of the dynamic linkages that differentiate climate from weather.

Proposition 3 assumed that each agent's average actions are $\bar{A}$. The following corollary establishes how relaxing this assumption changes the results.

Corollary 4. *Let the conditions given in Proposition 3 hold, except let each agent's average actions be strictly greater than $\bar{A}$. Then, for t large and $K > 2$,*

1. $\hat{\theta}_{f_t} = \bar{\pi}_1 \hat{\Gamma}_3$ *and, if either* $\bar{\pi}_{13} > 0$ *or* $\bar{\pi}_{23} > 0$, $\hat{\theta}_{w_t} > \bar{\pi}_3 + \bar{\pi}_1 \hat{\Gamma}_1$. *If* $\bar{\pi}_{12} = 0$, *then* $\hat{\theta}_{f_{t-1}} = \bar{\pi}_2 \hat{\Gamma}_3$, $\hat{\theta}_{w_{t-2}} = \bar{\pi}_2 \hat{\Gamma}_2$, *and, if either* $\bar{\pi}_{14} > 0$ *or* $\bar{\pi}_{24} > 0$, $\hat{\theta}_{w_{t-1}} > \bar{\pi}_4 + \bar{\pi}_2 \hat{\Gamma}_1 + \bar{\pi}_1 \hat{\Gamma}_2$.

2. *If Assumption 3 holds and at least one of* $\bar{\pi}_{13}$, $\bar{\pi}_{23}$, $\bar{\pi}_{14}$, *or* $\bar{\pi}_{24}$ *is strictly positive, then* $\mathrm{d}E_0[\pi_t]/\,\mathrm{d}C = \bar{\pi}_3 + \bar{\pi}_4 < \hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}}$.

3. *If* $\bar{\pi}_{12} = 0$ *and at least one of* $\bar{\pi}_{13}$, $\bar{\pi}_{23}$, $\bar{\pi}_{14}$, *or* $\bar{\pi}_{24}$ *is strictly positive, then* $\mathrm{d}E_0[\pi_t]/\,\mathrm{d}C < \hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{w_{t-2}} + \hat{\theta}_{f_t} + \hat{\theta}_{f_{t-1}}$.

The inequalities reverse if, instead, each agent's average actions are strictly less than $\bar{A}$.

Proof. See appendix. □

The first part of the corollary establishes that the bias from average actions not yet having reached the steady state enters through $\hat{\theta}_{w_t}$ and $\hat{\theta}_{w_{t-1}}$. The remaining parts of the corollary establish that the special cases that formerly sufficed to identify climate impacts from weather impacts now merely bound the effect of climate on payoffs. In particular, we obtain an upper bound if agents are approaching their steady-state actions from above and a lower bound otherwise. Intuitively, if climate shifts the steady-state action farther from the agent's current action, then any weather shocks incorporate transition costs that vanish from the effect of climate on long-run payoffs.

In practice, empirical researchers have not estimated equations like (10). Instead, empirical researchers have very rarely controlled for forecasts, and many also do not control for lagged weather outcomes when payoffs are the dependent variable.[28] Conventional regressions are closer to

$$\pi_{jt} = \alpha_j + \psi_t + X_{jt}\Phi + \delta_{jt}, \tag{11}$$

[28]Distributed lag models are instead estimated when mortality is the dependent variable (e.g., Deschênes and Moretti, 2009).

where the vector of covariates X_{jt} is now

$$\begin{bmatrix} w_{jt} & w_{j(t-1)} & \dots & w_{j(t-K)} \end{bmatrix}.$$

The error term δ_{jt} now includes not only actions but also current and past forecasts. We are interested in the vector of coefficients Φ. I denote each element with a subscript corresponding to the covariate it multiplies, and I again use a hat to denote the probability limit of each element. A superscript on each $\hat{\Phi}$ now denotes the value of K.

The following proposition relates these estimates to the desired effect of climate on payoffs.

Proposition 5. *Let Assumption 1 hold, or let Assumption 2 hold with the ϵ and ν normally distributed. Also let $(A_{j(t-1)} - \bar{A})^2$ and $(A_{jt} - \bar{A})^2$ be small for all observations, let each agent's average actions be $\bar{A}$, and let t be large.*

1. *Let Assumption 3 hold. Then:*
   (a) $\hat{\Phi}^K_{w_t} + \hat{\Phi}^K_{w_{t-1}} = \mathrm{d}E_0[\pi_t]/\,\mathrm{d}C$ *if* $K = 1$ *or* $K = 2$.
   (b) $\hat{\Phi}^0_{w_t} < \mathrm{d}E_0[\pi_t]/\,\mathrm{d}C$ *if and only if* $\bar{\pi}_4 > 0$.
2. *Let* $\bar{\pi}_{12} = 0$ *and* $\beta\bar{\pi}_{23} = 0$. *Then:*
   (a) $\hat{\Phi}^2_{w_t} + \hat{\Phi}^2_{w_{t-1}} + \hat{\Phi}^2_{w_{t-2}} = \mathrm{d}E_0[\pi_t]/\,\mathrm{d}C$.
   (b) *If* $\bar{\pi}_{14} = 0$, *then* $\hat{\Phi}^1_{w_t} + \hat{\Phi}^1_{w_{t-1}} = \mathrm{d}E_0[\pi_t]/\,\mathrm{d}C$.
   (c) *If* $\bar{\pi}_{14} = 0$, $\bar{\pi}_4 = 0$, *and* $\bar{\pi}_{13} > 0$, *then* $\hat{\Phi}^0_{w_t} < \mathrm{d}E_0[\pi_t]/\,\mathrm{d}C$ *if and only if* $\bar{\pi}_2 > 0$.

Proof. See appendix. □

The first result establishes that we can still recover the full effect of climate on payoffs if Assumption 3 holds and $K > 0$. Responses to forecasts generally identify ex-ante adaptation, but small changes in adaptation are irrelevant for payoffs when Assumption 3 holds. And Assumption 3 further implies that responses to forecasts do not induce omitted variables bias in (11). We can therefore recover the effect of climate from $\hat{\Phi}^K_{w_t} + \hat{\Phi}^K_{w_{t-1}}$.

However, the second result establishes that we can no longer recover the effects of climate on payoffs merely by assuming $\bar{\pi}_{12} = 0$. The reason is that by failing to control for forecasts, our estimates no longer fully capture the possibility of ex-ante adaptation based on expectations of future weather.[29] We must therefore introduce a further assumption—that there is no ex-ante adaptation ($\beta\bar{\pi}_{23} = 0$)—in order to recover the original result. If we estimate

[29] Some of this ex-ante adaptation is captured through omitted variables bias in the plausible case where weather and forecasts are positively correlated (i.e., where $\rho > 0$), but the proof shows that it can never be captured completely.

a model with only a single lag of weather, then we must also limit the interaction between actions and past weather by assuming that $\bar{\pi}_{14} = 0$. And if we follow much of the empirical literature in estimating a model without any lags of weather, then recovering the effects of climate change further requires the absence of adaptation to current weather ($\bar{\pi}_{13} = 0$) and the absence of lagged direct effects of weather ($\bar{\pi}_4 = 0$). These strong assumptions are unlikely to hold in many applications of interest.[30]

5 Long Differences

Rather than estimating either a cross-sectional or a panel model, some literature has instead estimated "long difference" specifications (e.g., Dell et al., 2012; Burke and Emerick, 2016). This approach averages weather and outcomes over two non-overlapping periods, differences the averages, and estimates how the differenced dependent variable changes with differenced temperature. To many, this approach's appeal rests in providing "plausibly credible causal estimates of climate impacts that account for adaptation" (Auffhammer, 2018b, 45): differencing removes the unobserved fixed factors that may covary with climate in a cross-sectional regression, and the variation induced by differential long-run changes in weather may identify the types of adaptations missing from standard panel regressions. I here derive the long difference estimator and show that it in fact relies on exactly the same variation in transient weather shocks as do the estimators already considered. Further, I show that the long difference estimator recovers the effects of climate change only in a subset of the cases in which previously described estimators can recover the effects of climate change.

Formally, assume, as before, that all agents are in the same climate and that this climate is stationary. The literature has claimed that the long difference estimator is identified by differential rates of climate change, but we will see that the estimator in fact has a clear interpretation even in the absence of climate change over the period of interest.[31] The econometric researcher averages outcomes over Δ periods, with the first averaging interval

[30] Rather than focusing on the $\hat{\Phi}$, Deryugina and Hsiang (2017) undertake a different calculation. Let $p(w_t; C)$ represent the probability density function for weather in climate C. They estimate $\pi(A_t(w_t), A_{t-1}(w_t), w_t, w_{t-1}(w_t)) - \pi(A_t(w^0), A_{t-1}(w^0), w^0, w_{t-1}(w^0))$ for each w_t, where w^0 indicates the omitted category and where we write $A_{t-1}(w_t)$ and $w_{t-1}(w_t)$ in order to focus on questions besides the evaluation point. They calculate the marginal effect of climate from the following expression: $\int_{-\infty}^{\infty} \left[\pi(A_t, A_{t-1}, w_t, w_{t-1}) - \pi(A_t, A_{t-1}, w^0, w_{t-1})\right] \frac{\mathrm{d}p(w_t;C)}{\mathrm{d}C} \mathrm{d}w_t \triangleq \Psi$. Analyzing, we find $\Psi = Cov\left[\pi(A_t, A_{t-1}, w_t, w_{t-1}), \frac{\frac{\mathrm{d}p(w_t;C)}{\mathrm{d}C}}{p(w_t;C)}\right]$. If w_t is normally distributed, then $\Psi = Cov\left[\pi(A_t, A_{t-1}, w_t, w_{t-1}),\ w_t\right]/Var[w_t]$, which, following the proof of Proposition 5, is equal to $\hat{\Phi}^0_{w_t}$. Proposition 5 shows that this estimator recovers the effects of climate change in only the most special of cases.

[31] The present setting can also be interpreted as one in which agents did not realize that the climate was changing, which is plausible when applying long differences to twentieth century data. See Dell et al. (2014) and Burke and Emerick (2016) for discussion.

starting at time 0 and the second averaging interval starting at some time T, where $T > \Delta$. Denote the averages with a tilde and use the time subscript to indicate the beginning of the averaging interval, so that, for instance, $\tilde{\pi}_{j0} \triangleq \sum_{t=0}^{\Delta-1} \pi_{jt}/\Delta$. A typical regression would have the form

$$\tilde{\pi}_{jT} - \tilde{\pi}_{j0} = \Lambda[\tilde{w}_{jT} - \tilde{w}_{j0}] + \tilde{u}_{jT} - \tilde{u}_{j0},$$

where the $\tilde{u}_{j0}$ and $\tilde{u}_{jT}$ are error terms. I assume that $\tilde{u}_{jT} - \tilde{u}_{j0}$ is uncorrelated with $\tilde{w}_{jT} - \tilde{w}_{j0}$.

The following proposition establishes properties of the estimator $\hat{\Lambda}$.

Proposition 6. *Let Assumption 1 hold, or let Assumption 2 hold with the ϵ and ν normally distributed. Also let $(A_{j(t-1)} - \bar{A})^2$ and $(A_{jt} - \bar{A})^2$ be small for all observations and let each agent's average actions be $\bar{A}$. Further, assume that $\bar{\pi}_{12} = 0$ and $\rho = 0$. Then, for t large and Δ large, the following conditions are individually sufficient for $\mathrm{d}E_0[\pi_t]/\mathrm{d}C \to \hat{\Lambda}$:*

1. *Assumption 3 holds.*
2. *$\bar{\pi}_{14} = 0$ and $\beta\bar{\pi}_{23} = 0$.*
3. *$\bar{\pi}_{14} = 0$ and τ^2/σ^2 is large.*

Proof. See appendix. □

The proposition considers an especially simple case, in which weather shocks are serially uncorrelated ($\rho = 0$) and actions are independent over time ($\bar{\pi}_{12} = 0$). The proof shows that the estimator is

$$\hat{\Lambda} = \hat{\Phi}^0_{w_t} + \frac{\Delta - 1}{\Delta}\Upsilon.$$

As Δ becomes small, the long difference estimator converges to $\hat{\Phi}^0_{w_t}$, which Proposition 5 showed can approximate the effect of climate change in only the most special of cases. As Δ becomes large, the coefficient on Υ goes to 1. This is the case considered by Proposition 6. The proposition shows that the long difference estimator can approximate the effect of climate change in a broader set of cases as Δ becomes large: the estimator $\hat{\Lambda}$ performs better than $\hat{\Phi}^0_{w_t}$ because the process of averaging picks up correlations between current payoffs and lags and leads of weather. Further, the proposition shows that the long difference estimator $\hat{\Lambda}$ does recover the desired effect of climate change if Assumption 3 holds: $\hat{\Lambda}$ can, like the other estimators, recover $\bar{\pi}_3 + \bar{\pi}_4$.

However, Proposition 6 also shows that the estimator $\hat{\Lambda}$ underperforms estimators analyzed in Section 4. In particular, if Assumption 3 does not hold, then the conditions under which $\hat{\Lambda}$ recovers the effect of climate change are more restrictive than in other cases. The proof shows that $\hat{\Lambda}$ fails to estimate $\mathrm{d}\bar{A}/\mathrm{d}C$ because it misses part of the delayed effect of past weather on actions and because it only imperfectly captures ex-ante adaptation. When $\bar{\pi}_{14} = 0$, the delayed effects vanish; when $\beta\bar{\pi}_{23} = 0$, ex-ante adaptation does not occur; and when τ^2/σ^2 is large, the estimator fully captures ex-ante adaptation because the variation in realized weather reflects only variance in forecasts. The estimator $\hat{\theta}_{w_t}+\hat{\theta}_{w_{t-1}}+\hat{\theta}_{w_{t-2}}+\hat{\theta}_{f_t}+\hat{\theta}_{f_{t-1}}$ required none of these assumptions, and the estimator $\hat{\Phi}^2_{w_t}+\hat{\Phi}^2_{w_{t-1}}+\hat{\Phi}^2_{w_{t-2}}$ required only that $\beta\bar{\pi}_{23}=0$.

The upshot of Proposition 6 is that long difference estimators may not bring us closer to estimating the consequences of climate change. Some have claimed that this estimator is identified by differential rates of climate change across locations, but we have seen that this estimator is in fact is also identified by random differences in sequences of transient weather shocks. At best, the estimator $\hat{\Lambda}$ conflates this variation with differential rates of climate change, and at worst, the estimator $\hat{\Lambda}$ instead captures nothing but the same transient weather shocks as more conventional estimators because the climate actually did not change differentially or because agents were not aware of such changes. Intuitively, averaging over several periods does not eliminate the old sources of variation and need not introduce new sources of variation. It therefore is unsurprising that Burke and Emerick (2016) obtain similar estimates in their panel and long difference approaches. They interpret this similarity as indicating the absence of long-run adaptation, but the similarity may in fact be mechanical.

6 Treatment Effects that Vary by Climate Zone

We have heretofore considered cases in which all agents exist in the same climate. However, much empirical work pools observations across broad geographic areas that include many climate zones. For instance, the base specifications in Deschênes and Greenstone (2007) pool counties over the whole United States.[32] Equation (4) implies that the effect of climate on actions is generically independent of the initial climate only if π is quadratic. And equation (8) implies that the effect of climate on payoffs is generically independent of the initial climate only if Assumption 3 holds and weather enters π linearly. Beyond these special cases, the treatment effect of climate is likely to be heterogeneous and pooling data across climate zones will, at best, recover a weighted treatment effect.

Recovering a weighted treatment effect changes not just the interpretation of the estimator but also its use. Following Deschênes and Greenstone (2007), many studies combine the estimated effects of weather shocks with spatially heterogeneous predictions from physical climate models in order to project how climate change will affect different regions. This approach aims to account for heterogeneity in the future climate treatment, but it does not make sense if researchers are in fact estimating a weighted treatment effect from weather shocks. Instead, empirical researchers should consider pooling observations only within climate zones in order to estimate the effect of marginal climate change within each region. These estimates may then be combined with physical climate models' predicted regional

[32] They explore state-by-state regressions in a supplemental analysis.

changes in climate in order to recover an effect of climate change that accounts for heterogeneity in the treatment effect as well as in the future treatment.[33]

7 Caveats and Potential Extensions

I have demonstrated how to estimate the effects of climate change from time series variation in weather. The setting is fairly general. Nonetheless, the results are subject to several caveats.

First, the present setting successfully captures the distinction between transient and permanent changes in weather, but global climate change also differs from most weather shocks in its spatial structure. A change in global climate affects weather in every location and thus will have general equilibrium consequences. The present setting has followed most empirical work in abstracting from such effects, but some recent empirical work has begun exploring the implications of changing the weather in many locations simultaneously (e.g., Costinot et al., 2016; Gouel and Laborde, 2018; Dingel et al., 2019). Future work should extend the present setting to account for general equilibrium effects.

Second, the present analysis has held the payoff function fixed over time. However, climate change should induce innovations that alter how weather affects payoffs, and many such innovations will arise even in the absence of climate change. Some historical studies have begun exploring the interaction between climate and agricultural innovation (e.g., Olmstead and Rhode, 2008, 2011; Roberts and Schlenker, 2011), but the potential for future innovation may be inherently unobservable. Future work should consider approaches to bounding the scope for innovation.

Third, the present analysis has considered only marginal changes in climate, but climate change over the next century is likely to be nonmarginal. One could approximate the consequences of nonmarginal changes in climate by summing the estimates from fixed effects regressions undertaken in different climate zones. Time series variation then identifies the consequences of marginal changes in climate and cross-sectional variation identifies the consequences of nonmarginal changes in climate. Similar approaches to combining panel and cross-sectional variation have recently been summarized by Auffhammer (2018b). However, two considerations call for caution when extrapolating reduced-form estimates to large

[33] Recently, researchers have begun estimating the consequences of weather in a semiparametric fashion that allows treatment effects to vary with the level of weather (see Carleton and Hsiang, 2016). For instance, the method used in footnote 30 requires estimating payoffs as a function of any realized w_t. This approach poses a further identification problem when data are pooled across climate zones: the effect of a given level of weather is identified from locations for which that weather shock is typical as well as from locations in which that weather shock is atypical. In the latter case, actions are likely to be farther from a relevant steady state, so the types of problems discussed in Corollary 4 now arise inside the weighted treatment effect. Further, omitted forecasts are likely to call for reversion to the mean in that same case, so the estimators are likely to capture even less ex-ante adaptation than did the estimators in (11) if weather is positively correlated over time.

changes in climate: the use of cross-sectional variation raises the usual concerns about identification, which becomes more severe as that cross-sectional variation is asked to do more work, and higher-order effects are likely to become relevant to nonmarginal climate change, even though absent from a summation of estimated marginal effects. Future work should explore whether nonlinear responses to weather shocks can inform nonmarginal consequences of climate change.

Finally, the present analysis has focused on identifying the long-run consequences of climate change, abstracting from the transition costs induced by climate change. In this regard, the present analysis matches the calculations undertaken by nearly all empirical work but omits a potentially critical aspect of climate change (see Quiggin and Horowitz, 1999, 2003; Kelly et al., 2005). Future work should consider whether imposing stronger assumptions on the decision-making environment can identify the costs of full-information transitions. Future work should also consider the potential to estimate structural models that could credibly simulate outcomes along counterfactual climate trajectories.

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A Proof of Lemma 1

Begin by considering the uniqueness of the steady state. The right-hand side of equation (2) monotonically decreases in $\bar{A}$ if and only if $(1+\beta)\pi_{12} < -\pi_{11} - \beta\pi_{22}$. Thus, the steady state is unique if $(1+\beta)\pi_{12} < -\pi_{11} - \beta\pi_{22}$, which is satisfied for all $\pi_{12} < 0$.

Now consider the stability of the steady state. Define $A^*_{t+1}(A_t, Z_t)$ from the Euler equation. Linearizing around $\bar{A}$ gives a first-order difference equation:

$$A_{t+1} - \bar{A} \approx \frac{-\bar{\pi}_{11} - \beta\bar{\pi}_{22}}{\beta\bar{\pi}_{12}}(A_t - \bar{A}) - \frac{1}{\beta}(Z_t - \bar{A}).$$

And we have $Z_{t+1} = A_t$. The product of the linearized-system's eigenvalues is $\frac{1}{\beta} > 1$, and the sum of the linearized system's eigenvalues is $\frac{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}{\beta\bar{\pi}_{12}}$, which is positive if and only if $\bar{\pi}_{12} > 0$.

First, assume that $\bar{\pi}_{12} > 0$. Both eigenvalues are positive and at least one is greater than 1. The characteristic equation is

$$\lambda^2 - \frac{-\bar{\pi}_{11} - \beta\bar{\pi}_{22}}{\beta\bar{\pi}_{12}}\lambda + \frac{1}{\beta},$$

where λ defines the eigenvalues. The smaller eigenvalue is less than 1 if and only if the characteristic equation is negative at $\lambda = 1$, and therefore if and only if

$$-\bar{\pi}_{11} - \beta\bar{\pi}_{22} > (1+\beta)\bar{\pi}_{12}.$$

In this case, the linearized system is saddle-path stable.

Now assume that $\bar{\pi}_{12} < 0$. Both eigenvalues are negative and at least one is less than -1. The characteristic equation is as before. The larger eigenvalue is greater than -1 if and only if the characteristic equation is negative at $\lambda = -1$, and therefore if and only if

$$\begin{aligned} \frac{-\bar{\pi}_{11} - \beta\bar{\pi}_{22}}{\bar{\pi}_{12}} + 1 + \beta &< 0 \\ \Leftrightarrow -\bar{\pi}_{11} - \beta\bar{\pi}_{22} &> -(1+\beta)\bar{\pi}_{12}. \end{aligned}$$

In this case, the linearized system is saddle-path stable.

The proposition follows from a standard application of the Hartman-Grobman theorem and from noticing that the conditions for saddle-path stability imply the condition for uniqueness.

B Proof of Lemma 2

Write A_{t+1} as $A(A_t, w_{t+1}, f_{t+1}, w_t; \zeta)$. Expanding the stochastic Euler equation around $\zeta = 0$ and noting that all terms of order ζ^2 or larger depend on at least the third derivative of π, either Assumption 1 or 2 ensures that we can drop all terms of order ζ^2 or larger. We therefore have:

$$\begin{aligned}
0 =&\pi_1(A_t, A_{t-1}, w_t, w_{t-1}) + \beta E_t\left[\pi_2(\tilde{A}_{t+1}, A_t, f_t, w_t) + \pi_{23}(\tilde{A}_{t+1}, A_t, f_t, w_t)\epsilon_{t+1}\zeta\right] \\
&+ \beta E_t\left[\pi_{12}(\tilde{A}_{t+1}, A_t, f_t, w_t)\left(\left.\frac{\partial A_{t+1}}{\partial \zeta}\right|_{\zeta=0} + \left.\frac{\partial A_{t+1}}{\partial w_{t+1}}\right|_{\zeta=0}\epsilon_{t+1} + \sum_{i=1}^{N}\left.\frac{\partial A_{t+1}}{\partial f_{i(t+1)}}\right|_{\zeta=0}\epsilon_{i(t+1)}\right)\zeta\right] \\
=&\pi_1(A_t, A_{t-1}, w_t, w_{t-1}) + \beta\pi_2(\tilde{A}_{t+1}, A_t, f_t, w_t) + \beta\pi_{12}(\tilde{A}_{t+1}, A_t, f_t, w_t)\left.\frac{\partial A_{t+1}}{\partial \zeta}\right|_{\zeta=0}\zeta, \quad \text{(B-1)}
\end{aligned}$$

where $\tilde{A}_{t+1} \triangleq A(A_t, f_t, C, w_t; 0)$.

We next establish two lemmas. The first one establishes that uncertainty does not have a first-order effect on policy:

Lemma 5. $\left.\frac{\partial A_{t+1}}{\partial \zeta}\right|_{(\bar{A},C,C,C;0)} = 0.$

Proof. Equation (B-1) defines A_t as a function of A_{t-1}, w_t, f_t, and ζ. Note that

$$\begin{aligned}
\left.\frac{\partial A_t}{\partial \zeta}\right|_{(\bar{A},C,C,C;0)} =&\frac{\beta\bar{\pi}_{12}\left(\left.\frac{\partial A_{t+1}}{\partial \zeta}\right|_{(\bar{A},C,C,C;0)} + \left.\frac{\partial^2 A_{t+1}}{\partial \zeta^2}\right|_{(\bar{A},C,C,C;0)}\zeta\right)}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{122}\left.\frac{\partial A_{t+1}}{\partial \zeta}\right|_{\zeta=0}\zeta - \beta\bar{\pi}_{12}\left.\frac{\partial^2 A_{t+1}}{\partial \zeta \partial A_t}\right|_{\zeta=0}\zeta} \\
=&\frac{\beta\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22}}\left.\frac{\partial A_{t+1}}{\partial \zeta}\right|_{(\bar{A},C,C,C;0)},
\end{aligned}$$

where the second equality applies $\zeta = 0$. Forward-substituting, we have:

$$\left.\frac{\partial A_t}{\partial \zeta}\right|_{(\bar{A},C,C,C;0)} = \left(\frac{\beta\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22}}\right)^j \left.\frac{\partial A_{t+j}}{\partial \zeta}\right|_{(\bar{A},C,C,C;0)}$$

for $j \in \mathcal{Z}^+$. The term in parentheses is < 1 by the condition imposed following Lemma 1. Because A_{t+j} evaluated around $A_{t+j-1} = \bar{A}$, $w_t = C$, $f_t = C$, and $\zeta = 0$ must be $\bar{A}$, we know that A_{t+j} is not infinite. The derivative on the right-hand side must also be finite, in which case the right-hand side goes to 0 as j becomes large. Therefore:

$$\left.\frac{\partial A_t}{\partial \zeta}\right|_{(\bar{A},C,C,C;0)} = 0.$$

Because the choice of t was arbitrary, we have established the lemma.

□

The second lemma solves for $\tilde{A}_{t+1}$:

Lemma 6. *If either Assumption 1 or 2 holds and $(A_t - \bar{A})^2$ is small, then there exists λ such that $|\lambda| < 1$ and*

$$\begin{aligned}\tilde{A}_{t+1} =&\bar{A} + \frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}(A_t - \bar{A}) + \frac{\bar{\pi}_{14}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}(w_t - C)\\ &+ \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}(f_t - C).\end{aligned}$$

Proof. For $\zeta = 0$, the weather in period $t+1$ matches the forecast in f_t, and the weather is always C after period $t+1$. Begin by solving for policy after period $t+1$. The characteristic equation given in the proof of Lemma 1 implies the following two eigenvalues:

$$\lambda, \mu = \frac{-\bar{\pi}_{11} - \beta\bar{\pi}_{22}}{2\beta\bar{\pi}_{12}} \pm \sqrt{\left(\frac{-\bar{\pi}_{11} - \beta\bar{\pi}_{22}}{2\beta\bar{\pi}_{12}}\right)^2 - \frac{1}{\beta}}.$$

The proof of Lemma 1 showed that the two eigenvalues have the same sign. Let λ be the eigenvalue that is smallest in absolute value. We seek the eigenvector corresponding to λ, which is the stable manifold. The eigenvector is defined by $(A_{t+2} - \bar{A}) - \lambda(A_{t+1} - \bar{A}) = 0$, and thus

$$\tilde{A}_{t+2} = \lambda A_{t+1} + (1-\lambda)\bar{A},$$

for some A_{t+1}.

Now consider policy at time $t+1$. The relevant Euler equation is:

$$0 = \pi_1(\tilde{A}_{t+1}, A_t, f_t, w_t) + \beta\pi_2(\tilde{A}_{t+2}, \tilde{A}_{t+1}, C, f_t),$$

where we recognize that $w_{t+1} = f_t$. A first-order approximation to $\tilde{A}_{t+1}$ around $\bar{A}$ and the solution for $\tilde{A}_{t+2}$ is exact when either Assumption 1 or 2 holds and $(A_t - \bar{A})^2$ is small. We then have the expression in the lemma. □

Applying Lemmas 5 and 6 to equation (B-1), we have:

$$0 =\pi_1(A_t, A_{t-1}, w_t, w_{t-1}) + \beta\pi_2(\tilde{A}_{t+1}(A_t, f_t, w_t), A_t, f_t, w_t). \tag{B-2}$$

We now have A_t implicitly defined as $A(A_{t-1}, w_t, f_t, w_{t-1}; 0)$. If $(A_{t-1} - \bar{A})^2$ is small and either Assumption 1 or 2 holds, then we have:

$$\begin{aligned}A_t =&\bar{A} + \left.\frac{\partial A_t}{\partial A_{t-1}}\right|_{(\bar{A},C,C,C;0)}(A_{t-1} - \bar{A}) + \left.\frac{\partial A_t}{\partial w_t}\right|_{(\bar{A},C,C,C;0)}(w_t - C) + \left.\frac{\partial A_t}{\partial f_t}\right|_{(\bar{A},C,C,C;0)}(f_t - C)\\ &+ \left.\frac{\partial A_t}{\partial w_{t-1}}\right|_{(\bar{A},C,C,C;0)}(w_{t-1} - C) + \left.\frac{\partial A_t}{\partial \zeta}\right|_{(\bar{A},C,C,C;0)}\zeta\\ =&\bar{A} + \frac{\bar{\pi}_{12}}{\chi}(A_{t-1} - \bar{A}) + \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}(w_t - C)\\ &+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}(f_t - C) + \frac{\bar{\pi}_{14}}{\chi}(w_{t-1} - C),\end{aligned}$$

where we use Lemma 5 in the first equality and where

$$\chi \triangleq -\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}.$$

The condition imposed following Lemma 1 and the fact that $|\lambda| < 1$ together ensure that $\chi > |\bar{\pi}_{12}|$.

C Proof of Lemma 4

Using Lemma 2 and standard regression properties, we have:

$$\begin{aligned}
\hat{\Gamma}_1 =& \omega\frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}},\\
\hat{\Gamma}_2 =& \omega\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}},\\
\hat{\Gamma}_3 =& \omega\frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}},\\
\hat{\Gamma}_4 =& \omega\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}},
\end{aligned}$$

where

$$\omega \triangleq \frac{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}}{\chi} > 0.$$

Note that $\omega > 1$ if $\bar{\pi}_{12} < 0$, $\omega = 1$ if $\bar{\pi}_{12} = 0$, and $\omega < 1$ if $\bar{\pi}_{12} > 0$. The lemma follows from defining

$$\Omega \triangleq \frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}.$$

The sign of Ω matches the sign of $\bar{\pi}_{12}$.

D Proof of Proposition 1

The proof of Lemma 4 provides expressions for $\hat{\Gamma}_1$, $\hat{\Gamma}_2$, $\hat{\Gamma}_3$, and $\hat{\Gamma}_4$. Now consider the parts of the proposition:

1. Follows directly from the expressions in the proof of Lemma 4.
2. If $\bar{\pi}_{12} = 0$, then $\omega = 1$. The result follows from Lemma 4.
3. If $\bar{\pi}_{23}, \bar{\pi}_{14}, \bar{\pi}_{24} = 0$ and $\bar{\pi}_{13} \neq 0$, then $\hat{\Gamma}_1/[\,\mathrm{d}\bar{A}/\,\mathrm{d}C] = \omega$. The result follows.

4. If $\bar{\pi}_{13}, \bar{\pi}_{14}, \bar{\pi}_{24} = 0$ and $\beta\bar{\pi}_{23} \neq 0$, then $\hat{\Gamma}_3/[\,\mathrm{d}\bar{A}/\,\mathrm{d}C] = \omega$. The result follows.

5. If $\beta = 0$ and either $\bar{\pi}_{13} \neq 0$ or $\bar{\pi}_{14} \neq 0$, then $[\hat{\Gamma}_1 + \hat{\Gamma}_2]/[\,\mathrm{d}\bar{A}/\,\mathrm{d}C] = \omega$. The result follows.

E Proof of Proposition 2

Begin by considering $\hat{\gamma}_1$. We apply the Frisch-Waugh theorem. The residuals from regressing w_{jt} on $w_{j(t-1)}$ are:

$$\tilde{w}_{jt} \triangleq w_{jt} - C - \frac{\rho}{\tau^2 + \sigma^2}(w_{j(t-1)} - C) = \zeta\epsilon_{jt} + \zeta\nu_{j(t-1)} - \zeta\frac{\rho}{\tau^2 + \sigma^2}[\epsilon_{j(t-1)} + \nu_{j(t-2)}].$$

We then have:

$$\begin{aligned}\hat{\gamma}_1 =& \frac{Cov[\tilde{w}_{jt}, A_{jt}]}{Var[\tilde{w}_{jt}]} \\ =& \omega\Bigg\{ \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \\ &+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \frac{\rho + \frac{\bar{\pi}_{12}}{\chi}\left(\tau^2 - \frac{\rho^2}{\tau^2+\sigma^2}\right)}{\sigma^2 + \tau^2 - \frac{\rho^2}{\tau^2+\sigma^2}} \\ &- \frac{\bar{\pi}_{14}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \frac{\bar{\pi}_{12}}{\chi} \frac{\frac{\rho^2}{\tau^2+\sigma^2}}{\sigma^2 + \tau^2 - \frac{\rho^2}{\tau^2+\sigma^2}} \Bigg\}.\end{aligned}$$

Now consider $\hat{\gamma}_2$. The residuals from regressing $w_{j(t-1)}$ on w_{jt} are:

$$\tilde{w}_{j(t-1)} \triangleq w_{j(t-1)} - C - \frac{\rho}{\tau^2 + \sigma^2}(w_{jt} - C) = \zeta\epsilon_{j(t-1)} + \zeta\nu_{j(t-2)} - \zeta\frac{\rho}{\tau^2 + \sigma^2}[\epsilon_{jt} + \nu_{j(t-1)}].$$

We then have:

$$\begin{aligned}\hat{\gamma}_2 =& \omega\Bigg\{ \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \frac{\bar{\pi}_{12}}{\chi} \\ &+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \frac{\frac{\bar{\pi}_{12}}{\chi}\frac{\sigma^2}{\tau^2+\sigma^2}\rho - \frac{\rho^2}{\tau^2+\sigma^2}}{\sigma^2 + \tau^2 - \frac{\rho^2}{\tau^2+\sigma^2}} \\ &+ \frac{\bar{\pi}_{14}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \left(1 + \frac{\bar{\pi}_{12}}{\chi} \frac{\rho}{\sigma^2 + \tau^2 - \frac{\rho^2}{\tau^2+\sigma^2}}\right) \Bigg\}.\end{aligned}$$

Now consider the parts of the proposition:

1. If $\bar{\pi}_{12} = 0$, then
$$\hat{\gamma}_1 = \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{23}\frac{\rho}{\sigma^2+\tau^2-\frac{\rho^2}{\tau^2+\sigma^2}}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22}}.$$
Because correlation coefficients are bounded above by 1,
$$\rho \leq \sigma\tau \leq \max\{\sigma^2, \tau^2\}$$
and
$$\sigma^2+\tau^2-\frac{\rho^2}{\tau^2+\sigma^2} \geq \sigma^2+\tau^2-\frac{\tau^2\sigma^2}{\tau^2+\sigma^2} = \max\{\sigma^2,\tau^2\}+\min\{\sigma^2,\tau^2\}\left[1-\frac{\max\{\sigma^2,\tau^2\}}{\tau^2+\sigma^2}\right].$$
Therefore
$$\frac{\rho}{\sigma^2+\tau^2-\frac{\rho^2}{\tau^2+\sigma^2}} \leq \frac{\max\{\sigma^2,\tau^2\}}{\max\{\sigma^2,\tau^2\}+\min\{\sigma^2,\tau^2\}\left[1-\frac{\max\{\sigma^2,\tau^2\}}{\tau^2+\sigma^2}\right]} < 1. \tag{E-3}$$
The results follow.

2. If $\bar{\pi}_{12} = 0$, then
$$\hat{\gamma}_2 = \frac{\bar{\pi}_{14} - \beta\bar{\pi}_{23}\frac{\frac{\rho^2}{\tau^2+\sigma^2}}{\sigma^2+\tau^2-\frac{\rho^2}{\tau^2+\sigma^2}}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}}$$
and
$$\hat{\gamma}_1 + \hat{\gamma}_2 = \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \bar{\pi}_{14} + \beta\bar{\pi}_{23}\frac{\rho\left(1-\frac{\rho}{\tau^2+\sigma^2}\right)}{\sigma^2+\tau^2-\frac{\rho^2}{\tau^2+\sigma^2}}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}}.$$
From equation (E-3),
$$\hat{\gamma}_1 + \hat{\gamma}_2 \leq \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \bar{\pi}_{14} + \beta\bar{\pi}_{23}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}},$$
with strict inequality if $\beta\bar{\pi}_{23} > 0$ and equality if $\beta\bar{\pi}_{23} = 0$. Because
$$1 - \frac{\rho}{\tau^2+\sigma^2} \geq 1 - \frac{\sigma\tau}{\tau^2+\sigma^2} > 0,$$
we have
$$\hat{\gamma}_1 + \hat{\gamma}_2 > \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \bar{\pi}_{14}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}}$$
if $\rho\beta\bar{\pi}_{23} > 0$. Finally, note that, for $\bar{\pi}_{12} = 0$,
$$\hat{\Gamma}_1 + \hat{\Gamma}_2 = \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \bar{\pi}_{14}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}}.$$

The results follow.

3. If $\beta = 0$, then

$$\hat{\gamma}_1 = \omega \left\{ \frac{\bar{\pi}_{13}}{-\bar{\pi}_{11} - \bar{\pi}_{12}} - \frac{\bar{\pi}_{14}}{-\bar{\pi}_{11} - \bar{\pi}_{12}} \frac{\bar{\pi}_{12}}{\chi} \frac{\frac{\rho^2}{\tau^2+\sigma^2}}{\sigma^2 + \tau^2 - \frac{\rho^2}{\tau^2+\sigma^2}} \right\}.$$

$\omega > 1$ if and only if $\bar{\pi}_{12} < 0$. If $\bar{\pi}_{12} > 0$ then

$$\hat{\gamma}_1 < \frac{\bar{\pi}_{13}}{-\bar{\pi}_{11} - \bar{\pi}_{12}} \leq \frac{\mathrm{d}\bar{A}}{\mathrm{d}C}.$$

If $\bar{\pi}_{12} < 0$ and $\bar{\pi}_{14} = 0$, then

$$\hat{\gamma}_1 > \frac{\bar{\pi}_{13}}{-\bar{\pi}_{11} - \bar{\pi}_{12}} = \frac{\mathrm{d}\bar{A}}{\mathrm{d}C}.$$

We have established the result.

F Proof of Proposition 3

The estimated coefficients are $\hat{\theta} = E[\tilde{X}_K^T \tilde{X}_K]^{-1} E[\tilde{X}_K \pi_t]$, where each row of $\tilde{X}_k$ is

$$\begin{bmatrix} w_{jt} - C & f_{jt} - C & w_{j(t-1)} - C & f_{j(t-1)} - C & \ldots & w_{j(t-K)} - C & f_{j(t-K)} - C \end{bmatrix}$$

and the rows correspond to the J observations. Subtracting C demeans each covariate, as implied by the fixed effects. The following lemma establishes that we need only analyze a case with $K = 3$ in order to derive the coefficients on each $w_{j(t-n)}$ and $f_{j(t-n)}$ for $n \in \{0, 1, 2\}$:

Lemma 7. *It $K \geq 3$, then the first six elements of $E[\tilde{X}_K^T \tilde{X}_K]^{-1} E[\tilde{X}_K \pi_t]$ are identical to the first six elements of $E[\tilde{X}_3^T \tilde{X}_3]^{-1} E[\tilde{X}_3 \pi_t]$.*

Proof. The goal is to show that the first six rows of $E[\tilde{X}_K^T \tilde{X}_K]^{-1}$ are equal to the first six rows of $E[\tilde{X}_3^T \tilde{X}_3]^{-1}$ extended to have zeros in columns 9 through $2(K+1)$.

The case with $K = 3$ holds trivially, so assume that $K > 3$. First, note that

$$E[\tilde{X}_K^T \tilde{X}_K] = \begin{bmatrix} E[\tilde{X}_{K-1}^T \tilde{X}_{K-1}] & C_{K-1} \\ C_{K-1}^T & D \end{bmatrix},$$

where C_{K-1} is a $2K \times 2$ matrix with the only nonzero entries being in row $2K - 1$, which is $[J\zeta^2\rho \quad J\zeta^2\tau^2]$, and where

$$D = J\zeta^2 \begin{bmatrix} \sigma^2 + \tau^2 & \rho \\ \rho & \tau^2 \end{bmatrix}.$$

Define

$$B_K \triangleq \begin{bmatrix} E[\tilde{X}_{K-1}^T \tilde{X}_{K-1}] & C_{K-1} \\ C_{K-1}^T & \hat{D} \end{bmatrix},$$

where

$$\hat{D} = J\zeta^2 \begin{bmatrix} \sigma^2 & \rho \\ \rho & \tau^2 \end{bmatrix}.$$

Note that[34]

$$B_{K-1} = E[\tilde{X}_{K-1}^T \tilde{X}_{K-1}] - C_{K-1} D^{-1} C_{K-1}^T.$$

Then, using standard results for block matrix inversion,

$$E[\tilde{X}_K^T \tilde{X}_K]^{-1} = \begin{bmatrix} B_{K-1}^{-1} & -B_{K-1}^{-1} C_{K-1} D^{-1} \\ -D^{-1} C_{K-1}^T B_{K-1}^{-1} & D^{-1} + D^{-1} C_{K-1}^T B_{K-1}^{-1} C_{K-1} D^{-1} \end{bmatrix}.$$

We are concerned with the top row. Note that $C_{K-1}D^{-1}$ has zeros up to row $2K-1$, so it has zeros in its first six rows. It remains to consider B_{K-1}^{-1}. We prove by induction that B_{K-1}^{-1} has the desired form. The basis step considers B_3^{-1}. This is

$$B_3^{-1} = \frac{1}{J\zeta^2[\sigma^2\tau^2 - \rho^2]} \begin{bmatrix} \tau^2 & -\rho & 0 & -\tau^2 & 0 & 0 & 0 & 0 \\ -\rho & \sigma^2 & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau^2 & -\rho & 0 & -\tau^2 & 0 & 0 \\ -\tau^2 & \rho & -\rho & \sigma^2+\tau^2 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau^2 & -\rho & 0 & -\tau^2 \\ 0 & 0 & -\tau^2 & \rho & -\rho & \sigma^2+\tau^2 & 0 & \rho \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau^2 & -\rho \\ 0 & 0 & 0 & 0 & -\tau^2 & \rho & -\rho & \sigma^2+\tau^2 \end{bmatrix}.$$

The first six rows are identical to $E[\tilde{X}_3^T \tilde{X}_3]^{-1}$. Our desired result therefore holds for the basis step. We now turn to the induction step. The induction hypothesis is that the first six rows of B_N^{-1} are equal to the first six rows of $E[\tilde{X}_3^T \tilde{X}_3]^{-1}$ extended to have zeros in columns 9 through $2(N+1)$, for some $N \geq 3$. We want to establish this result for B_{N+1}^{-1}. We have[35]

$$B_N = E[\tilde{X}_N^T \tilde{X}_N] - C_{K-1} \hat{D}^{-1} C_{K-1}^T$$

and thus

$$B_{N+1}^{-1} = \begin{bmatrix} B_N^{-1} & -B_N^{-1} C_N \hat{D}^{-1} \\ -\hat{D}^{-1} C_N^T B_N^{-1} & \hat{D}^{-1} + \hat{D}^{-1} C_N^T B_N^{-1} C_N \hat{D}^{-1} \end{bmatrix}.$$

[34] The only nonzero entries in the $2K \times 2$ matrix $C_{K-1}D^{-1}$ are in row $2K-1$, which is $[0 \quad 1]$, and thus the only nonzero entry in the $2K \times 2K$ matrix $C_{K-1}D^{-1}C_{K-1}^T$ is entry $(2K-1, 2K-1)$, which is $\zeta^2\tau^2$. Subtracting this entry from $E[\tilde{X}_{K-1}^T \tilde{X}_{K-1}]$ yields the result.

[35] The only nonzero entries in the $2K \times 2$ matrix $C_{K-1}\hat{D}^{-1}$ are in row $2K-1$, which is $[0 \quad 1]$, and thus the only nonzero entry in the $2K \times 2K$ matrix $C_{K-1}\hat{D}^{-1}C_{K-1}^T$ is entry $(2K-1, 2K-1)$, which is $\zeta^2\tau^2$. Subtracting this entry from $E[\tilde{X}_N^T \tilde{X}_N]$ yields the result.

Note that $C_N \hat{D}^{-1}$ has zeros up to row $2(N+1)-1$, so it has zeros in its first six rows. Applying the induction hypothesis, the first six rows of B_{N+1}^{-1} are equal to the first six rows of $E[\tilde{X}_3^T \tilde{X}_3]^{-1}$ extended to have zeros in columns 9 through $2(N+2)$. As a result, the first six rows of $E[\tilde{X}_K^T \tilde{X}_K]^{-1}$ are equal to the first six rows of $E[\tilde{X}_3^T \tilde{X}_3]^{-1}$ extended to have zeros in columns 9 through $2(K+1)$. We have proved the desired result.

□

We therefore focus on $K=3$ when deriving coefficients on lags of up to two periods. Note that:

$$E[\tilde{X}_3^T \tilde{X}_3] = J\zeta^2 \begin{bmatrix} \sigma^2+\tau^2 & \rho & \rho & \tau^2 & 0 & 0 & 0 & 0 \\ \rho & \tau^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho & 0 & \sigma^2+\tau^2 & \rho & \rho & \tau^2 & 0 & 0 \\ \tau^2 & 0 & \rho & \tau^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & \sigma^2+\tau^2 & \rho & \rho & \tau^2 \\ 0 & 0 & \tau^2 & 0 & \rho & \tau^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho & 0 & \sigma^2+\tau^2 & \rho \\ 0 & 0 & 0 & 0 & \tau^2 & 0 & \rho & \tau^2 \end{bmatrix}$$

and

$$E[\tilde{X}_3^T \tilde{X}_3]^{-1}$$
$$= \frac{1}{J\zeta^2} \begin{bmatrix} \frac{\tau^2}{\sigma^2\tau^2-\rho^2} & \frac{-\rho}{\sigma^2\tau^2-\rho^2} & 0 & \frac{-\tau^2}{\sigma^2\tau^2-\rho^2} & 0 & 0 & 0 & 0 \\ \frac{-\rho}{\sigma^2\tau^2-\rho^2} & \frac{\sigma^2}{\sigma^2\tau^2-\rho^2} & 0 & \frac{\rho}{\sigma^2\tau^2-\rho^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\tau^2}{\sigma^2\tau^2-\rho^2} & \frac{-\rho}{\sigma^2\tau^2-\rho^2} & 0 & \frac{-\tau^2}{\sigma^2\tau^2-\rho^2} & 0 & 0 \\ \frac{-\tau^2}{\sigma^2\tau^2-\rho^2} & \frac{\rho}{\sigma^2\tau^2-\rho^2} & \frac{-\rho}{\sigma^2\tau^2-\rho^2} & \frac{\sigma^2+\tau^2}{\sigma^2\tau^2-\rho^2} & 0 & \frac{\rho}{\sigma^2\tau^2-\rho^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\tau^2}{\sigma^2\tau^2-\rho^2} & \frac{-\rho}{\sigma^2\tau^2-\rho^2} & 0 & \frac{-\tau^2}{\sigma^2\tau^2-\rho^2} \\ 0 & 0 & \frac{-\tau^2}{\sigma^2\tau^2-\rho^2} & \frac{\rho}{\sigma^2\tau^2-\rho^2} & \frac{-\rho}{\sigma^2\tau^2-\rho^2} & \frac{\sigma^2+\tau^2}{\sigma^2\tau^2-\rho^2} & 0 & \frac{\rho}{\sigma^2\tau^2-\rho^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{\tau^2(\sigma^2+\tau^2)-\rho^2} & \frac{-\rho}{\tau^2(\sigma^2+\tau^2)-\rho^2} \\ 0 & 0 & 0 & 0 & \frac{-\tau^2}{\sigma^2\tau^2-\rho^2} & \frac{\rho}{\sigma^2\tau^2-\rho^2} & \frac{-\rho}{\tau^2(\sigma^2+\tau^2)-\rho^2} & \frac{\tau^2(\sigma^2+\tau^2)^2-\rho^2(\sigma^2+2\tau^2)}{\rho^4+\sigma^2\tau^4(\sigma^2+\tau^2)-\rho^2\tau^2(2\sigma^2+\tau^2)} \end{bmatrix}.$$

We also have:

$$E[\tilde{X}_3^T \pi_{jt}] = J \begin{bmatrix} Cov[w_{jt}-C, \pi_{jt}] \\ Cov[f_{jt}-C, \pi_{jt}] \\ Cov[w_{j(t-1)}-C, \pi_{jt}] \\ Cov[f_{j(t-1)}-C, \pi_{jt}] \\ Cov[w_{j(t-2)}-C, \pi_{jt}] \\ Cov[f_{j(t-2)}-C, \pi_{jt}] \\ Cov[w_{j(t-3)}-C, \pi_{jt}] \\ Cov[f_{j(t-3)}-C, \pi_{jt}] \end{bmatrix}.$$

From here, drop the j subscript to save on unnecessary notation. Consider $Cov[w_t - C, \pi_t]$. Expanding π around $A_t = \bar{A}$, $A_{t-1} = \bar{A}$, $w_t = C$, and $w_{t-1} = C$, applying either Assumption 1 or 2, and assuming that $(A_t - \bar{A})^2$ and $(A_{t-1} - \bar{A})^2$ are small, we have:

$$\begin{aligned}\pi(A_t, A_{t-1}, w_t, w_{t-1}) =& \bar{\pi} + \bar{\pi}_1(A_t - \bar{A}) + \bar{\pi}_2(A_{t-1} - \bar{A}) + \bar{\pi}_3(w_t - C) + \bar{\pi}_4(w_{t-1} - C)\\ &+ \frac{1}{2}\bar{\pi}_{33}(w_t - C)^2 + \bar{\pi}_{13}(A_t - \bar{A})(w_t - C) + \bar{\pi}_{23}(A_{t-1} - \bar{A})(w_t - C)\\ &+ \frac{1}{2}\bar{\pi}_{44}(w_{t-1} - C)^2 + \bar{\pi}_{14}(A_t - \bar{A})(w_{t-1} - C) + \bar{\pi}_{24}(A_{t-1} - \bar{A})(w_{t-1} - C)\\ &+ \bar{\pi}_{34}(w_t - C)(w_{t-1} - C).\end{aligned}$$

As a result,

$$\begin{aligned}Cov[w_t - C, \pi_t] =& \bar{\pi}_1 Cov[A_t, w_t] + \bar{\pi}_2 Cov[A_{t-1}, w_t] + \bar{\pi}_3 Var[w_t] + \bar{\pi}_4 Cov[w_t, w_{t-1}]\\ &+ \frac{1}{2}\bar{\pi}_{33} Cov[w_t - C, (w_t - C)^2] + \frac{1}{2}\bar{\pi}_{44} Cov[w_t - C, (w_{t-1} - C)^2]\\ &- C\bar{\pi}_{13} Cov[A_t, w_t] - \bar{A}\bar{\pi}_{13} Var[w_t] + \bar{\pi}_{13} Cov[w_t, A_t w_t]\\ &- C\bar{\pi}_{23} Cov[A_{t-1}, w_t] - \bar{A}\bar{\pi}_{23} Var[w_t] + \bar{\pi}_{23} Cov[w_t, A_{t-1} w_t]\\ &- C\bar{\pi}_{14} Cov[w_t, A_t] - \bar{A}\bar{\pi}_{14} Cov[w_t, w_{t-1}] + \bar{\pi}_{14} Cov[w_t, A_t w_{t-1}]\\ &- C\bar{\pi}_{24} Cov[w_t, A_{t-1}] - \bar{A}\bar{\pi}_{24} Cov[w_t, w_{t-1}] + \bar{\pi}_{24} Cov[w_t, A_{t-1} w_{t-1}]\\ &- C\bar{\pi}_{34} Var[w_t] - C\bar{\pi}_{34} Cov[w_t, w_{t-1}] + \bar{\pi}_{34} Cov[w_t, w_t w_{t-1}].\end{aligned}$$

If the ϵ and ν are normally distributed, then $Cov[w_t - C, (w_t - C)^2] = 0$, or if Assumption 1 holds, then $Cov[w_t - C, (w_t - C)^2] \approx 0$. Using results from Bohrnstedt and Goldberger (1969), we have:

$$Cov[w_t - C, (w_{t-1} - C)^2] = E[(w_t - C)(w_{t-1} - C)^2],$$

which is zero if either the ϵ and ν are normally distributed or Assumption 1 holds. Again using results from Bohrnstedt and Goldberger (1969), we also have:

$$Cov[w_t, A_t w_t] = E[A_t]\, Var[w_t] + C\, Cov[A_t, w_t] + E[(w_t - C)^2(A_t - E[A_t])].$$

If either the ϵ and ν are normally distributed or Assumption 1 holds, then this becomes :

$$Cov[w_t, A_t w_t] = E[A_t]\, Var[w_t] + C\, Cov[A_t, w_t].$$

Analogous derivations yield:

$$\begin{aligned}Cov[w_t, A_{t-1} w_t] =& E[A_{t-1}]\, Var[w_t] + C\, Cov[w_t, A_{t-1}],\\ Cov[w_t, A_t w_{t-1}] =& E[A_t]\, Cov[w_t, w_{t-1}] + C\, Cov[w_t, A_t],\\ Cov[w_t, A_{t-1} w_{t-1}] =& E[A_{t-1}]\, Cov[w_t, w_{t-1}] + C\, Cov[w_t, A_{t-1}]\end{aligned}$$

if either the ϵ and ν are normally distributed or Assumption 1 holds. Substituting these results in, we find:

$$\begin{aligned}Cov[w_t - C, \pi_t] =&\bar{\pi}_1 Cov[A_t, w_t] + \bar{\pi}_2 Cov[A_{t-1}, w_t] + \bar{\pi}_3 Var[w_t] + \bar{\pi}_4 Cov[w_t, w_{t-1}]\\ &+ \left(E[A_t] - \bar{A}\right)\left(\bar{\pi}_{13} Var[w_t] + \bar{\pi}_{14} Cov[w_t, w_{t-1}]\right)\\ &+ \left(E[A_{t-1}] - \bar{A}\right)\left(\bar{\pi}_{23} Var[w_t] + \bar{\pi}_{24} Cov[w_t, w_{t-1}]\right).\end{aligned}$$

The assumption that actions are on average around $\bar{A}$ implies $E[A_t] = \bar{A}$ and $E[A_{t-1}] = \bar{A}$. Using that and Lemma 2, we obtain:

$$\begin{aligned}\frac{1}{\zeta^2} Cov[w_t - C, \pi_t] =&(\sigma^2 + \tau^2)\bar{\pi}_3 + \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}\left((\sigma^2 + \tau^2)\bar{\pi}_1 + \bar{\pi}_2\rho + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\rho\right)\\ &+ \bar{\pi}_4\rho + \frac{\bar{\pi}_{14}}{\chi}\bar{\pi}_1\rho + \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_1\rho + \bar{\pi}_2\tau^2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\tau^2\right).\end{aligned}$$

Analogous derivations yield:

$$\begin{aligned}\frac{1}{\zeta^2} Cov[f_t - C, \pi_t] =&\frac{1}{\zeta^2}\left(\bar{\pi}_1 Cov[A_t, f_t] + \bar{\pi}_3 Cov[w_t, f_t]\right)\\ =&\bar{\pi}_3\rho + \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}\bar{\pi}_1\rho\\ &+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}\bar{\pi}_1\tau^2,\end{aligned}$$

$$\begin{aligned}\frac{1}{\zeta^2} Cov[w_{t-1} - C, \pi_t] =&\frac{1}{\zeta^2}\left(\bar{\pi}_1 Cov[A_t, w_{t-1}] + \bar{\pi}_2 Cov[A_{t-1}, w_{t-1}] + \bar{\pi}_3 Cov[w_t, w_{t-1}] + \bar{\pi}_4 Var[w_{t-1}]\right)\\ =&\bar{\pi}_3\rho + \bar{\pi}_4(\sigma^2 + \tau^2) + \bar{\pi}_1\frac{\bar{\pi}_{14}}{\chi}(\sigma^2 + \tau^2) + \left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\frac{\bar{\pi}_{14}}{\chi}\rho\\ &+ \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi}\left[\bar{\pi}_1\rho + \left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left(\sigma^2 + \tau^2 + \frac{\bar{\pi}_{12}}{\chi}\rho\right)\right]\\ &+ \left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left(\rho + \frac{\bar{\pi}_{12}}{\chi}\tau^2\right)\frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{\chi},\end{aligned}$$

$$
\begin{aligned}
\frac{1}{\zeta^2}Cov[f_{t-1}-C,\pi_t] =& \frac{1}{\zeta^2}\left(\bar{\pi}_1 Cov[A_t,f_{t-1}] + \bar{\pi}_2 Cov[A_{t-1},f_{t-1}] + \bar{\pi}_3 Cov[w_t,f_{t-1}] + \bar{\pi}_4 Cov[w_{t-1},f_{t-1}]\right)\\
=& \bar{\pi}_3\tau^2 + \bar{\pi}_4\rho + \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_1\tau^2 + \bar{\pi}_2\rho + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\rho\right)\\
&+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_2\tau^2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\tau^2\right)\\
&+ \bar{\pi}_1\frac{\bar{\pi}_{14}}{\chi}\rho,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\zeta^2}Cov[w_{t-2}-C,\pi_t] =& \frac{1}{\zeta^2}\left(\bar{\pi}_1 Cov[A_t,w_{t-2}] + \bar{\pi}_2 Cov[A_{t-1},w_{t-2}] + \bar{\pi}_3 Cov[w_t,w_{t-2}] + \bar{\pi}_4 Cov[w_{t-1},w_{t-2}]\right)\\
=& \bar{\pi}_4\rho + \bar{\pi}_1\frac{\bar{\pi}_{14}}{\chi}\rho + \frac{\bar{\pi}_{14}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left[(\sigma^2+\tau^2) + \rho\frac{\bar{\pi}_{12}}{\chi}\right]\\
&+ \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left[\rho + \frac{\bar{\pi}_{12}}{\chi}(\sigma^2+\tau^2) + \left(\frac{\bar{\pi}_{12}}{\chi}\right)^2\rho\right]\\
&+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\frac{\bar{\pi}_{12}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left[\rho + \frac{\bar{\pi}_{12}}{\chi}\tau^2\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\zeta^2}Cov[f_{t-2}-C,\pi_t] =& \frac{1}{\zeta^2}\left(\bar{\pi}_1 Cov[A_t,f_{t-2}] + \bar{\pi}_2 Cov[A_{t-1},f_{t-2}] + \bar{\pi}_3 Cov[w_t,f_{t-2}] + \bar{\pi}_4 Cov[w_{t-1},f_{t-2}]\right)\\
=& \bar{\pi}_4\tau^2 + \bar{\pi}_1\frac{\bar{\pi}_{14}}{\chi}\tau^2 + \frac{\bar{\pi}_{14}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\rho\\
&+ \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left[\tau^2 + \frac{\bar{\pi}_{12}}{\chi}\rho\right]\\
&+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13} + \beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\frac{\bar{\pi}_{12}}{\chi}\tau^2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\zeta^2}Cov[w_{t-3}-C,\pi_t] =& \frac{1}{\zeta^2}\left(\bar{\pi}_1 Cov[A_t,w_{t-3}] + \bar{\pi}_2 Cov[A_{t-1},w_{t-3}] + \bar{\pi}_3 Cov[w_t,w_{t-3}] + \bar{\pi}_4 Cov[w_{t-1},w_{t-3}]\right)\\
=& \frac{\bar{\pi}_{14}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left\{\rho + \frac{\bar{\pi}_{12}}{\chi}\left[(\sigma^2+\tau^2) + \rho\frac{\bar{\pi}_{12}}{\chi}\right]\right\}\\
&+ \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\frac{\bar{\pi}_{12}}{\chi}\\
&\quad\left[\rho + \frac{\bar{\pi}_{12}}{\chi}(\sigma^2+\tau^2) + \left(\frac{\bar{\pi}_{12}}{\chi}\right)^2\rho\right]\\
&+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13}+\beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\frac{\bar{\pi}_{12}}{\chi}\right)^2\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left[\rho + \frac{\bar{\pi}_{12}}{\chi}\tau^2\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\zeta^2}Cov[f_{t-3}-C,\pi_t] =& \frac{1}{\zeta^2}\left(\bar{\pi}_1 Cov[A_t,f_{t-3}] + \bar{\pi}_2 Cov[A_{t-1},f_{t-3}] + \bar{\pi}_3 Cov[w_t,f_{t-3}] + \bar{\pi}_4 Cov[w_{t-1},f_{t-3}]\right)\\
=& \frac{\bar{\pi}_{14}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left\{\tau^2 + \frac{\bar{\pi}_{12}}{\chi}\rho\right\}\\
&+ \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\frac{\bar{\pi}_{12}}{\chi}\left[\tau^2 + \frac{\bar{\pi}_{12}}{\chi}\rho\right]\\
&+ \frac{\beta\bar{\pi}_{23} + \beta(\bar{\pi}_{13}+\beta\bar{\pi}_{24})\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}-\beta\bar{\pi}_{12}\lambda}}{\chi}\left(\bar{\pi}_2 + \bar{\pi}_1\frac{\bar{\pi}_{12}}{\chi}\right)\left(\frac{\bar{\pi}_{12}}{\chi}\right)^2\tau^2.
\end{aligned}
$$

Now consider the regression coefficients, from $E[\tilde{X}_3^T\tilde{X}_3]^{-1}E[\tilde{X}_3^T\pi_t]$. The coefficient on w_{jt} is:

$$
\begin{aligned}
\hat{\theta}_{w_t} =& \frac{1}{\zeta^2}\left(\frac{\tau^2}{\sigma^2\tau^2-\rho^2}Cov[w_t-C,\pi_t] + \frac{-\rho}{\sigma^2\tau^2-\rho^2}Cov[f_t-C,\pi_t] + \frac{-\tau^2}{\sigma^2\tau^2-\rho^2}Cov[f_{t-1}-C,\pi_t]\right)\\
=& \bar{\pi}_3 + \hat{\Gamma}_1\bar{\pi}_1.
\end{aligned}
$$

The coefficient on f_{jt} is:

$$
\begin{aligned}
\hat{\theta}_{f_t} =& \frac{1}{\zeta^2}\left(\frac{-\rho}{\sigma^2\tau^2-\rho^2}Cov[w_t-C,\pi_t] + \frac{\sigma^2}{\sigma^2\tau^2-\rho^2}Cov[f_t-C,\pi_t] + \frac{\rho}{\sigma^2\tau^2-\rho^2}Cov[f_{t-1}-C,\pi_t]\right)\\
=& \hat{\Gamma}_3\bar{\pi}_1.
\end{aligned}
$$

The coefficient on $w_{j(t-1)}$ is:

$$
\begin{aligned}
\hat{\theta}_{w_{t-1}} =& \frac{1}{\zeta^2}\left(\frac{\tau^2}{\sigma^2\tau^2-\rho^2}Cov[w_{t-1}-C,\pi_t] - \frac{\rho}{\sigma^2\tau^2-\rho^2}Cov[f_{t-1}-C,\pi_t] - \frac{\tau^2}{\sigma^2\tau^2-\rho^2}Cov[f_{t-2}-C,\pi_t]\right)\\
=& \bar{\pi}_4 + \hat{\Gamma}_1\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2 + \hat{\Gamma}_2\bar{\pi}_1.
\end{aligned}
$$

The coefficient on $f_{j(t-1)}$ is:

$$\begin{aligned}\hat{\theta}_{f_{t-1}} =& \frac{1}{\zeta^2}\Bigg(- \frac{\tau^2}{\sigma^2\tau^2-\rho^2}Cov[w_t - C, \pi_t] - \frac{\rho}{\sigma^2\tau^2-\rho^2}Cov[w_{t-1} - C, \pi_t] + \frac{\rho}{\sigma^2\tau^2-\rho^2}Cov[f_t - C, \pi_t] \\ &+ \frac{\sigma^2+\tau^2}{\sigma^2\tau^2-\rho^2}Cov[f_{t-1} - C, \pi_t] + \frac{\rho}{\sigma^2\tau^2-\rho^2}Cov[f_{t-2} - C, \pi_t]\Bigg) \\ =& \hat{\Gamma}_3\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2.\end{aligned}$$

And the coefficient on $w_{j(t-2)}$ is:

$$\begin{aligned}\hat{\theta}_{w_{t-2}} =& \frac{1}{\zeta^2}\Bigg(\frac{\tau^2}{\sigma^2\tau^2-\rho^2}Cov[w_{t-2} - C, \pi_t] - \frac{\rho}{\sigma^2\tau^2-\rho^2}Cov[f_{t-2} - C, \pi_t] - \frac{\tau^2}{\sigma^2\tau^2-\rho^2}Cov[f_{t-3} - C, \pi_t]\Bigg) \\ =& \hat{\Gamma}_1\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\frac{\bar{\pi}_{12}}{\chi}\bar{\pi}_2 + \hat{\Gamma}_2\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2.\end{aligned}$$

We now prove the parts of the proposition.

1. Directly follows from the foregoing.
2. If Assumption 3 holds, then the Euler equation requires $\bar{\pi}_2 = \bar{\pi}_1 = 0$. The result follows by inspection.
3. By Proposition 1, $\bar{\pi}_{12} = 0$ implies $\mathrm{d}\bar{A}/\mathrm{d}C = \hat{\Gamma}_1 + \hat{\Gamma}_2 + \hat{\Gamma}_3$. The result follows from the foregoing and equation (8).
4. Under the given assumptions, we have:

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} = \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_1\bar{\pi}_1 + \hat{\Gamma}_1\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2,$$

so

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} \leq \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_1\bar{\pi}_1 + \hat{\Gamma}_1\bar{\pi}_2 \quad \Leftrightarrow \quad \bar{\pi}_2\bar{\pi}_{12} > 0.$$

Using Proposition 1 and equations (2) and (8),

$$\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} > \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_1\bar{\pi}_1 + \hat{\Gamma}_1\bar{\pi}_2 \quad \Leftrightarrow \quad \bar{\pi}_2\bar{\pi}_{12} > 0,$$

using $\bar{\pi}_{13} > 0$. Therefore

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} < \frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} \quad \Leftrightarrow \quad \bar{\pi}_2\bar{\pi}_{12} > 0.$$

We have established the result.

5. Under the given assumptions, we have:

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{f_t} + \hat{\theta}_{f_{t-1}} = \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_3 \bar{\pi}_1 + \hat{\Gamma}_3 \left(1 - \beta \frac{\bar{\pi}_{12}}{\chi}\right) \bar{\pi}_2,$$

so

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{f_t} + \hat{\theta}_{f_{t-1}} < \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_3 \bar{\pi}_1 + \hat{\Gamma}_3 \bar{\pi}_2 \quad \Leftrightarrow \quad \bar{\pi}_2 \bar{\pi}_{12} > 0,$$

using $\beta \bar{\pi}_{23} > 0$. Using Proposition 1 and equations (2) and (8),

$$\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} > \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_3 \bar{\pi}_1 + \hat{\Gamma}_3 \bar{\pi}_2 \quad \Leftrightarrow \quad \bar{\pi}_2 \bar{\pi}_{12} > 0,$$

using $\beta \bar{\pi}_{23} > 0$. Therefore

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{f_t} + \hat{\theta}_{f_{t-1}} < \frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} \quad \Leftrightarrow \quad \bar{\pi}_2 \bar{\pi}_{12} > 0.$$

We have established the result.

6. It is clear that $\beta = 0$ implies $\hat{\theta}_{f_t} = 0$ and $\hat{\theta}_{f_{t-1}} = 0$. Further, the Euler equation implies $\bar{\pi}_1 = 0$. We then have:

$$\begin{aligned}
\hat{\theta}_{w_t} &= \bar{\pi}_3, \\
\hat{\theta}_{w_{t-1}} &= \bar{\pi}_4 + \hat{\Gamma}_1 \bar{\pi}_2, \\
\hat{\theta}_{w_{t-2}} &= \hat{\Gamma}_2 \bar{\pi}_2 + \hat{\Gamma}_1 \frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}} \bar{\pi}_2.
\end{aligned}$$

Therefore:

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{w_{t-2}} = \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_1 \left(1 + \frac{\bar{\pi}_{12}}{-\bar{\pi}_{11}}\right) \bar{\pi}_2 + \hat{\Gamma}_2 \bar{\pi}_2.$$

We then have:

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{w_{t-2}} \leq \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_1 \bar{\pi}_2 + \hat{\Gamma}_2 \bar{\pi}_2 \quad \Leftrightarrow \quad \bar{\pi}_2 \bar{\pi}_{12} > 0.$$

Using Proposition 1 and equations (2) and (8),

$$\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} > \bar{\pi}_3 + \bar{\pi}_4 + \hat{\Gamma}_1 \bar{\pi}_2 + \hat{\Gamma}_2 \bar{\pi}_2 \quad \Leftrightarrow \quad \bar{\pi}_2 \bar{\pi}_{12} > 0,$$

using either $\bar{\pi}_{13} > 0$ or $\bar{\pi}_{14} > 0$. Therefore

$$\hat{\theta}_{w_t} + \hat{\theta}_{w_{t-1}} + \hat{\theta}_{w_{t-2}} < \frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} \quad \Leftrightarrow \quad \bar{\pi}_2 \bar{\pi}_{12} > 0.$$

We have established the result.

G Proof of Corollary 4

We follow the proof of Proposition 1 but now do not impose the assumption that $E[A_t] = \bar{A}$ and $E[A_{t-1}] = \bar{A}$. We now have:

$$\begin{aligned} Cov[w_t - C, \pi_t] =& \bar{\pi}_1 Cov[A_t, w_t] + \bar{\pi}_2 Cov[A_{t-1}, w_t] \\ &+ \left(\bar{\pi}_3 + \bar{\pi}_{13}(E[A_t] - \bar{A}) + \bar{\pi}_{23}(E[A_{t-1}] - \bar{A})\right) Var[w_t] \\ &+ \left(\bar{\pi}_4 + \bar{\pi}_{14}(E[A_t] - \bar{A}) + \bar{\pi}_{24}(E[A_{t-1}] - \bar{A})\right) Cov[w_t, w_{t-1}], \end{aligned}$$

$$Cov[f_t - C, \pi_t] = \bar{\pi}_1 Cov[A_t, f_t] + \left(\bar{\pi}_3 + \bar{\pi}_{13}(E[A_t] - \bar{A}) + \bar{\pi}_{23}(E[A_{t-1}] - \bar{A})\right) Cov[w_t, f_t],$$

$$\begin{aligned} Cov[w_{t-1} - C, \pi_t] =& \bar{\pi}_1 Cov[A_t, w_{t-1}] + \bar{\pi}_2 Cov[A_{t-1}, w_{t-1}] \\ &+ \left(\bar{\pi}_3 + \bar{\pi}_{13}(E[A_t] - \bar{A}) + \bar{\pi}_{23}(E[A_{t-1}] - \bar{A})\right) Cov[w_t, w_{t-1}] \\ &+ \left(\bar{\pi}_4 + \bar{\pi}_{14}(E[A_t] - \bar{A}) + \bar{\pi}_{24}(E[A_{t-1}] - \bar{A})\right) Var[w_{t-1}], \end{aligned}$$

$$\begin{aligned} Cov[f_{t-1} - C, \pi_t] =& \bar{\pi}_1 Cov[A_t, f_{t-1}] + \bar{\pi}_2 Cov[A_{t-1}, f_{t-1}] \\ &+ \left(\bar{\pi}_3 + \bar{\pi}_{13}(E[A_t] - \bar{A}) + \bar{\pi}_{23}(E[A_{t-1}] - \bar{A})\right) Cov[w_t, f_{t-1}] \\ &+ \left(\bar{\pi}_4 + \bar{\pi}_{14}(E[A_t] - \bar{A}) + \bar{\pi}_{24}(E[A_{t-1}] - \bar{A})\right) Cov[w_{t-1}, f_{t-1}], \end{aligned}$$

and so on. Following that analysis, we obtain the regression coefficients:

$$\hat{\theta}_{w_t} = \bar{\pi}_3 + \bar{\pi}_{13}(E[A_t] - \bar{A}) + \bar{\pi}_{23}(E[A_{t-1}] - \bar{A}) + \omega \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \bar{\pi}_1,$$

$$\begin{aligned} \hat{\theta}_{w_{t-1}} =& \bar{\pi}_4 + \bar{\pi}_{14}(E[A_t] - \bar{A}) + \bar{\pi}_{24}(E[A_{t-1}] - \bar{A}) \\ &+ \omega \frac{\bar{\pi}_{13} + \beta\bar{\pi}_{24} + \beta\bar{\pi}_{14}\frac{\bar{\pi}_{12}}{-\bar{\pi}_{11} - \beta\bar{\pi}_{22} - \beta\bar{\pi}_{12}\lambda}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \left(1 - \beta\frac{\bar{\pi}_{12}}{\chi}\right) \bar{\pi}_2 + \omega \frac{\bar{\pi}_{14}}{-\bar{\pi}_{11} - (1+\beta)\bar{\pi}_{12} - \beta\bar{\pi}_{22}} \bar{\pi}_1. \end{aligned}$$

The other coefficients are unchanged. Under the assumption that at least one of $\bar{\pi}_{13}, \bar{\pi}_{14}, \bar{\pi}_{23}, \bar{\pi}_{24}$ is strictly positive, we have increased $\hat{\theta}_{w_t}$ and $\hat{\theta}_{w_{t-1}}$ if average actions are above $\bar{A}$ and have decreased $\hat{\theta}_{w_t}$ and $\hat{\theta}_{w_{t-1}}$ if average actions are below $\bar{A}$. The results follow.

H Proof of Proposition 5

We derive estimators in the case of $K = 0$, $K = 1$, and $K = 2$. The superscript on the estimators will indicate K.

Begin with $K = 0$. Demeaned to account for fixed effects, each row of the matrix $\tilde{X}$ of covariates is now simply $w_{jt} - C$, with the rows corresponding to the J observations. Therefore

$$E[\tilde{X}^T \tilde{X}] = J\zeta^2(\sigma^2 + \tau^2)$$

and

$$E[\tilde{X}^T \tilde{X}]^{-1} = \frac{1}{J\zeta^2(\sigma^2 + \tau^2)}.$$

We also have:

$$E[\tilde{X}^T \pi_{jt}] = J\, Cov[w_{jt} - C, \pi_{jt}].$$

We analyzed this covariance in the proof of Proposition 3 under the same assumptions. Using those results, we find that

$$\begin{aligned}\hat{\Phi}^0_{w_t} =& \bar{\pi}_3 + \bar{\pi}_4 \frac{\rho}{\sigma^2 + \tau^2} + \hat{\Gamma}_1 \left[\bar{\pi}_1 + \left(1 - \beta\frac{\bar{\pi}_{12}}{\chi}\right) \frac{\rho}{\sigma^2 + \tau^2}\bar{\pi}_2\right] \\ &+ \hat{\Gamma}_2 \bar{\pi}_1 \frac{\rho}{\sigma^2 + \tau^2} + \hat{\Gamma}_3 \left[\bar{\pi}_1 \frac{\rho}{\sigma^2 + \tau^2} + \left(1 - \beta\frac{\bar{\pi}_{12}}{\chi}\right) \frac{\tau^2}{\sigma^2 + \tau^2}\bar{\pi}_2\right].\end{aligned}$$

Now consider $K = 1$. Demeaned to account for fixed effects, each row of the matrix $\tilde{X}$ of covariates is

$$\begin{bmatrix} w_{jt} - C & w_{j(t-1)} - C \end{bmatrix},$$

with the rows corresponding to the J observations. Thus,

$$E[\tilde{X}^T \tilde{X}] = J\zeta^2 \begin{bmatrix} \sigma^2 + \tau^2 & \rho \\ \rho & \sigma^2 + \tau^2 \end{bmatrix}$$

and

$$E[\tilde{X}^T \tilde{X}]^{-1} = \frac{1}{J\zeta^2[(\sigma^2 + \tau^2)^2 - \rho^2]} \begin{bmatrix} \sigma^2 + \tau^2 & -\rho \\ -\rho & \sigma^2 + \tau^2 \end{bmatrix}.$$

We also have:

$$E[\tilde{X}^T \pi_{jt}] = J \begin{bmatrix} Cov[w_{jt} - C, \pi_{jt}] \\ Cov[w_{j(t-1)} - C, \pi_{jt}] \end{bmatrix}.$$

We analyzed these covariances in the proof of Proposition 3 under the same assumptions. Using those results, we find that

$$\hat{\Phi}^1_{w_t} = \bar{\pi}_3 + \hat{\Gamma}_1\left\{\bar{\pi}_1 - \frac{\rho^2}{(\sigma^2+\tau^2)^2-\rho^2}\frac{\bar{\pi}_{12}}{\chi}\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2\right\} - \hat{\Gamma}_2\frac{\rho^2}{(\sigma^2+\tau^2)^2-\rho^2}\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2$$
$$+ \hat{\Gamma}_3\left\{\frac{\rho(\sigma^2+\tau^2)}{(\sigma^2+\tau^2)^2-\rho^2}\bar{\pi}_1 + \left(\frac{\tau^2(\sigma^2+\tau^2)-\rho^2}{(\sigma^2+\tau^2)^2-\rho^2} - \frac{\rho\tau^2}{(\sigma^2+\tau^2)^2-\rho^2}\frac{\bar{\pi}_{12}}{\chi}\right)\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2\right\},$$

$$\hat{\Phi}^1_{w_{t-1}} = \bar{\pi}_4 + \hat{\Gamma}_1\left(1+\frac{\rho(\sigma^2+\tau^2)}{(\sigma^2+\tau^2)^2-\rho^2}\frac{\bar{\pi}_{12}}{\chi}\right)\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2 + \hat{\Gamma}_2\left[\bar{\pi}_1 + \frac{\rho(\sigma^2+\tau^2)}{(\sigma^2+\tau^2)^2-\rho^2}\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2\right]$$
$$+ \hat{\Gamma}_3\left\{\frac{-\rho^2}{(\sigma^2+\tau^2)^2-\rho^2}\bar{\pi}_1 + \left(\frac{\rho\sigma^2}{(\sigma^2+\tau^2)^2-\rho^2} + \frac{\tau^2(\sigma^2+\tau^2)}{(\sigma^2+\tau^2)^2-\rho^2}\frac{\bar{\pi}_{12}}{\chi}\right)\left(1-\beta\frac{\bar{\pi}_{12}}{\chi}\right)\bar{\pi}_2\right\}.$$

Finally, consider $K = 2$. Demeaned to account for fixed effects, each row of the matrix $\tilde{X}$ of covariates is

$$\begin{bmatrix} w_{jt} - C & w_{j(t-1)} - C & w_{j(t-2)} - C \end{bmatrix},$$

with the rows corresponding to the J observations. Thus,

$$E[\tilde{X}^T\tilde{X}] = J\zeta^2\begin{bmatrix} \sigma^2+\tau^2 & \rho & 0 \\ \rho & \sigma^2+\tau^2 & \rho \\ 0 & \rho & \sigma^2+\tau^2 \end{bmatrix}$$

and

$$E[\tilde{X}^T\tilde{X}]^{-1} = \frac{1}{J\zeta^2[(\sigma^2+\tau^2)^2-2\rho^2]}\begin{bmatrix} \sigma^2+\tau^2-\frac{\rho^2}{\sigma^2+\tau^2} & -\rho & \frac{\rho^2}{\sigma^2+\tau^2} \\ -\rho & \sigma^2+\tau^2 & -\rho \\ \frac{\rho^2}{\sigma^2+\tau^2} & -\rho & \sigma^2+\tau^2-\frac{\rho^2}{\sigma^2+\tau^2} \end{bmatrix}.$$

We also have:

$$E[\tilde{X}^T\pi_{jt}] = J\begin{bmatrix} Cov[w_{jt}-C, \pi_{jt}] \\ Cov[w_{j(t-1)}-C, \pi_{jt}] \\ Cov[w_{j(t-2)}-C, \pi_{jt}] \end{bmatrix}.$$

We analyzed these covariances in the proof of Proposition 3 under the same assumptions.

Using those results, we find that

$$\begin{aligned}
\hat{\Phi}^2_{w_t} =& \bar{\pi}_3 + \hat{\Gamma}_1 \Bigg\{ \bar{\pi}_1 + \frac{\rho}{\sigma^2+\tau^2} \frac{\rho^2}{(\sigma^2+\tau^2)^2 - 2\rho^2} \left(\frac{\bar{\pi}_{12}}{\chi} \right)^2 \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \Bigg\} \\
&+ \hat{\Gamma}_3 \frac{1}{(\sigma^2+\tau^2)^2 - 2\rho^2} \Bigg\{ \left(\sigma^2 + \tau^2 - \frac{\rho^2}{\sigma^2+\tau^2} \right) \left(\bar{\pi}_1 \rho + \bar{\pi}_2 \tau^2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \tau^2 \right) \\
&\qquad + \left(\frac{\rho^2}{\sigma^2+\tau^2} \frac{\bar{\pi}_{12}}{\chi} - \rho \right) \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \left(\rho + \frac{\bar{\pi}_{12}}{\chi} \tau^2 \right) \Bigg\} \\
&+ \hat{\Gamma}_2 \frac{\rho^2}{(\sigma^2+\tau^2)^2 - 2\rho^2} \frac{\rho}{\sigma^2+\tau^2} \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \frac{\bar{\pi}_{12}}{\chi},
\end{aligned}$$

$$\begin{aligned}
\hat{\Phi}^2_{w_{t-1}} =& \bar{\pi}_4 + \hat{\Gamma}_1 \Bigg\{ 1 - \frac{\rho^2}{(\sigma^2+\tau^2)^2 - 2\rho^2} \left(\frac{\bar{\pi}_{12}}{\chi} \right)^2 \Bigg\} \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \\
&+ \hat{\Gamma}_3 \frac{1}{(\sigma^2+\tau^2)^2 - 2\rho^2} \Bigg\{ -\rho^2 \bar{\pi}_1 + \sigma^2 \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \rho + [\tau^2(\sigma^2+\tau^2) - \rho^2] \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \frac{\bar{\pi}_{12}}{\chi} \\
&\qquad - \rho\tau^2 \left(\frac{\bar{\pi}_{12}}{\chi} \right)^2 \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \Bigg\} \\
&+ \hat{\Gamma}_2 \Bigg\{ \bar{\pi}_1 - \frac{\rho^2}{(\sigma^2+\tau^2)^2 - 2\rho^2} \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \frac{\bar{\pi}_{12}}{\chi} \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\hat{\Phi}^2_{w_{t-2}} =& \hat{\Gamma}_1 \Bigg\{ \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \frac{\bar{\pi}_{12}}{\chi} + \frac{\rho\,(\sigma^2+\tau^2)}{(\sigma^2+\tau^2)^2 - 2\rho^2} \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \left(\frac{\bar{\pi}_{12}}{\chi} \right)^2 \Bigg\} \\
&+ \hat{\Gamma}_3 \frac{1}{(\sigma^2+\tau^2)^2 - 2\rho^2} \\
&\quad \Bigg\{ \frac{\rho^2}{\sigma^2+\tau^2} \bar{\pi}_1 \rho - \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \frac{\sigma^2\rho^2}{\sigma^2+\tau^2} + \rho \left(\sigma^2 - \frac{\rho^2}{\sigma^2+\tau^2} \right) \frac{\bar{\pi}_{12}}{\chi} \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \\
&\qquad + \tau^2 \left(\sigma^2 + \tau^2 - \frac{\rho^2}{\sigma^2+\tau^2} \right) \left(\frac{\bar{\pi}_{12}}{\chi} \right)^2 \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right) \Bigg\} \\
&+ \hat{\Gamma}_2 \Bigg\{ 1 + \frac{(\sigma^2+\tau^2)^2 - \rho^2}{(\sigma^2+\tau^2)^2 - 2\rho^2} \frac{\rho}{\sigma^2+\tau^2} \frac{\bar{\pi}_{12}}{\chi} \Bigg\} \left(\bar{\pi}_2 + \bar{\pi}_1 \frac{\bar{\pi}_{12}}{\chi} \right).
\end{aligned}$$

We now prove the parts of the proposition:

1. If Assumption 3 holds, then the Euler equation requires $\bar{\pi}_2 = \bar{\pi}_1 = 0$. The result follows by inspection and by recognizing that $\rho \leq \sigma\tau$ (because correlation coefficients are bounded above by 1).

2. If $\bar{\pi}_{12} = 0$, then

$$\hat{\Phi}^0_{w_t} = \bar{\pi}_3 + \bar{\pi}_4 \frac{\rho}{\sigma^2+\tau^2} + \hat{\Gamma}_1 \left[\bar{\pi}_1 + \frac{\rho}{\sigma^2+\tau^2}\bar{\pi}_2\right] + \hat{\Gamma}_2 \bar{\pi}_1 \frac{\rho}{\sigma^2+\tau^2} + \hat{\Gamma}_3 \left[\bar{\pi}_1 \frac{\rho}{\sigma^2+\tau^2} + \frac{\tau^2}{\sigma^2+\tau^2}\bar{\pi}_2\right],$$

$$\hat{\Phi}^1_{w_t} = \bar{\pi}_3 + \hat{\Gamma}_1 \bar{\pi}_1 - \hat{\Gamma}_2 \frac{\rho^2}{(\sigma^2+\tau^2)^2-\rho^2}\bar{\pi}_2 + \hat{\Gamma}_3 \left\{ \frac{\rho(\sigma^2+\tau^2)}{(\sigma^2+\tau^2)^2-\rho^2}\bar{\pi}_1 + \frac{\tau^2(\sigma^2+\tau^2)-\rho^2}{(\sigma^2+\tau^2)^2-\rho^2}\bar{\pi}_2 \right\},$$

$$\hat{\Phi}^1_{w_{t-1}} = \bar{\pi}_4 + \hat{\Gamma}_1 \bar{\pi}_2 + \hat{\Gamma}_2 \left[\bar{\pi}_1 + \frac{\rho(\sigma^2+\tau^2)}{(\sigma^2+\tau^2)^2-\rho^2}\bar{\pi}_2\right] + \hat{\Gamma}_3 \left\{ \frac{-\rho^2}{(\sigma^2+\tau^2)^2-\rho^2}\bar{\pi}_1 + \frac{\rho\sigma^2}{(\sigma^2+\tau^2)^2-\rho^2}\bar{\pi}_2 \right\},$$

$$\hat{\Phi}^2_{w_t} = \bar{\pi}_3 + \hat{\Gamma}_1 \bar{\pi}_1 + \hat{\Gamma}_3 \frac{1}{(\sigma^2+\tau^2)^2-2\rho^2} \left\{ \frac{(\sigma^2+\tau^2)^2-\rho^2}{\sigma^2+\tau^2} \left(\bar{\pi}_1\rho + \bar{\pi}_2\tau^2\right) - \rho^2\bar{\pi}_2 \right\},$$

$$\hat{\Phi}^2_{w_{t-1}} = \bar{\pi}_4 + \hat{\Gamma}_1 \bar{\pi}_2 + \hat{\Gamma}_2 \bar{\pi}_1 + \hat{\Gamma}_3 \frac{\rho}{(\sigma^2+\tau^2)^2-2\rho^2} \left\{ -\rho\bar{\pi}_1 + \sigma^2\bar{\pi}_2 \right\},$$

$$\hat{\Phi}^2_{w_{t-2}} = \hat{\Gamma}_2 \bar{\pi}_2 + \hat{\Gamma}_3 \frac{\rho}{\sigma^2+\tau^2} \frac{\rho}{(\sigma^2+\tau^2)^2-2\rho^2} \left\{ \bar{\pi}_1\rho - \bar{\pi}_2\sigma^2 \right\}.$$

By Proposition 1, $\bar{\pi}_{12} = 0$ implies $\mathrm{d}\bar{A}/\,\mathrm{d}C = \hat{\Gamma}_1 + \hat{\Gamma}_2 + \hat{\Gamma}_3$. If, in addition, $\beta\bar{\pi}_{23} = 0$, then $\hat{\Gamma}_3 = 0$. Result (a) follows from equation (8). $\bar{\pi}_{14} = 0$ implies $\hat{\Gamma}_2 = 0$. Result (b) follows from equation (8).

If $\bar{\pi}_{12} = 0$, $\beta\bar{\pi}_{23} = 0$, $\bar{\pi}_{14} = 0$, and $\bar{\pi}_4 = 0$, then

$$\hat{\Phi}^0_{w_t} = \bar{\pi}_3 + \hat{\Gamma}_1 \left[\bar{\pi}_1 + \frac{\rho}{\sigma^2+\tau^2}\bar{\pi}_2\right]$$

and

$$\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} = \bar{\pi}_3 + \hat{\Gamma}_1 \left[\bar{\pi}_1 + \bar{\pi}_2\right].$$

We then have:

$$\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} > \hat{\Phi}^0_{w_t} \quad \Leftrightarrow \quad \hat{\Gamma}_1 \bar{\pi}_2 > \hat{\Gamma}_1 \frac{\rho}{\sigma^2+\tau^2}\bar{\pi}_2.$$

If $\bar{\pi}_{13} > 0$, then $\hat{\Gamma}_1 > 0$ and

$$\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} > \hat{\Phi}^0_{w_t} \quad \Leftrightarrow \quad \bar{\pi}_2 > \frac{\rho}{\sigma^2+\tau^2}\bar{\pi}_2.$$

Because correlation coefficients are bounded above by 1, $\rho \leq \sigma\tau$. Therefore

$$\bar{\pi}_2 > \frac{\rho}{\sigma^2+\tau^2}\bar{\pi}_2 \quad \Leftrightarrow \quad \bar{\pi}_2 > 0.$$

We have established result (c).

I Proof of Proposition 6

Observe that

$$\hat{\Lambda} = \frac{Cov[\tilde{\pi}_{jT}, \tilde{w}_{jT} - \tilde{w}_{j0}]}{Var[\tilde{w}_{jT} - \tilde{w}_{j0}]} + \frac{Cov[\tilde{\pi}_{j0}, \tilde{w}_{j0} - \tilde{w}_{jT}]}{Var[\tilde{w}_{jT} - \tilde{w}_{j0}]}.$$

Under the assumptions that actions are independent of past actions ($\bar{\pi}_{12} = 0$) and that weather is not serially correlated ($\rho = 0$), time t actions are independent of weather before time $t-1$ and after $t+1$. We then have:

$$\begin{aligned}
Cov[\tilde{\pi}_{jT}, \tilde{w}_{jT} - \tilde{w}_{j0}] =& \frac{1}{\Delta^2}\Bigg\{\sum_{t=T+1}^{T+\Delta-2}\Big(Cov[\pi_{jt}, w_{jt}] + Cov[\pi_{jt}, w_{j(t-1)}] + Cov[\pi_{jt}, f_{jt}]\Big)\\
&+ Cov[\pi_{jT}, w_{jT}] + Cov[\pi_{jT}, f_{jT}]\\
&+ Cov[\pi_{j(T+\Delta-1)}, w_{j(T+\Delta-1)}] + Cov[\pi_{j(T+\Delta-1)}, w_{j(T+\Delta-2)}]\Bigg\},\\
Cov[\tilde{\pi}_{j0}, \tilde{w}_{j0} - \tilde{w}_{jT}] =& \frac{1}{\Delta^2}\Bigg\{\sum_{t=1}^{\Delta-2}\Big(Cov[\pi_{jt}, w_{jt}] + Cov[\pi_{jt}, w_{j(t-1)}] + Cov[\pi_{jt}, f_{jt}]\Big)\\
&+ Cov[\pi_{j0}, w_{j0}] + Cov[\pi_{j0}, f_{j0}]\\
&+ Cov[\pi_{j(\Delta-1)}, w_{j(\Delta-1)}] + Cov[\pi_{j(\Delta-1)}, w_{j(\Delta-2)}]\Bigg\}.
\end{aligned}$$

Using previous results with $\bar{\pi}_{12} = 0$ and $\rho = 0$, those covariances are

$$\frac{1}{\zeta^2}Cov[w_{jt}, \pi_{jt}] = (\sigma^2+\tau^2)\bar{\pi}_3 + \frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}(\sigma^2+\tau^2)\bar{\pi}_1 + \frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_2\tau^2,$$

$$\frac{1}{\zeta^2}Cov[f_{jt}, \pi_{jt}] = \frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_1\tau^2,$$

$$\frac{1}{\zeta^2}Cov[w_{j(t-1)},\pi_{jt}] = \bar{\pi}_4(\sigma^2+\tau^2) + \bar{\pi}_1\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}(\sigma^2+\tau^2) + \frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_2\left(\sigma^2+\tau^2\right).$$

Summing these, we have:

$$\begin{aligned}&Cov[w_{jt},\pi_{jt}] + Cov[f_{jt},\pi_{jt}] + Cov[w_{j(t-1)},\pi_{jt}]\\ =&(\sigma^2+\tau^2)\left\{\bar{\pi}_3+\bar{\pi}_4+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}[\bar{\pi}_1+\bar{\pi}_2]+\bar{\pi}_1\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}[\bar{\pi}_1+\bar{\pi}_2]\right\}.\end{aligned}$$

In addition,

$$Cov[w_{jt},\pi_{jt}] + Cov[f_{jt},\pi_{jt}] = (\sigma^2+\tau^2)\left\{\bar{\pi}_3+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_1+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}[\bar{\pi}_1+\bar{\pi}_2]\right\}$$

and

$$\begin{aligned}Cov[w_{jt},\pi_{jt}] + Cov[w_{j(t-1)},\pi_{jt}] =&(\sigma^2+\tau^2)\left\{\bar{\pi}_3+\bar{\pi}_4+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}[\bar{\pi}_1+\bar{\pi}_2]+\bar{\pi}_1\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\right.\\ &\left.+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}\bar{\pi}_2\right\}.\end{aligned}$$

The estimator is then:

$$\begin{aligned}\hat{\Lambda} =&\frac{\Delta-2}{\Delta}\left\{\bar{\pi}_3+\bar{\pi}_4+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}[\bar{\pi}_1+\bar{\pi}_2]+\bar{\pi}_1\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}[\bar{\pi}_1+\bar{\pi}_2]\right\}\\ &+\frac{1}{\Delta}\left\{\bar{\pi}_3+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_1+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}[\bar{\pi}_1+\bar{\pi}_2]\right\}\\ &+\frac{1}{\Delta}\left\{\bar{\pi}_3+\bar{\pi}_4+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}[\bar{\pi}_1+\bar{\pi}_2]+\bar{\pi}_1\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}\bar{\pi}_2\right\}\\ =&\bar{\pi}_3+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_1+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}\bar{\pi}_2\\ &+\frac{\Delta-1}{\Delta}\left\{\bar{\pi}_4+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_2+\bar{\pi}_1\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}\bar{\pi}_1\right\}\\ =&\hat{\Phi}^0_{w_t}+\frac{\Delta-1}{\Delta}\left\{\bar{\pi}_4+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_2+\bar{\pi}_1\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}\bar{\pi}_1\right\}.\end{aligned}$$

As Δ becomes large, this goes to

$$\bar{\pi}_3+\bar{\pi}_4+\frac{\bar{\pi}_{13}+\beta\bar{\pi}_{24}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}[\bar{\pi}_1+\bar{\pi}_2]+\frac{\beta\bar{\pi}_{23}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\frac{\tau^2}{\sigma^2+\tau^2}[\bar{\pi}_1+\bar{\pi}_2]+\frac{\bar{\pi}_{14}}{-\bar{\pi}_{11}-\beta\bar{\pi}_{22}}\bar{\pi}_1.$$

The results follow by inspection, proceeding as in previous proofs to show that Assumption 3 implies $\bar{\pi}_1 = \bar{\pi}_2 = 0$.