Efficient and Incentive Compatible Mediation: An Ordinal Market Design Approach*

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Abstract

Mediation is an alternative dispute resolution method, which has gained increasing popularity over the last few decades and become a multi-billion dollar industry. When two or more parties are in a disagreement, they can take the case to a court and let the judge make a binding final decision. Alternatively, the disputing parties can get assistance from an experienced, neutral third party, i.e., a mediator, who facilitates the negotiation and help them voluntarily reach an agreement short of litigation. The emphasis in mediation is not upon who is right or wrong, but rather on exploring mutually satisfactory solutions. Employment disputes, patent/copyright violations, construction disputes, and family disputes are some of the most common mediated disputes. The rising popularity of mediation is often attributed to the increasing workload of courts, its cost effectiveness and speed relative to litigation, and disputants' desire to have control over the final decision. Many traditional "cardinal" settings of bargaining and mechanism design, starting with the seminal work of Myerson and Satterhwaite (1983), have shown the incompatibility between efficiency and incentives, even in Bayesian sense. This paper uses an "ordinal" market/mechanism design approach, where the mediator seeks a resolution over (at least) two issues in which negotiators have diametrically opposed rankings over the alternatives. Each negotiator has private information about her own ranking of the outside option, e.g., the point beyond which the negotiator would rather take the case to the court. We construct a simple theoretical framework that is rich and practical enough allowing for optimal mechanisms that the mediators can use for efficient resolution of disputes. We propose and characterize the class of strategy-proof, efficient, and individually rational mediation mechanisms. A central member of this class, the "constrained shortlisting" mechanism stands out as the unique strategy-proof, efficient, and individually rational mechanism that minimizes rank variance. We also provide analogous mechanisms when the issues consist of a continuum of alternatives.

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"*Mediation has rapidly become, with precious little fanfare, the ocean we swim in and the air we breathe. It would now be hard to imagine a world where it wasn't.*" Jim Melamed (founder and CEO of Mediate.com, recipient of American Bar Association Institutional Problem Solver Award.)

Introduction

The best-seller book, "Getting to Yes," by Roger Fisher and William Ury is arguably one of, if not the most famous, works on the topic of negotiation. They identify conflict as a growth industry, and the last few decades proved them right. Courts in all US states offer some form of ADR (alternative dispute resolution) for the cases filed in state courts. 17 states require mandatory mediation: 11% of civil cases in Northern California courts in 2011, 35.6% of civil and 21.6% of divorce cases in New York state courts in 2016 have been mediated.[1] Total value of mediated cases in UK is estimated to be £10.5bn. in 2011, excluding mega-cases, family and community disputes.[2] In addition to face-to-face mediation practices, online dispute resolution, aiming to resolve disputes that arise online, has also gained increasing popularity over the last decade. These are small disputes in size but large in number. Dispute resolution centers of E-bay, PayPal, Uber and Amazon tackles more than a billion disputes a year. Many online dispute resolution web sites use automated mechanisms to help parties resolve their disputes. Empirical studies and mediation program evaluations suggest 60-90% success rate, 90-95% satisfaction by the disputants and higher rate of compliance relative to court-imposed orders.

Unlike litigation and arbitration, mediation does not search for truth, rather searches for satisfaction. In mediation a neutral third party facilitates communication and negotiation, promotes exploration of mutually acceptable alternatives. Namely, the emphasis is not on who is right or wrong, but rather upon establishing a workable solution that meets the participants' needs. Disputants prefer mediation over it's alternatives because it is cost effective. According to Hadfield (2000) it costs a minimum of $100,000 to litigate a straightforward business claim, whereas a mediation session varies from few hours to a day and even the most reputable mediators charge around $10,000 - $15,000 for a day. In addition, disputants do not have to pay any fees for expert, witness, document preparation, investigation or paralegal, which would easily pile up the costs. Airline companies and hospitals, for example, prefer mediation because mediation sessions are private and confidential. It is impossible to discuss a legally "irrelevant" issue in litigation/arbitration and some disputes are not just about money or being right. For example, an employee

[1] Sources: dispute resolution centers of State of New York and California.

[2] The Seventh Mediation Audit, Centre for effective dispute resolution.

would be suing her company for sexual assault and reinstatement or a good reference letter would be more important for her than compensation. However, in mediation, parties can discuss and negotiate issues that are not directly linked to the case. This infinite flexibility of bringing any issue on the table can be used to transform a competitive, zero-sum negotiation problem into a "multi-issue" negotiation problem that enlarges the set of acceptable outcomes.

Market design has been fruitful in many applications, most notably in auctions and matching theory. The goal of this paper is to offer a first market design setting to analyze dispute resolution via mediation, which is simple enough to be practically relevant while maintaining the informational richness and complexities faced in practical disputes. To this end, our modeling significantly departs from the traditional mechanism design approach to bargaining that builds on the seminal work of Myerson and Satterthwaite (1983) in the context of bilateral trade over a good for which traders have private valuations each drawn from pre-specified distributions and commonly known utility functions. This type of "cardinal approach" has however been the subject of the famous Wilson critique for it lacks "detail-freeness" and does not provide robust incentives to participants. In a similar vein, Ausubel, Crampton, and Denecker voice a similar concern:

> "*... Despite these virtues, mechanism design has two weaknesses. First, the mechanisms depend in complex ways on the traders' beliefs and utility functions, which are assumed to be common knowledge. Second, it allows too much commitment. In practice, bargainers use simple trading rules—such as a sequence of offers and counteroffers—that do not depend on beliefs or utility functions.*" Handbook of Game Theory

The ordinal approach, whereby the designer elicits only ordinal preference information, has already lead to quite notable success in applications of matching and assignment such as medical residency, school choice, kidney exchange, and course assignment, where a plethora of strategy-proof and efficient mechanisms have been obtained, extensively studied, and even adopted in practice.

Our model assumes that two negotiators are in a dispute and aim to reach a resolution through a mediator. There is a main issue, issue X, consisting of a finite number of alternatives, which is relevant for both parties' welfare.[3] The negotiators have diametrically opposed preferences over alternatives in the sense that if one negotiator prefers one alternative over another, then the other negotiator has exactly opposite ranking of the two alternatives. However, not all alternatives are acceptable for any given negotiator. When offered one such alternative for her, a negotiator rejects the mediator's proposal and

[3] In the paper we later relax the finiteness and discreteness assumptions on X.

pursues alternative ways of resolution, e.g., litigation. We capture such circumstances by assuming an outside option whose ranking is each negotiator's private information. The mediator's objective is to truthfully elicit negotiators' private information about the position of their outside options and propose an efficient and mutually acceptable, i.e., individually rational, outcome.

We first show that if there is a single issue, i.e., no other issues than issue X, then there is no strategy-proof, efficient, and strategy-proof mechanism. Furthermore, we show that this impossibility extends to multiple issues if each issue has an outside option similarly defined, i.e., in each issue each negotiator has an outside option whose ranking is her private information. This motivates the need for a setting that asymmetrically treats different issues: Consider a second issue, issue Y, where the outside option is the least preferred alternative for both negotiators. In other words, litigation for the second issue is always inefficient. This asymmetric treatment of the outside options can be motivated by various employment, family, construction or patent/copyright infringement disputes. Litigation is naturally the default option if the issue is compensation or division of property and it is a very long and costly process, and so, inefficient relative to other potential divisions (alternatives). Although money is an important issue in disputes, it is rarely the only issue (Malhotra and Bazerman, 2008). Disputes over change orders and extra work or disputes over the contract scope of work would be alternative issues in construction disputes. Child custody and visitation would be alternative issues in family disputes. In this two-issue mediation problem, the mediator recommends a bundle (x, y) of outcomes from $X \times Y$. A mediation rule/mechanism is a systematic way of choosing an outcome for any given preferences of the negotiators.

Since the mediator asks negotiators to report their outside options over alternatives in issue X (recall that there is no uncertainty regarding negotiators' preferences over alternatives in issue Y), one needs to invoke extension mappings to obtain the possible set of negotiators' underlying preferences over bundles. Alternatively, it is conceivable that the mediator elicits preferences over bundles of alternatives. This approach, which we do not pursue, however, has two drawbacks: First, the number of bundles to rank increases quadratically with the number of alternatives in each issue, which in turn makes asking for full-fledged rankings over bundles highly impractical. Second, a similar impossibility to the single-issue mediation would arise in this case.

In this paper we ask if there is an impartial and dominant strategy incentive compatible, i.e., strategy-proof, way of soliciting true preferences so that mediation outcomes are efficient and individually rational. A sufficient and almost necessary condition for obtaining a positive answer to this question is the so-called "logrolling (quid pro qu)" condition on negotiators' preferences which implies a form of substitution between issues X and Y.

More specifically, logrolling requires preferences to be rich enough such that a negotiator is able to compromise issue X for a more preferred alternative in issue Y, e.g., for a given (x, y) bundle, there exists a (weakly) more preferred bundle which involves getting a worse alternative in X combined with a better alternative in Y. In the continuous version of our model, we show that many commonly used utility functions satisfy this assumption.

Our main result is a complete characterization of the class of strategy-proof, efficient, and individually rational mediation rules. These rules operate through an exogenously specified precedence order over a set of special bundles, which we call as the logrolling bundles and always make selections among these bundles. The logrolling bundles form a simple lattice structure with respect to the negotiators' preferences: given any set of mutually acceptable alternatives, for each negotiator there is always an optimal-logrolling bundle that she prefers over all other acceptable bundles; this bundle is the pessimal-logrolling bundle for the opposite negotiator. The characterized class of rules nest interesting extremal members. When the precedence order over coincides with the preference ranking of a given negotiator over the logrolling bundles, we obtain the corresponding negotiator-optimal rule.

In keeping with our main objective of finding impartial mediation rules, we search for members of this class of rules that satisfy sensible fairness criteria. To this end, we define the "rank variance" of an outcome as the sum of the square of each negotiator's ranking of each alternative in each issue. It turns out there is a unique member of the family of strategy-proof, efficient, and individually rational mediation rules that minimizes rank variance. This is the so-called "constrained shortlisting" rule, which recommends the median logrolling bundle when it is mutually acceptable, or the closest mutually acceptable logrolling bundle to it when it is not mutually acceptable. This rule is intuitive and simple enough to be used as a standardized protocol for finding the middle ground between disputing parties in practice.

Related Literature

Our paper and modeling approach connects and spans four different types of literature:

1) Bargaining and Mechanism Design: Mediation is a part of the bargaining literature, which is primarily based on the cardinal approach discussed above. The more broadly-defined mechanism design approach to bargaining in the presence of private outside options, started with the classic paper by Myerson and Satterthwaite (1983) [MS henceforth], has generally emphasized the difficulty/impossibility of reaching efficient outcomes even in Bayesian settings let alone dominant strategies. Specifically, for the mediation context,

there are very few papers: Bester and Warneryd (2006) show that the news are even worse than MS in this setting. The MS result depends on there being a positive probability of trade being inefficient ex post. Bester and Warneryd (2006), in a model featuring continuum of types, show that asymmetric information about relative strengths as an outside option in a conflict may cause agreement to be impossible even if the agreement is always efficient. In their model, conflict shrinks the pie and thus agreement on a peaceful settlement is always ex post efficient. Following Bester and Warneryd (2006), Horner et al (2015) compare the optimal mechanisms, with two types of negotiators, under arbitration, mediation and unmediated communication. For both models, there is no ex post efficient and Bayesian incentive compatible mechanism. Namely, optimal mechanism is necessarily inefficient.

In our model, we adopt an ordinal mechanism design approach in the sense that negotiators rank finitely many available options in opposite ways, which is common knowledge, but the outside option of each negotiator is her private information as in the Compte and Jehiel (2007) model, which adopts a cardinal utility approach much like the rest of this literature. In our benchmark model of single-issue mediation, a conflict situation which is defined as the mediators recommending negotiators to exercise their outside options, is also ex post inefficient so long as the mediators have a mutually acceptable outcome (clearly, when there is no mutually acceptable outcome, mediation is hopeless). Also, for the second issue Y, the conflict situation (outside option) is always inefficient in our model.

By contrast to this literature, our ordinal approach together with our modeling specifications enables us to obtain positive results: Indeed, we are able to achieve ex post efficiency in dominant strategies and argue that the proposed rules can potentially be convenient and simple enough to use in practice.

2) Political Economy: Our benchmark model (but not the main, two-issue model where preferences over bundles are not necessarily single-peaked) resembles a voting model with single-peaked preferences where a number of voters have single-peaked preferences over the single-dimensional political spectrum and a voting rule aggregates individual preferences (Black 1948, Moulin 1980, Barbera, Gul and Stachetti 1991, and Ching 1997). In this type of models, the famous median voter theorem states that majority-rule voting system will select the outcome most preferred by the median voter.

In our model, preferences can also be thought to be single-peaked with each negotiator preferring the opposite extremes of the spectrum. There are a number of differences in our model from a voting model. In a voting model, there are several voters whose bliss point (peak value) is their private information. In our model, peaks are publicly known. What is private information is the two negotiators' outside options, which do not have any analogues in a voting model. Because of this difference, the voting model admits a class of strategy-proof rules for which efficiency and individual rationality vacuously hold. In our setup, however, such rules are either inefficient or violate individual rationality. In our setup, even dictatorship rules, despite being efficient, violate individual rationality.

In our benchmark model with single issue, there is no strategy-proof, efficient and individually rational rule. This in turn motivates to consider multi-issue models where single-peakedness does not necessarily hold and there is asymmetry in terms of outside options are treated for each issue. In fairness, there are also multi-dimensional voting models where people vote on multiple issues but this literature also concludes that strategy-proofness effectively requires each dimension to be treated independently for other. In our setup, by contrast we exploit a kind of exchangeability between the two issues, together with an asymmetric treatment of outside options, to arrive at strategy-proof rules

A logically independent but similar result that we find to the median voter theorem is that although the family of strategy-proof, efficient and individually rational rules we characterize are much different than those strategy-proof rules characterized generalized Condorcet rules, our family also nests a central rule dubbed the “constrained shortlisting” rule that chooses the “median bundle” when preferences of the negotiators are symmetric and tries to choose outcomes as close to the median bundle as possible. In our setup, however, the class of rules need not even include the impartial median-type rule. Indeed, we also identify rules that may also be partial toward either negotiator.

3) Matching/Assignment: Matching models and applications have championed the ordinal mechanism design approach (see, for example, Gale and Shapley 1962, Shapley and Shubik 1971, Crés and Moulin 2001, and especially recent applications of ordinal assignment mechanisms Balinski and Sönmez 1998, Abdulkadiroglu and Sönmez 2003, Roth, Sönmez and Ünver 2005.) Ordinal rankings over objects together with an outside option is a common feature of matching/assignment models. Given that both negotiators end up consuming the same bundle, our model with ordinal preferences can be though to be a “public good” assignment version of a matching problem. This connection to matching is important in two regards: 1. Ordinal mechanisms may be more practical and convenient than cardinal ones as supported by experimental work. In this regard ordinal mechanisms coupled with strategy-proofness can help avoid the Wilson’s critique often imposed on the “cardinal/Bayesian” mechanism design approach.

A second connection that surfaces to matching type models as a result of our analysis is that we find that the class of strategy-proof, efficient and individually rational rules also contain the negotiator-optimal rules, much in the same spirit as the proposing-side optimal deferred acceptance mechanisms or the buyer/seller optimal core assignments in the Shapley-Shubik assignment game.

4) Non-dictatorial strategy-proof mechanisms escaping the Arrow - Gibbard - Satterthwaite impossibilities: With the hope of arriving at possibility results, there is a tradition of identifying strategy-proof rules in restricted economic environments: see, for example, Vickrey (1961), Groves (1973), Clarke (1971) [VCG] for public goods and private assignment with transfers, uniform rule (Benassy 1982, Sprumont 1991) for division of divisible private good under single-peaked preferences, generalized median-voters (Moulin 1980), proportional-budget exchange rules (Barbera and Jackson 1995) that allow for trading from a finite number of pre-specified proportions (budget sets), deferred acceptance (Gale and Shapley, 1962) and top trading cycles (David Gale, 1974 and Abdulkadiroglu and Sönmez, 2003); hierarchical exchange and brokerage (Papai 2001 and Pycia and Ünver 2015). We also add to this literature in the sense that one may draw a conceptual parallel with the VCG mechanisms, though our rules look nothing like the above rules. In the VCG model, preferences over objects are private info and the preferences over money is common knowledge. This is much like negotiator's preferences over issue X versus issue Y. This connection is only superficial since VCG mechanisms are cardinal, and assignments and transfers depend on reported utilities.

The Model

We begin to describe the environment with a simple example and a short discussion about why the assumption of diametrically opposed preferences is without loss of generality.

A simple example: single-issue mediation

Negotiators 1 and 2 are in dispute over a single issue that is important for both. Let x_1 and x_2 denote the available alternatives (solutions) for the dispute. The negotiators are also entitled to the outside option, o, in case one or both of them reject to accept one of the alternatives. Therefore, the set $X = \{x_1, x_2, o\}$ denotes the set of all possible outcomes of the dispute.

It is common knowledge that negotiator 1 (strictly) prefers alternative x_1 to x_2 and negotiator 2 prefers x_2 to x_1. That is, the negotiators have diametrically opposed preferences over the alternatives x_1 and x_2. The ranking of the outside option, however, is the negotiators' private information. Therefore, each negotiator has two types[4]:

$\theta_1^{x_1}$	$\theta_1^{x_2}$
x_1	x_1
o	x_2
x_2	o

$\theta_2^{x_2}$	$\theta_2^{x_1}$
x_2	x_2
o	x_1
x_1	o

[4]We suppose, without loss of generality, that there is at least one acceptable alternative for each negotiator.

Consider the mediation process, denoted by f, as a mechanism with veto rights that maps the negotiators' private information to an outcome in X. Then, it would be represented by the following matrix

	$\theta_2^{x_1}$	$\theta_2^{x_2}$
$\theta_1^{x_1}$	$f_{1,1}$	$f_{1,2}$
$\theta_1^{x_2}$	$f_{2,1}$	$f_{2,2}$

where $f_{\ell,j} \in X$ for all $\ell, j \in \{1,2\}$.

We can assign $f_{1,2} = o$, without loss of generality, because there is no mutually acceptable alternative when the negotiators' types are $\theta_1^{x_1}$ and $\theta_2^{x_2}$, and thus, the outside option o is effectively the only result in all voluntary mediation processes. If the outcomes of the mediation process are (Pareto) efficient, then $f_{1,1}$ should be x_1 or x_2. Moreover, if the process produces individually rational outcomes, then we must have $f_{1,1} = x_1$. Likewise, an efficient and individually rational mediation process suggests $f_{2,2} = x_2$ and $f_{2,1} \in \{x_1, x_2\}$.

Therefore, we can construct several efficient and individually rational mechanisms for this simple example. However, none of these processes are immune to strategic manipulation (strategy-proofness). To prove this point, suppose that $f_{2,1} = x_1$. In this case, type $\theta_2^{x_1}$ of negotiator 2 would deviate and declare his type as $\theta_2^{x_2}$ to obtain x_2, contradicting with strategy-proofness. Alternatively, if $f_{2,1} \neq x_1$, then type $\theta_1^{x_2}$ of negotiator 1 would deviate and declare his type as $\theta_1^{x_1}$ to obtain x_1, again contradicting with strategy-proofness.

It is easy to extend this example to the case with more than two alternatives, and so extrapolate that there exists no efficient, individually rational and strategy-proof single-issue mediation process.[5]

Modeling conflicting preferences

Using diametrically opposed preferences over alternatives, when describing a dispute, is intuitive because it resembles the standard bargaining problem, which is modeled as a zero sum game, and unavoidable when the number of available alternatives is just two. However, intuition suggests that when there are more than two alternatives many other preference profiles, which are not diametrically opposed, would also depict a dispute. Consider, for example, the case where the set of available alternatives (other than the outside option) is $A = \{x_1, x_2, x_3, x_4, x_5\}$ and the negotiators' preferences are

[5] However, there are efficient, individually rational and Bayesian incentive compatible mediation rules when negotiators are sufficiently risk averse (Kesten and Ozyurt, 2018).

θ_1	θ_2
x_1	x_3
x_2	x_5
x_3	x_4
x_4	x_2
x_5	x_1

These preferences are not diametrically opposed but they are certainly conflicting—to some degree—as the agents cannot agree on their best alternative. Notice, however, that alternatives x_4 and x_5 are (Pareto) dominated by x_3, and so, if selecting an efficient outcome by the negotiation protocol is desired, then the presence of these two alternatives is irrelevant for the negotiation problem. Knowing whether or not these two alternatives are acceptable, i.e., better than the outside option, is also an "irrelevant" piece of information because these alternatives are acceptable by a negotiator whenever x_3 is acceptable. Thus, this particular dispute problem can be transformed into a simplified and "outcome equivalent" version where the only available alternatives are x_1, x_2 and x_3 and the negotiators' preferences over these three are diametrically opposed. We can generalize this observation for any (discrete) set of alternatives and for any preference profiles, where negotiators cannot agree upon their first best.

Let A be non-empty set of available alternatives and Θ be the set of all complete, transitive and antisymmetric preference relations on A. Define $max(\theta)$ to be the maximal element of the preference ordering $\theta \in \Theta$, namely if $x^\star = max(\theta)$, then $x^\star \ \theta \ x$ for all $x \in A \setminus \{x^\star\}$. Therefore, a ***two-person, single-issue dispute*** (dispute in short) problem is a list $D = (\theta_1, \theta_2, A)$ where $\theta_i \in \Theta$ for $i = 1, 2$ and $max(\theta_1) \neq max(\theta_2)$.

For any non-empty subset $\widetilde{A}$ of A, let $\theta|_{\widetilde{A}}$ denote the restriction of the preference ordering $\theta \in \Theta$ on $\widetilde{A}$. Therefore, define $\widetilde{D} = (\widetilde{\theta}_1, \widetilde{\theta_2}, \widetilde{A})$ to be a dispute reduced from $D = (\theta_1, \theta_2, A)$ whenever $\widetilde{A} \subseteq A$ and $\widetilde{\theta}_i = \theta_i|_{\widetilde{A}}$ for $i = 1, 2$.

Proposition 1. *By deleting all the Pareto inefficient alternatives, any two-person, single-issue dispute problem D can be reduced into a two-person, single-issue dispute problem $\widetilde{D}$ where the negotiators preferences are diametrically opposed.*

A similar result holds for two-person, multi-issue dispute problems whenever preferences over bundles satisfy monotonicity.[6]

Proof. Let $\widetilde{A} \subseteq A$ is the set of alternatives that survive the elimination of Pareto inefficient alternatives. Namely, none of the alternatives in $\widetilde{A}$ is Pareto inefficient. Re-number the elements in $\widetilde{A}$, and so suppose, without loss of generality, that $\widetilde{A} = \{x_1, ..., x_m\}$ where $m \geq 2$, and negotiator 1 ranks alternatives as $x_k \ \widetilde{\theta}_1 \ x_{k+1}$. If x_m is not the best alternative

[6]See next section for the formal definition of monotonicity.

for $\widetilde{\theta}_2$ on $\widetilde{A}$, then there must exist some x_k where $k < m$ such that $x_k \; \widetilde{\theta}_2 \; x_m$. But this contradicts with the assumption that x_m is not Pareto inefficient. Thus, negotiator 2 must rank x_m as the top alternative. With a similar reasoning, if x_{m-1} is not negotiator 2's second best alternative, then it must be Pareto inefficient, contradicting with the assumption that x_{m-1} survives after deletion of Pareto inefficient alternatives. Iterating this logic implies that the rankings of the negotiators must be diametrically opposed. □

The Main Model: Multi-Issue Mediation

This section proves that we may escape from the impossibility result akin to Myerson and Satterthwaite (1983) if the negotiators are in dispute over multiple issues and two issues are enough to make our point. There are two agents, $\mathrm{I} = \{1, 2\}$, in a dispute who aim to reach a resolution through mediation. Without loss of generality, there are two **issues** that are important for the negotiators.[7] Let the sets $X = \{x_1, ..., x_m, o_X\}$ and $Y = \{y_1, ..., y_m, o_Y\}$ denote the finite sets of potential **outcomes** for each issue. The sets $X \setminus \{o_X\}$ and $Y \setminus \{o_Y\}$ are the available **alternatives**. The negotiators are entitled to an **outside option** (disagreement point) for each issue, o_X and o_Y, in case one or both of them reject to accept an alternative that is available for that issue. Negotiators have at least two available alternatives for each issue, and so $m \geq 2$.

Preferences over Outcomes: The negotiators' preferences over outcomes for each individual issue satisfy the following three condition:

1. The negotiators' preferences over alternatives (not including the outside option) for each individual issue are diametrically opposed and public information.

2. Both negotiators' rankings of the outside option (relative to other alternatives) are private information in one of the issues.

3. It is public information that both negotiators rank the outside option as their worst outcome in one of the issues,.

More formally, for any issue $Z \in \{X, Y\}$, where $Z = \{z_1, ..., z_m, o_Z\}$, let Θ_i^z denote the set of all complete, transitive and antisymmetric preference relations of negotiator $i \in \mathrm{I}$ over issue Z and θ_i^z denote an ordinary element of the set Θ_i^z. It is public information

[7] If there are more than 2 issues, we can easily regroup these issues as those that fall under the category of issue X and category Y. Please see the distinction between these two categories next. Under this re-grouping, the alternatives would be vectors of alternatives, one for each issue. The negotiators' preferences over these vectors of alternatives need not be diametrically opposed. However, in light of Proposition 1, two issues with diametrically opposed preferences is without loss of generality.

that $z_k \; \theta_1^Z \; z_{k+1}$ and $z_{k+1} \; \theta_2^Z \; z_k$ for all $k = 1, ..., m-1$. Namely, the negotiators' preferences over the alternatives for each issue are diametrically opposed (the first condition). The ranking of the outside option in issue X, o_X, is the negotiators' private information (the second condition). Finally, it is common knowledge that $y \; \theta_i^Y \; o_Y$ for all i and $y \in Y \setminus \{o_Y\}$ (the third condition). Therefore, the set of acceptable alternatives for issue X is privately known by the negotiators, and it is unknown to them whether there is a mutually acceptable alternative for that issue. However, all alternatives in issue Y are acceptable by both negotiators and efficient. Note that there is a unique preference ordering in Θ_i^Y and $m+1$ orderings in Θ_i^X. Therefore, let $\Theta_i = \Theta_i^X$ denote the set of all **types** for negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ be the set of all type profiles.

This asymmetric treatment of the outside options can be motivated by various employment, family, construction or patent/copyright infringement disputes. Litigation would naturally be the default option if the issue is compensation or division of property. It usually is the case in such disputes that litigation is a very long and costly process, and so, inefficient relative to other potential divisions (alternatives). Such issues would be mapped into the issue Y in our framework. Although money is an important component in disputes, it is not the only issue: In employment disputes, for example, the quality of the reference letter that the former employer would be willing to write could be another issue for the disputants, or child custody or visitation would be the alternative issues in family disputes. Such issues, where the disputants' ranking of the outside option is not clear to all the parties, would be represented by the issue X in our setup. Nonetheless, it is natural to find examples, where the ranking of the outside option in all issues are the disputants' private information. For that reason, the symmetric treatment of the outside option is formally investigated at the end of this section.

Preferences over Bundles: A **bundle** (x, y) is a vector of outcomes, one for each issue, and the set $X \times Y$ denotes the set of all bundles. Let $\Re$ denote the set of all complete and transitive binary relations over the bundles. R is a standard element of the set $\Re$ and for any two bundles $b, b' \in X \times Y$, $b \; R \; b'$ means "b is at least as good as b'." We denote P for the strict counterpart of R.[8] An extension map is a rule Λ which assigns to every negotiator i and type $\theta_i \in \Theta_i$ a non-empty set $\Lambda(\theta_i) \subseteq \Re$ of admissible orderings over bundles.

For any negotiator i and type $\theta_i \in \Theta_i$, let $A(\theta_i) = \{x \in X | \; x \; \theta_i^X \; o_X\}$ denote the set of **acceptable** alternatives in issue X. For any type profile $(\theta_1, \theta_2) \in \Theta$, the set $A(\theta_1, \theta_2) = \{x \in X | \; x \; \theta_i^X \; o_X \text{ for all } i \in N\}$ denote the set of **mutually acceptable** alternatives in issue X. In case we need to specify a type's acceptable alternatives, we use $\theta_i^x \in \Theta_i^X$: It

[8] That is, $b \; P \; b'$ if and only if $b \; R \; b'$ holds but $b' \; R \; b$ does not.

denotes the preference relation (type) of negotiator i in which alternative $x \in X$ is the worst acceptable alternative. Namely, for any $x' \in X \setminus \{o_X\}$, $x \ \theta_i^x \ x' \Longrightarrow o_X \ \theta_i^x \ x'$.

Definition 1. *The* ***extension map*** Λ *is* ***consistent*** *if the followings hold for all* i, $\theta_i \in \Theta_i$ *and all* $R_i \in \Lambda(\theta_i)$*:*

i. [Monotonicity] For any $x, x' \in X$ *and* $y, y' \in Y$ *with* $(x, y) \neq (x', y')$*,*

$$(x, y) \ P_i \ (x', y') \text{ whenever } \left[x \ \theta_i^X \ x' \text{ or } x = x'\right] \text{ and } \left[y \ \theta_i^Y \ y' \text{ or } y = y'\right].$$

ii. [Deal Breakers] For any $y, y' \in Y \setminus \{o_Y\}$*,*

$$(x, y) \ R_i \ (x', y') \text{ whenever } x \in A(\theta_i) \cup \{o_X\}, \ x' \notin A(\theta_i) \text{ and } x \neq x'.$$

iii. [Logrolling] For any i*, there exists a one-to-one mapping* $t_i : X \to Y$ *such that for all* $\theta_i \in \Theta_i$*,* $R_i \in \Lambda(\theta_i)$ *and all* $x, x' \in A(\theta_i)$ *with* $x \ \theta_i \ x'$*,*

$$\big(x', t_i(x')\big) \ R_i \ \big(x, t_i(x)\big).$$

Monotonicity is a standard assumption. The second condition suggests that unacceptable alternatives in issue X are "deal-breakers" for the negotiators: regardless of the alternative in the second issue, a bundle with an unacceptable alternative is never preferred to a bundle with an acceptable alternative. Put differently, alternatives in issue Y are not "too important" for the negotiators, and so, an unacceptable alternative can never become a part of an acceptable bundle.

Logrolling or quid pro quo allows trading of favors. It requires two things. First, for any two acceptable alternatives x, x' in X where x is ranked above x' for type θ_i, there must exist two alternatives y, y' in Y s.t (x', y') is ranked at least as high as (x, y) at all consistent orderings over bundles, $R_i \in \Lambda(\theta_i)$. Second, types must be "consistent." Namely, order reversing mapping, t_i, is independent of types. Logrolling implies that alternatives in Y are "important enough" to reverse the rankings of (acceptable) alternatives in X when they are bundled with alternatives in the second issue. Logrolling rules out lexicographic preferences and many standard utility functions satisfy it. We discuss the last point later in detail. Furthermore, it is sufficient and "almost necessary" for the possibility result. We prove this point next after an example and some important definitions.

Example 1 (logrolling): Suppose that $X = \{x_1, x_2, x_3, o_X\}$ and $Y = \{y_1, y_2, y_3, o_Y\}$. Because the number of alternatives in issues X and Y are equal, there is a unique one-to-one mapping t (which is the same for both negotiators), where $t(x_k) = y_{4-k}$ for $k = 1, 2, 3$, which satisfies the requirements of Definition 1.[9] Therefore, logrolling implies that the type $\theta_1^{x_3}$ of negotiator 1 who deems all three alternatives in issue X acceptable, i.e., $x_1\ \theta_1^{x_3}\ x_2\ \theta_1^{x_3}\ x_3\ \theta_1^{x_3}\ o_X$, will rank (x_3, y_1) at least as high as the bundle (x_2, y_2) and rank (x_2, y_2) at least as high as the bundle (x_1, y_3) for all consistent orderings $R \in \Lambda(\theta_1^{x_3})$. The consistency of the mapping t over the types implies, for example, that type $\theta_1^{x_2}$ of negotiator 1 who deems only x_1 and x_2 acceptable, i.e., $x_1\ \theta_1^{x_2}\ x_2\ \theta_1^{x_2}\ o_X\ \theta_1^{x_2}\ x_3$, will rank (x_2, y_2) at least as high as the bundle (x_1, y_3). Logrolling imposes no restriction on consistent orderings $R \in \Lambda(\theta_1^{x_2})$ regarding how they rank the bundle (x_3, y_1) relative to the bundles (x_2, y_2) and (x_1, y_3).

For the rest of the paper, we let $\mathbf{B}$ denote the set of logrolling bundles. Namely, $\mathbf{B} = \left\{(x_k, y_{m+1-k}) \in X \times Y \middle| k = 1, ..., m\right\}$.

Direct Mechanisms with Veto Rights: Mediation would be a very complicated, multi-stage game between the negotiators and the mediator. The mediation protocol, whatever the details are, produces proposals for agreement that are always subject to unanimous approval by the negotiators. That is, before finalizing the protocol, each negotiator has the right to veto the proposal and the option to receive the outside options.

A version of the revelation principle that we prove in the appendix guarantees that we can stipulate the following direct mechanism with veto rights without loss of generality, when representing mediation. The direct mediation mechanism consists of two stages: an *announcement* stage and a *ratification* stage; and it is characterized by the mediation rule $f : \Theta \to X \times Y$. After being informed of its type, each negotiator i privately reports his type, $\widehat{\theta}_i$, to the mediator, who then proposes $f(\widehat{\theta}_1, \widehat{\theta}_2) \in X \times Y$. In the ratification stage, each party simultaneously decides whether to accept or veto the proposed bundle. In case both negotiators accept the proposed bundle, then it becomes the final outcome. In case one or both negotiators veto the proposal, each party gets the outside option for both issues, i.e., (o_X, o_Y). Such direct mechanisms will be called direct truthful mechanisms with veto rights.

Definition 2. *The mediation rule f is **strategy-proof** if for all i and all $\theta_i \in \Theta_i$, $f(\theta_i, \theta_{-i})\ R_i\ f(\theta_i', \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.*

Definition 3. *The mediation rule f is **individually rational** if for all i and all $(\theta_i, \theta_{-i}) \in \Theta$, $f(\theta_i, \theta_{-i})\ R_i\ (o_X, o_Y)$ for all $R_i \in \Lambda(\theta_i)$.*

[9] Note that logrolling is a well-defined concept only if the number of alternatives in issue Y is greater than or equal to the number of alternatives in issue X. In case the number of alternatives in Y greater than that of X, one of many one-to-one mappings suffices.

Definition 4. *The mediation rule f is **efficient** if there exists no $(\theta_i, \theta_{-i}) \in \Theta$ and $(x', y') \in X \times Y$ such that $(x', y')\ R_i\ f(\theta_i, \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$ and all $i \in I$, and for at least one $i \in I$, $(x', y')\ P_i\ f(\theta_i, \theta_{-i})$ for some $R_i \in \Lambda(\theta_i)$.*

We seek direct mechanisms with veto rights in which, it is a *dominant strategy equilibrium* to report the true private information at the announcement stage, and in which, in equilibrium, proposals are not vetoed. It immediately follows from the definitions that such an equilibrium exists if and only if the mediation rule f is strategy-proof and (ex-post) individually rational.[10]

For convenience, we present a mediation rule f as an $m \times m$ matrix $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$, where $M = \{1, ..., m\}$. The rows indicate all the types of negotiator 1 and the columns are for all the types of negotiator 2. We ignore, without loss of generality, the types that deems no alternative acceptable from our matrix representation.

$f =$

	$\theta_2^{x_1}$	$\cdots$	$\theta_2^{x_m}$
$\theta_1^{x_1}$	$f_{1,1}$	$\cdots$	$f_{1,m}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\theta_1^{x_m}$	$f_{m,1}$	$\cdots$	$f_{m,m}$

In this matrix, row (column) ℓ represents the preference of negotiator 1 (2) that finds all alternatives $\{x_k | k \leq \ell\}$ $\left(\{x_k | k \geq \ell\}\right)$ acceptable. Therefore, there is a unique mutually acceptable alternative in the main (first) diagonal of the matrix, i.e., $\{f_{\ell,\ell} \mid \ell \in M\}$. Note that there is no mutually acceptable alternative in the upper half of the matrix.

Theorem 1. *The mediation rule f is efficient, individually rational and strategy-proof if and only if the following hold:*

(i) *If $\ell < j$, then $f_{\ell,j} = (o_X, y)$ for some $y \in Y$.*

(ii) *If $\ell = j$, then $f_{\ell,j} = (x_\ell, y_{m+1-\ell})$.*

(iii) *(Adjacency) If $\ell > j$, then $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ and there exists a complete, transitive and strict precedence order $\rhd$ on $\mathbf{B}$ such that*

$$f_{\ell,j} = \begin{cases} f_{\ell-1,j}, & \text{if } f_{\ell-1,j} \rhd f_{\ell,j+1} \\ f_{\ell,j+1}, & \text{oth.} \end{cases}$$

[10] The proof is omitted as it directly follows from similar arguments in our revelation principle result.

Example 2 (Adjacent rules): Let $X = \{x_1, x_2, x_3, x_4, x_5, o_X\}$, $Y = \{y_1, y_2, y_3, y_4, y_5, o_Y\}$, so the set of logrolling bundles is $\mathbf{B} = \big\{(x_1, y_5), (x_2, y_4), (x_3, y_3), (x_4, y_2), (x_5, y_1)\big\}$. A standard member of the family of adjacent rules is constructed by the following steps. It gives the outside option in issue X, bundled with some alternative from issue Y, whenever the negotiators have no mutually acceptable alternative in issue X. For our example, it is the bundle (o_X, y_3).

We fill the main diagonal with the members of the set of logrolling bundles, $\mathbf{B}$. In the first row and column, for example, we have (x_1, y_5). A reason for this is that the only mutually acceptable alternative is x_1 for the types in the first row and column. Therefore, deal-breakers property of the preferences imply that an individually rational rule must suggest a bundle with x_1. Thus, we must have (x_1, y_5) in the first row and column because the adjacent rules always suggest a bundle from the set of logrolling bundles—a critical property of the adjacent rules that is necessary for strategy-proofness, and we explain this point shortly. For the rest of the matrix, i.e., the lower half of it, we need a strict precedence order over the logrolling bundles, $\mathbf{B}$. One example is

$$\rhd : (x_5, y_1) \;\rhd\; (x_1, y_5) \;\rhd\; (x_4, y_2) \;\rhd\; (x_2, y_4) \;\rhd\; (x_3, y_3)$$

Because the bundle (x_5, y_1) is ranked first, it beats all the other bundles in $\mathbf{B}$ in a binary comparison. Therefore, starting from the row and column of the bundle (x_5, y_1), all the rows below it and all the columns to the left of it should be filled with (x_5, y_1). Then the second bundle in the precedence order is (x_1, y_5), and it beats all the other bundles in $\mathbf{B}$ except (x_5, y_1). Thus, starting from the row and column of the bundle (x_1, y_5) on the main diagonal, all the empty rows below it and all the empty columns to the left of it should be filled with (x_5, y_1). Iterating this process for all the bundles in the precedence order will yield the following matrix:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_4}$	(x_1, y_5)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(o_X, y_3)
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

Figure 1:*A standard member of the adjacent rules family*

Special members of the adjacent rules family

There are some special members of the adjacent rules family. Negotiator 1 (2)-optimal adjacent rule, for example, is constructed by using the strict counterpart of the preference of negotiator 1 (2) over the logrolling bundles, **B**, as the precedence order. For the same example above, the negotiator 1-optimal rule takes

$$\rhd^1\colon (x_5, y_1) \;\rhd^1\; (x_4, y_2) \;\rhd^1\; (x_3, y_3) \;\rhd^1\; (x_2, y_4) \;\rhd^1\; (x_1, y_5)$$

whereas the negotiator 2-optimal rule takes

$$\rhd^2\colon (x_1, y_5) \;\rhd^2\; (x_2, y_4) \;\rhd^2\; (x_3, y_3) \;\rhd^2\; (x_4, y_2) \;\rhd^2\; (x_5, y_1)$$

and they look as follow:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)				
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)			
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)		
$\theta_1^{x_4}$	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

Figure 2-a: *Negotiator 1-optimal rule*

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)				
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)			
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)		
$\theta_1^{x_4}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	
$\theta_1^{x_5}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Figure 2-b: *Negotiator 2-optimal rule*

Negotiator 1-optimal rule always picks negotiator 1's most preferred bundle among the *mutually acceptable logrolling bundles*. Although these rules are efficient, individually rational and strategy-proof, they are not impartial (symmetric). There is another special member of the adjacent rules family that treats negotiators symmetrically whenever the mediation problem is symmetric.[11]

[11] The mediation problem is symmetric if the number of alternatives in issues is odd number. The problem is symmetric in this case because there is a unique median alternative in each issue, and thus, the number of alternatives (not including the outside option) better than the median alternative is the same for both negotiators. However, if there are two median alternatives, which is the case when the number of alternatives is even, then the mediation problem is not symmetric.

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)				
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)			
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)		
$\theta_1^{x_4}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(x_4, y_2)	
$\theta_1^{x_5}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Figure 3: *constrained shortlisting rule*

Constrained shortlisting and its characterization

In the **"constrained shortlisting" (CS)** rule the mediator picks one of the negotiators and asks her least acceptable alternative in issue X, say x_k. Then the mediator proposes three bundles to the other negotiator to pick one of them as the final outcome. One of these bundles is the logrolling bundle with x_k, namely (x_k, y_{m-k+1}). The other bundle is the logrolling bundle with the median alternatives in both issues, namely (x_n, y_n) where n is the indices of the median alternative in each issue. More formally, $n \in \{\overline{n}, \underline{n}\}$, where $\overline{n} = \lceil \frac{m+1}{2} \rceil$ and $\underline{n} = \lfloor \frac{m+1}{2} \rfloor$. If m is odd, then there is a unique median alternative in each issue because $\overline{n} = \underline{n} = \frac{m+1}{2}$. If there are two median alternatives, namely m is even, then the mediator picks one of them at all times. Finally, the third bundle is the one with outside option in issue X, i.e., (o_X, y_n).

CS rule is a special member of the adjacent rule family. It acts as though it is a negotiator 1 or 2-optimal rule if the median alternative in issue X is not mutually acceptable, and suggests the "median" bundle, (x_n, y_n), otherwise. In addition to being efficient, individually rational and strategy-proof, CS rule minimizes rank variance within the class of efficient, individually rational and strategy-proof rules. We prove this point next.

Given the negotiators' fixed preferences over alternatives (not including the outside option), let $r_i(z) \in M$ denote negotiator i's ranking of the alternative $z \in Z \in \{X, Y\}$.[12] Given a mediation rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$, let $f_{\ell,j} = (f^X_{\ell j}, f^Y_{\ell j}) \in X \times Y$ denote the bundle the mediation rule f proposes when the negotiators' types are $\theta_1^{x_\ell}$ and $\theta_2^{x_j}$. Therefore, the *rank variance* of the bundle $f_{\ell,j}$ is defined by[13]

$$var(f_{\ell,j}) \equiv \sum_{i\in I} \left(r_i(f^X_{\ell j})\right)^2 + \left(r_i(f^Y_{\ell j})\right)^2 .$$

[12]We ignore the outside option from the rank calculations without loss of generality because we will restrict our attention to individually rational and efficient rules.

[13]One may assign different weights to the issues in the definition of rank variance. The results still go through without any loss.

Thus, the rank variance of the mediation rule f is

$$Var(f) = \sum_{\ell=1}^{m} \sum_{j=1}^{m} var(f_{\ell,j}).$$

A bundle including a/the median alternative in both issues has the smallest rank variance and bundles (x_1, y_1) and (x_m, y_m) have the highest rank variance. Intuitively, rank variance of a bundle is a measure of the extent to which that bundle favors one negotiator over the other negotiator. In this sense, the higher the rank variance of a bundle or a mediation rule, the more biased its treatment is. Alternately, the lower the rank variance of a mediation rule, the more impartial it is. Normatively speaking, a rule that aims to minimize rank variance can be viewed as one choosing "the center of gravity" or the "middle ground" along the tradeoffs the negotiators are facing.

Definition 5. *For any $k \in M$, let the bundle $b_k = (x_k, y_{m-k+1})$ be the logrolling bundle in $\mathbf{B}$. A rule is a "constrained shortlisting" rule, denoted $f^{CS} = [f_{\ell,j}]_{(\ell,j)\in M^2}$, if it is an adjacent rule (as described in Theorem 1) that is associated with a precedence order $\rhd^{CS}$, where $b_n \rhd^{CS} b_{n-1} \rhd^{CS} \ldots \rhd^{CS} b_1$ and $b_n \rhd^{CS} b_{n+1} \rhd^{CS} \ldots \rhd^{CS} b_m$ with n being the indices of the median alternative in both issues, and $f^{CS}_{\ell,j} = (o_X, y_n)$ whenever $\ell < j$.*

Note that there is a unique CS rule if m is odd. If m is even, however, a CS rule prescribes one of four types of outcomes depending on whether $b_{\overline{n}}$ or $b_{\underline{n}}$ has the highest precedence order and whether $y_{\overline{n}}$ or $y_{\underline{n}}$ is chosen when no mutually acceptable alternative in issue X exists.

Theorem 2. *A mediation rule minimizes rank variance within the class of efficient, individually rational and strategy-proof rules if and only if it is a CS rule.*

Proof of Theorem 2: Clearly, a CS rule belongs to the adjacent rule family. To see that the rank variance of a CS rule is lower than any other member of the adjacent rule family, we simply consider two cases about the number of possible alternatives. First, when m is odd, $var(b_n) = (m+1)^2$. For any $b_{n-t}, b_{n+t} \in \mathbf{B}$ with $t < n$, we have $var(b_{n-t}) = var(b_{n+t}) = 2(\frac{(m+1)}{2} - t)^2 + 2(\frac{(m+1)}{2} + t)^2 = (m+1)^2 + 4t^2$. Thus, $var(b_n) < var(b)$ for any $b \in \mathbf{B} \setminus \{b_n\}$.

Since any member of the adjacent rule family must pick an element of $\mathbf{B}$ whenever there is a mutually acceptable alternative in issue X (by cases *(ii)* and *(iii)* of Theorem 1), minimization of rank variance requires that $b_n \rhd b$ for any $b \in \mathbf{B} \setminus \{b_n\}$. Also observe that $var(b_n) < var(b_{n-1}) < \ldots < var(b_1)$ and $var(b_n) < var(b_{n+1}) < \ldots < var(b_m)$. Thus, minimization of rank variance subsequently requires that $b_{n-1} \rhd \ldots \rhd b_1$ and $b_{n+1} \rhd \ldots \rhd b_m$. By case (i) of Theorem 1, the outcome for issue X is fixed to o_X

whenever there is no mutually acceptable alternative in this issue. Therefore, (o_X, y_n) is the rank minimizing bundle. Note that when m is odd, rank variance of the unique CS rule is strictly less than any other member of the adjacent rule family.

On the other hand, when m is even, $var(b_{\overline{n}}) = var(b_{\underline{n}}) = \frac{1}{2}(m^2 + (m+2)^2)$. For any $b_{\underline{n}-t}, b_{\overline{n}+t} \in \mathbf{B}$ with $t < n$, we have $var(b_{\underline{n}-t}) = var(b_{\overline{n}+t}) = 2(\frac{m}{2} - t)^2 + 2(\frac{(m+2)}{2} + t)^2 = \frac{1}{2}(m^2 + (m+2)^2) + 4t^2$. Hence, $var(b_{\overline{n}}) = var(b_{\underline{n}}) < var(b)$ for any $b \in \mathbf{B} \backslash \{b_{\overline{n}}, b_{\underline{n}}\}$. Note that we also have $var(b_{\underline{n}}) = var(b_{\overline{n}}) < var(b_{\underline{n}-1}) < \ldots < var(b_1)$ and $var(b_{\underline{n}}) = var(b_{\overline{n}}) < (b_{\overline{n}+1}) < \ldots < var(b_m)$. Then, minimization of rank variance subsequently requires that either $b_{\overline{n}} \rhd b_{\underline{n}}$ or $b_{\underline{n}} \rhd b_{\overline{n}}$ together with $b_{\underline{n}-1} \rhd \ldots \rhd b_1$ and $b_{\overline{n}+1} \rhd \ldots \rhd b_m$. By case (i) of Theorem 1, the outcome for issue X is o_X and both $(o_X, y_{\overline{n}})$ and $(o_X, y_{\underline{n}})$ are rank minimizing bundles. Consequently, any one of the four types of CS rules are rank minimizing. Note that when m is even, rank variance of a CS rule is weakly less than any other member of the adjacent rule family.

Extension: Continuous Mediation Problem

Suppose now that the issues X and Y are two closed and convex intervals of the real line with the same measure. The outside options, o_Y and o_Y, may or may not be the elements of these sets. We assume, without loss of generality, that $X = Y = [0, 1]$, with the interpretation that the negotiators aim to divide a unit surplus in each issue. In order to keep the notation consistent with the previous section, let a bundle $b = (x, y)$ indicate that negotiator 2 gets x and y in issues X and Y, respectively, and thus, negotiator 1 gets $1 - x$ and $1 - y$, respectively. Namely, each alternative in each issue indicates what negotiator 2 receives. Agents having diametrically opposing preferences on each issue means that for any issue $Z \in \{X, Y\}$ and two alternatives $z, z' \in Z$, negotiator 1 (2) prefers z to z' whenever $z < z'$ ($z > z'$). The value/ranking of the outside option o_X in issue X is each negotiators' private information. However, the value/ranking of the outside option o_Y in issue Y is common knowledge, and both negotiators prefer all $y \in Y$ to o_Y.

For any $\ell \in [0, 1]$, type ℓ of negotiator 1 (2), denoted by θ_1^ℓ (θ_2^ℓ), prefers the outside option o_X to all alternatives $k \in [0, 1]$ with $\ell < k$ ($\ell > k$).[14] Parallel to the discrete case, we denote the mediation rule $f = [f_{\ell,j}]_{(\ell,j) \in [0,1]^2}$ where $f_{\ell,j} = f(\theta_1^\ell, \theta_2^j)$ for all $0 \leq \ell, j \leq 1$.[15] The negotiators have no mutually acceptable alternative in issue X at type profile $(\theta_1^\ell, \theta_2^j)$ when $\ell < j$. The set of mutually acceptable alternatives is $A(\theta_1^\ell, \theta_2^j) = \{j, ..., \ell\}$ whenever $\ell \geq j$. The consistency assumption in the previous section can directly be applied to the

[14] Therefore, all k with $\ell \geq k$ ($\ell \leq k$) are deemed acceptable by type θ_1^ℓ (θ_2^ℓ) of negotiator 1 (2).

[15] We assume, without loss of generality, that each negotiator has at least one acceptable alternative. Therefore, there is no type profile where a negotiator deems all alternatives unacceptable.

continuous case.[16] Therefore, the set of logrolling bundles is

$$\mathbf{B} = \left\{(x, y) \in [0,1]^2 \mid y = 1 - x\right\}.$$

Thus, for all values of $\ell, j \in [0,1]$ with $j \le \ell$, $\mathbf{B}_{\ell j} = \{(k, 1-k) \in \mathbf{B} \mid j \le k \le \ell\}$ denotes the set of all mutually acceptable logrolling bundles at type profile $(\theta_1^\ell, \theta_2^j)$.

Define $\rhd$ to be a complete, transitive and antisymmetric binary relation over the set of logrolling bundles. When $(\mathbf{B}, d)$ is a metric space with a proper metric d, $\mathbf{B}_{\ell j}$ with $\ell \ge j$ is a non-empty and compact subset of the set of logrolling bundles.

Definition 7. *The binary relation $\rhd$ is said to be* ***quasi upper-semicontinuous over*** $\mathbf{B}_{\ell j}$ *with $\ell \ge j$ if for all $a, c \in \mathbf{B}_{\ell j}$ with $a \ne c$, $a \rhd c$ implies that there exists a bundle $a' \in \mathbf{B}_{\ell j}$ and a neighborhood $\mathcal{N}(c)$ of c such that $a' \rhd b$ for all $b \in \mathcal{N}(c) \cap \mathbf{B}_{\ell j}$.*[17]

Therefore, the binary relation $\rhd$ is quasi upper-semicontinuous if it is quasi upper-semicontinuous over all compact subsets $\mathbf{B}_{\ell j}$ of $\mathbf{B}$. A bundle $b^\star \in \mathbf{B}_{\ell j}$ is said to be a maximal element of the binary relation $\rhd$ on $\mathbf{B}_{\ell j}$ if $b^\star \rhd b$ for all $b \in \mathbf{B}_{\ell j}$. Theorem 1 in Tian and Zhoub (1995) proves that quasi upper-semicontinuity is both necessary and sufficient for $\rhd$ to attain its maximum on all compact subsets $\mathbf{B}_{\ell j}$ of $\mathbf{B}$. Therefore, the analogous version of Theorem 1 in the continuous case reads as follows.

Theorem 3. *The mediation rule f is efficient, individually rational and strategy-proof if and only if there exists a complete, transitive, antisymmetric, and quasi upper-semicontinuous binary relation $\rhd$ over the set of logrolling bundles $\mathbf{B}$ and $y \in Y \setminus \{o_Y\}$ such that*

$$f_{\ell,j} = \begin{cases} (o_X, y), & \text{if } \ell < j, \\ \underset{\mathbf{B}_{\ell j}}{\operatorname{argmax}} \rhd, & \text{oth.} \end{cases}$$

Analogous to the discrete case, we use the following continuously indexed matrix to describe a mediation rule f.

[16] The same is true for the definition of strategy-proofness, individual rationality and efficiency.

[17] This is Definition 2 in Tian and Zhoub (1995).

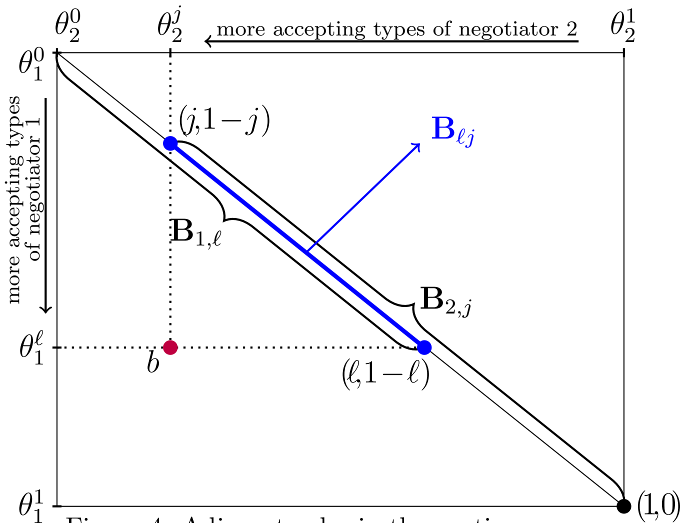


Figure 4: Adjacent rules in the continuous case

The rows, i.e., the vertical axis, correspond to the types of negotiator 1 and columns, i.e., the horizontal axis, indicate all possible types of negotiator 2. Each point on the main diagonal represents a logrolling bundle for the mediation rule that is described in Theorem 3, and each logrolling bundle appears only once on this diagonal. The bundle b, for example, represents the value of f when the true type of negotiator 1 and 2 are θ_1^ℓ and θ_2^j, respectively. When the true type profile is (θ_1^1, θ_2^1), negotiator 1 finds all alternatives acceptable and negotiator 2 deems all alternatives except one unacceptable, and thus, the only mutually acceptable logrolling bundle is $(1, 0)$.

The set of all acceptable logrolling bundles for type θ_1^ℓ of negotiator 1 is denoted by $\mathbf{B}_{1,\ell}$, which consists of all the logrolling bundles on the upper portion of the main diagonal, starting from the north west corner bundle, $(0, 1)$, and goes all the way down to the bundle $(\ell, 1-\ell)$. That is, $\mathbf{B}_{1,\ell} = \{(k, 1-k) \in \mathbf{B} \mid 0 \leq k \leq \ell\}$. Similarly, the set of all acceptable logrolling bundles for type θ_2^j of negotiator 2 is represented by $\mathbf{B}_{2,j}$ and it consists of all the bundles on the lower portion of the main diagonal, i.e., all bundles from $(j, 1-j)$ to $(1, 0)$. Namely, $\mathbf{B}_{2,j} = \{(k, 1-k) \in \mathbf{B} \mid j \leq k \leq 1\}$. Thus, the set of all mutually acceptable logrolling bundles at the type profile $(\theta_1^\ell, \theta_2^j)$ is the intersection of these two sets, i.e., $\mathbf{B}_{\ell j} = \mathbf{B}_{1,\ell} \cap \mathbf{B}_{2,j}$. Theorem 3 states that bundle b is the logrolling bundle that maximizes $\rhd$ within the set $\mathbf{B}_{\ell j}$ (see Figure 4). A maximal bundle uniquely exists because $\rhd$ is antisymmetric.

Preferences that Satisfy Logrolling

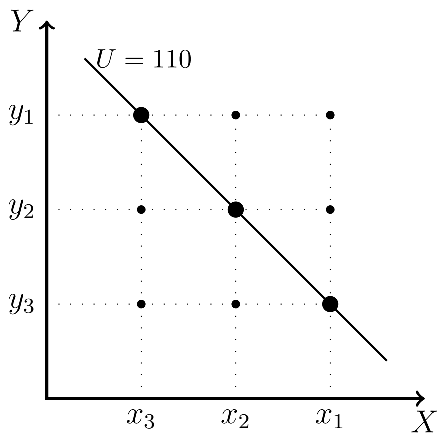
The critical requirement for our results is logrolling. Simply put, it requires that alternatives in issue Y are “important enough” to reverse the ranking of the alternatives in issue X when they are bundled together. Absolute utility values of the alternatives are irrelevant for logrolling. To see this point, consider the following simple example:

Example 3: Suppose that preferences over the bundles are additively separable and each issue has three alternatives. Let $u(.)$ and $v(.)$ represent preferences over issues X and Y, respectively. Therefore, $U(x, y) = u(x) + v(y)$ is the utility function over the bundles.[18]

X	$u(.)$	Y	$v(.)$	$X \times Y$	$U(.)$
x_1	100	y_1	20	(x_3, y_1)	110
x_2	98	y_2	12	(x_2, y_2)	110
x_3	90	y_3	10	(x_1, y_3)	110

The utility functions (preferences) in this simple example satisfy logrolling although the worst alternative in issue X is 4.5 times more valuable, in absolute terms, than the most valuable alternative in issue Y.

In standard consumer theory, we represent preferences over bundles by drawing corresponding indifference curves on commodity space, where each axis corresponds the quantity of a particular commodity. In the current model, issues serve the same role with commodities. However, distance between two alternatives is irrelevant in our setup as we abstract away from quantities. In our discrete setup, marginal rate of substitution is the rate at which a negotiator can give up some *number of alternatives* in one issue in exchange for the other issue while maintaining the same level of utility. Therefore, without loss of generality, we can place all alternatives equidistantly. Also, we place less preferred alternatives closer to the origin, implying (together with monotonicity) higher indifferent curves as we move northeastern direction. For the previous numerical example, therefore, preferences of the type that deems all alternatives acceptable can be pictured as follows:



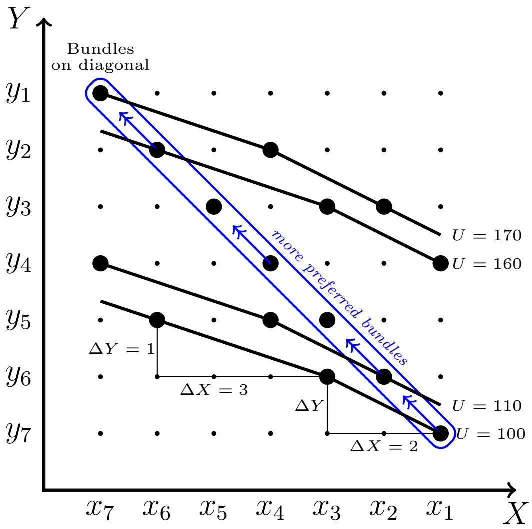
As it is evident from this graph, logrolling is a property of the bundles that are placed on the *diagonal*. Logrolling requires that these bundles are either lying on the same indifference curve or on higher indifference curves as we move along the diagonal in

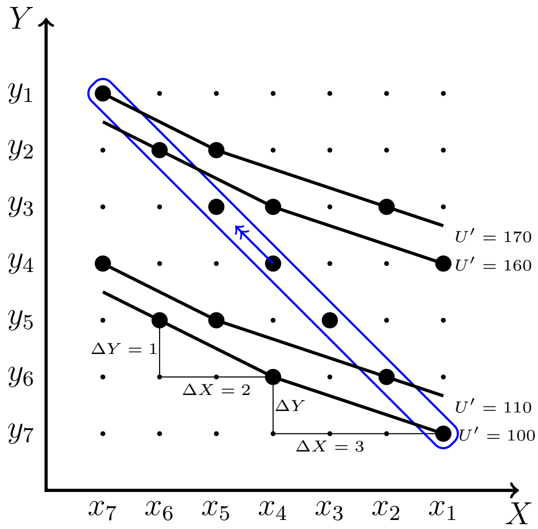
[18]For completeness, one may assume that all types get very large disutility from unacceptable alternatives, including the outside option.

northwestern direction (and southeast direction for the other negotiator). The marginal rate of substitution at diagonal bundles is one for our example because it requires one alternative to trade between the issues to keep the negotiator's utility the same. The utility function we picked behaves *as if* issues are perfect substitutes. But it is hardly possible to make a concrete statement with only nine bundles. On the other hand, logrolling is consistent with all range of utility functions that have "convex" or "concave" indifference curves. Consider the following two utility functions, U and U':

X	$u(.)$	$u'(.)$	Y	$v(.)$	$X \times Y$	$U(.) = u(.) + v(.)$	$U'(.) = u'(.) + v(.)$
x_1	100	100	y_1	120	(x_7, y_1)	170	170
x_2	90	90	y_2	100	(x_6, y_2)	160	160
x_3	80	85	y_3	80	(x_5, y_3)	145	150
x_4	70	80	y_4	60	(x_4, y_4)	130	140
x_5	65	70	y_5	40	(x_3, y_5)	120	125
x_6	60	60	y_6	20	(x_2, y_6)	110	110
x_7	50	50	y_7	0	(x_1, y_7)	100	100

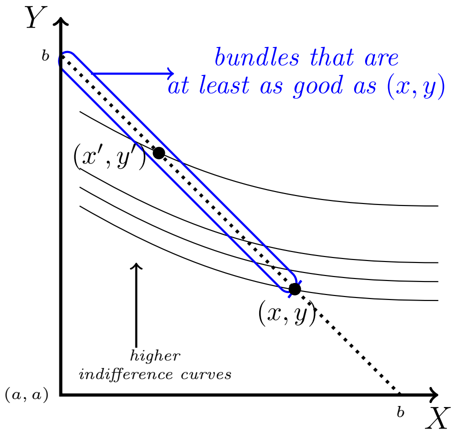
Both utility functions satisfy logrolling and their indifference curves are drawn in the following two graphs. As it is also clear from these graphs, the marginal rate of substitution of the utility function U (U') is increasing (decreasing) as we move to the right in the X axis, which are interpreted as indifference curves for U (U') being concave (convex).

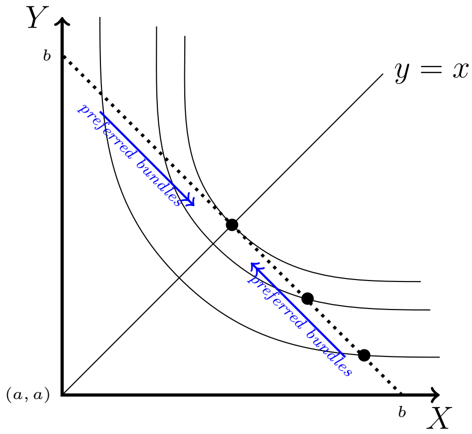




In a more standard setup where alternatives for issues X and Y represent quantities of two commodities, ranging over some interval $[a, b] \subseteq \mathbb{R}^2$, logrolling will be satisfied for all utility functions with marginal rate of substitution (MRS) that is less than or equal to one at all points of the diagonal. For example, a utility function $U(x, y) = \sqrt{x} + y$

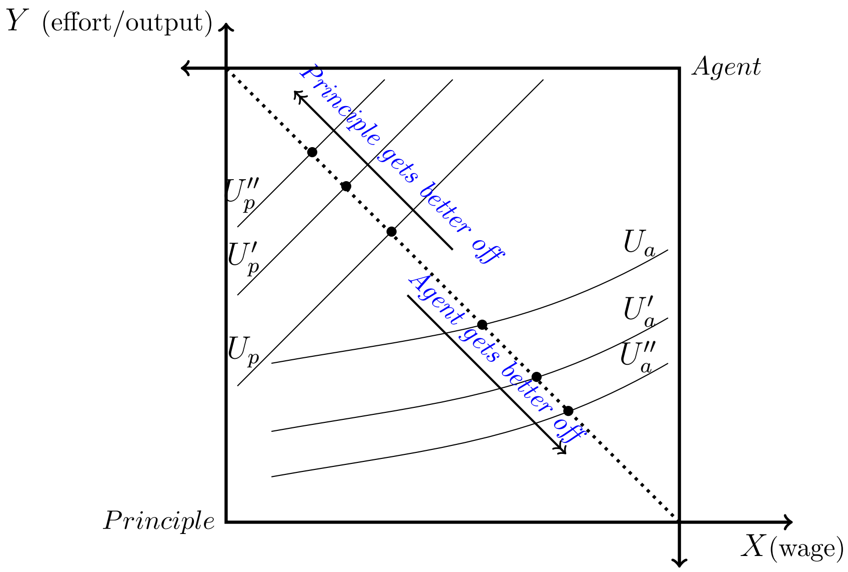
does satisfy this condition whenever $1/4 \leq a$. This is true because upper counterset of a bundle (x, y) that is on diagonal includes all the other bundles (x', y') on the diagonal that are situated northwest of the original bundle (x, y).





In fact, all utility functions with convex indifference curves with $MRS_{x=y} \leq 1$ satisfy a weaker condition of logrolling, where logrolling holds only for the first half of the alternatives in issue X. This weaker condition is sufficient to guarantee strategy-proof rules. An example for these utility functions would be $U(x, y) = x^{\alpha} y^{\beta}$ where $\alpha/\beta \leq 1$. The second graph above demonstrates why such convex utility functions satisfy this weaker condition.

Alternatively, one may consider a moral hazard situation between a principle and an negotiator, where X denotes the domain for wage and Y denotes different levels of effort/output. Consistent to this framework, let the agent's and the principle's utility functions are $U_a(x, y) = u_a(x) - v_a(y)$ and $U_p(x, y) = v_p(y) - u_p(x)$, respectively, where all u's and v's are increasing functions. Along the diagonal, the principle's indifference curves increase as we move in northwestern direction (and the worker's indifference curves increase as we move in southeastern direction) as required by the logrolling condition. For a simple example, one may consider $U_w(x, y) = x - y^2$ and $U_p = y - x$, which we depict below.



Relating to the impossibility result of Myerson and Satterthwaite (83)

This section explores the underlying factors that are potentially absent in Myerson and Satterthwaite (83) model, which leads to the possibility result in our case. It is already well-known in the literature that denseness of the type space is one reason for the impossibility result in Myerson and Satterthwaite (83) [MS]. However, this is not the driving force for our strategy-proof mediation rules in multi issue mediation case. The main factor seems to be that MS is effectively not a multi-issue negotiation problem.

MS considers a bilateral trade between a seller, who owns an indivisible good, and a buyer, who likes to buy this good, as a mechanism design problem. The mechanism (p, x) has two components; the probability of trade, p, and the transfer, x, both of which are functions of the players' reports. If no trade occurs, then $x = p = 0$ (the outside option), and so both players receive zero utility. The utility functions are $U_b = v_b p - x$ for the buyer and $U_s = x - v_s p$ where the valuations v_b, v_s are the players' private information. Consider for simplicity that both players' valuations are distributed over the unit interval $[0, 1]$ according to some probability distribution.

One may map this setup to our two-issue framework, with continuum of types, where the first issue is the probability of trade, i.e., p, and the second issue is the amount of transfer, i.e., x. It is clear from the utility functions that agents preferences over the individual issues are diametrically opposed. That is, for any fixed value of x, the buyer gets better off as p decreases from 1 to 0 and the seller gets worse off as p decreases from 1 to 0. Similarly, for any fixed value of p, the buyer gets better off as x decreases from 1 to 0 and the seller gets worse off as x decreases from 1 to 0. In addition to this, it is easy to verify that the preferences over the bundles satisfy logrolling.

In our setup, each issue has separate outside option whereas MS assumes joint outside option for the issues (no trade). However, this is not directly the main driving force for the difference between these two papers. Aside from this divergence of these two models, MS corresponds to our symmetric two issue case. Although the utility of the outside option in MS in each issue is 0, the ranking of the outside option is the negotiators' private information. Namely, the set of acceptable alternatives for the agents' is their private information. This is true because for the buyer, for example, the set of acceptable alternatives in issue p must satisfy $p \geq \frac{x}{v_b}$ for any fixed value of x, and this set is the buyer's private information as v_b is not common knowledge. Therefore, the set of acceptable outcomes (or the ranking of the outside option in each individual issue) are the players' private information, as it is the case in our symmetric case. We show, in the symmetric treatment of the outside option, that there is no individually rational and ex-post efficient strategy-proof mediation rules. We prove this point next.

Symmetric Treatment of the Outside Options

In this section, we relax the assumption that $y\ \theta_i^Y o_Y$ for all $i \in \mathrm{I}$ and $y \in Y \setminus \{o_Y\}$. Instead, the negotiators' ranking of the outside option, o_Y, is their private information, as is the case for issue X. Thus, $\Theta_i = \Theta_i^X \times \Theta_i^Y$ denotes the set of all **types** of negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ is the set of all type profiles. We also relax our assumption of consistency for the negotiators' ranking over the bundles, and suppose that they satisfy monotonicity, i.e., condition (i) of Definition 1, and the following modification of condition (ii). We need a modified version of the second condition of Definition 1 because now both issues X and Y have unacceptable alternatives.

Definition 6. *Under the symmetric treatment of the outside options, the extension map Λ satisfies for all i, $\theta_i \in \Theta_i$ and all $R_i \in \Lambda(\theta_i)$:*

i. [Monotonicity] For any $x, x' \in X$ and $y, y' \in Y$ with $(x, y) \neq (x', y')$,

$$(x, y)\ P_i\ (x', y') \text{ whenever } \left[x\ \theta_i^X\ x' \text{ or } x = x'\right] \text{ and } \left[y\ \theta_i^Y\ y' \text{ or } y = y'\right].$$

ii. [DB] $(o_X, o_Y)\ P_i\ (x, y)$ whenever $o_X\ \theta_i^X\ x$ or $o_Y\ \theta_i^Y\ y$.

Proposition 2. *Under the symmetric treatment of the outside options, there is no mediation rule f that is strategy-proof, individually rational and efficient.*

Note that this impossibility result can easily be carried out to a single issue or more than two issue contexts. A rule that always picks the pair (o_X, o_Y) is strategy-proof but not efficient. A dictatorship is efficient and strategy-proof but not individually rational.

Proof of Proposition 2: Consider the (true) preference profile $(\theta_1, \theta_2) = (\theta_1^{x_m}, \theta_1^{y_m}, \theta_2^{x_1}, \theta_2^{y_1})$. That is, both negotiators find all alternatives acceptable. Let $(x, y) = f(\theta_1, \theta_2)$. Because negotiators preferences over alternatives are diametrically opposed for each single issue, there is at least one negotiator $i \in \mathrm{I}$ and an issue for which negotiator i does not get her top alternative for that issue. Suppose, without loss of generality, that this negotiator is 1 and the issue is X: that is, $x \neq x_1$. Consider the new profile where only negotiator 1's preferences are different, $(\theta_1', \theta_2) = (\theta_1^{x_1}, \theta_1^{y_1}, \theta_2^{x_1}, \theta_2^{y_1})$.

We claim that $f(\theta_1', \theta_2) = (x_1, y_1)$. Suppose for a contradiction that $f(\theta_1', \theta_2) = (x', y') \neq (x_1, y_1)$. I will only show that $x' = x_1$ because similar arguments also prove $y' = y_1$, yielding the desired contradiction. To show $x' = x_1$, suppose for a contradiction that $o_X\ \theta_1^{x_1}\ x'$. Since Λ satisfies DB, $(o_X, o_Y)\ P_1\ (x', y')$ for all $R_1 \in \Lambda(\theta_1')$, and thus $f(\theta_1', \theta_2) = (x', y')$ contradicts with the individual rationality of f. Now suppose for a contradiction that $x' = o_X$. Then, since Λ satisfies Monotonicity, $(x_1, y')\ P_i\ (x', y')$ for $i = 1, 2$ and all $R_1 \in \Lambda(\theta_1^{x_1})$ and all $R_2 \in \Lambda(\theta_2^{x_1})$. Therefore, (x', y') is an inefficient bundle at (θ_1', θ_2), and thus $f(\theta_1', \theta_2) = (x', y')$ contradicts with the efficiency of f. Hence, we must have $x' = x_1$.

To conclude, we already know that $f(\theta_1, \theta_2) = (x, y)$ and $x \neq x_1$, which implies $x_1\ \theta_1^{xm}\ x$. Because y_1 is negotiator 1's best alternative in issue Y, either $y = y_1$ or $y_1\ \theta_1^{y_1}\ y$ is true. In either case, Monotonicity and transitivity of preferences imply $(x_1, y_1)\ P_1\ (x, y)$ for all $R_1 \in \Lambda(\theta_1)$. Finally, we showed in the previous paragraph that by misrepresenting his preferences at profile (θ_1, θ_2), negotiator 1 can achieve the bundle (x_1, y_1), which is strictly better than (x, y) for all $R_1 \in \Lambda(\theta_1)$, contradicting that f is strategy-proof.

Appendix

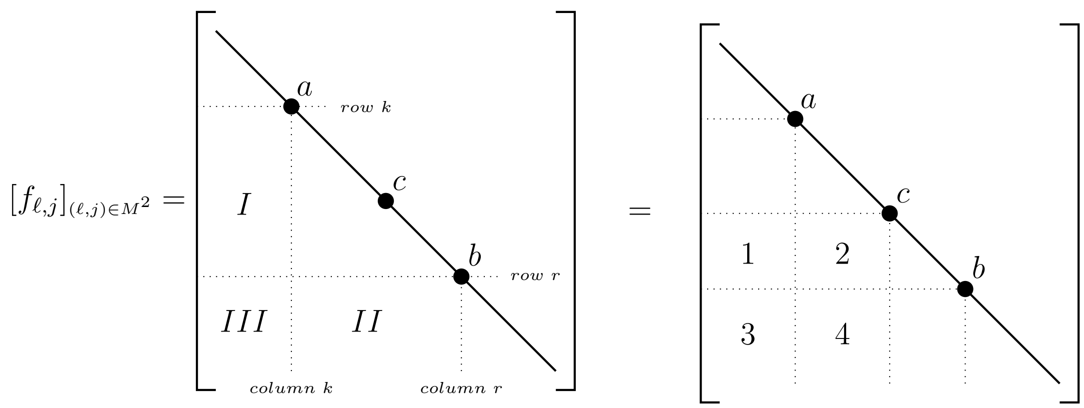
Proof of Theorem 1: *Proof of 'if' part*:

It is relatively easy to verify that an adjacent rule f is individually rational: It never suggests an alternative for an issue that is worse than the outside option of that issue, and thus, it is individually rational by the consistency of preferences. To show efficiency, consider the type profile where both negotiators deem all alternatives acceptable in issue X. At that profile, an adjacent rule proposes a bundle from the set of logrolling bundles $\mathbf{B}$. Let us call this bundle as b. If instead the negotiators receive another bundle from $\mathbf{B}$ at that profile, one of the negotiators will certainly get worse off. The reason for this is the fact that for any two logrolling bundles $a, b \in \mathbf{B}$, if $a\ R_1\ b$ then $b\ R_2\ a$ for all consistent preferences R_1, R_2. If the negotiators receive a bundle with the outside option in issue X, then both negotiators get worse off because of the deal-breakers assumption. Finally, if the negotiators receive any other bundle, say c, which is neither a logrolling bundle nor a bundle with the outside option in issue X, then the consistency assumption puts no restriction on how negotiators compare bundle b with c. Therefore, there exists at least one negotiator, i, and a consistent preference ordering, R_i, such that $b\ P_i\ c$. That is, the bundle c makes negotiator i worse off at some consistent preference ordering.

Thus, no other bundle would make one negotiator better off without hurting the other when both of the negotiators deem all alternatives acceptable. We can directly apply the same logic to all type profiles that the negotiators deem less alternatives acceptable. Finally, for those type profiles where there is no mutually acceptable alternative in issue X, in which case the rule suggests (o_X, y) for some $y \in Y \setminus \{o_Y\}$, any other bundle will include an alternative that is unacceptable in issue X by at least one of the negotiators because their preferences over each individual issue are diametrically opposed. Thus, by monotonicity and deal-breakers assumptions, at least one negotiator gets worse off if f proposes something other than (o_X, y). Hence, the adjacent rule f is efficient.

We next prove that adjacent rules are strategy-proof. Before going through the details of the proof, we first need to establish some facts about the structure of the adjacent rules defined in Theorem 1. We start with relevant jargon. Let $a = f_{\ell,j}$ and $b = f_{r,s}$ be two bundles, namely bundle a appears on row ℓ and column j whereas bundle b appears on row r and column s. We say bundle a appears above (below) bundle b whenever $\ell < r$ ($\ell > r$). Likewise, bundle a appears the right (left) of bundle b whenever $j > s$ ($j < s$).

Given a mediation rule f and a bundle a that appears on the main diagonal, i.e., $a = f_{k,k}$ for some $k \in M$, define $V(a)$ to be the **value region of bundle** a, which is the sub-matrix of $[f_{\ell,j}]_{(\ell,j)\in M^2}$ excluding all the rows lower than row k and all the columns higher than column k. Namely, $V(a) = [f_{\ell,j}]_{(\ell,j)\in(M^k, M_k)}$ where $M^k = \{k, ..., m\}$ and $M_k = \{1, ..., k\}$. Furthermore, if bundle $b = f_{r,r}$ appears on the main diagonal with $r \in M$ and $r > k$, then $V(a) \cap V(b) = [f_{\ell,j}]_{(\ell,j)\in(M^r, M_k)}$ where $M^r = \{r, ..., m\}$. In the following figure, the value region of bundle a is region I and III, value region of bundle b, $V(b)$, is region II and III, and $V(a) \cap V(b)$ is region III.



Lemma 1. *If the mediation rule f is an adjacent rule that is described in Theorem 1, then for any two bundles $a, b \in \mathbf{B}$*

(*i*) *a never appears outside of its value region* $V(a)$,

(*ii*) *a and b both never appear in* $V(a) \cap V(b)$, *and*

(*iii*) *if both a and b appear on the same column (or row), where a is above b (or a is on the left of b), then on the main diagonal, bundle a appears above bundle b.*

Proof. The first claim directly follows from the last two conditions of Theorem 1. The existence of complete, transitive and strict order $\rhd$ on $\mathbf{B}$ implies the second claim and deserves a proof. Suppose first that a and b appear on the same column in region III, say column s, and a is located above bundle b on this column, namely a is on row r_a and b is on row r_b where $r \leq r_a < r_b \leq m$. Starting from column and row r, i.e., from bundle b, as we move from column r to column s along the row r, adjacency and transitivity of $\rhd$ imply that the bundles on the row r are either ranked higher than b (with respect to $\rhd$) or equal to b, which includes the bundle $f_{r,s}$. Now starting from column s and row r, i.e., the bundle $f_{r,s}$, and move towards row r_a along column s. Adjacency and transitivity of $\rhd$ imply that the bundle on the row r_a and column s, i.e., the bundle a is ranked higher than b with respect to $\rhd$. Namely, $a \rhd b$ must hold.

Continue iterating from where we left. Starting from column s and row r_a, i.e., the bundle a, as we move from row r_a to r_b along the column s, adjacency and transitivity of $\rhd$ imply that all the bundles are either ranked above a or equal to a, including the bundle at row r_b, i.e., b. Thus, we must have $b \rhd a$, contradicting with the fact that $\rhd$ is strict. If bundle b is above bundle a on column s, then we start the iteration from $f_{k,k} = a$. Therefore, a and b cannot appear on the same column in region III. Symmetric arguments suffice to show that they cannot appear on the same row in region III either.

Therefore, suppose that a and b appear on different rows and columns. With similar arguments above, if we start iteration from $f_{r,r} = b$ and go left on the same row and then go down to bundle a in region III, we conclude that $a \rhd b$ by adjacency and transitivity of $\rhd$. However, when we start iteration from $f_{k,k} = a$ and go down on the same column and then go left to bundle b in region III, we conclude that $b \rhd a$, which yields the desired contradiction. Hence, either bundle a or b, whichever is ranked first with respect to $\rhd$, may appear in region III, but not both.

The proof of condition (*iii*) uses (*ii*). Suppose for a contradiction that a and b appear on the same column s, where b is above a (i.e., $r_b < r_a$) and a appears above b on the main diagonal. If we refer back to the previous figure, a and b can appear on the same column with $r_b < r_a$ only in region III, which contradicts with what we just proved above. We can make symmetric arguments for rows as well. □

We now ready to show that an adjacent rule $f = [f_{\ell,j}]_{(\ell,j) \in M^2}$ is strategy-proof. Consider, without loss of generality, deviations of negotiator 1 only. If $\ell < j$, then

$A(\theta_1^{x_\ell}, \theta_2^{x_j}) = \emptyset$. Negotiator 1 may receive a different bundle by deviating to a type that is represented by a higher (numbered) row, say $\theta_1^{x_k}$ where $k > \ell$. $A(\theta_2^{x_j})$ is fixed because negotiator 2's type is fixed. Because the negotiators' preferences over issue X are diametrically opposed and f is individually rational, the alternative in issue X at type profile $(\theta_1^{x_k}, \theta_2^{x_j})$ will be unacceptable for negotiator 1's true type, $\theta_1^{x_\ell}$. Thus, by deal-breakers property, negotiator 1 has no profitable deviation from a type profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ with $\ell < j$.

On the other hand, if $\ell = j$, then negotiator 1 can deviate to (1) a lower row and receive (o_X, y), which is worse than $f_{\ell,i} = (x_\ell, y_{m-\ell+1})$ by deal-breakers, or (ii) a higher row and receive a bundle that suggests an unacceptable alternative in issue X. Thus, deal-breakers property imply that negotiator 1 has no profitable deviation in that case either.

Finally, suppose that $\ell > j$. Let $c \in \mathbf{B}$ denote the bundle negotiator 1 gets if he truthfully reports his type. If negotiator 1 deviates to a row where f takes the value (o_X, y), then he clearly get worse off by deal-breakers property. If he deviates to a lower numbered row and receives bundle, say, a, then a appears on the first diagonal above bundle c, by the third condition of Lemma 1. The last observation and logrolling property of the preferences imply that negotiator 1 prefers bundle c to a at all consistent preferences. Hence, there is no profitable deviation for negotiator 1 by declaring a lower numbered row. However, if he declares a higher numbered row and gets a different bundle, say, b, then c appears on the first diagonal above bundle b, again by the third condition of Lemma 1. As it is clearly visible in the last figure, Lemma 1 implies that negotiator 1's true preferences must give him the bundle c in region 1 or 2 and the deviation bundle b must be in region 3 or 4 because they cannot coexists in region 3 or 4. However, bundle b includes alternative x_r from issue X, which is an unacceptable alternative for all types that lie above row r, including negotiator 1's true type. Thus, by deal-breakers property, negotiator 1 has no profitable deviation in that case either. Hence, f is strategy-proof.

***Proof of 'only if'*:**

***Proof of Part i*:** By individual rationality and consistency of preferences, the alternative for issue X must be o_X whenever $\ell < j$. Then by efficiency, $f_{\ell,j} = (o_X, y)$ for some $y \in Y \setminus \{o_Y\}$. By strategy-proofness and monotonicity, we must have $f_{\ell',j} = (o_X, y)$ for all $\ell' < j$. Similarly, $f_{\ell,j'} = (o_X, y)$ for all $\ell < j'$. Fixing j (and ℓ) and applying the same argument for all remaining rows and columns yield $f_{\ell,j} = (o_X, y)$ whenever $\ell < j$.

***Proof of Part ii*:** Consider the main diagonal where $\ell = j = k$. Row and column k correspond to preference profile $(\theta_1^{x_k}, \theta_2^{x_k})$ where the only mutually acceptable alternative in issue X is x_k. Therefore, by efficiency, individual rationality and consistency of preferences, $f_{k,k}^X = x_k$. Now, we will show that $f_{k,k} = (x_k, \hat{y}_k)$ for every $k = 1, ..., m$ and

$\hat{y}_k = y_{m+1-k}$. We start from $k=1$. If $f^Y_{1,1} = y \neq \hat{y}_1$, then $f_{2,1}$ must also be (x_1, y) by strategy-proofness. This is true because (1) negotiator 1 is not able to unambiguously rank (x_1, y) against any other bundle (x', y') with $x' \neq x_1$ and $y' \; \theta^Y_1 \; y$, and (2) negotiator 1 deviates from $\theta^{x_2}_1$ if $y \; \theta^Y_1 \; y'$ by monotonicity. With exactly the symmetric arguments, we must have $f_{2,2} = (x_1, y)$. However, $f^X_{2,2} = x_1$ contradicts with individual rationality and efficiency of f. Hence, we must have $f_{1,1} = (x_1, \hat{y}_1)$.

By induction, suppose that the claim is true for all entries on the main diagonal up to $k-1$. We now show that it must also hold for k. Suppose for a contradiction that $f^Y_{k,k} = y \neq \hat{y}_k$. Then $f_{k,k-1}$ must also be (x_k, y) by strategy-proofness. This is true because (1) by efficiency and individual rationality $f^X_{k,k-1} \notin \{x_{k+1}, ..., x_m, o_X\}$, (2) negotiator 2 is not able to unambiguously rank (x_k, y) against any other bundle (x', y') with $x_k \; \theta_2(x_k) \; x'$ and $y' \; \theta^Y_2 \; y$ and (3) negotiator 2 deviates from $\theta^{x_k}_2$ if $y \; \theta^Y_2 \; y'$ by monotonicity. With symmetric arguments, we must have $f_{k-1,k-1} = (x_k, y)$, contradicting with our induction hypothesis that $f_{k-1,k-1} = (x_{k-1}, \hat{y}_{k-1})$.

***Proof of Part iii*:** We refer to bundles $\{f_{k,1}, f_{k+1,2}, ..., f_{m,m-k+1}\}$ where $k = 1, ..., m$ as those on the $k-$th diagonal. Note that each diagonal has one less bundle than its immediate predecessor and the $m-$th diagonal consists of a single bundle, namely $f_{m,1}$.

Lemma 2. *Suppose that adjacency holds for all bundles on all diagonals $t = 2, ..., k$ where $k \leq m$. That is, for all $t \in \{2, ..., k\}$ and $m \geq \ell > j$ with $\ell = j + t - 1$, $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$. Consider two bundles $a, b \in \mathbf{B}$ that appear on some diagonal $t \in \{2, ..., k\}$. If bundle a lies on a higher row than b on the first diagonal, then a also lies on a higher row than b on all diagonals up to (and including) diagonal t.*

Proof. Since both a and b appear on diagonal t, by adjacency, they both must also appear on every diagonal from the second through $(t-1)-$st diagonal. Suppose that a lies above b on the first diagonal. From the first diagonal to the second, adjacency implies that a bundle can either move by one cell horizontally to the left or drop by one cell down. If a moves horizontally, clearly it will remain above b on the second diagonal. If a drops by one cell, it remains above b or on the same row with b (which happens when a and b are diagonally adjacent on the first diagonal). In the former case, b is clearly below a on the second diagonal. In the latter case, for b to also appear on the second diagonal it must also have dropped one cell below, in which case it is again below a on the second diagonal. Iterating this argument for rows 3 through t yields the desired result. □

STEP 1 (Adjacency): We first show the following: Take a bundle on some diagonal except the first one. This bundle is equal to the bundle immediately above it or immediately to its right. Lemma 3 states this more formally.

Lemma 3. *For all $m \geq \ell > j$, $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$.*

Proof. Part (ii) of Theorem 1 proves that the set of bundles on the first diagonal is equal to the set of logrolling bundles, $\mathbf{B}$. We first prove our claim for the second diagonal. That is, take any $m \geq \ell > j$ where $\ell = j+1$ we have $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$. Suppose for a contradiction that this is not true for some such ℓ and j. That is, $f_{\ell,j} \notin \{f_{\ell,j+1}, f_{\ell-1,j}\} = \{(x_\ell, \hat{y}_\ell), (x_{\ell-1}, \hat{y}_{\ell-1})\}$. Note that row ℓ and column j correspond to the profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ and by efficiency and individual rationality $f^X_{\ell,j} \in \{x_\ell, x_{\ell-1}\}$.

If $f_{\ell,j} = (x_\ell, y)$ where $y\ \theta_2^Y\ \hat{y}_\ell$, then negotiator 2 unambiguously prefers $f_{\ell,j}$ to $f_{\ell,j+1} = (x_\ell, \hat{y}_\ell)$ by monotonicity, and so deviates from $\theta_2^{x_{j+1}}$, contradicting strategy-proofness. If $f_{\ell,j} = (x_\ell, y)$ where $\hat{y}_\ell\ \theta_2^Y\ y$, then negotiator 2 unambiguously prefers $f_{\ell,j+1}$ to $f_{\ell,j}$ by monotonicity, and so deviates from $\theta_2^{x_j}$, contradicting strategy-proofness. Similarly, if $f_{\ell,j} = (x_{\ell-1}, y)$ where $y\ \theta_1^Y\ \hat{y}_{\ell-1}$, then player 1 unambiguously prefers $f_{\ell,j}$ to $f_{\ell-1,j} = (x_{\ell-1}, \hat{y}_{\ell-1})$ by monotonicity, and so deviates from $\theta_1^{x_{\ell-1}}$, contradicting strategy-proofness. Finally, if $f_{\ell,j} = (x_{\ell-1}, y)$ where $\hat{y}_{\ell-1}\ \theta_1^Y\ y$, then negotiator 1 unambiguously prefers $f_{\ell-1,j}$ to $f_{\ell,j}$, and so deviates from $\theta_1^{x_\ell}$, contradicting strategy-proofness. Hence, our claim holds for the second diagonal.

Now by induction, suppose that our claim holds for all diagonals up to k and show that the claim also holds for diagonal $k+1$. That is, for any $m \geq \ell > j$ where $\ell = j+k$ we have $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$. Once again, suppose for a contradiction that for some ℓ, j with $m \geq \ell > j$ and $\ell = j+k+1$, $f_{\ell,j} \notin \{f_{\ell-1,j}, f_{\ell,j+1}\}$. There are two exhaustive cases that we need to consider:

Case 1: Suppose that $f_{\ell,j} \in \mathbf{B}$. First note that by efficiency and individual rationality of f, all three bundles, $f_{\ell-1,j}$, $f_{\ell,j+1}$ and $f_{\ell,j}$, are unambiguously ranked over (o_X, o_Y) for both negotiators at profile $(\theta_1^{x_\ell}, \theta_j^{x_j})$. This is true because both agents are more accepting at this profile. Next we claim that $f_{\ell-1,j} \neq f_{\ell,j+1}$. Suppose not. By strategy-proofness $f_{\ell,j}\ R_1\ f_{\ell-1,j}$ for all consistent R_1 in $\Lambda(\theta_1^{x_\ell})$. Because the negotiators' preferences over $\mathbf{B}$ are diametrically opposed, then $f_{\ell-1,j} = f_{\ell,j+1}\ R_2\ f_{\ell,j}$ for all consistent R_2 in $\Lambda(\theta_2^{x_j})$. Therefore, there exists a consistent R_2 such that $f_{\ell,j+1}\ P_2\ f_{\ell,j}$, and thus negotiator 2 prefers to deviate from $\theta_2^{x_j}$, contradicting with strategy-proofness of f.

Thus, we have $f_{\ell,j} \notin \{f_{\ell-1,j}, f_{\ell,j+1}\}$ and $f_{\ell-1,j} \neq f_{\ell,j+1}$. By Lemma 2, we have $f_{\ell,j+1}\ R_1\ f_{\ell-1,j}$ for all R_1 consistent with $\Lambda(\theta_1^{x_\ell})$ because the former is below the latter in the $k-$th diagonal. There are three exhaustive subcases that we need to consider regarding how agent 1 ranks $f_{\ell,j}$ relative to $f_{\ell-1,j}$ and $f_{\ell,j+1}$:

Case 1A: Suppose that $f_{\ell,j}\ R_1\ f_{\ell,j+1}\ R_1\ f_{\ell-1,j}$ for all consistent R_1 in $\Lambda(\theta_1^{x_\ell})$. Because negotiators' preferences over $\mathbf{B}$ are diametrically opposed, there is a consistent ordering R_2 for negotiator 2 such that $f_{\ell,j+1}\ P_2\ f_{\ell,j}$. Thus, negotiator 2 deviates from $\theta_2^{x_j}$, contradicting with strategy-proofness of f.

Case 1B: Suppose that $f_{\ell,j+1}\ R_1\ f_{\ell-1,j}\ R_1\ f_{\ell,j}$ for all consistent R_1 in $\Lambda(\theta_1^{x_\ell})$. Then, there exists a consistent ordering R_1 such that $f_{\ell-1,j}\ P_1\ f_{\ell,j}$, and so negotiator 1 deviates from $\theta_1^{x_\ell}$, contradicting with strategy-proofness of f.

Case 1C: Suppose that $f_{\ell,j+1}\ R_1\ f_{\ell,j}\ R_1\ f_{\ell-1,j}$ for all consistent R_1 in $\Lambda(\theta_1^{x_\ell})$. Because these three bundles are in $\mathbf{B}$ and different from one another, we must have $\left(f_{\ell-1,j}^X = x_s\right)\ \theta_1^X\ \left(x_r = f_{\ell,j}^X\right)\ \theta_1^X\ \left(x_h = f_{\ell,j+1}^X\right)$ for some $s < r < h$. Acceptable alternatives in issue X for type $\theta_1^{x_{\ell-1}}$ of negotiator 1 is $\{x_1, ..., x_{\ell-1}\}$ and by individual rationality x_s is in this set, i.e., $s \leq \ell - 1$. Similarly, acceptable alternatives for type $\theta_1^{x_\ell}$ of negotiator 1 is $\{x_1, ..., x_\ell\}$ and by individual rationality both x_h and x_r are in this set, i.e., $r < h \leq \ell$. The last inequality implies that $r \leq \ell - 1$, and thus x_r is an acceptable alternative for type $\theta_1^{x_{\ell-1}}$. Namely, $f_{\ell,j}$ is acceptable for type $\theta_1^{x_{\ell-1}}$. Therefore, there exists a consistent preference R_1 in $\Lambda(\theta_1^{x_{\ell-1}})$ in which $f_{\ell,j}\ P_1\ f_{\ell-1,j}$. Thus, negotiator 1 deviates from $\theta_1^{x_{\ell-1}}$, contradicting with strategy-proofness of f.

Case 2: Suppose now that $f_{\ell,j} \notin \mathbf{B}$. For notational simplicity, for any $z, z' \in Z \in \{X, Y\}$ and $i \in$ I, we denote $z \succeq_i^Z z'$ whenever $z\ \theta_i^Z\ z'$ or $z = z'$. Because the bundle $f_{\ell-1,j}$ and $f_{\ell,j+1}$ are lying on diagonal k and $f_{\ell-1,j}$ is above $f_{\ell,j+1}$, we have $f_{\ell,j+1}\ R_1\ f_{\ell-1,j}$ for all consistent R_1 in $\Lambda(\theta_1^{x_\ell})$ by Lemma 2. Because these two bundles are in $\mathbf{B}$, we must have $\left(f_{\ell-1,j}^X = x_s\right)\ \theta_1^X\ \left(x_h = f_{\ell,j+1}^X\right)$ for some $1 \leq s \leq h \leq m$. Strategy-proofness of f implies that $f_{\ell,j}$ and $f_{\ell-1,j}$ are unambiguously ranked and $f_{\ell,j}\ R_1\ f_{\ell-1,j}$ at all consistent R_1 in $\Lambda(\theta_1^{x_\ell})$. Because $f_{\ell,j} \notin \mathbf{B}$, $f_{\ell,j}$ is at least as good as $f_{\ell-1,j}$ at all consistent preferences whenever $\left(f_{\ell,j}^X = x_r\right) \succeq_1^X x_s$ and $f_{\ell,j}^Y \succeq_1^Y f_{\ell-1,j}^Y$, by monotonicity. Thus, we have $r \leq s$.

Similarly, strategy-proofness of f implies $f_{\ell,j}$ and $f_{\ell,j+1}$ are unambiguously ranked and $f_{\ell,j}\ R_2\ f_{\ell,j+1}$ for all consistent R_2 in $\Lambda(\theta_2^{x_j})$, and thus $x_r \succeq_2^X x_h$ and $f_{\ell,j}^Y \succeq_2^Y f_{\ell,j+1}^Y$ by monotonicity. Thus, $h \leq r$. These three conditions, $s \leq h$, $h \leq r$, and $r \leq s$, hold simultaneously if and only if $s = h = r$, implying that $f_{\ell,j}^X = f_{\ell-1,j}^X = f_{\ell,j+1}^X$. The last condition requires $f_{\ell,j+1} = f_{\ell-1,j}$ because they both are in $\mathbf{B}$. Let $f_{\ell,j+1} = f_{\ell-1,j} = (x, y)$. On the other hand, because the negotiators' preferences on issue Y are diametrically opposed, $f_{\ell,j}^Y \succeq_1^Y y$ and $f_{\ell,j}^Y \succeq_2^Y y$ hold simultaneously if and only if $f_{\ell,j}^Y = y$ or $f_{\ell,j}^Y = o_Y$. The second case is not possible because f is efficient. Therefore, we have $f_{\ell,j} = (x, y)$, contradicting with the assumption that $f_{\ell,j} \notin \mathbf{B}$. □

STEP 2 (Construction of a precedence order $\rhd$): By step 1, we know that $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ for all $\ell > j$. To construct $\rhd$, perform a pairwise comparison for all the entries $f_{\ell,j}, f_{\ell-1,j}, f_{\ell,j+1}$. More formally, $f_{\ell-1,j} \vartriangleright f_{\ell,j+1}$ whenever $f_{\ell,j} = f_{\ell-1,j}$ and $f_{\ell,j+1} \vartriangleright f_{\ell-1,j}$ whenever $f_{\ell,j} = f_{\ell,j+1}$. We obtain a partial order $\vartriangleright$ on $\mathbf{B}$, which may not be complete at this point. Next, we will show that $\vartriangleright$ is asymmetric and transitive.

Lemma 4. *Order $\vartriangleright$ is asymmetric. That is, if $a \vartriangleright b$ for any pair $a, b \in \mathbf{B}$, then $\neg b \vartriangleright a$.*

Proof. Suppose for a contradiction that there is $a, b \in \mathbf{B}$ such that both $a \vartriangleright b$ and $b \vartriangleright a$. Let $t \geq 1$ be the smallest diagonal on which a and b are diagonally adjacent and a is "chosen" according to $\vartriangleright$. That is, let $f_{\ell-1,j} = a$, $f_{\ell,j+1} = b$, and so $f_{\ell,j} = a$. By lemma 2, bundle a must appear above bundle b at all diagonals, including the first one, which means $b \ R_1 \ a$ for all consistent $R_1 \in \Lambda(\theta_1^{x_\ell})$. Furthermore, because f is efficient, individually rational and taking values a and b when negotiator 1 announces his type as $\theta_1^{x_\ell}$, both bundles must be acceptable for negotiator 1 of types $\theta_1^{x_k}$ where $k \geq \ell$. Because $b \vartriangleright a$ by assumption, there must exist another diagonal $t' > t$ in which a and b are diagonally adjacent and b is chosen. By Lemma 2, bundles a and b cannot be adjacent to one another more than once on the same diagonal, and thus $t' > t$. Therefore, let $f_{s-1,r} = a$, $f_{s,r+1} = b = f_{s,r}$. By Lemma 2 and Lemma 3, we have $s > \ell$. Given the previous arguments, we know that negotiator 1 unambiguously ranks bundle b over a and both bundles are acceptable for type $\theta_1^{x_{s-1}}$, where negotiator 1 receives bundle a. Therefore, negotiator 1 would profitably deviate from $\theta_1^{x_{s-1}}$ to $\theta_1^{x_s}$ and get b, given that negotiator 2 is of type $\theta_2^{x_r}$, contradicting strategy-proofness of f. □

Let two bundles a and b be diagonally adjacent. If a lies on a higher row than b, the we say that a is *diagonally adjacent to b from below*. Equivalently, we say that b is *diagonally adjacent to a from above*.

Lemma 5. (*i*) *Let bundle $a = f_{\ell,j} \in \mathbf{B}$ be diagonally adjacent to some bundle $b \in \mathbf{B}$ from below and $a \vartriangleright b$. Then, bundle b never appears on or below row ℓ, i.e., there is no $k \geq \ell$ and r such that $f_{k,r} = b$. Additionally, bundle a never appears (strictly) above row ℓ and (strictly) to the left of column j, i.e., there is no $\ell' < \ell$ and $j' < j$ such that $f_{\ell',j'} = a$.*

(*ii*) *Let bundle $c = f_{\ell,j} \in \mathbf{B}$ be diagonally adjacent to some bundle $d \in \mathbf{B}$ from above and $c \vartriangleright d$. Then, bundle d never appears on column j or any lower column, i.e., there is no $k \leq j$ and r such that $f_{r,k} = d$. Additionally, bundle c never appears (strictly) below row ℓ and (strictly) to the right of column j, i.e., there is no $\ell' > \ell$ and $j' > j$ such that $f_{\ell',j'} = c$.*

Proof. We prove the first part, i.e., (*i*), as symmetric arguments will suffice to prove part (*ii*). First part of (*i*): The bundle b must be above a on the first diagonal because b

is above a at some diagonal. Thus, by logrolling, $b\ R_2\ a$ for all types of negotiator 2 that deem both bundles acceptable. Moreover, negotiator 2 may receive bundles a and b (depending on negotiator 1's type) when he declares his type as $\theta_2^{x_{j-1}}$, and so by efficiency and individual rationality of f, both these bundles must be acceptable for type $\theta_2^{x_{j-1}}$ of negotiator 2. Suppose for a contradiction that b occurs below row ℓ. By the adjacency property, this b should be coming all the way from the main diagonal, and so b must also appear on row ℓ. Let $f_{\ell,k} = b$ for some $k \neq j, j-1$. But then at some consistent preference ordering in which $b\ P_2\ a$, negotiator 2 would deviate from $\theta_2^{x_{j-1}}$ to $\theta_2^{x_k}$ to get the bundle b, contradicting strategy-proofness of f.

Second part of (i): Because b is above a, then logrolling implies $a\ R_1\ b$ for all types of negotiator 1 that deem both bundles acceptable. Because $f_{\ell-1,j-1} = b$, the bundle b must appear on the first diagonal on column j or higher. Because a is below b on main diagonal as well, it also can appear on the main diagonal on column $j+1$ or higher. Therefore, if bundle a appears in the region, for a contradiction, then by the adjacency property bundle a must appear on column $j+1$ as well. Let $f_{k,j+1} = a$ for some $k \leq \ell - 1$. But if a is acceptable for type $\theta_1^{x_k}$ of negotiator 1, it must also be acceptable for type $\theta_1^{x_{\ell-1}}$ of negotiator 1, when he gets the bundle b. Because negotiator 1 prefers a to b, he would deviate from $\theta_1^{x_{\ell-1}}$ to $\theta_1^{x_k}$ to get the bundle a, contradicting strategy-proofness of f. □

Lemma 6. *Order $\triangleright$ is transitive. That is, for any triple $a, b, c \in \mathbf{B}$ such that $a \triangleright b$ and $b \triangleright c$, we have $\neg c \triangleright a$.*

Proof. Suppose, for a contradiction, that $a \triangleright b$ and $b \triangleright c$, but $c \triangleright a$. Without loss of generality, suppose b is diagonally adjacent to a from above. Let $t \geq 1$ be the smallest diagonal on which a and b are adjacent where $f_{\ell,j} = a$, $f_{\ell-1,j-1} = b$ and $f_{\ell,j-1} = a$ because $a \triangleright b$. By Lemma 5 part (i), b never appears on row ℓ or below. Let t' be the smallest diagonal on which b and c are adjacent. We consider two cases:

Case 1: $t' \geq t$: This case has two subcases:

Case 1A: Suppose first that c is adjacent to b from below on diagonal t': Consider diagonal t. Clearly, c should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t' \geq t$. Then by Lemma 3, since c is adjacent to b from below on diagonal t', it must appear below b on row $\ell + 1$ or below on diagonal t. Then by Lemma 2 and adjacency, c can appear only on $\ell + 1$ or below on diagonal $t' \geq t$ as well. However, By Lemma 5 part (i), $f_{\ell,j} = a \triangleright b$ implies that b can never appear on row ℓ or below. This means b and c cannot be adjacent on diagonal $t' \geq t$; a contradiction.

Case 1B: Suppose now that c is adjacent to b from above on diagonal t'. Let $f_{p,q} = b$ and $f_{p-1,q-1} = c$. Because b never appears on row ℓ or below, $p \leq \ell - 1$. By Lemma 4, $b \triangleright c$ implies $f_{p,q-1} = b$. By Lemma 5 part (i), $b \triangleright c$ implies that c never appears on row p or below. Because b is diagonally adjacent to a from above and c is adjacent to b from above, by Lemma 3, $c \rhd a$ implies that c must be adjacent to a from above on some diagonal t''. By Lemma 3, there is no b on diagonal t'' for otherwise it would be either below a or above c. Then $t'' > t'$. Thus, let $f_{r,s} = a$ and $f_{r-1,s-1} = c$ on diagonal t'', and so $f_{r,s-1} = c$ by $c \rhd a$. Because there is no c on or below row $p \leq \ell - 1$, a and c must then be adjacent above row p on diagonal $t'' > t'$. That is, $r < p$. Then $t'' > t'$ implies that $s \leq q-2$. Because there is no c on row $p \leq \ell - 1$ or below and $t' \geq t$, $f_{r,s} = a$ lies on row above row p, i.e., $r < p$ and on column $j - 2$ or to the left, i.e., $s \leq j - 2$. However, by Lemma 5 part (i), $f_{\ell,j} = a \rhd b$ implies that bundle a should never appear in the box (strictly) above row ℓ and (strictly) to the left of column j; a contradiction.

Case 2: $t' < t$: This case also has two subcases.

Case 2A: Suppose c is adjacent to b from above on diagonal t'. Consider diagonal t'. Clearly, a should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t > t'$. Since a lies below b on diagonal t, it must again be below b on diagonal t'. Let k be the row on which b lies on diagonal t'. Clearly, a lies below row k on diagonal t' or any other diagonal $t'' > t$. Since c is adjacent to b from above on diagonal t' and $b \rhd c$, Lemma 5 part (i) implies that c never appears on row k or below. Thus, a and c cannot be diagonally adjacent on any diagonal $t'' > t'$. But they cannot be diagonally adjacent on any diagonal $t''' < t'$ either because that would mean that there is no b on diagonal t''' for otherwise b would be above c or below a, contradicting Lemma 3; a contradiction.

Case 2B: Suppose c is adjacent to b from below on diagonal t'. Consider diagonal t'. Clearly, a should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t > t'$. Because a lies below b on diagonal t, it must lie below both b and c on diagonal t'. Suppose a and c are diagonally adjacent on some diagonal t''. Let $f_{p,q} = c$ on diagonal t''. Clearly, c must lie above a on diagonal t''. Because b is diagonally adjacent to a from above on diagonal t, there is no c on diagonal t (or on any higher numbered diagonal) for otherwise c would be above b or below a on diagonal t, contradicting Lemma 2. Thus, $t'' < t$. Since $a = f_{p+1,q+1}$ and $c = f_{p,q}$ are diagonally adjacent on t'' and $c \rhd a$, Lemma 5 part (ii) implies that a never appears on column q or any lower numbered column. Since $f_{\ell,j} = a$, we need $q < j-1$. Since $t'' < t$ and $q < j-1$, bundle $a = f_{p+1,q+1}$ must lie above row ℓ. But, recall that Lemma 5 part (i) and $f_{\ell,j} = a \rhd b$ implies that a should never appear in the box (strictly) above row ℓ and (strictly) to the left of column j; a contradiction. □

Finally, we stipulate that any incomplete portions of partial order $\rhd$ are chosen in any arbitrary manner without violating transitivity. This and Lemmas 4-6 give us a complete, asymmetric and transitive order $\rhd$ satisfying statement (iii) of Theorem 1.

The Revelation Principle

We prove the revelation principle for the symmetric treatment of the outside options. The same logic applies directly to the case with heterogeneous treatment of the outside options. A mediation mechanism $\Gamma = \big(S_1, S_2, g(.)\big)$ with veto rights is a collection of strategy sets (S_1, S_2) and an outcome function $g : S_1 \times S_2 \to X \times Y$. The mechanism Γ combined with possible types (Θ_1, Θ_2) and preferences over bundles (R_1, R_2) with $R_i \in \Lambda(\theta_i)$ for all i defines a game of incomplete information. A strategy for negotiator i in the game of incomplete information created by a mechanism Γ is a function $s_i : \Theta_i \to S_i$.

Lemma 7 (Revelation Principle in Dominant Strategies). *Suppose that there exists a mechanism $\Gamma = \big(S_1, S_2, g(.)\big)$ that implements the mediation rule f in dominant strategies. Then f is strategy-proof and individually rational.*

Proof. If Γ implements f in dominant strategies, then there exists a profile of strategies $s^*(.) = \big(s_1^*(.), s_2^*(.)\big)$ such that $g\big(s^*(\theta)\big) = f(\theta)$ for all $\theta \in \Theta$, and for all $i \in I$ and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}(\theta_{-i})\big) \; R_i \; g\big(s_i'(\theta_i'), s_{-i}(\theta_{-i})\big) \tag{1}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ and all $s_i'(.), s_{-i}(.)$. Condition 1 must also hold for s^*, meaning that for all i and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big) \; R_i \; g\big(s_i^*(\theta_i'), s_{-i}^*(\theta_{-i})\big) \tag{2}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last inequality implies that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i}) \; R_i \; f(\theta_i', \theta_{-i}) \tag{3}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$.

On the other hand, because the mechanism Γ always allows negotiators to veto proposed bundle before the mediation game ends, there exists a deviation strategy $\hat{s}_i(.)$ for any strategy $s_i(.)$ such that $g(\hat{s}_i(\theta_i), s_{-i}) = \big(o_X, o_Y\big)$ for all $\theta_i \in \Theta_i$ and all $s_{-i} \in S_{-i}$. The idea is that the negotiator i plays in $\hat{s}_i(.)$ exactly the same way in $s_i(.)$ (for all θ_i's) until the ratification stage and vetoes the proposed bundle.

Therefore, if $\hat{s}_i(.)$ is such a deviation strategy for $s_i^*(.)$, then condition 1 must also hold for $\hat{s}_i(.)$, implying that for all i and $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big) \; R_i \; g\big(\hat{s}_i(\theta_i'), s_{-i}^*(\theta_{-i})\big) = \big(o_X, o_Y\big)$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last condition means that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i}) \; R_i \; \big(o_X, o_Y\big) \tag{4}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Hence, conditions 3 and 4 imply that f is strategy-proof and individually rational. □

Proof of Theorem 3:

Proof of 'if': Same arguments in the proof of Theorem 1 suffice to verify that the mediation rule described in Theorem 3 is individually rational and efficient. Lemma 1 also holds in the continuous case. The proof of part (i) of Lemma 1 is straightforward; given the location of a logrolling bundle a on the main diagonal, $f_{\ell,j}$ can be a only if $a \in \mathbf{B}_{\ell j}$, and so, a can never appear outside of its value region $V(a)$. To prove part (ii), let $f_{\ell,j} = a$ and $f_{s,r} = b$ and suppose for a contradiction that $a, b \in V(a) \cap V(b)$. Therefore, we have $a, b \in \mathbf{B}_{\ell j} \cap \mathbf{B}_{sr}$. The bundle a beats b with respect to $\rhd$ because a wins over $\mathbf{B}_{\ell j}$. Likewise, b beats a with respect to $\rhd$ because b wins over $\mathbf{B}_{sr}$. The last two observations contradict with the assumption that $\rhd$ is strict. To prove part (iii), suppose that $f_{\ell,s} = a$ and $f_{j,s} = b$ where $\ell < j$, whereas a appears below b on the main diagonal. This is possible only when $a, b \in V(a) \cap V(b)$, contradicting with the second part. Similar arguments prove the claim when bundles a and b are on the same row.

Now we prove that f is strategy-proof. It suffices to consider the deviations of one negotiator to prove that f is strategy-proof. Take any $\ell, j \in [0,1]$ such that $f(\theta_1^\ell, \theta_2^j) = f_{\ell,j} = (o_X, y)$ (see figure 5-a). Deviating from θ_1^ℓ does not benefit negotiator 1 if he deviates to θ_1^s where $s < j$ because the outcome of f will not change. However, if negotiator 1 deviates to some $s \geq j$ and get some b, we know that b is one of the logrolling bundles in $\mathbf{B}_{sj}$. However, all of the bundles in $\mathbf{B}_{sj}$ are unacceptable for type θ_1^ℓ of negotiator 1 since $\ell < s$, and so, not preferable to (o_X, y) by the deal-breaker property.

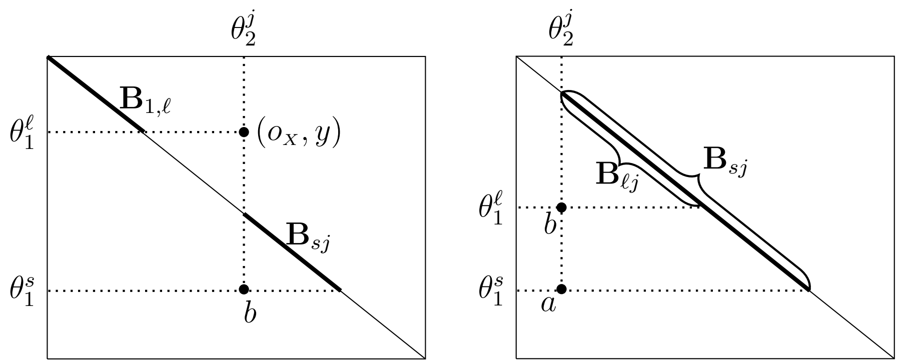


Figure 5-a　　Figure 5-b

Now take any $\ell, j \in [0,1]$ such that $\ell \geq j$ and $f(\theta_1^\ell, \theta_2^j) = f_{\ell,j} = b \in \mathbf{B}$. Deviating from

θ_1^ℓ does not benefit negotiator 1 if he deviates to θ_1^s where $s < j$ because the outcome of f would be (o_X, y), which is not better than $b \in \mathbf{B}$ by the deal-breaker property. If negotiator 1 deviates to some $\ell > s \geq j$ and get some a, then a must appear above b on the main diagonal (part (ii) of Lemma 1). Logrolling implies that negotiator 1 finds b at least as good as a at all consistent preferences, and thus, deviating to s is not profitable.

Finally, suppose that negotiator 1 deviates to some $s > \ell \geq j$ and get some a (see figure 5-b). Therefore, a beats b with respect to $\rhd$ because both a and b are in $\mathbf{B}_{sj}$ and a is chosen. Thus, a cannot be an element of $\mathbf{B}_{\ell j}$ as b is the maximizer of $\rhd$ over this set. Thus, $a \in \mathbf{B}_{sj} \setminus \mathbf{B}_{\ell j}$, implying that a is not acceptable for type θ_1^ℓ, and so, deviating to θ_1^s is not profitable by the deal-breaker property. Hence, f is strategy-proof.

Proof of 'only if': The same arguments in the proof of Theorem 1 suffices to show that there must exist some $y \in Y \setminus \{o_Y\}$ such that $f_{\ell j} = (o_X, y)$ for all $\ell, j \in [0, 1]$ with $\ell < j$. Consider now for $\ell \geq j$.

STEP 1 (Adjacency):

Lemma 8. *If f is a strategy-proof, individually rational and efficient mediation rule, then $f_{\ell,j} \in \mathbf{B}_{\ell j}$ for all $\ell \geq j$.*

Proof. First consider the case where $\ell > j$ and suppose for a contradiction that $f_{\ell,j} = (x, y) \notin \mathbf{B}$. By individual rationality we have $x \in [j, \ell]$. Moreover, strategy-proofness implies $x = j$. Suppose not, i.e., $x > j$. If $y \geq j$, then there is a consistent preference ordering of negotiator 1 such that the bundle $f_{j,j}$ is preferred to the bundle $f_{\ell,j} = (x, y)$ by monotonicity, and so type θ_1^ℓ would profitably deviate to type θ_1^j, contradicting with strategy-proofness. On the other hand, if $y < j$, then bundles $f_{j,j}$ and (x, y) are not unambiguously comparable, namely there exists a consistent preference ordering of negotiator 1 where the bundle $f_{j,j}$ is preferred to the bundle $f_{\ell,j}$ and another consistent ordering where $f_{\ell,j}$ is preferred to $f_{j,j}$. Therefore, type θ_1^ℓ would profitably deviate to θ_1^j, contradicting again with strategy-proofness. Symmetric arguments suffice to prove that $x = \ell$ because otherwise negotiator 2 would profitably deviate. The last two claims lead to the desired contradiction because we must have $x = j$ and $x = \ell$, but $\ell > j$.

Now consider the case where $j = \ell$. Similar to the previous arguments $f_{s,j} = (x, y)$ must suggest j in issue X for all $s > j$, i.e., $x = j$. This is true because otherwise negotiator 1 would profitably deviate either because of monotonicity of preferences or because of the fact that $f_{\ell,j} \notin \mathbf{B}$, and so the bundles $f_{j,j}$ and $f_{s,j}$ are not unambiguously comparable. Similarly, for all $s > \ell$, $f_{s,j}$ must suggest s in issue X, i.e., $x = s$, because otherwise negotiator 2 would profitably deviate. The last two claims lead to the desired contradiction because we must have $x = j$ and $x = s$, but $s > j$.

Finally, given that $f_{\ell,j} \in \mathbf{B}$, individual rationality requires $f_{\ell,j} \in \mathbf{B}_{\ell j}$. □

Lemma 8 and individual rationality suffice to prove that efficient, individually rational and strategy-proof mediation rule f must satisfy $f_{kk} = (k, 1-k) \in \mathbf{B}$ for all $k \in [0,1]$.

STEP 2 (Construction of a precedence order): To construct *vartriangleright*, we perform the following pairwise comparison: Let $f_{\ell,\ell} = a \in \mathbf{B}$ and $f_{j,j} = b \in \mathbf{B}$ for some $\ell, j \in [0,1]$ with $\ell > j$ and define $a \rhd b$ whenever $f_{\ell,j} = a$ and $b \rhd a$ whenever $f_{\ell,j} = b$. Lemma 9 below proves that this binary relation is not empty. Namely, there exists some such a and b where either $a \rhd b$ or $b \rhd a$.

Lemma 9. *Let the mediation rule f be strategy-proof, individually rational, efficient and $f_{\ell,j} = a \in \mathbf{B}$ where $\ell > j$. Then there exists some $k \leq j$ such that $f_{k,k} = a$ and $f_{\ell,k} = a$.*

Proof. Given that $f_{\ell,j} = a \in \mathbf{B}$ where $\ell > j$, Lemma 8 implies that $a \in \mathbf{B}_{\ell j}$, and so there is some $k \in [j, \ell]$ such that $f_{k,k} = a$. To prove the second part, suppose for a contradiction that $f_{\ell,k} = z$ where $z \neq a$. Again by Lemma 8, we know that $z \in \mathbf{B}_{\ell k}$ and so there is some $k' \in [k, \ell]$ such that $f_{k',k'} = z$. By the way the logrolling bundles are ranked by negotiator 2, $f_{k,k} = a$ is preferred to $f_{k',k'} = z$ because $k < k'$. Therefore, type θ_2^k would profitably deviate to θ_2^j to get a instead of z, contradicting with strategy-proofness. □

Because the logrolling bundles a and b can appear on the main diagonal only once, i.e., ℓ and j's are unique, the binary relation $\rhd$ is asymmetric by definition. It is not necessarily complete. The next result shows that it is transitive.

Lemma 10. *Order $\rhd$ is transitive. That is, for any triple $a, b, c \in \mathbf{B}$ such that $a \rhd b$ and $b \rhd c$, we have $\neg c \rhd a$.*

Proof. Suppose for a contradiction that there exists $a, b, c \in \mathbf{B}$ such that $a \rhd b$, $b \rhd c$ and $c \rhd a$. There are six possible cases to consider regarding how these three bundles are placed on the main diagonal and the readers can refer to figure 6 for all these cases:

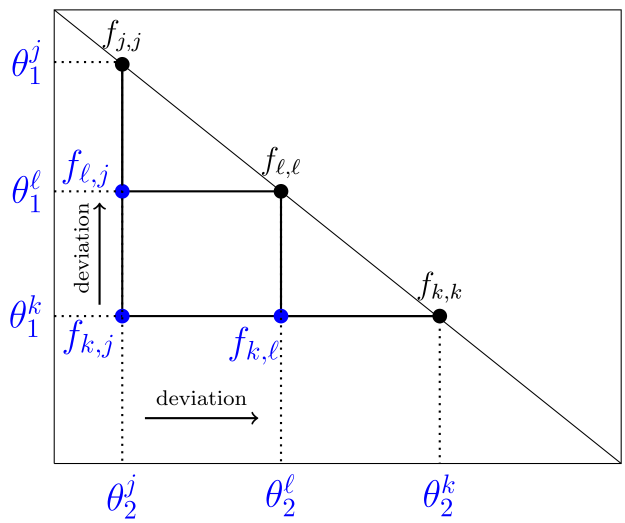


Figure 6

□

Case 1: Suppose, a appears above bundle b and b appears above bundle c on the main diagonal. Namely, $f_{j,j} = a$, $f_{\ell,\ell} = b$ and $f_{k,k} = c$. Therefore, negotiator 2 prefers a to b and b to c, and type θ_2^j finds all three bundles acceptable. Moreover, $a \rhd b$, $b \rhd c$ and $c \rhd a$ implies $f_{\ell,j} = a$, $f_{k,j} = c$ and $f_{k,\ell} = b$. Given that player 1 is of type θ_1^k, θ_2^j would profitably deviate to type θ_2^ℓ because b is more preferred than c, contradicting with strategy-proofness.

Case 2: Suppose, a appears above bundle c and c appears above bundle b on the main diagonal. Namely, $f_{j,j} = a$, $f_{\ell,\ell} = c$ and $f_{k,k} = b$. Therefore, negotiator 1 prefers b to c and c to c, and type θ_1^k finds all three bundles acceptable. Moreover, $a \rhd b$, $b \rhd c$ and $c \rhd a$ implies $f_{\ell,j} = c$, $f_{k,j} = a$ and $f_{k,\ell} = b$. Given that player 2 is of type θ_2^j, θ_1^k would profitably deviate to type θ_1^ℓ because c is more preferred than a, contradicting with strategy-proofness.

Case 3: Suppose, b appears above bundle a and a appears above bundle c on the main diagonal. Namely, $f_{j,j} = b$, $f_{\ell,\ell} = a$ and $f_{k,k} = c$. Therefore, negotiator 1 prefers c to a and a to b, and type θ_1^k finds all three bundles acceptable. Moreover, $a \rhd b$, $b \rhd c$ and $c \rhd a$ implies $f_{\ell,j} = a$, $f_{k,j} = b$ and $f_{k,\ell} = c$. Given that player 2 is of type θ_2^j, θ_1^k would profitably deviate to type θ_1^ℓ because a is more preferred than b, contradicting with strategy-proofness.

Case 4: Suppose, b appears above bundle c and c appears above bundle a on the main diagonal. Namely, $f_{j,j} = b$, $f_{\ell,\ell} = c$ and $f_{k,k} = a$. Therefore, negotiator 2 prefers b to c and c to a, and type θ_2^j finds all three bundles acceptable. Moreover, $a \rhd b$, $b \rhd c$ and $c \rhd a$ implies $f_{\ell,j} = b$, $f_{k,j} = a$ and $f_{k,\ell} = c$. Given that player 1 is of type θ_1^k, θ_2^j would profitably deviate to type θ_2^ℓ because c is more preferred than a, contradicting with strategy-proofness.

Case 5: Suppose, c appears above bundle a and a appears above bundle b on the main diagonal. Namely, $f_{j,j} = c$, $f_{\ell,\ell} = a$ and $f_{k,k} = b$. Therefore, negotiator 2 prefers c to a and a to b, and type θ_2^j finds all three bundles acceptable. Moreover, $a \rhd b$, $b \rhd c$ and $c \rhd a$ implies $f_{\ell,j} = c$, $f_{k,j} = b$ and $f_{k,\ell} = a$. Given that player 1 is of type θ_1^k, θ_2^j would profitably deviate to type θ_2^ℓ because a is more preferred than b, contradicting with strategy-proofness.

Case 6: Suppose, c appears above bundle b and b appears above bundle a on the main diagonal. Namely, $f_{j,j} = c$, $f_{\ell,\ell} = b$ and $f_{k,k} = a$. Therefore, negotiator 1 prefers a to b and b to c, and type θ_1^k finds all three bundles acceptable. Moreover, $a \rhd b$, $b \rhd c$ and $c \rhd a$ implies $f_{\ell,j} = b$, $f_{k,j} = c$ and $f_{k,\ell} = a$. Given that player 2 is of type θ_2^j, θ_1^k would profitably deviate to type θ_1^ℓ because b is more preferred than c, contradicting with strategy-proofness.

Lemma 11. *Let f be strategy-proof, efficient, individually rational and $f_{\ell,j} = a$. Then, $a \rhd b$ for all $b \in \mathbf{B}_{\ell j}$ with $b \neq a$.*

Proof. Suppose for a contradiction that there exists some $b \in \mathbf{B}_{\ell j}$ with $b \neq a$ such that $b \rhd a$. Consider the case where the bundle a is located above the bundle b on the main diagonal. Suppose that $f_{s,s} = a$ and $f_{r,r} = b$, and so $f_{r,s} = b$. Strategy-proofness and IR imply that $f_{r,j} = a$: This is true because if $f_{r,j} \in \mathbf{B}_{sr}$, then type θ_1^ℓ would profitable deviate to θ_1^r, and if $f_{r,j} \in \mathbf{B}_{js}$, then θ_1^r would deviate to θ_1^ℓ, all of which contradict with strategy-proofness. With a similar reasoning, we must have $f_{r,s} = a$ given that $f_{r,j} = a$, which contradicts with $a \neq b$: This is true because when $f_{r,s} \in \mathbf{B}_{rs} \setminus \{a\}$, then type θ_2^s would deviate to θ_2^j, contradicting with strategy-proofness.

Similar arguments will yield a contradiction when a is located below the bundle b on the main diagonal. Thus, $a \rhd b$ for all $b \in \mathbf{B}_{\ell j}$ with $b \neq a$.

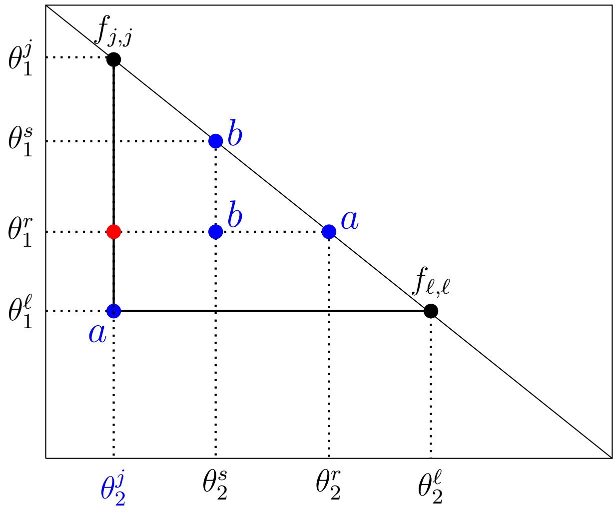


Figure 7

□

The last lemma proves that a strategy-proof, efficient and individually rational mediation rule picks the maximal element of $\rhd$ on $\mathbf{B}_{\ell j}$ for all $0 \leq \ell, j \leq 1$ with $\ell \geq j$. Namely, $f_{\ell,j} = \underset{\mathbf{B}_{\ell j}}{\text{argmax}} \rhd$ for all $\ell \geq j$. By the Sziplrajn's extension theorem, one can extend $\rhd$ to a complete ordering. This extension will clearly preserve the maximal elements in every compact subset $\mathbf{B}_{\ell j}$ as the maximal element in every set had already complete relation with all the elements in that set. Finally, Theorem 1 in Tian and Zhoub (1995) proves that quasi upper-semicontinuity is both necessary and sufficient for $\rhd$ to attain its maximum on all compact subsets $\mathbf{B}_{\ell j}$, which completes the proof.

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