Fair Matching under Constraints: Theory and Applications

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Daycare allocation as a matching problem

Daycare seats over-demand in Japan

- Matching mechanisms are used for daycare allocation.
- Policy experiments to improve matching mechanisms.
- Teacher-child ratio (& space-child ratio) varies with age (Okumura, 2016)

→not standard “capacity”; instead, **matching with constraints**

Markets with constraints

- Many other matching markets are subject to **constraints** too
  - Affirmative action (diversity constraints)
  - Gender composition in workplace
  - More real-life examples (later)
- **Question: Desirable outcomes & mechanisms?**

Main Results

- Stable matching does not always exist
- **Fair matchings** are characterized via fixed points of a function
- Necessary and sufficient condition for existence of a **student-optimal fair matching (SOFM)**
    - **general upper-bound**
- Application to daycare allocation with data

Model

- **Students** (denoted *i*, *I*) and **schools** (denoted *s*, *S*)
  - Many-to-one matching
- Each Student has strict **preferences** over schools (& outside option, ∅)
- Each school has a strict **priority order** over students
  - Generalizable to weak priority (i.e., ties)

Constraints

- Each school s is subject to a **constraint**
  - For each subset *I'* of students, a constraint tells **"feasible"** or **"infeasible"**
  - c.f. Constraints at the level of *sets* of schools (Biro et al. 2010, Kamada and Kojima, 2015, 2016a,b, Kojima et al. 2016, Goto et. al 2016)
  - For each school, assume there is at least one feasible set of students.

Desirable properties

- **Feasibility** (students feasible at every school), **IR** (students should be matched to ∅ or better)
- **Non-wastefulness:** there are no *i, s,* such that
  - *i* prefers *s* to her own assignment,
  - moving *i* to s results in a feasible matching
- **Fairness** (elimination of justified envy): there are no *i*, *i'*, *s*, such that
  - *i* prefers s to her own assignment,
  - *i'* is matched to *s* and *i* has a higher priority than *i'* at *s*

Discussion on fairness

- **Fairness** (elimination of justified envy): there are no *i*, *i'*, *s*, such that
  Weak fairness
  - *i* prefers s to her own assignment,
  - *i'* is matched to *s* and *i* has a higher priority than *i'* at *s*
  - and replacing *i'* with *i* is feasible at s
- Appropriate fairness concept depends on applications
  - Labor markets (medical match): weak fairness
  - College admission with disability, disaster relief material: fairness
- Non-existence problem robust to fairness concepts employed

Preliminary Facts

- **Fact 1:** feasibility & IR & fairness & non-wastefulness ⇔ **stability**

- **Fact 2:** Stability (=Feasibility & IR & Fairness & Non-wastefulness) leads to ***non-existence***

  - "Necessary and sufficient" condition turns out to be capacity constraints (later)

Fair matching

- Approach: Don't insist on (exact) non-wastefulness but require **fairness** (+ feasibility, IR)
- Existence? Structure?
    - Characterization via a mapping

Cutoff adjustment function

- ***P_s*** : the **cutoff** (=lowest priority/“score” to be admitted) at school *s*;
  - regarded as an element in *{1,…,n,n+1}*, where *n:=number of students*.
- ***$P=(P_s)_s$***: a cutoff profile at all schools.
- ***$D(P)=(D_s(P))_s$***: the **demand profile** at P
  - each student chooses favorite available school given P (or ∅)
- **Cutoff adjustment function *T*** from cutoff profiles to themselves:
  - ***$T_s(P)=P_s+1$*** (mod n+1) if *$D_s(P)$* is infeasible (i.e., “over-demanded”)
  - ***$T_s(P)=P_s$*** otherwise.
- *T* is like Walrasian tatonnement but doesn’t try to eliminate under-demand

Characterization

Theorem: If a cutoff profile P is a fixed point of T, then the induced matching is feasible, individually rational, and fair. Moreover, if a matching is feasible, individually rational, and fair, then there exists a cutoff profile that induces it.

- Proof: Given P induces matching $D(P)=(D_s(P))_s$,
  - there is no guarantee that $D(P)$ is feasible, but
  - $D(P)$ is IR and fair
- $P=T(P)$ iff $D(P)$ is feasible by definition of T.

Problem with fairness

- An arbitrary fair matching may be undesirable.
- Is there a “(most) desirable” fair matching?

SOFM

- A matching is a **student-optimal fair matching (SOFM)** if
    1. fair, IR, feasible, and
    2. weakly preferred by every student to any matching satisfying (1).
- Similar to “student-optimal stable matching” in standard case
    - note a stable matching may not exist

General upper bound

- We say constraints are **general upper-bound** if every subset of a feasible subset is also feasible
    - subsume standard settings like (1) capacity constraints and (2) type-specific quotas (diversity in schools), but exclude e.g., minimum (floor) constraints
- More examples of general upper-bound; next

General upper bound

- Recall **general upper-bound**; every subset of a feasible subset is also feasible
- More (less standard) examples of general upper-bound
  - College admission with students with disability (budget constraint)
  - Refugee match (Delacretaz et al. 2016)
  - School Choice and bullying (Kasuya 2016)
  - Separating conflicting groups in refugee match
  - Daycare/nursery school matching: more on this later

Sufficiency for SOFM

Theorem: If each school's constraint is a general upper bound, then there exists an SOFM**.**

- Similar to the existence of SOSM in standard case
    - note a stable matching may not exist
- Computation is easy (c.f. proof)

Proof (1)

- Given our characterization theorem, we study fixed points of T.
- Under general upper bound, use Tarski's fixed point theorem (below)
- A set is called a **lattice** if for any pair of elements, their "join" (least upper bound) and "meet" (greatest lower bound) both exist.
  - Example: "set of cutoff profiles"$=\{1,\ldots n+1\}^m$ with the product order.
  - In particular, there is a "largest" and "smallest" elements

Tarski's Theorem (special case): Let X be a finite lattice and f: X→X be weakly increasing, i.e., $x \le x'$ implies $f(x) \le f(x')$.

Then the set of the fixed points of *f* is a finite lattice. In particular, there are largest and smallest fixed points.

Proof (2)

Tarski's Theorem (from last slide): Let X be a finite lattice and f: $X \to X$ be weakly increasing, i.e., $x \leq x'$ implies $f(x) \leq f(x')$.

Then the set of the fixed points of f is a finite lattice. In particular, there are largest and smallest fixed points.

- Back to proof: We'll show T is weakly increasing. Suppose $P \leq P'$.
  1. If $P_s < P'_s$, then $T_s(P) \leq P_s + 1 \leq P'_s \leq T_s(P')$.
  2. Suppose $P_s = P'_s$.
     - Demand for s is (weakly) larger if students face higher cutoffs at all other schools, so $D_s(P)$ is a subset of $D_s(P')$.
     - So, $T_s(P) = P_s + 1$ *implies* $T_s(P') = P'_s + 1$, thus $T_s(P) = T_s(P')$.
- So $T(P) \leq T(P')$.
  - Smallest fixed point induces SOFM. **QED**

Algorithm

- Tarski's theorem gives an intuitive (and polynomial-time) algorithm.
- Start with lowest possible cutoff profile, $\underline{P}$ (i.e., every student is above the cutoff at every school)
  - Then $\underline{P} \leq T(\underline{P})$
  - Apply T repeatedly and get: $\underline{P} \leq T(\underline{P}) \leq T(T(\underline{P})) \leq T^3(\underline{P}) \leq T^4(\underline{P}) \leq \ldots$
  - At some point it stops at some P^*, and
    - $T(P^*)=P^*$; so it induces a feasible, IR, and fair matching
    - For any fixed point P, $P^* \leq P$; P^* corresponds to SOFM

More general constraints?

- The “general upper-bound” includes many practical cases, but not all (e.g., minimum constraints)
- Does SOFM exist more generally?
- Answer: “no” in a specific sense

Theorem: Suppose the constraint of a school *s* is **not** a general upper bound. Then there exist student preferences and capacity constraints at other schools s.t., SOFM does not exist.

Proof (1)

- Suppose the constraint at s is not a general upper bound.
- Consider two cases:

Proof (2)

- Case 1 (“easy” case): Suppose the empty matching (i.e., no one is matched) is infeasible at s.
- Assume all students find s unacceptable.
- Clearly, there is no feasible and IR matching.

Proof (3)

- Case 2 ("less easy" case): Suppose the empty matching is feasible at *s*.
- Note there is some set *I'* of students and its subset *I''* such that *I'* is feasible but *I''* is not (and both are nonempty).
- Fix $s' \neq s$ and assume preferences
  - students in *I'': s, s'*
  - students in *I'\I'': s', s*
  - all other students find all schools unacceptable
  - *s'* has a large capacity

Proof (4)

- Recall (from last slide)
  - students in *I''*: *s, s'*
  - students in *I'**I''*: *s', s*
  - all other students find all schools unacceptable
  - *s'* has a large capacity

- Two fair (&feasible and IR) matchings:
  1. everyone in *I'* is matched to *s* and everyone else is unmatched
  2. everyone in *I'* is matched to *s'* and everyone else is unmatched.
- If there is SOFM, then it should
  - match everyone in *I''* to *s*, *I'**I''* to *s'* and un-match everyone else

→infeasible! **QED**

Application: Daycare Match

- Japan: daycare is greatly over-demanded
- Municipal governments are under great pressure to accommodate more children

- Some resources (teachers, rooms, etc.) can be used for kids of different ages (Okumura 2017)
- Resource demand per kid varies across ages (younger kids need more teachers and space per capita)

  → **general upper bound** (while not capacity)

- Centralized matching algorithms.
  - flexible assignments tried in several municipalities (but in ad hoc manners)

Comparative statics

Proposition: SOFM under flexible constraints is Pareto superior for students to SOFM under rigid constraints.

- Easy to prove, true more generally for arbitrary “relaxation of constraints”
- c.f. Results for SOSM in standard models (e.g., Crawford 1991; Konishi and Unver 2006)
- Flexibility across different ages will help.
  - How about the magnitude?

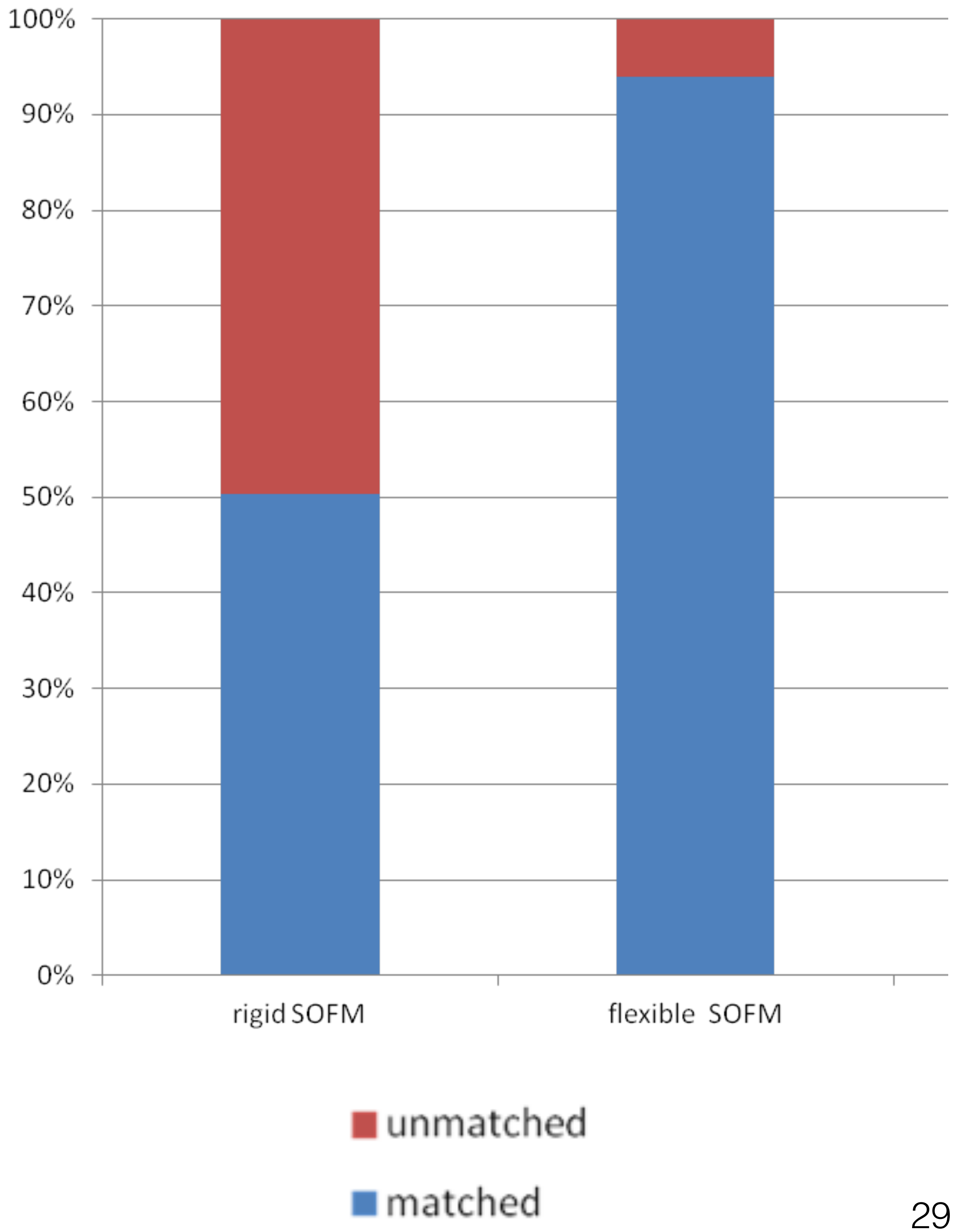
Daycare Match Data Analysis

- Data from Yamagata City (Yamagata) and Bunkyo City (Tokyo), Japan:
  - preferences (mechanism is strategy-proof)
  - priorities
  - outcomes
- We simulate SOFM under "flexible" and "rigid" constraints

Recall: SOFM under flexible constraints is Pareto superior to SOFM under rigid constraints.

Match Rate

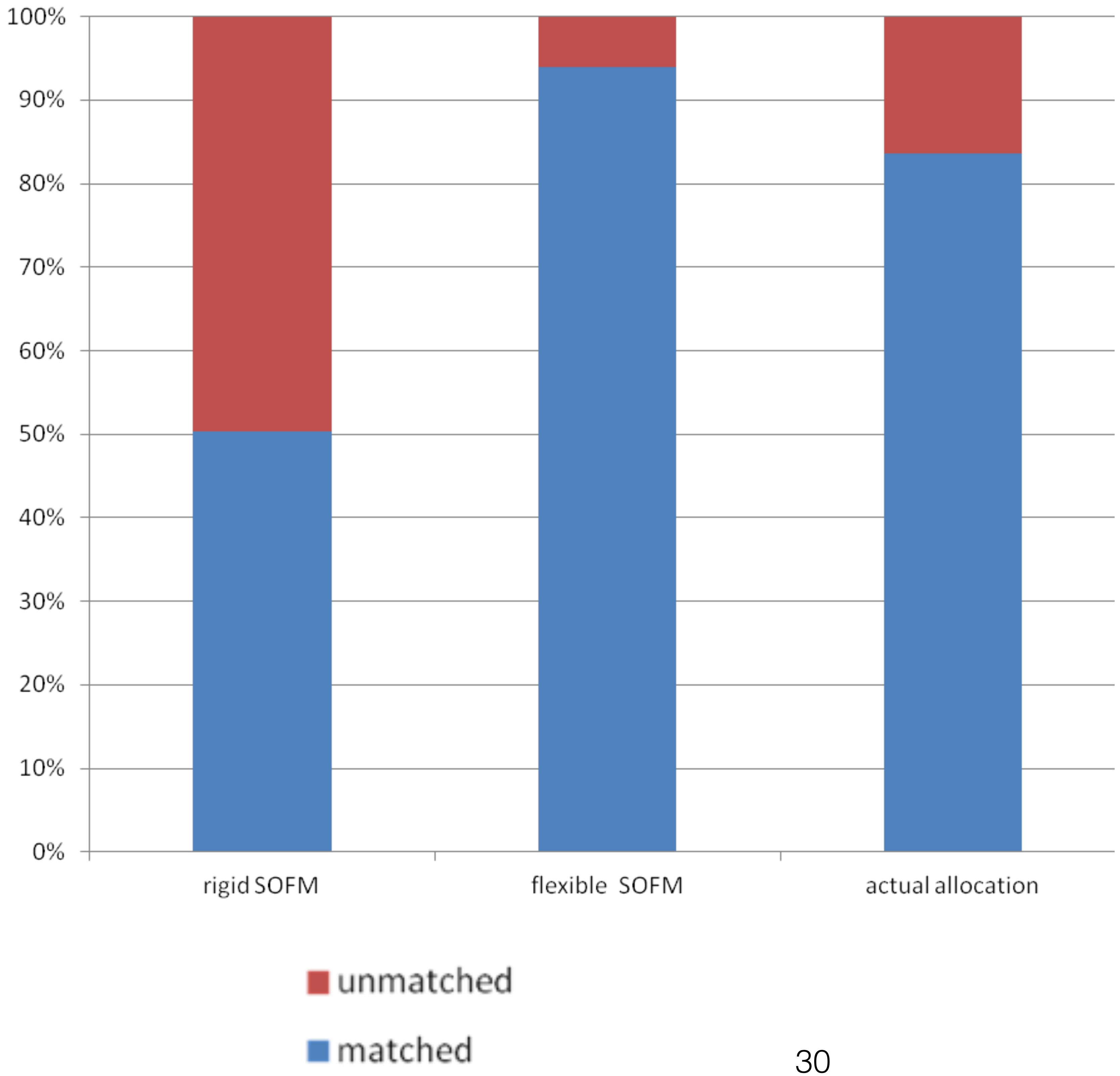
(1437 applicants in total)



(Data: Yamagata)

Match Rate

(1437 applicants in total)



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Match Rate

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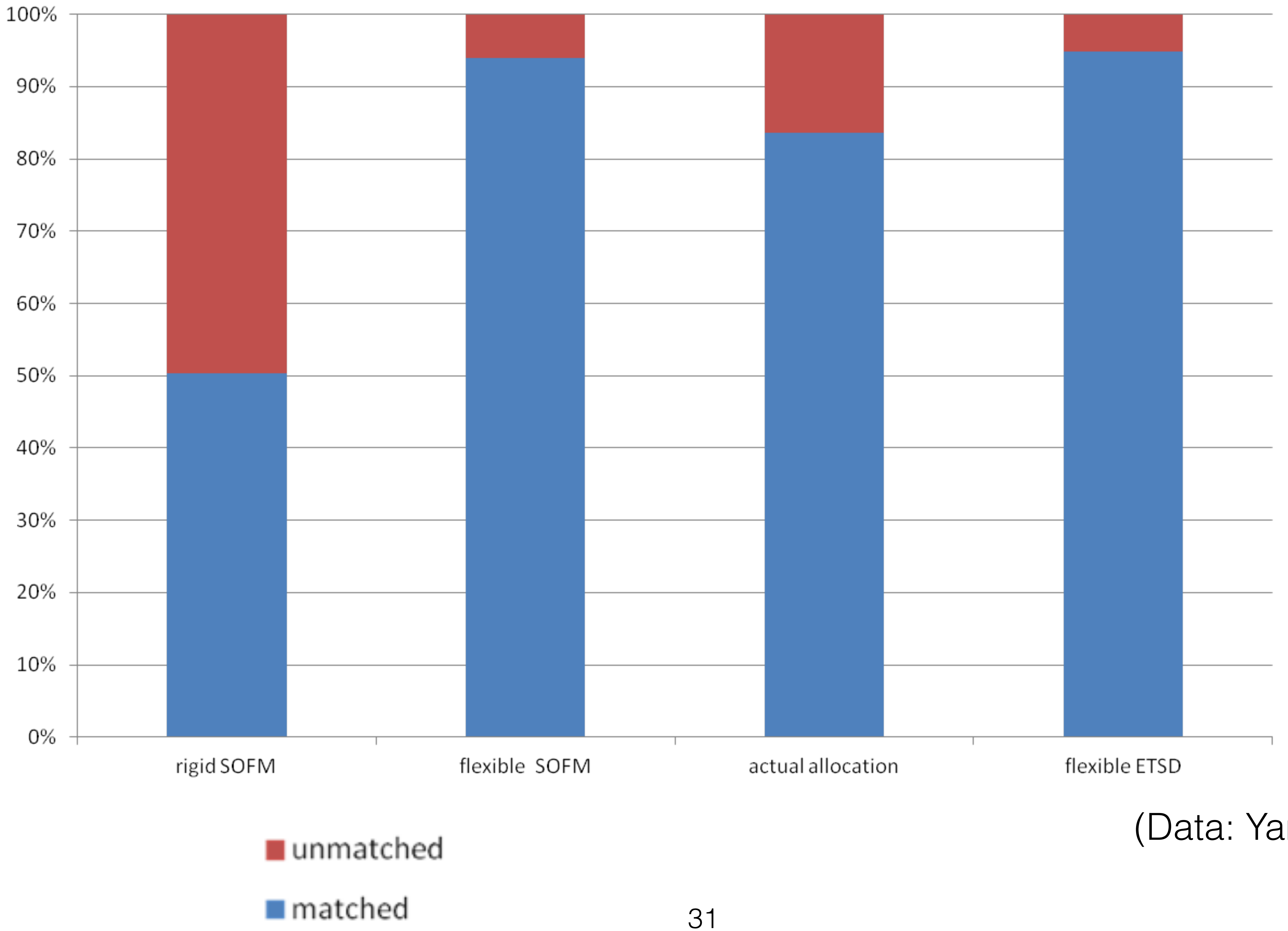


(Data: Yamagata)

How many people are better off?

(1437 applicants in total)

From/To	rigid SOFM	flexible SOFM	actual allocation	flexible ETSD
rigid SOFM	0	867.272 (60.35 %)	658.456 (45.82 %)	881.944 (61.37 %)
flexible SOFM	0	0	72.132 (5.02 %)	49.78 (3.46 %)
actual allocation	13.188 (0.92 %)	237.944 (16.56 %)	0	248.676 (17.31 %)
flexible ETSD	0	0	62.876 (4.38 %)	0

(Data: Yamagata)

Justified Envy

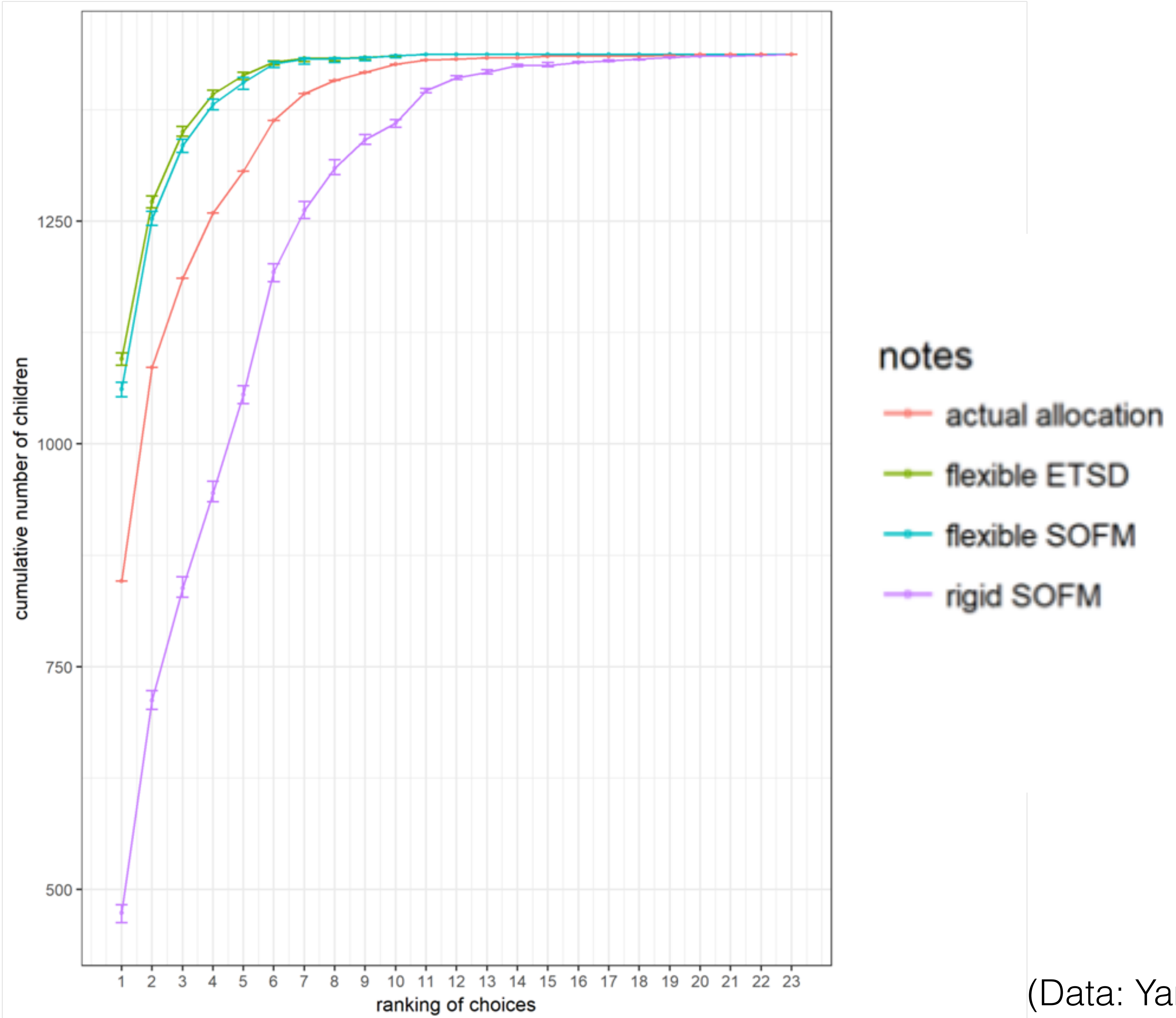
(1437 applicants, 93 daycares in total)

	rigid SOFM	flexible SOFM	actual allocation	flexible ETSD
pairs with envy	0	0	989 (0.74 %)	157.188 (0.12 %)
students with envy	0	0	475 (33.05 %)	129.956 (9.04 %)
daycares with envy	0	0	62 (66.67 %)	22.164 (23.83 %)

(Data: Yamagata)

Number of justified envy is comparable to TTC in Boston and New Orleans (Abdulkadiroglu et al. 2018)

Rank distribution



(Data: Yamagata)

Extension: Tie in priority

- College admission in Hungary (Biro 2010) uses a mechanism like deferred acceptance, but
  - Ranking over students are based on test score → ties
  - Admitting all students with a score is infeasible → reject **all** students of that score
- Disaster shelter in Kobe and Tohoku earthquakes (Hayashi 2003, Hayashi 2011)
  - Priorities include lots of ties (e.g., own house livable or not)
  - Insufficient food supply was not allocated

Problems with ties:

1	*2*
A	*A*
∅	∅

A
1,2
∅

- *A* has capacity of 1
- *A* ranks *1* and *2* equally
- But our theory extends: SOFM exists, etc.
    - Characterization: fair and non-wastefulness are compatible iff capacity constraints **and no ties**.

Stability: Maximal domain

- Recall stability (=Feasibility & IR & Fairness & Non-wastefulness) leads to ***non-existence.*** In fact,

> **Theorem:** Suppose the constraint of a school *s* is **not** a capacity constraint (while being a general upper-bound). Then there exist a priority at *s* and student preferences s.t. there exists no stable matching.

- Note: "necessary and sufficient" condition for stable matching existence

- The conclusion holds for *any* priorities and constraints at other schools.

Strategic issues

- SOFM mechanism isn't necessarily strategy-proof for students
  - Capacity constraints → SP for students
  - Turns out this is "necessary" as well.

Theorem: Suppose the constraint of a school *s* is **not** a capacity constraint. Then there are school priorities and standard capacity constraints at other schools such that the SOFM mechanism isn't strategy-proof for students.

- But
  - The same impossibility holds for any mechanism with feasibility, fairness, and **unanimity**.
  - Approximate incentive compatibility holds in large markets.

Related literature

- **Distributional Constraints:** Sonmez-Unver (2006), Biro-Fleiner-Irving-Manlove (2010 TCS), Kamada-Kojima (2015 AER, 2016 JET, 2017 TE), Milgrom (2009 AEJ Micro), Budish-Che-Kojima-Milgrom (2013 AER), Che-Kim-Mierendorff (2013 ECMA), Akbarpour-Nikzad (2016), Fragiadakis-Troyan (2016 TE), Goto-Hashimoto-Iwasaki-Kawasaki-Ueda-Yasuda-Yokoo (2014 AAMAS), Goto-Kojima-Kurata-Tamura-Yokoo (2016 AEJ), Kojima-Tamura-Yokoo (2018 JET),

- **Affirmative action/diversity:** Roth (1991 AER), Abdulkadiroglu-Sonmez (2003 AER), Abdulkadiroglu (2005 IJGT), Aygun-Turhan (2016), Dur-Pathank-Sonmez (2016), Ergin-Sonmez (2006 JPubE), Abdulkadiroglu-Pathak-Roth (2009 AER), Kojima (2012 GEB), Ehlers-Hafalir-Yenmez-Yildirim (2014 JET), Echenique-Yenmez (2015 AER), Hafalir-Yenmez-Yildirim (TE 2013), Westkamp (2010 ET), Sonmez (2013 JPE), Sonmez-Switzer (2013 ECMA), Dur-Kominers-Pathak-Sonmez (2016 JPE), Delacretaz-Kominers-Teytelboym (2016), Hassidim-Romm-Shorrer (2016), Milgrom-Segal (2016), Okumura (2017)

- **Fairness:** Foley (1967), Balinski-Sonmez (1999 JET), Sotomayor (1996 GEB), Blum-Roth-Rothblum (JET 1997), Wu-Roth (2017 GEB), Kesten-Yacizi (2010 ET), Biro (2010)

Conclusion

- Characterization of fair matchings via a cutoff adjustment function
- The general upper-bound is the most general condition to guarantee existence of SOFM
- Daycare match application
- Future research
  - Solution under non-general upper bounds
  - More numerical and empirical study (new data just granted)
  - Implementing the new design