Abstract

We analyze the payout decision of a financially-constrained firm that cannot raise external funds. Exogenous cash flows are generated by a two-state Markov regime-switching process and are positive in one regime and negative in the other regime. The firm is motivated to pay out dividends to impatient shareholders but is also motivated to accumulate cash within the firm to make required payments when cash flow is negative. If cash on hand is insufficient to make these payments, the firm terminates, thereby losing its claim on future cash flows. The optimal payout policy can be described as a form of precautionary saving. However, contrary to conventional wisdom about precautionary saving, we find that such saving falls in response to a mean-preserving increase in the variance of cash flows.

We extend the model to include a capital investment decision. Instead of smooth and convex adjustment costs, we introduce an upper bound on the investment-capital ratio, which leads to a bang-bang solution for investment. Thus optimal investment and dividends are each governed by trigger policies. If a particular myopic value of a unit of capital exceeds one, the firm can ignore the financing constraint and invest in capital whenever cash flow from operations is positive. Alternatively, when this myopic value of capital is less than one, the trigger for optimal investment equates marginal $q$ to the "static cost of funds," which is the marginal valuation of a dollar of cash within the firm. In this situation, average $q$ and marginal $q$ are equal to each other at the investment trigger, and their common value is one if and only if the financial constraint is not binding at the investment trigger. However, if the financial constraint binds at the investment trigger, then average $q$ exceeds marginal $q$ which exceeds one.
It is an enduring and robust fact that firms smooth their dividends relative to their earnings. We offer a simple neoclassical explanation of dividend smoothing arising from the optimal payout policy of a firm that manages it cash holdings as a precaution against being unable to make required cash payments. This explanation does not rely on asymmetric information, signaling, or concave utility. Instead payout policy is governed by the desire to maximize the expected present of dividends paid to shareholders who are impatient in that their rate of time preference, $\rho$, exceeds the interest rate, $r$, that can be earned on cash held by the firm.\footnote{One rationale for $\rho > r$ is that the firm is subject to an exogenous catastrophic shock that terminates the firm. If this shock is a Poisson shock with arrival intensity $\lambda > 0$, then the effective discount rate of shareholders is $\rho = \rho^* + \lambda$, where $\rho^*$ is the pure rate of time preference. In this case, if the $r$ is a riskless rate equal to the pure rate of time preference, $\rho^*$, then $\rho$ exceeds $r$. An alternative rationale given by Bolton, Chen, and Wang (2011) is based on agency costs associated with free cash flow.} The impatience of shareholders pushes the firm to pay dividends rather than to retain earnings. Working in the opposite direction is a financial constraint that prevents the firm from raising additional funds through debt or equity issuance. Formally, this constraint is a non-negativity contraint on dividends. If a firm is required to make payments, say if revenues fall short of costs, and if the firm does not have sufficient cash on hand to make these payouts, then the firm is forced to terminate. Termination eliminates the claim of existing shareholders on any future profits of the firm. The desire to avoid termination pushes the firm toward retaining earnings rather than paying dividends. Optimal payout policy reflects the interplay of shareholders’ impatience to receive dividends soon and the precautionary demand for retained earnings to avoid termination.

We examine the cash management problem of a firm that has exogenous stochastic inflows and outflows of cash from operations. We begin with the simplest framework in which the only decision of the firm is the payout decision—specifically how much of its cash to pay out to shareholders and how much to hold for a rainy day. In this formulation, a rainy day is an episode of exogenous negative cash flow. If the firm does not have enough cash on hand to pay the exogenous outflow, then the firm is forced to terminate, thereby suffering the loss of any future net cash flows. This framework gives rise to a precautionary demand for cash.

With a two-state Markov regime-switching process governing whether cash flow takes on a given positive value (Regime $H$) or a given negative value (Regime $L$), optimal cash management can be characterized by a target level of cash on hand. When cash on hand is less than the target level, the firm does not pay dividends. During episodes of positive cash
flow, the firm accumulates cash until its cash on hand reaches its optimally-chosen target. Then the firm pays out additional positive cash flows from operations as dividends. When the regime switches to negative cash flow, the firm ceases dividends and makes its required payments from its cash on hand. If the episode of negative cash flow lasts long enough so that the firm exhausts its cash on hand while facing negative cash flows, it will be forced to terminate. But if the regime switches to positive cash flow before the firm exhausts its cash on hand, the process described above will repeat itself. The time series of dividends associated with this optimal cash management policy will be smoother than the time series of cash flows from operations. Specifically, the dividends will have a lower variance and will change values less frequently than cash flows from operations.

A novel feature of our model is that the firm’s cash flows follow a persistent process unlike typical formulations in the dynamic corporate finance literature, where the cash flows follow a serially uncorrelated process. Surprisingly, in our framework with persistence, a mean-preserving increase in the variance of cash flows reduces the optimal target level of cash on hand. Specifically an increase in the positive cash flow in Regime $H$ accompanied by an offsetting decrease in the (negative) cash flow in Regime $L$ hastens the time at which a given target level of cash will fail to prevent the firm from having to terminate in the face of sustained negative cash flows. Therefore, an increase in variance increases the likelihood of termination when entering Regime $L$ with any given amount of cash on hand and thus increases shareholders incentive to receive dividends rather than increase cash on hand. This effect in which higher variance is associated with a reduced level of precautionary saving is contrary to the standard finding in theoretical models of precautionary saving.

After exploring the optimal target level of cash, we introduce a real investment decision. The firm can choose to invest irreversibly in fixed capital at a constant price that we normalize to equal one. Net cash flow from operations is proportional to the capital stock. To allow for an analytic solution, we do not include any explicit convex adjustment costs. Instead we impose an upper bound on the investment-capital ratio. Cash flow from operations less expenditure on capital investment is linearly homogeneous in investment and the capital stock. This linear homogeneity is the essence of the Hayashi condition, which is widely used to guarantee that average $q$ and marginal $q$ are identically equal to each other. However, despite this linear homogeneity in the current framework, average $q$ exceeds marginal $q$ because cash on hand is effectively a second capital asset that contributes to the value of the firm.

With a real investment decision as well as a payout decision, the optimal behavior of the
firm is characterized by two thresholds, or trigger values, of the cash-to-capital ratio \( x \equiv \frac{X}{K} \), where \( X \) is cash on hand and \( K \) is the capital stock. These thresholds are operative in Regime \( H \).\(^2\) One threshold triggers capital investment and the other threshold triggers the payment of dividends. The dependence of optimal dividends and capital investment on the cash-to-capital ratio is, of course, a deviation from Modigliani-Miller. For low values of \( x \equiv \frac{X}{K} \), the marginal valuation of capital is lower than the marginal valuation of cash on hand. That is, the firm would prefer to keep a dollar of cash on hand rather than spend that dollar to buy a unit of capital; optimal investment is zero in this case. However, for high enough values of \( x \), the marginal valuation of a unit of capital exceeds the marginal valuation of a dollar of cash, so the firm prefers to use a dollar of cash to buy capital; optimal investment is positive.

The marginal valuation of cash on hand, \( V_X \), can be interpreted as the “static cost of funds.” For an ongoing firm that follows an optimal cash management program, the static cost of funds will be greater than or equal to one. When the static cost of funds equals one, the firm pays dividends to its shareholders, but when the static cost of funds exceeds one, a dollar is worth more inside the firm as cash on hand than it is in the hands of shareholders. Therefore, the firm will not pay dividends when the static cost of funds exceeds one. However, if the marginal valuation of capital, \( V_K \), often called “marginal \( q \)” (since the replacement cost of a unit of capital is normalized to equal one), is greater than or equal to the static cost of funds, \( V_X \), then the firm will find it optimal to use some of its funds to purchase new capital, even if \( V_X > 1 \). Put differently, the investment trigger can be described in terms of the ratio of marginal \( q \) to the static cost of funds, \( V_K/V_X \), as emphasized by Bolton, Chen, and Wang (\( \)\). Specifically, if this ratio is greater than or equal to one, it is optimal to invest.

The power of the Hayashi condition stems from the fact that marginal \( q \) is the relevant measure of \( q \) for investment decisions, but marginal \( q \) is not directly observable. However, average \( q \), which is \( V/K \) (in typical formulations that do not include cash on hand), is generally observable and can be used as a perfect proxy for marginal \( q \) under the Hayashi conditions because average \( q \) and marginal \( q \) are typically equal in that situation. But, as mentioned above, average \( q \) and marginal \( q \) are potentially no longer equal in this non-MM framework in which cash on hand affects the optimal investment decision. In fact, average \( q \), \( V/K \), exceeds marginal \( q \) by \( \frac{X}{K} V_X \geq 0 \), with strict inequality if \( X > 0 \). Remarkably, however,

\(^2\)In Regime \( L \), optimal dividends are always zero in an ongoing firm (Proposition 2). We confine attention to \( \phi^L > \gamma(\phi^H) \), defined in Proposition 10, to ensure that optimal investment is always negative for an ongoing firm in Regime \( L \).
when the cash-to-capital ratio, $x$, equals the value that triggers investment, it turns out that marginal $q$ equals an alternative measure of average $q$, namely $V/(K + X)$, which is the ratio of the value of the firm to the replacement cost of its total assets, consisting of capital and cash on hand. This ratio is observable.

Although shareholders have rational expectations over the indefinite future, a particular myopic valuation, which we will call the *myopic valuation of capital in Regime H*, turns out to be a key element of the optimal capital investment decision. The model is configured so that positive capital investment is optimal only during Regime $H$, which has positive net operating profits. Moreover, even in Regime $H$, it is optimal to undertake positive investment only if the cash-to-capital ratio, $x$, is greater than or equal to the investment threshold value. As it turns out, the investment threshold value equals zero—that is, positive investment is optimal whenever the firm is in Regime $H$—if and only if the myopic value of capital in Regime $H$ exceeds one. This myopic value is simply the expected present value of operating profits per unit of capital over the remaining (stochastic) duration of the current Regime $H$. In this case, there is no need to compare marginal $q$ to the static cost of funds described above. If the myopic value of a unit of capital in the current Regime $H$ is high enough to cover the cost, normalized to one, of acquiring the capital, then it is optimal to invest. In this situation, the financial imperfection arising from the inability to raise additional funds through debt or equity issuance is irrelevant for capital investment decisions.

In parameter configurations in which the myopic value of a unit of capital in Regime $H$ is less than one, the non-negativity constraint on dividends my bind or not. Remarkably, if the non-negativity constraint on dividends is not binding, then average $q, \frac{V - X}{K}$, equals marginal $q$ and both measures equal one at the investment trigger. However, if the non-negativity constraint on dividends is binding, then average $q, \frac{V - X}{K}$, exceeds marginal $q$, which, in turn, exceeds one. Therefore, if the myopic value of a unit of capital in Regime $H$ is less than one, $\frac{V - X}{K} > 1$, or, equivalently, $V > K + X$, if and only if the negativity constraint is binding.

## 1 The Firm’s Decision Problem

Consider a firm that operates in continuous time and has an exogenous stochastic stream of net cash flow from operations $\phi_t$ at time $t$. For now, there is no investment decision nor depreciation, so the capital stock is constant and normalized to equal one. Suppose that the cash flow $\phi_t$ evolves according to a two-state Markov regime-switching process. Specifically, in Regime $H$, cash flow is $\phi_t = \phi^H > 0$, and in Regime $L$, cash flow is $\phi_t = \phi^L < 0$. 


The negative cash flows in Regime L can arise, for instance, from an unavoidable cost of owning and maintaining a fixed capital stock or from a commitment to make purchases from suppliers that exceed revenues.

The transitions between regimes $H$ and $L$ are arrivals of Poisson events with possibly different arrival intensities. When $\phi_t = \phi^H$, cash flow remains equal to $\phi^H$ until the regime switches (from $H$ to $L$) and $\phi$ changes to $\phi^L$. The instantaneous probability of switching to Regime $L$ from Regime $H$ is $\mu^L > 0$, and the instantaneous probability of switching to Regime $H$ from Regime $L$ is $\mu^H > 0$. Assume that

\[
(\rho + \mu^L) \phi^L + \mu^H \phi^H > 0, \tag{1}
\]

which implies that the conditional expected present value of $\phi$, discounted at the shareholders’ common discount rate $\rho > 0$ over the infinite future, is positive even when conditioning on $\phi_t = \phi^L < 0$ in Regime $L$.

**Definition 1** Define the roundtrip discount factor $\Gamma \equiv \frac{\mu^H}{\rho + \mu^L} < 1$.

The roundtrip discount factor $\Gamma$ is the expected present value, discounted at rate $\rho$, of one dollar at future time $t^*$, which is the first time that the firm returns to its current regime after leaving current regime.

**Definition 2** Define the myopic value of a unit of capital as the expected present value of operating profit $\phi$, over the remaining duration of the current regime. The myopic value of a unit of capital is $\nu^H \equiv \frac{\phi^H}{\rho + \mu^L} > 0$ in Regime $H$ and is $\nu^L \equiv \frac{\phi^L}{\rho + \mu^H} < 0$ in Regime $L$.

Even though shareholders have rational expectations that extend indefinitely far into the future, these myopic values have an important role in characterizing many features of the firm’s decision problem.

---

3 $E_\tau \left\{ \int_0^\tau \phi_s e^{-\rho(s-t)} ds \mid \phi_t = \phi^L \right\}$ equals $\tilde{V}^L(0, 1)$ in Lemma 4 in the situation in which $i^H = i^L = 0$ so that $\tilde{V}^L(0, 1) = (\rho + \mu^L - \rho^H) \phi^L + \rho^H \phi^H$, which is positive if and only if the condition in equation (1) holds.

4 Suppose that Regime $j$ prevails at time 0, continues to prevail until $t_1 > 0$, when the other regime $(-j)$ arrives; the next arrival of Regime $j$ is at $t^* > t_1$. Then $\Gamma \equiv E \{ e^{-\rho t^*} \} = \int_0^{t_1} \mu(-j) e^{-\rho t} dt \int_{t_1}^{t^*} \mu(j) e^{-\rho t} dt = \frac{\mu^H}{\rho + \mu^H} \mu^L = \frac{\mu^H}{\rho + \mu^H} \mu^L$.

5 The myopic value of a unit of capital in a Regime $H$ that prevails from time 0 to a random date $t_1 > 0$ is $V^H_M = E \left\{ \phi^H e^{-\rho t} dt \right\} = \phi^H E \left\{ \frac{1 - e^{-\rho t} t}{\rho} \right\} = \phi^H E \left\{ \frac{1}{\rho} (1 - \frac{\mu^L}{\rho + \mu^H}) \right\} = \frac{\phi^H}{\rho + \mu^H}$. Similarly, $V^L_M = \frac{\phi^L}{\rho + \mu^L}$. 

5
The roundtrip discount factor and the myopic values of the firm can be used to rewrite the assumption in equation (1) as
\[ \frac{-v^L}{v^H} < \frac{\mu^H}{\rho + \mu^H} < 1, \]
which implies that the ratio of the myopic value of the losses in a single Regime L is smaller than the myopic value of positive cash flow in a single Regime H by a discrete amount.

The only decision facing the firm is the payout decision, that is, how much cash to retain and how much to distribute to shareholders. The firm holds a stock of cash on hand, \( X_t \geq 0 \), and earns an interest rate \( 0 \leq r < \rho \) on this cash. The firm can use the cash on hand to pay cash outflows, \(-\phi^L > 0\), required in Regime L. The shareholders of the firm cannot inject any new funds into the firm to make these required payments. Therefore, any required cash payments must be paid from \( X_t \). If \( X_t = 0 \) and \( \phi_t < 0 \), then the firm fails to make required payments and therefore immediately and permanently terminates with zero salvage value.

The shareholders of the firm are risk-neutral. Their objective is to maximize the expected present value of dividends received from the firm, discounted at rate \( \rho \). The assumption that shareholders cannot inject any additional funds into the firm implies that dividends cannot be negative. In principle, dividends can be paid as a finite flow per unit of time or as lump sums at discretely-spaced points of time. However, a firm that has been in Regime H at some point in the past, and has followed an optimal payout policy since that time, will never find it optimal to pay a lump-sum dividend at the current time. We will call such a firm an "ongoing firm" and confine attention to ongoing firms in this paper.\(^6\)

The equality of sources and uses of funds for an ongoing firm is given by
\[ \dot{X}_t + D_t = rX_t + \phi_t, \]
where the sources of funds are the interest receipts on cash, \( rX_t \), and the net cash flow from operations, \( \phi_t \); the uses of funds are to accumulate cash at rate \( \dot{X}_t \) and to pay dividends at rate \( D_t \geq 0 \).

Let
\[ V_t \equiv E_t \left\{ \int_t^\tau D_s e^{-\rho(s-t)} ds \right\} \]
be the conditional expected present value of the flow of dividends paid from time \( t \) until the endogenous time \( \tau \geq t \) when the firm runs out of cash while facing a negative cash flow from

---

\(^6\)A firm that does not have a history of optimal payout (and thus is not an ongoing firm, as defined here) may find itself in a position in which it has so much cash that it is optimal to pay an immediate one-time lump-sum dividend.
operations. Thus, $\tau$ is the time at which the firm is forced to terminate because it cannot make payments required when $\phi_t < 0$. Formally,

$$\tau \equiv \min \{ s \geq t : X_s = 0 \text{ and } \phi_s < 0 \}.$$  \hspace{1cm} (5)

Because the firm cannot borrow, nor raise additional funds by issuing equity or paying negative dividends, it must terminate at time $\tau$.

Let $V^H(X_t)$ and $V^L(X_t)$ be the maximized expected present values, in regimes $H$ and $L$, respectively, of the flows of dividends from time $t$ until the termination time $\tau$. The Hamilton-Jacoby-Bellman (HJB) equation during Regime $H$ is

$$\rho V^H = D + \left( rX_t + \phi^H - D \right) V^H_X + \mu^L \left( V^L - V^H \right),$$  \hspace{1cm} (6)

which along with the complementary slackness condition

$$(V^H_X - 1)D = 0,$$  \hspace{1cm} (7)

implies

$$\rho V^H = \left( rX_t + \phi^H \right) V^H_X + \mu^L \left( V^L - V^H \right).$$  \hspace{1cm} (8)

Similarly, during Regime $L$, the HJB equation is

$$\rho V^L = \left( rX_t + \phi^L \right) V^L_X + \mu^H \left( V^H - V^L \right).$$  \hspace{1cm} (9)

The term on the left hand side of equation (8) is the required return on $V^H$ per unit of time and the two terms on the right hand side of this equation comprise the expected return per unit of time. Equation (3) states that $rX_t + \phi_t = \dot{X}_t + D_t$. Therefore, the first term on the right hand side of equation (8) is $\left( rX_t + \phi^H \right) V^H_X = \left( \dot{X}_t + D_t \right) V^H_X$, which is the return to shareholders in the forms of current dividends, $D_t$, and accumulation of cash on hand, $\dot{X}_t$, both multiplied by the marginal valuation of cash on hand $V^H_X$. The second term on the right hand side of equation (8) is the change in the firm’s value if the regime switches to $L$ from $H$, multiplied by $\mu^L$, the instantaneous probability of such a switch. The interpretation of equation (9) is symmetric.

The ODEs in equations (8) and (9) must satisfy the following boundary conditions

$$V^L(0) = 0$$  \hspace{1cm} (10)

$$V^H_X(X^*) = 1, \text{ if } X^* > 0$$  \hspace{1cm} (11a)

$$V^H_X(0) \leq 1, \text{ if } X^* = 0$$  \hspace{1cm} (11b)
where $X^*$ is the value of cash on hand, $X$, that triggers the payment of dividends in Regime $H$. The boundary condition in equation (10) states that if $\phi_t = \phi^L < 0$ and the firm has zero cash on hand, then the value of the firm is zero, because it must terminate immediately.

The boundary condition in equation (11a) states that in Regime $H$ if $X_t = X^* > 0$, then an extra dollar of cash on hand is worth a dollar to shareholders. That is, shareholders are indifferent at the margin about whether to retain an additional dollar of cash within the firm or to pay out that dollar as a current dividend. If $X = X^* > 0$ in Regime $H$, then the firm pays out dividends at rate $rX^* + \phi^H$ to keep $X$ equal to $X^* > 0$. Alternatively, if $X^* = 0$, then $V_{XX}^H(0) \leq 1$ (equation 11b) and the firm always pays dividends in Regime $H$ because a dollar is worth at least as much in the hands of shareholders as it is worth as cash on hand within the firm.

The boundary condition in equation (12) insures that $V_{XX}^H(X^*)$, which can be interpreted as the "static cost of funds" in Regime $H$, is minimized when $X = X^*$.\footnote{The modifier "static" distinguishes the static cost of funds from the conventional intertemporal cost of funds equal to the relevant interest rate.}

We will use the ODEs in equations (8) and (9) and the boundary conditions in equations (10) - (12) to derive the value functions in Regimes $H$ and $L$ and the optimal target level of cash on hand, $X^*$. Before proceeding to a more complete analysis, we present $V^H(X)$ for the case in which $X^* = 0$.

**Proposition 1** If $X^* = 0$, then for all $X \geq 0$, $V^H(X) = \nu^H + X$, so $V_{XX}^H(X) = 1$ and $V_{XX}^H(X) = 0$, where $\nu^H \equiv \frac{\phi^H}{\rho + \mu}$ is the myopic value of a unit of capital in Regime $H$.

Since the proof of Proposition 1 is both straightforward and instructive, we present here rather than in the appendix.

**Proof. of Proposition 1:** If $X^* = 0$, the firm maintains a zero balance of cash on hand. If, for some reason, the firm is holding cash, $X_0 > 0$, at an initial time 0, it immediately pays this entire amount to its shareholders as dividends and then, for the remainder of the current Regime $H$, pays all net inflows of cash from operations, $\phi^H > 0$, as dividends as soon as they arrive. When the current Regime $H$ ends and the next Regime $L$ arrives at, say, time $t_1$, the firm terminates. Therefore, the expected present value of dividends is $X_0$ plus the expected present value of cash flows from operations over the duration of Regime
$H$. That is, $\nu^H (X_0)$ equals $X_0$ plus the myopic value of a unit of capital in Definition 2, $\nu^H \equiv \frac{\phi^H}{\rho + \mu^H}$. Therefore,

$$V^H (X_0) = \nu^H + X_0,$$

which implies $V^H_X (X_0) = 1$ and $V^H_{XX} (X_0) = 0$. \[ \blacksquare \]

Proposition 1 states that when the optimal target level of cash is $X^* = 0$, the value function $V^H (X)$ is a linear function of $X \geq 0$ with slope equal to one. The intercept of this function is the myopic value of a unit of capital in Regime $H$, $\nu^H \equiv \frac{\phi^H}{\rho + \mu^H}$.

### 1.1 Marginal Value of Cash on Hand

The firm will not pay dividends when the marginal value of cash on hand exceeds one, that is, when an additional dollar of cash inside the firm is worth more to shareholders than an additional dollar of dividends. To compute the marginal value of a dollar of cash on hand in Regime $L$, differentiate equation (8) with respect to $X$ and rearrange to obtain

$$V^L_X = \frac{1}{\mu^L} \left[ (\rho - r + \mu^L) V^H_X - (r X + \phi^H) V^H_{XX} \right].$$

Evaluate equation (14) at $X = X^*$ and use the boundary conditions in equations (11a) and (12) to obtain

$$V^L_X (X^*) = 1 + \frac{\rho - r}{\mu^L} > 1, \text{ if } X^* > 0. \quad (15)$$

An ongoing firm that has been pursing optimal payout policy will always have $X \leq X^*$ because it will not accumulate additional $X$ beyond $X^*$ in Regime $H$. When Regime $L$ arrives, the firm will have $X \leq X^*$, and since profits from operations, $\phi^L$, are negative and

$rX^* \leq -\phi^L$, it cannot increase $X$. The concavity of $V^L (X)$, which we verify later, together with equation (15), implies that $V^L_X (X) \geq V^L_X (X^*) > 1$ for all $X \leq X^*$, so an ongoing firm prefers to retain earnings instead of pay dividends at all values of $X$ when it is in Regime $L$. The argument in this paragraph proves the following proposition.\[ \blacksquare \]

\[ ^8 \text{If } rX^* \text{ were greater than } -\phi^L > 0, \text{ then } rX^* + \phi^L \text{ would always be positive and once the firm reaches } X_t = X^*, \text{ it would be able to (1) pay positive dividends at every point in time, even if Regime } L \text{ persists forever, and (2) still maintain a cushion of cash on hand forever. Since the discount rate of shareholders, } \rho, \text{ exceeds the riskless interest rate, } r, \text{ shareholders would prefer to have the superfluous cushion of cash paid out as dividends.} \]

\[ ^9 \text{Proposition 2 does not address } V^L_X (X^*) \text{ in the case in which } X^* = 0 \text{ because ongoing firm would terminates immediately upon entering Regime } L \text{ if } X^* = 0. \]
**Proposition 2** If \( X^* > 0 \), then \( V^L_X (X^*) = 1 + \frac{r}{\mu^L} > 1 \), which implies that an ongoing firm never pays dividends in Regime \( L \).

The optimal target value of cash on hand in Regime \( H \), \( X^* \), can be either positive or zero, depending on the configuration of values of the fundamental parameters \( \rho, r, \phi^H, \phi^L, \mu^H, \) and \( \mu^L \). The following definition provides a function of these fundamental parameters that determines whether \( X^* \) is zero or positive.

**Definition 3** Define \( \Lambda \equiv \Lambda (\rho, r, \phi^H, \phi^L, \mu^H, \mu^L) \equiv \frac{\phi^H}{\rho + \mu^L} - \frac{\phi^L}{\rho - r + \mu^L} \).

**Proposition 3** \( X^* > 0 \iff V^H_X (0) < 0 \iff \Lambda \equiv \frac{\phi^H}{\rho + \mu^L} - \frac{\phi^L}{\rho - r + \mu^L} > 1 \).

Proposition 3 implies that if \( \Lambda > 1 \), then \( X^* > 0 \) so that an ongoing firm accumulates cash whenever it enters Regime \( H \).\(^{10}\) If \( \Lambda \leq 1 \), then \( X^* = 0 \) and the firm never accumulates cash; it always pays dividends \( \phi^H \) in Regime \( H \) and then terminates when Regime \( L \) arrives.

The locus of parameter values for which \( \Lambda = 1 \) is the border between the region of parameter space where \( X^* = 0 \) and the region where \( X^* > 0 \). The following corollary states that the marginal value of cash on hand, \( V^H_X (0) \), equals \( \Lambda \) on this border.

**Corollary 1** If \( \Lambda = 1 \), then \( X^* = 0 \) and \( V^H_X (0) = \Lambda \).

The intuition underlying Corollary 1 is straightforward. Consider a firm in Regime \( H \) at time \( 0 \) with \( X_0 = \varepsilon \) for small \( \varepsilon > 0 \). Suppose that the firm pays dividends at rate \( \phi^H + r\varepsilon \), thereby maintaining \( X_t = \varepsilon \) as long as the current Regime \( H \) prevails. The current Regime \( H \) ends at time \( t_1 > 0 \) when the regime switches to \( L \) and \( X_{t_1} = \varepsilon \). While the firm is in Regime \( L \), it makes the required payments \( -\phi^L > 0 \) and does not pay dividends. If Regime \( L \) persists long enough, the firm will run out of cash and will terminate. Alternatively, if Regime \( L \) ends before the firm runs out of cash, the firm will continue into the next Regime \( H \).

Figure 1 illustrates the evolution of \( X_t \) over time in the case in which \( r = 0 \). During Regime \( H_1 \), which prevails from time \( 0 \) to \( t_1 \), the firm maintains \( X_t = \varepsilon \) by paying dividends \( \phi^H \). When the firm enters Regime \( L_1 \) at time \( t_1 \), its cash on hand, \( X_{t_1} = \varepsilon \), allows it to makes its required payments \( -\phi^L > 0 \) per unit of time until time \( \tau = t_1 + \frac{\varepsilon}{-\phi^L} \). If Regime \( H_2 \) arrives at time \( t_2 < \tau \), the firm is still viable and resumes dividends at rate \( \phi^H \) until time

\(^{10}\)Whenever a firm transitions from Regime \( L \) to Regime \( H \), it will have \( X < X^* \) because it always decumulates cash in Regime \( L \).
Figure 1: Marginal Value of Cash at X=0

$t_3$ when Regime $L_2$ arrives. As illustrated in Figure 1, $V_X^H(0)$ is the product of three terms: (1) $\frac{\mu^L}{\rho + \mu^L}$ is the expected present value, at time 0, of a dollar at time $t_1 > 0$ when the current Regime $H$ ends; (2) $\frac{\mu^H}{\phi^H} \varepsilon$ is the probability of surviving until Regime $H_2$ once the firm enters Regime $L_1$, because the firm has a window of time from $t_1$ to $t_1 + \frac{\varepsilon}{\phi^L}$ before running out of cash, and $\mu^H$ is the instantaneous probability of arrival of Regime $H$; and (3) $\frac{\phi^H}{\rho + \mu^L} \equiv \nu^H$, which is the myopic value of a unit of capital that will be obtained if the firm survives until Regime $H_2$ arrives. The myopic valuation is appropriate because there is essentially zero probability that after having emerged from Regime $L_1$ in a very short period of time, the firm will also emerge from Regime $L_2$ in a short enough period of time to survive.

The following corollary expresses $\Lambda$ as the product of the roundtrip discount factor $\Gamma$ and the ratio of the myopic values of capital in Regimes $H$ and $L$, respectively.

**Corollary 2** If $r = 0$, then

1. $\Lambda \equiv \Gamma \frac{\nu^H}{\nu^L}$

2. if $\Gamma \nu^H + \nu^L = 0$, then $X^* = 0$ and $V_X^H(0) = 1$. 

11
1.2 Effects of Parameters on $X^*$

We next examine the effects of changes in the primitive parameters $\rho$, $r$, $\phi^H$, $\phi^L$, $\mu^H$, and $\mu^L$ on the optimal target level of cash $X^*$. We focus the comparative statics on parameter values such that $\Lambda - 1$ is in a positive neighborhood of zero. The following proposition shows the impact of changing various parameters on $X^*$, starting from $\Lambda = 1$ where $X^* = 0$.

**Proposition 4** Starting from a parameter configuration for which $\Lambda \equiv \Lambda (\rho, r, \phi^H, \phi^L, \mu^H, \mu^L) = 1$, and hence $X^* = 0$, the following changes in parameters increase $X^*$ to a positive value:

1. a decrease in $\rho$
2. an increase in $r$
3. an increase in $\mu^H$
4. an increase in $\mu^L$ if $\mu^L < \sqrt{\rho (p - r)}$
5. a decrease in $\mu^L$ if $\mu^L > \sqrt{\rho (p - r)}$
6. an increase in $\phi^H$
7. an increase in $\phi^L$, and
8. a rightward translation of the unconditional distribution of $\phi$, that is, increases in $\phi^H$ and $\phi^L$ by equal amounts.

Starting from a parameter configuration for which $\Lambda = 1$ so that $V_X^H (0) = 1$ and $X^* = 0$, any change that increases $\Lambda$ will increase $V_X^H (0)$ to a value greater than one, and hence will increase $X^*$ to a positive value. A decrease in $\rho$ increases $V_X^H (0)$, both by increasing the present value of a dollar at time $t_1$, $\frac{\mu^L}{\rho + \mu^L}$, and by increasing the myopic valuation of the firm in Regime $H$, $\nu^H \equiv \frac{\phi^H}{\rho + \mu^L}$. Therefore, a decrease in $\rho$ increases $X^*$ (Statement 1 of Proposition 4). An increase in $r$, which is the rate of return earned on cash on hand, $X$, leads to a positive optimal target level of cash on hand, $X^*$ (Statement 2). An increase in $\mu^H$ increases the probability $\frac{\mu^H}{\phi^L} \varepsilon$ that a firm that enters Regime $L$ with $X = \varepsilon$ will emerge from Regime $L$ with cash on hand and hence can continue to operate in the next Regime $H$. Therefore, an increase in $\mu^H$ increases $X^*$ (Statement 3).

An increase in $\mu^L$ has two opposing effects on the marginal valuation of a unit of cash. On one hand, an increase in $\mu^L$ reduces the expected time until the arrival of the next Regime
and hence the next Regime $H$, thereby increasing the marginal valuation of a unit of cash and increasing $X^*$. On the other hand, an increase in $\mu^L$ reduces the myopic valuation $\frac{\phi^H}{\rho + \mu^L}$ and thus reduces $X^*$. If $\mu^L < \sqrt{\rho (\rho - r)}$, then the first effect dominates (Statement 4), but if $\mu^L > \sqrt{\rho (\rho - r)}$, then the second effect dominates (Statement 5).

An increase in $\phi^H$ increases the dividend that will be paid per unit of time during Regime $H$ when $X = X^*$ and therefore increases the value of being able to emerge from Regime $L$ with cash on hand to enter the next Regime $H$. Therefore, an increase in $\phi^H$ increases the marginal valuation of a unit of cash and increases $X^*$ (Statement 6). An increase in $\phi^L$, that is, a reduction in $|\phi^L|$ reduces the rate at which cash on hand is depleted during Regime $L$ and thus increases the probability $\mu^H \frac{\phi^L}{\phi^L}$ that the firm emerges from the next Regime $L$ with cash on hand to enter the next Regime $H$. Therefore, the marginal valuation of a unit of cash increases and hence $X^*$ increases (Statement 7). Since the marginal valuation of a unit of cash increases in response both to an increase in $\phi^H$ and an increase in $\phi^L$, a variance-preserving increase in the unconditional mean of $\phi$ (which is an increase in $\phi^H$ and $\phi^L$ by equal amounts) increases $X^*$ (Statement 8).

1.3 Mean-Preserving Change in Variance

Proposition 5 Starting from a parameter configuration for which $\Lambda = 1$ and hence $X^* = 0$, a mean-preserving change in $\phi^H$ and $\phi^L$ that decreases the unconditional variance$^{11}$ of $\phi$ increases $X^*$ to a positive value.

A mean-preserving decrease in the variance of the unconditional distribution of $\phi$ increases $\phi^L$ and decreases $\phi^H$ while keeping $\mu^L \phi^L + \mu^H \phi^H$ unchanged. The increase in $\phi^L$, which decreases $-\phi^L > 0$, slows down the depletion of cash during Regime $L$ and thus lengthens the window of time in Regime $L$ before exhausting a given amount of cash balances. Thus an increase in $\phi^L$ increases the probability that the firm will emerge from Regime $L$ to a subsequent Regime $H$, which increases $V_X^H (0)$ and $X^*$. Working in the opposite direction, a decrease in $\phi^H$ decreases the myopic valuation of the firm in Regime $H$, which decreases the value of emerging from Regime $L$. Therefore, a decrease in $\phi^H$ decreases $V_X^H (0)$ and $X^*$. To see which of these opposing effects is dominant, observe from Definition 3 that $\Lambda$ can be written as

$$\Lambda = \frac{\mu^L}{\rho + \mu^L} \frac{\mu^L}{\rho - r + \mu^L} \frac{\mu^H \phi^H}{\mu^L \phi^L \phi^H - M},$$

$^{11}$Formally, a decrease in $\phi^H$ and an increase in $\phi^L$ that leaves $\mu^H \phi^H + \mu^L \phi^L$ unchanged and maintains $\phi^H > 0 > \phi^L$. 

13
where $0 < M \equiv \mu^L \phi^L + \mu^H \phi^H < \mu^H \phi^H$ is unchanged by a mean-preserving change in $\phi^H$ and $\phi^L$. Starting from an initial parameter configuration in which $\Lambda = 1$, a mean-preserving decrease in $\phi^H$ increases the ratio $\frac{\mu^H \phi^H}{\mu^H \phi^H - M}$ and hence increases $\Lambda$ to a value greater than one, thereby increasing $X^*$ to a positive number.

When $X^* > 0$, the firm holds cash as a precaution against the possibility of being unable to make required payments in Regime $L$, when $\phi_t = \phi^L < 0$. Proposition 5 implies that a mean-preserving decrease in the variance of $\phi$ increases $X^*$; that is, a mean-preserving decrease in variance increases precautionary saving, which is the opposite of the standard result for precautionary saving by households. In models of household precautionary saving, risk averse households undertake precautionary saving if the third derivative of the utility function is positive. However, in the current model, shareholders are risk-neutral, thereby having utility functions with both second and third derivatives identically equal to zero. The precautionary saving is induced by the desire to avoid termination and the consequent loss of future dividends. A reduction in the variance of $\phi$ increases $\phi^L < 0$, which reduces the speed at which cash is depleted during Regime $L$, and thus increases the probability that a firm that enters Regime $L$ with a given amount of cash on hand will avoid running out of cash before the next Regime $H$ arrives. This effect dominates the opposing effect associated with a decrease in $\phi^H$ and thus a mean-preserving decrease in variance increases the marginal value of a unit of cash on hand and increases $X^*$.

## 2 Zero interest earned on cash held by firm: a closed-form solution

We will assume for remainder of this paper that the interest rate earned on cash held by the firm, $r$, equals zero. This assumption allows derivation of a closed-form solution to the system of ODEs in equations (8) and (9) and boundary conditions in equations (10) - (12). Setting $r = 0$ in the ODEs in equations (8) and (9) yields a system of first-order linear constant-coefficient homogeneous ordinary differential equations

$$
\begin{bmatrix}
  V^H_X \\
  V^L_X
\end{bmatrix} = A
\begin{bmatrix}
  V^H \\
  V^L
\end{bmatrix}
$$

where

$$
A \equiv \begin{bmatrix}
  \eta_2 & -\frac{\mu^L}{\phi^H} \\
  -\frac{\mu^H}{\phi^H} & \eta_1
\end{bmatrix}
$$
\[ \eta_1 \equiv \frac{\mu^H + \rho}{\phi^L} < 0 \quad (19) \]

and

\[ \eta_2 \equiv \frac{\phi^H}{\rho + \mu^L} > 0. \quad (20) \]

The parameter restriction in equation (1) implies that

\[ \eta_1 + \eta_2 < 0. \quad (21) \]

The parameters \( \eta_1 \) and \( \eta_2 \) have simple economic interpretations. Specifically, \( \eta_1^{-1} = \frac{\phi^H}{\mu + \rho} \equiv \nu^H > 0 \) and \( \eta_2^{-1} = \frac{\phi^L}{\rho + \mu^L} \equiv \nu^L < 0 \) are the myopic values of a unit of capital in Regimes \( H \) and \( L \), respectively, in Definition 2.

It is straightforward to solve the system of ODEs in equation (17). The details of the solution procedure are contained in Appendix B, where it is shown that the general solution of the system of ODEs is

\[
\begin{bmatrix}
V^H(X) \\
V^L(X)
\end{bmatrix} = c_1 \begin{bmatrix}
1 \\
1
\end{bmatrix} e^{\omega_1(X-X^*)} + c_2 \begin{bmatrix}
1 \\
1
\end{bmatrix} e^{\omega_2(X-X^*)},
\]

where \( \omega_1 \) and \( \omega_2 \) are the eigenvalues of \( A \) and satisfy

\[ \eta_1 < \omega_1 < 0 < \omega_2 < \eta_2. \quad (23) \]

Lemma 1 The eigenvalues \( \omega_1 \) and \( \omega_2 \) have the following properties

1. \( \omega_1 + \omega_2 = \frac{1}{\nu^H} + \frac{1}{\nu^L} < 0 \)

2. \( \frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{1}{\nu^H} (\nu^H + V^H_L) > 0. \)

The three undetermined constants, \( c_1, c_2, \) and \( X^* \) in the general solution in equation (22) are pinned down by the three boundary conditions in equations (10) - (12). As shown in Appendix B,

\[ c_1 = \frac{1}{\omega_2 - \omega_1} \frac{\omega_2}{\omega_1} < 0 \quad (24) \]

\[ c_2 = \frac{1}{\omega_1 - \omega_2} \frac{\omega_1}{\omega_2} > 0 \quad (25) \]

and

\[ X^* = \max \{ x_m, 0 \} \quad (26) \]
where
\[ x_m \equiv \frac{1}{\omega_2 - \omega_1} \ln \frac{\omega_1^2 \eta_2 - \omega_2}{\omega_2^2 \eta_2 - \omega_1}. \] (27)

Equation (26) implies that \( X^* > 0 \) if and only if \( x_m > 0 \). The following lemma provides alternative conditions for \( x_m > 0 \).

**Lemma 2** If \( r = 0 \), then \( \text{sign} (x_m) = \text{sign} (\Lambda - 1) = \text{sign} (\Gamma \nu^H + \nu^L) \).

Lemma 2 combined with the assumption in equation (2) implies that \( x_m > 0 \), and hence \( X^* > 0 \), if
\[ -\frac{\nu^L}{\nu^H} < \Gamma < \frac{\mu^H}{\rho + \mu^H} < 1. \]

Thus, if the myopic value of losses in Regime \( L \) are small enough relative the myopic profits in Regime \( H \) to make \( -\frac{\nu^L}{\nu^H} < \Gamma \), then \( X^* > 0 \). However, if \( \Gamma \leq -\frac{\nu^L}{\nu^H} < \frac{\mu^H}{\rho + \mu^H} \), which is consistent with an admissible parameter configuration, then \( X^* = 0 \). If \( -\frac{\nu^L}{\nu^H} \geq \frac{\mu^H}{\rho + \mu^H} \), then the losses in Regime \( L \) are large enough to violate the assumption in equation (2).

### 2.1 Properties of the Value of the Firm

This subsection presents useful properties of the value of the firm.

**Proposition 6** The value function \( V^H (X) \) has the following properties.

1. If \( X \geq X^* \), then \( V^H (X) = V^H (X^*) + X - X^* \).

2. \( V^H (X) \geq \nu^H + X = \frac{\phi^H}{\rho + \mu^H} + X \) and \( V^L (X) \geq X \).

3. If \( x_m \geq 0 \), then \( V^H (X^*) = \frac{1}{1 - \Gamma} (\nu^H + \nu^L) \) and \( V^L (X^*) = \frac{1}{1 - \Gamma} \left( \frac{\mu^H}{\rho + \mu^H} \nu^H + \frac{\rho + \mu^L}{\mu^L} \nu^L \right) < V^H (X^*) \).

4. \( V^H (X) < \frac{\phi^H}{\rho} + X \).

Statement 1 is that for values of cash on hand, \( X \), greater than \( X^* \), the value function in Regime \( H \) is a linear function of \( X \) with slope equal to one. When the firm is in Regime \( H \) with cash on hand \( X \), it can pay an immediate lump-sum dividend equal to \( X \) and then pay a flow of dividends \( \phi^H \) until the current Regime \( H \) ends. Following this feasible policy, the expected present value of dividends is the sum of \( X \) and the myopic expected present value of dividends, \( \nu^H = \frac{\phi^H}{\rho + \mu^H} \). Therefore, \( V^H (X) \geq \nu^H + X \). If the firm has cash on
hand, $X$, in Regime $L$, it can simply pay a liquidating dividend $X$ and then immediately terminate. Therefore, $V^L(X) \geq X$. (Statement 2). Statement 3 expresses $V^H(X^*)$ as a simple function of the roundtrip discount factor and the myopic values $\nu^H$ and $\nu^L$, which implies a simple expression for $V^L(X^*)$. Finally, if Regime $H$ were known with certainty to prevail forever, then the firm with cash on hand $X$ could pay an immediate dividend of $X$ in addition to a perpetuity of $\phi^H$ forever in which case the present value of dividends would be $\frac{\phi^H}{\rho} + X$. However, if there is a nonzero chance of a transition to Regime $L$, the expected present value of dividends will be less than $\frac{\phi^H}{\rho} + X$. (Statement 4).

Define $\hat{V}^H(X)$ and $\hat{V}^L(X)$ to be the values of a firm, in Regimes $H$ and $L$, respectively, that can pay negative dividends whenever it chooses and has cash on hand $X \geq 0$. Continuing to assume $r = 0$, it is straightforward to show that

$$\hat{V}^H(X) = X + \frac{1}{1-\Gamma} \left( \frac{\nu^H + \frac{\mu^L}{\rho + \mu^L} \nu^L}{\rho + \mu^H} \right) > 0$$

$$\hat{V}^L(X) = X + \frac{1}{1-\Gamma} \left( \frac{\mu^H}{\rho + \mu^H} \nu^H + \nu^L \right) > 0. \quad (28)$$

The expressions for $\hat{V}^H(X)$ and $\hat{V}^L(X)$ in equations (28) and (28) express the value of a firm that can pay negative dividends as the sum of three components. The first component is simply the current amount of cash on hand, $X$, since the firm can pay an immediate dividend of $X$ and operate forever without any cash on hand, because it can pay its required outflows $-\phi^L > 0$ whenever it is in Regime $L$ by paying negative dividends. The second component is the expected present value of the positive operating profits whenever the firm is in Regime $H$. If the firm is currently in Regime $H$, that expected present value is $\frac{\nu^H}{1-\Gamma}$. However, if the firm is currently in Regime $L$ at time 0, it will enter the next Regime $H$ at time $t_H > 0$ and $E \{ e^{-\rho t_H} \} = \frac{\mu^H}{\rho + \mu^H}$; in that case, the expected present value of positive operating profits whenever $\phi_t = \phi^H > 0$ is $\frac{1}{1-\Gamma} \frac{\mu^H}{\rho + \mu^H} \nu^H$. The third component is the expected present value of the negative operating profits whenever the firm is in Regime $L$. If the firm is currently in Regime $L$, that expected present value is $\frac{\nu^L}{1-\Gamma}$. However, if the firm is currently in Regime $H$ at time 0, it will enter the next Regime $L$ at time $t_L > 0$ and $E \{ e^{-\rho t_L} \} = \frac{\mu^L}{\rho + \mu^L}$; in that case, the expected present value of negative operating profits whenever $\phi_t = \phi^L < 0$ is $\frac{1}{1-\Gamma} \frac{\mu^L}{\rho + \mu^L} \nu^L$.

---

12 If $x_m = 0$, then $X^* = 0$ and the value of the firm in Regime $H$ is simply the myopic value $\nu^H$. This valuation is implied by Statement 3 since Lemma 2 implies that if $x_m = 0$, then $\nu^L = -\Gamma \nu^H$, so $\frac{1}{1-\Gamma} (\nu^H + \nu^L) = \nu^H$. 

17
Now compare the value of a firm that can pay negative dividends with the value of a firm that cannot pay negative dividends. First, consider the value $V^H(X^*)$ in Statement 3 of Proposition 6 for a firm in Regime $H$ that cannot pay negative dividends. Use the three components of $\tilde{V}^H(X)$ described above to interpret the expression for $V^H(X^*)$. The first component of $\tilde{V}^H(X^*)$, which is the amount of cash on hand, $X^*$, is absent from the expression for $V^H(X^*)$, thereby reducing $V^H(X^*)$ relative to $\tilde{V}^H(X)$. The second component of $\tilde{V}^H(X)$, which is $\frac{1}{1-\rho}\nu^H$, appears exactly the same as a component of $V^H(X^*)$. The third component of $\tilde{V}^H(X)$, which is $\frac{1}{1-\rho+\mu}v^L$, appears as $\frac{1}{1-\rho}v^L$ in the expression for $V^H(X^*)$. Since $\nu^L < 0$, the component $\frac{1}{1-\rho}v^H$ in the expression for $V^H(X^*)$ is less than the component $\frac{1}{1-\rho+\mu}v^L$ in the expression for $\tilde{V}^H(X)$. This component of $V^H(X^*)$ for a firm that is constrained from paying negative dividends is $\frac{\mu}{\mu^L} > 1$ times as large as the corresponding component for a firm that can pay negative dividends.. It is as if all of the negative cash flows in Regimes $L$ are multiplied by $\frac{\mu}{\mu^L} > 1$. Equivalently, it is as if all of the negative cash flows in Regimes $L$ are moved forward in time by an amount $t_L$, hence having a larger expected present value in absolute value.

The comparison of $V^L(X^*)$ and $\tilde{V}^L(X^*)$ is similar to that of $V^H(X^*)$ and $\tilde{V}^H(X^*)$. In particular, the first component of $\tilde{V}^L(X^*)$, which is the amount of cash on hand, $X^*$, is absent from the expression for $V^L(X^*)$, thereby reducing $V^L(X^*)$ relative to $\tilde{V}^L(X)$ by $X^*$. The second component of $\tilde{V}^L(X)$, which is $\frac{1}{1-\rho}p^H\nu^L$, appears exactly the same as a component of $V^L(X^*)$. The third component of $\tilde{V}^L(X)$, which is $\frac{1}{1-\rho+\mu}v^L$, appears as $\frac{1}{1-\rho}p^L\nu^L$ in the expression for $V^L(X^*)$. Since $\nu^L < 0$, the component $\frac{1}{1-\rho}p^L\nu^H$ in the expression for $V^L(X^*)$ is less than the component $\frac{1}{1-\rho+\mu}v^L$ in the expression for $\tilde{V}^L(X)$. This component of $V^L(X^*)$ for a firm that is constrained from paying negative dividends is $\frac{p^L}{\mu^L} > 1$ times as large (in absolute value) as the corresponding component for a firm that can pay negative dividends.. As in the case of a firm in Regime $H$, it is as if all of the negative cash flows in Regimes $L$ are multiplied by $\frac{p^L}{\mu^L} > 1$. That is, it is as if all of the negative cash flows in Regimes $L$ are moved forward in time by an amount $t_L$.

To summarize, for a firm with $X = X^*$, the cost of imposing the non-negativity constraint on dividends has two components. Starting with a firm that can pay negative dividends and has cash on hand equal to $X^*$, the imposition of a non-negativity constraint reduces the value of the firm by an amount equivalent to $X^*$ plus the expected present value of stream $-\frac{p}{\mu^L}\phi^L > 0$ whenever the firm is in Regime $L$.

Since the component of the valuation of the firm that involves current and future negative cash flows $\phi^L$ effectively multiplies $\phi^L$ by $\frac{p^L}{\mu^L}$, and since $V^L_X(X^*) = \frac{p^L}{\mu^L}$ (from Proposition
Remarkably, even though the firm will eventually, in finite time, terminate, its value equals the expected present value of weighted operating profits over the infinite future. Also remarkable is that the weights on the operating profits are simply $V^H_X(X^*) = 1$ in Regime $H$ and $V^L_X(X^*) = \frac{\rho + \mu^L}{\mu^L} > 1$ even though $X$ is not always $X^*$. Since the weight is larger than one in Regime $L$, when profits are negative, the value of the firm is less than the expected present value of (unweighted) profits over the infinite future. The weight $V^L_X(X^*) = \frac{\rho + \mu^L}{\mu^L}$ is just the right amount to adjust for the fact that eventually the firm will terminate and will not reap the continuation value at the termination date.

### 2.2 Symmetric Persistence

Proposition 4 describes the effects of persistence separately for Regime $H$ and Regime $L$. The effects in Proposition 4 are local effects, confined to a (one-sided) neighborhood of $X^* = 0$. This subsection analyzes the effects of persistence beyond the neighborhood near $X^* = 0$. For tractability, this analysis focuses on symmetric persistence, which is the situation with $\mu^L = \mu^H = \mu$.

**Proposition 7** Suppose $\mu^L = \mu^H = \mu$.

1. If $\mu \leq \frac{\rho}{\sqrt{\frac{\rho^H}{\mu^L} - 1}}$, then $X^* = 0$.

2. If $\frac{\rho}{\sqrt{\frac{\rho^H}{\mu^L} - 1}} \leq \mu < \infty$, then $X^* > 0$.

3. $\lim_{\mu \to \infty} X^* = 0$.

Proposition 7 implies that $X^*$ is non-monotonic in $\mu$. For sufficiently small $\mu$ (as described in Statement 1 of the proposition), $X^* = 0$. Increasing $\mu$ from that small value will increase $X^*$ to a positive value. The increase in $\mu^H$ ($= \mu$) hastens the expected transition from Regime $L$ to Regime $H$, thereby increasing the benefit of entering Regime $L$ with positive cash on hand, which tends to increase $X^*$. The increase in $\mu^L$ ($= \mu$) increases the expected present value of a dollar at the beginning of the next Regime $L$, which tends to increase $X^*$, but it reduces the myopic value of the firm, $\frac{\rho H^H}{\rho + \mu^L}$, in the subsequent Regime $H$, 

where $H^H$ is the transition from Regime $H$ to Regime $H$. The increase in $\mu^L$ hastens the expected transition from Regime $L$ to Regime $H$, thereby increasing the benefit of entering Regime $L$ with positive cash on hand, which tends to increase $X^*$. The increase in $\mu^L$ increases the expected present value of a dollar at the beginning of the next Regime $L$, which tends to increase $X^*$, but it reduces the myopic value of the firm, $\frac{\rho H^H}{\rho + \mu^L}$, in the subsequent Regime $H$,
which tends to decrease $X^*$. For high enough $\mu^L (= \mu)$, the second effect dominates the first effect (as suggested by Statement 5 of Proposition 4), and for sufficiently high $\mu^H = \mu^L = \mu$, an increase in $\mu$ eventually reduces $X^*$ toward zero.

3 Moments of Profits and Dividends

The long-run distributions of profits and dividends are degenerate because eventually a Regime $L$, with $\phi_t = \phi^L < 0$, will persist sufficiently long to exhaust the cash on hand before it ends, thereby terminating the firm.\(^{13}\) So, instead of focusing on steady-state or stationary distributions of profits and dividends, we focus on the distributions of profits and dividends over long periods of time.

Consider profits and dividends over the period of time from time 0 until time $\hat{t}$ for a firm that has cash on hand $X_0$, $0 \leq X_0 \leq X^*$, at time 0 and remains in operation at time $\hat{t}$, at which time it has cash on hand $X_{\hat{t}}$, $0 \leq X_{\hat{t}} \leq X^*$. Continue to assume that the firm earns zero interest on its cash on hand. Therefore, set $r = 0$ in equation (3) and integrate both sides of the resulting equation with respect to time from 0 to $\hat{t}$ to obtain

$$X_{\hat{t}} - X_0 = \int_0^{\hat{t}} \phi_t dt - \int_0^{\hat{t}} D_t dt.$$ (31)

Equation (31) states that the change in cash on hand between time 0 and time $\hat{t}$ equals the amount by which accumulated operating profits, $\phi_t$, exceed accumulated dividends, $D_t$, over this period of time. Since both $X_0$ and $X_{\hat{t}}$ are in $[0, X^*]$, the change in cash on hand, in either direction, cannot exceed $X^*$. That is, $|X_{\hat{t}} - X_0| \leq X^*$ so

$$|\bar{\phi}_{\hat{t}} - \overline{D}_{\hat{t}}| \leq \frac{X^*}{\hat{t}},$$ (32)

where $\overline{D}_{\hat{t}} \equiv \frac{1}{\hat{t}} \int_0^{\hat{t}} D_t dt$ is the average dividend over the period from time 0 to time $\hat{t}$ and $\bar{\phi}_{\hat{t}} \equiv \frac{1}{\hat{t}} \int_0^{\hat{t}} \phi_t dt$ is the average operating profit over the period from time 0 to time $\hat{t}$.

For sufficiently large $\hat{t}$, the average value of dividends is arbitrarily close to the average value of profits. The analysis in this section confines attention to $\hat{t}$ sufficiently large that approximately (1) $\overline{D}_{\hat{t}} = \bar{\phi}_{\hat{t}}$, (2) $\phi_t = \phi^H$ a fraction $\frac{\mu^H}{\mu^H + \mu^L}$ of the time, (3) and $\phi_t = \phi^L$ a

\(^{13}\)When the firm is in Regime $L$ with cash on hand $X_t > 0$, the probability that the firm will run out of cash before the current regime $L$ ends is $1 - \exp\left(\frac{\mu^H X_t}{\phi_t}\right) > 0$, so there is a positive probability that the firm will terminate in each regime $L$. Eventually, termination will occur.
fraction \( \frac{\mu^L}{\mu^L + \mu^H} \) of the time. The following lemma presents the distribution of dividends for sufficiently large \( \hat{t} \).

**Lemma 3** For sufficiently large \( \hat{t} \), \( \Pr \{ D_t = \phi^H \} \approx \alpha \) and \( \Pr \{ D_t = 0 \} \approx 1 - \alpha \), where \( 0 < \alpha \equiv \frac{\mu^H + \mu^L \phi^L}{\mu^L + \mu^H} < \frac{\mu^H}{\mu^L + \mu^H} \).

The firm pays positive dividends only when \( \phi_t = \phi^H \), but not always when \( \phi_t = \phi^H \). Therefore, the fraction of time that the firm pays positive dividends, \( \alpha \), is smaller than the fraction of the time that the firm has positive profits, \( \phi_t = \phi^H \).

It is straightforward to calculate the variance of the exogenous operating profit, \( \phi_t \), and to use Lemma 3 to calculate the variance of the endogenous dividends over the interval of time from 0 to \( \hat{t} \).

**Proposition 8** For sufficiently large \( \hat{t} \), \( \frac{\text{Var}(D)}{\text{Var}(\phi)} \approx \frac{1 + \frac{\mu^L \phi^L}{\mu^L + \mu^H}}{1 - \frac{\mu^L \phi^L}{\mu^L + \mu^H}} < 1 \).

Proposition 8 indicates that optimal payout policy is characterized by "dividend smoothing." Dividend smoothing arises in this simple framework with linear utility as a result of the financing constraint that prevents an ongoing firm from raising additional funds by issuing equity.

## 4 Capital Investment

To introduce a capital investment decision in addition to a payout decision, assume that the operating profit of the firm at time \( t \) is \( \phi_t K_t \), where \( K_t \) is the firm’s capital stock at time \( t \), and \( \phi_t \) is the operating profit per unit of capital, which is either \( \phi^H > 0 \) or \( \phi^L < 0 \), where \( \phi^H \) and \( \phi^L \) continue to satisfy equation (1). In previous sections, the capital stock was implicitly held fixed and set equal to one. Now assume that the capital stock evolves endogenously as

\[
\dot{K}_t = I_t - \delta K_t, \tag{33}
\]

where \( I_t \geq 0 \) is the rate of gross investment at time \( t \) and \( \delta \geq 0 \) is the constant rate at which physical capital depreciates. In addition to the non-negativity constraint on gross investment, there is an upper bound on the rate of investment specified as

\[
I_t \leq \tilde{I} K_t, \tag{34}
\]
where $\bar{r} < \min \{\phi^H, \rho\}$ is the upper bound on the rate of gross investment relative to the capital stock.\footnote{One can view upper bound $\bar{r}$ as arising from an extreme form of convex adjustment costs, $c(i) K$, with a kink at $i = \bar{r}$. For instance, let $c(i) = \theta \left( \frac{i}{\psi} \right)^\psi$, where $\theta > 0$ and $\psi > 1$. In the limit as the convexity parameter $\psi$ grows without bound, $c(i) = 0$ for $i \leq \bar{r}$ and the right hand derivative of $c(i)$ is infinite at $i = \bar{r}$.} For simplicity that admits a closed-form solution for the value of the firm, continue to assume that the interest rate is $r = 0$ and also assume that the depreciation rate of capital is $\delta = 0$.

The lemma that follows provides the value of a firm that does not face the financing constraint that prevents the payment of negative dividends. This lemma helps provide a sufficient condition to guarantee that the maximized expected present value of future net cash flow, $(\phi_t - i_t) K_t$, is positive even when the firm is currently in Regime $L$.

**Lemma 4** Define $\widetilde{V}^{(j)}(X, K)$ to be the maximized expected present value of dividends payable by a firm that can undertake non-negative capital investment and can pay negative, as well as positive, dividends. Then $\widetilde{V}^L(X, K) = X + \frac{(\rho + \mu^L - i^L)(\phi^L - i^L) + \mu^H(\phi^H - i^H)}{(\rho + \mu^H - i^L)(\rho + \rho - \mu^L) - \mu^H \mu^L} K$ and $\widetilde{V}^H(X, K) = X + \frac{(\rho + \mu^H - i^H)(\phi^H - i^H) + \mu^L(\phi^L - i^L)}{(\rho + \mu^L - i^L)(\rho + \mu^H - i^L) - \mu^L \mu^H} K$ where $i^H \in [0, \bar{r}]$ and $i^L \in [0, \bar{r}]$ are the optimal values of the investment-capital ratio in Regime $H$ and Regime $L$, respectively, and $\bar{r} < \rho$.

**Corollary 3** If $(\rho + \mu^L - i^H)(\phi^L - i^L) + \mu^H(\phi^H - i^H) > 0$, then $\widetilde{V}^H(X, K) > \widetilde{V}^L(X, K) > 0$ for $K > 0$ and $X \geq 0$.

The condition in Corollary 3 can be written as

$$- (\phi^L - i^L) < \alpha (\phi^H, i^H) \equiv \frac{\phi^H - i^H}{\rho + \mu^L - i^H}.$$

Henceforth we assume that equation (35) holds so that the maximized expected present value of net cash flows is always positive for an unconstrained firm that can pay negative as well as positive dividends.

With the inclusion of capital investment, there are three uses of funds—capital investment $I_t$, dividends $D_t$, and accumulation of cash on hand $\bar{X}_t$—so the accumulation of cash by an ongoing firm is

$$\bar{X}_t = \phi_t K_t - D_t - I_t.$$

The optimal values of investment and dividends satisfy

$$\rho V(X_t, K_t) = \max_{D_t \geq 0, \tau K_t \geq 0, I_t \geq 0} \left( D_t + \frac{1}{\delta t} E_t \{dV(X_t, K_t)\} \right)$$
where the expected capital gain, \( \frac{1}{dt} E_t \{dV(X_t, K_t) \} \) in regime \( j, j \in \{ L, H \} \), is given by

\[
\frac{1}{dt} E_t \{dV^{(j)}\} = V^{(j)}_X \dot{X}_t + V^{(j)}_K \dot{K}_t + \mu^{(-j)} (V^{(-j)} - V^{(j)}). \tag{38}
\]

Substitute the expressions for \( \dot{K}_t \) and \( \dot{X}_t \) from equation (33) with \( \delta = 0 \) and equation (36), respectively, into the expected capital gain in equation (38) to rewrite equation (37) as

\[
\rho V^{(j)} = \max_{D \geq 0, iK \geq I \geq 0} \left( 1 - V^{(j)}_X \right) D + \left( V^{(j)}_K - V^{(j)}_X \right) I + \phi^j K V^{(j)}_X + \mu^{(-j)} V^{(-j)} - \mu^{(-j)} V^{(j)} \tag{39}
\]

The maximand on the right-hand side of equation (39) is linear in both dividends and capital investment. The optimal value of dividends satisfies the first-order condition

\[
V^{(j)}_X \geq 1 \tag{40}
\]

and the complementary slackness condition

\[
\left( 1 - V^{(j)}_X \right) D = 0. \tag{41}
\]

Equation (40) implies that optimal dividend policy prevents the firm from ever getting into a situation where the marginal valuation of a dollar of cash inside the firm, \( V^{(j)}_X \), is less than one. Equation (41) implies that if the marginal valuation of a dollar of cash inside the firm, \( V^{(j)}_X \), exceeds one, the firm will not pay dividends. The firm will pay dividends only when \( V^{(j)}_X \) equals one.

The optimal rate of investment satisfies the following two equations

\[
\left( V^{(j)}_K - V^{(j)}_X \right) I \geq 0 \tag{42}
\]

\[
\left( V^{(j)}_K - V^{(j)}_X \right) (iK - I) \leq 0 \tag{43}
\]

Equation (42) implies that if the marginal valuation of capital, \( V^{(j)}_K \), is less than the marginal valuation of cash inside the firm, \( V^{(j)}_X \), optimal gross investment equals zero. Equation (43) implies that if the marginal valuation of capital, \( V^{(j)}_K \), is greater than the marginal valuation of cash inside the firm, \( V^{(j)}_X \), the firm will invest at the highest possible rate, \( iK \).

The value function \( V(X, K) \) is linearly homogeneous in \( X \) and \( K \) so it may be written as

\[
V^{(j)}(X, K) \equiv v^{(j)}(x) K, \tag{44}
\]

\footnote{To reduce notational clutter, the arguments \((X_t, K_t)\) are suppressed.}
where \( x \equiv \frac{x}{R} \) and

\[
V_X^{(j)} = v_x^{(j)}(x), \quad V_K^{(j)} = v^{(j)}(x) - v_x^{(j)}(x) x, \quad \text{and} \quad V_{XX}^{(j)} = v_x^{(j)}(x) \frac{1}{K}.
\] (45)

Use equations (44) and (45) to rewrite equations (40) - (43) as

\[
v_x^{(j)}(x) \geq 1
\] (46)

\[
(1 - v_x^{(j)}(x)) d^{(j)} = 0
\] (47)

\[
(v^{(j)}(x) - (1 + x) v_x^{(j)}(x)) i^{(j)} \geq 0
\] (48)

\[
(v^{(j)}(x) - (1 + x) v_x^{(j)}(x)) (i - i^{(j)}) \leq 0
\] (49)

where \( d \equiv \frac{D}{K} \) is the ratio of dividends to the capital stock and \( i \equiv \frac{I}{K} \) is the investment-capital ratio.

Now define \( d^* \) and \( i^* \) as the values of \( d \) and \( i \) that attain the maximum on the right hand side of equation (39). Divide both sides of that equation by \( K \) and use equations (44), (45), and the complementary slackness condition from equation (47) to obtain the ODEs that prevail in Regimes \( H \) and \( L \), respectively,

\[
\begin{align*}
(\rho + \mu^L) v^H(x) &= (v^H(x) - (1 + x) v_x^H(x)) i^{H*} + \phi^H v_x^H(x) + \mu^L v^L(x) \quad (50a) \\
(\rho + \mu^H) v^L(x) &= (v^L(x) - (1 + x) v_x^L(x)) i^{L*} + \phi^L v_x^L(x) + \mu^H v^H(x). \quad (50b)
\end{align*}
\]

### 4.1 The Investment Trigger in Regime \( H \)

Define \( x^*_I \geq 0 \) to be the value of \( x_I \) that triggers capital investment in Regime \( H \). Specifically, in Regime \( H \), optimal \( i_t = 0 \) for \( x_t < x^*_I \) and optimal \( i_t > 0 \) for \( x_t > x^*_I \). In Regime \( H \), optimal capital investment is positive if \( V_K^H(X, K) > V_K^H(X, K) \), or equivalently, if \( v^H(x) - x v_x^H(x) > v_x^H(x) \). Therefore, optimal investment is positive when \( x = 0 \) in Regime \( H \) if \( v^H(0) > v_x^H(0) \); optimal investment is zero when \( x = 0 \) in Regime \( H \) if \( v^H(0) \leq v_x^H(0) \). To compare the values of \( v^H(0) \) and \( v_x^H(0) \), evaluate equation (50a) at \( x = 0 \) and use the fact that \( v^L(0) = 0 \) to obtain

\[
(\rho + \mu^L) v^H(0) = (v^H(0) - v_x^H(0)) i^{H*} + \phi^H v_x^H(0).
\] (51)

Equation (51) implies the following lemma.

**Lemma 5** \( \frac{v^H(0)}{v_x^H(0)} = \frac{\phi^H - i^{H*}}{\rho + \mu^L - i^{H*}} \geq 1 \) as \( \frac{\phi^H}{\rho + \mu^L} \leq 1 \).
The expression for \( \frac{v^H(0)}{v_x^H(0)} \) that follows the first equality in Lemma 5 is simply the expected present value of net cash flows accruing to a current unit of capital over the remaining life of the current Regime \( H \). Specifically, the numerator, \( \phi^H - i^{H^*} \), is the net cash flow from operations per unit of capital, \( \phi^H \), minus optimal investment expenditure per unit of capital, \( i^{H^*} \). Because the depreciation rate of capital is assumed to be zero, the capital stock grows at rate \( i^{H^*} \). Therefore, the denominator, \( \rho + \mu^L - i^{H^*} \), is the discount rate adjusted for the instantaneous probability that the current Regime \( H \) ends, \( \rho + \mu^L \), less the growth rate of capital during Regime \( H \), \( i^{H^*} \). Therefore, the ratio \( \frac{\phi^H}{\rho + \mu^L - i^{H^*}} \) is the expected present value of a stream that begins at rate \( \phi^H - i^{H^*} \), and grows at rate \( i^{H^*} \), until the current Regime \( H \) ends. Lemma 5 states that whether the ratio \( \frac{v^H(0)}{v_x^H(0)} \) is greater or less than one depends solely on whether the myopic value of capital in Regime \( H \), \( \frac{\phi^H}{\rho + \mu^L} \), is greater or less than one. This lemma implies the following proposition.

**Proposition 9** \( x^*_t = 0 \) if and only if \( \frac{\phi^H}{\rho + \mu^L} \geq 1 \).

Proposition 9 implies that optimal investment is always positive in Regime \( H \), regardless of the amount of cash on hand, \( x \), if and only if the myopic value of a unit of capital, \( \frac{\phi^H}{\rho + \mu^L} \), is greater than or equal to one. When this myopic value is greater than or equal to one, the expected payoff to an additional unit of capital is at least as large as its cost, even if the firm terminates at the end of the current regime \( H \). Therefore, optimal investment is positive if \( \frac{\phi^H}{\rho + \mu^L} \geq 1 \), which explains the "if" part of Proposition 9.

To understand the "only if" part of Proposition 9, observe that the investment-capital ratio enters the ODE in equation (51) only through the term \( (v^H(0) - v_x^H(0)) i^{H^*} \). The optimal rate of investment will be positive at \( x = 0 \) in Regime \( H \) only if \( v^H(0) - v_x^H(0) > 0 \) so that \( v^H(0) i^{H^*} > v_x^H(0) i^{H^*} \). The term on the left hand side of this inequality, \( v^H(0) i^{H^*} \), reflects the fact that \( i^{H^*} \) is the growth rate of the capital stock, and since each unit of capital is worth \( v^H(0) \), investment contributes \( v^H(0) i^{H^*} \) to the growth in the value of the firm. The term on the right hand side, \( v_x^H(0) i^{H^*} \), reflects the fact that investment at rate \( i^{H^*} \) reduces the cash holdings of the firm by \( i^{H^*} \) per unit of capital, and since the marginal valuation of cash is \( v_x^H(0) \), this use of cash holdings reduces the value of the firm by \( v_x^H(0) i^{H^*} \). Comparing the growth in the value of the firm arising from the increase in the capital stock, \( v^H(0) i^{H^*} \), with the valuation of the reduction in cash induced by undertaking this investment, \( v_x^H(0) i^{H^*} \), the firm will strictly prefer a positive rate of investment to zero investment if and only if \( v^H(0) > v_x^H(0) \). Finally, Lemma 5 implies that \( v^H(0) > v_x^H(0) \)
if and only if the myopic value of capital in Regime $H$, $\frac{\phi^H}{\rho+\mu^L}$, is greater than one, so that $x_i^* = 0$ if and only if $\frac{\phi^H}{\rho+\mu^L} \geq 1$.

**Corollary 4** Suppose that $\frac{\phi^H}{\rho+\mu^L} = 1$, which implies $x_i^* = 0$. Then

1. $x_i^*$ increases in response to an increase in $\rho + \mu^L$,
2. $x_i^*$ increases in response to a decrease in $\phi^H$, and
3. $x_i^*$ is invariant to any change in a single parameter or a joint change in multiple parameters that leaves the myopic value of capital in Regime $H$, $\frac{\phi^H}{\rho+\mu^L}$, unchanged.

Corollary 4 is based on the fact that $x_i^*$ will be positive only if the myopic value of a unit of capital in Regime $H$ is less than one. Starting from a situation in which the myopic value equals one, the expected present value can be reduced below one by an increase in the discount rate, $\rho$, the instantaneous probability of switching to Regime $L$ from Regime $H$, $\mu^L$, or by a reduction in the instantaneous flow of profit per unit of capital in Regime $H$, $\phi^H$.

### 4.2 Investment in Regime L

In Regime $L$, the firm has negative operating profits. Proposition 2 states that the optimal payout for an ongoing firm in Regime $L$ is zero dividends, but this proposition was proved for the case without the possibility of capital investment. This section extends the zero-dividend result to the case with investment and then derives a sufficient condition for optimal investment to be zero whenever an ongoing firm is in Regime $L$.

First differentiate equation (50a) with respect to $x$ to obtain

$$
(\rho + \mu^L) v_x^H (x) = (\phi^H - (1 + x) i^H) v_{xx}^H (x) + \mu^L v_x^L (x). \tag{52}
$$

Now evaluate equation (52) at $x = x_D^*$ and use the boundary conditions $v_x^H (x_D^*) = 1$ and $v_{xx}^H (x_D^*) = 0$ to obtain

$$
v_x^L (x_D^*) = 1 + \frac{\rho}{\mu^L} > 1. \tag{53}
$$

\[\text{16}\] Since $\frac{dx^*}{dx} = 0$ except at $x = x_i^*$, and since $v^H (x_i^*) - (1 + x_i^*) v_x^H (x_i^*) = 0$, $(v^H (x) - (1 + x) v_x^H (x)) \frac{dx^*}{dx} = 0$.  

---

26
Since $v^L_x(x^*_D) > 1$, the firm will not pay dividends in Regime $L$ when $x = x^*_D$. Since $x$ never exceeds $x^*_D$ for an ongoing firm, it is never optimal for an ongoing firm to pay dividends when it is in Regime $L$.

To analyze optimal investment when $x = x^*_D$ in Regime $L$, evaluate equation (50a) at $x = x^*_D$ and use the boundary condition $v^H_x(x^*_D) = 1$ to obtain

$$\left(\rho + \mu^L\right) v^H(x^*_D) = \left(v^H(x^*_D) - (1 + x^*_D)\right)i^H + \phi^H + \mu^L v^L(x^*_D).$$

(54)

Use equations (53) and (54) to obtain

$$v^L(x^*_D) - (1 + x^*_D) v^L_x(x^*_D) = \frac{1}{\mu^L} \left[\left(\rho + \mu^L - i^H\right) \left(v^H(x^*_D) - (1 + x^*_D)\right) - \phi^H\right].$$

(55)

Optimal investment will be zero in Regime $L$ for all $x \leq x^*_D$ if the left hand side of equation (55) is negative. The following proposition provides a sufficient condition for the right hand side of equation (55) to be negative, and hence for optimal investment to be zero whenever an ongoing firm is in Regime $L$.

**Proposition 10** Define the linear function $\gamma(\phi^H) \equiv \frac{\mu^H}{\rho + \mu^L - \bar{i}} (\phi^H - \xi)$, where

$$\xi \equiv \left(1 + \frac{\rho - \bar{i}}{\mu^L}\right) \left(1 + \frac{\rho + \mu^L - \bar{i}}{\mu^L}\right) \rho > \rho > \bar{i}. \text{ If } -\phi^L > \gamma(\phi^H), \text{ then optimal investment is zero whenever an ongoing firm is in Regime } L.$$

Proposition 10 implies that if the operating profit in Regime $H$ is sufficiently small, that is, if $\phi^H \leq \xi$, then optimal investment is zero whenever Regime $L$ prevails in an ongoing firm. Alternatively, if $\phi^H > \xi$, then if the operating loss in Regime $L$, $-\phi^L$, is sufficiently large, optimal investment is zero whenever Regime $L$ prevails.\(^{17}\) Henceforth, we confine attention to $-\phi^L > \gamma(\phi^H)$ so that optimal investment is always zero for an ongoing firm in Regime $L$.

### 5 Analytic Expressions for Dividend and Investment Triggers in Regime H

This section describes the conditions that lead the firm to pay dividends and to undertake capital investment in Regime $H$. To keep the number of cases manageable, the analysis is

\(^{17}\)In Figure 2, all four regions will be feasible if $\gamma(\rho + \mu^L) < \frac{\mu^L}{\rho + \mu^L - \bar{i}} \mu^H$. This condition will be satisfied if $\frac{\mu^H}{\rho + \mu^L - \bar{i}} (\rho + \mu^L - \xi) < \frac{\mu^L}{\rho + \mu^L - \bar{i}} \mu^H$, which is equivalent to $\rho + \mu^L - \xi < \frac{\rho + \mu^L - \bar{i}}{\rho + \mu^L - \bar{i}} \mu^L = \left(1 - \frac{\bar{i}}{\rho + \mu^L - \bar{i}}\right) \mu^L$, which is equivalent to $\xi - \rho > \frac{\bar{i}}{\rho + \mu^L - \bar{i}} \mu^L$. 27
Proposition 11 results on the optimal trigger values for investment and dividends. The following proposition summarizes the optimal triggers for investment and dividends. This departure from constant coefficients complicates the solution of the ODE. Appendix C derives an analytic solution for $v^H(x)$, which includes Kummer functions, and derives the optimal triggers for investment and dividends. The following proposition summarizes the results on the optimal trigger values for investment and dividends.

**Proposition 11** Define $a \equiv \frac{\rho + \mu L}{\rho + \mu L - \frac{1}{\tau}} > 0$, $b \equiv \frac{\rho + \mu L - \frac{1}{\tau}}{\rho + \mu L} > a > 0$, and for $x < \frac{\phi^H}{\tau} - 1$, define $z(x) \equiv \eta_1 \left(1 + x - \frac{\phi^H}{\tau}\right) > 0$. Also define $x_m \equiv \frac{1}{\omega_2 - \omega_1} \ln \frac{\eta_2 - \omega_2}{\eta_2 - \omega_1} > 0$.

1. If $\frac{\phi^H}{\rho + \mu L} \geq 1$, then $x_I^* = 0$ and
   
   (a) if $0 < \beta \left(\phi^H, \tau\right) \equiv \frac{\phi^H - \frac{1}{\tau}}{\rho + \mu L - \frac{1}{\tau}} \mu L \frac{1}{\rho + \mu L} < -\phi^L$, then $x_D^* = 0$ so $I = \tau K$ and $D = (\phi^H - \frac{1}{\tau}) K$ whenever Regime $H$ prevails.
   
   (b) if $\beta \left(\phi^H, \tau\right) \equiv \frac{\phi^H - \frac{1}{\tau}}{\rho + \mu L - \frac{1}{\tau}} \mu L \frac{1}{\rho + \mu L} > -\phi^L$, then $x_D^* > x_I^* = 0$ so $I = \tau K$ whenever Regime $H$ prevails and $D = (\phi^H - (1 + x_D^*) \frac{1}{\tau}) K$ whenever $x = x_D^*$.

   - $x_D^*$ given by the unique positive root of $U(a + 2, b + 3, z(x_D^*)) \equiv b_{a,b} \frac{a_{a,b,z(0)}}{M(a + 2, b + 3, z(x_D^*))}$.

2. If $\frac{\phi^H}{\rho + \mu L} < 1$, then $x_I^* > 0$ and
   
   (a) if $x_m \leq 0$, then $x_D^* = 0$ and an ongoing firm never invests in capital.
   
   (b) if $x_m > 0$, then $x_D^* > 0$ and
      
      i. if $x_m > \max \left[0, \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1\right]$, then $x_D^* = x_m$ and an ongoing firm never invests in capital.
      
      ii. if $x_m < \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1$, then $x_D^* > x_I^* > 0$.

   - $x_I^*$ is the unique root in $(0, x_m)$ of $e^{(\omega_2 - \omega_1)x_I^*} = \frac{\eta_2 - \omega_2 - 1}{\eta_2} \frac{1}{\omega_1} \omega_2$ and $0 < x_I^* < \frac{1}{\omega_2} - 1$.

---

18 The assumption that $0 < \tau < \rho$ implies that $a > 0$ and $b > 0$.  

28
\[ \bullet x_D^* \text{ is the unique root in } \left( x^*_D, \frac{\phi_H}{\rho + \mu L} - 1 \right) \text{ of } \]
\[ \frac{U(a+2, b+3, z(x_D^*))}{M(a+2, b+3, z(x_D^*))} = \frac{1}{(b+1)(b+2)} \frac{U(a, b+1, z(x_D^*)) + (1 + \gamma_1)(a + 1, b + 2, z(x_D^*))}{M(a, b+1, z(x_D^*)) - (1 + \gamma_1) \frac{\phi_H}{\rho + \mu L} M(a+1, b+2, z(x_D^*))} \]

As stated in Proposition 9, the myopic value of a unit of capital in Regime \( H \), \( \frac{\phi_H}{\rho + \mu L} \), is the sole factor that determines whether the investment threshold, \( x_I^* \), is positive or zero. Statement 1 in Proposition 11 describes the case in which the myopic value of a unit of capital in Regime \( H \) is greater than one. In this case, the firm invests whenever it is in Regime \( H \) so the investment threshold is \( x_I^* = 0 \). The conditions in Statements 1a and 1b that determine whether the dividend threshold, \( x_D^* \), is zero or positive are straightforward adaptations of Proposition 3. Specifically, the condition in Statement 1a can be written as \( V_X^H (0) = \frac{\mu_L}{\rho + \mu L} \times \frac{\phi_H}{\rho + \mu L} \times \frac{\phi_H^P}{\rho + \mu L^*} < 1 \), which has the same marginal valuation of cash on hand as in Corollary 1, except that (1) \( r = 0 \) and (2) the myopic valuation in Corollary 1, \( \frac{\phi_H}{\rho + \mu L} \), is replaced by an investment-augmented myopic valuation \( \frac{\phi_H^P}{\rho + \mu L^*} \), which is the expected present value of the flow of operating profits less investment expenditures that begin equal to \( \phi_H - \bar{t} \) and grow at rate \( \bar{t} \) until the current Regime \( H \) ends, with hazard rate \( \mu L^* \). When the product of these three terms, which equals \( V_X^H (0) \), is less than one, the firm always pays dividends in Regime \( H \) and \( x_D^* = 0 \) (Statement 1a). When the product of these three terms is greater than one, \( V_X^H (0) > 1 \) and the firm accumulates cash on hand in Regime \( H \) whenever \( x < x_D^* \). The value of \( x_D^* \) is the unique value of \( x > 0 \) for which \( v_X^H (x) = 1 \) (Statement 1b).

Statement 2 of Proposition 11 addresses the case in which the myopic value of a unit of capital in Regime \( H \), \( \frac{\phi_H}{\rho + \mu L} \), is less than one, so optimal investment is zero when \( x = 0 \) in Regime \( H \) and hence \( x_I^* > 0 \). If \( x_D^* < x_I^* \), then the ODE in equation (17) along with the boundary conditions in equations (10) - (12) continue to characterize \( x_D^* \), so that \( x_D^* = \max [0, x_m] \) as in equation (26). Furthermore, when \( x_D^* < x_I^* \), an ongoing firm never invests in physical capital. The dividend trigger, \( x_D^* \), will be less than the investment trigger, \( x_I^* \), when \( \max [0, x_m] < x_I^* \). This situation arises in two cases. In one case, \( x_m \leq 0 \), which implies that the dividend trigger is \( x_D^* = 0 \) so the firm never accumulates cash on hand and never invests in capital (Statement 2a). In the other case, \( 0 < x_m < x_I^* \), which arises if \( x_m > \max \left[ 0, \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 \right] \), so \( x_D^* = x_m \) and an ongoing firm never invests in capital (Statement 2(b)i). If, however, \( \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 - x_m > 0 \), then \( V_K (x_D^* K, K) - V_X (x_D^* K, K) > 0 \) so optimal investment is positive when \( x = x_m \) and hence \( x_I^* < x_D^* \) (Statement 2(b)ii).

The following lemma leads to a sufficient condition for the situation in 2(b)i to prevail.
Lemma 6 : $\text{sign} \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 \right) = -\text{sign} \left( -\phi^L - \delta \left( \phi^H \right) \right)$, where $\delta \left( \phi^H \right) \equiv \frac{\phi^H}{\rho + \mu^R} \left( \rho + \mu^H \right) - (\rho + \mu^H) + \frac{\mu^L}{\rho + \mu^R} \mu^H$.

Each of the two threshold levels, $x^*_D$ and $x^*_L$, can be either zero or positive, so there are four possible combinations of positive and zero thresholds corresponding to Statements 1a, 1b, 2a, and 2b of Proposition 11. Figure 2 illustrates four regions in parameter space that yield these four different combinations and indicates the role of each of the six parameters $(\phi^H, \phi^L, \mu^H, \mu^L, \rho, \text{and } \tau)$ in determining which of the four combinations of zero and positive triggers for dividends and investment prevail. In this figure, the horizontal axis is the flow of net operating profit, $\phi^H > 0$, in Regime $H$, the vertical axis is the flow of net operating losses, $-\phi^L > 0$, in Regime $L$, and the remaining four parameters $(\mu^H, \mu^L, \rho, \text{and } \tau)$ are held constant at values indicated on the axes. This figure is a two-dimensional slice of the 6-dimensional parameter space. Remarkably, the boundaries between regions in the $(\phi^H, \phi^L)$ plane are linear in the parameters $\phi^H$ and $\phi^L$. Table 1 lists the various linear equations that are delineate these boundaries and identifies two points on each line to facilitate drawing Figure 2.19

6 Dynamic Behavior of Cash and Capital

Figure 3 illustrates the dynamic behavior of cash on hand, $X_t$, and the capital stock, $K_t$. The ratio of cash to the capital stock, that is, $x_t \equiv \frac{X_t}{K_t}$, at any point in the $(K_t, X_t)$ plane is the slope of the line from the origin through that point. Two particular values of $x$ are highlighted in the figure: the line with slope equal to $x^*_D$ is the locus of combinations of $X_t$ and $K_t$ for which $x_t = x^*_D$, the value of $x_t$ that triggers positive capital investment; the line with slope equal to $x^*_L$ is the locus of combinations of $X_t$ and $K_t$ for which $x_t = x^*_L$, the value of $x_t$ that triggers positive dividends. This figure is drawn for the case in which (1) Statement 2(b)ii in Proposition 11 holds; (2) the condition in Proposition 10 holds so

19In addition to identifying two points on the line $-\phi^L = \delta \left( \phi^H \right)$, where $\delta \left( \phi^H \right)$ is defined in Lemma 6, Table 1 provides a sufficient condition for $\delta \left( \xi \right) > 0$, which implies that the line $-\phi^L = \delta \left( \phi^H \right)$ crosses the horizontal axis to the left of point D. To derive this condition, $\delta \left( \xi \right) = \frac{1}{\rho + \mu^R} \left[ (\rho + \mu^H) \xi - (\rho + \mu^H) (\rho + \mu^L) + \mu^L \mu^H \right] = \frac{1}{\rho + \mu^R} \left[ (\rho + \mu^H) \xi - (\rho + \mu^H) + \mu^H \right] = \frac{\mu^H}{\rho + \mu^R} \left[ 1 + \xi + \frac{\mu^H - \mu^L + \mu^H}{\mu^H} \right]$. The definition of $\xi$ and the assumption that $\rho > \tilde{\tau}$ imply that $\frac{1}{\rho} \xi \equiv \left( 1 + \frac{\rho}{\mu^H} \right) \left( 1 + \frac{\mu^H - \mu^L + \mu^H}{\mu^H} \right) > \frac{\mu^H + \rho + \mu^H - \tau}{\mu^H}$, so $\frac{1}{\rho} \xi - \frac{\mu^H + \rho + \mu^H - \tau}{\mu^H} > -\frac{7}{\mu^H}$. Therefore, $\delta \left( \xi \right) > \frac{\mu^H}{\rho + \mu^R} \left[ \frac{1}{\mu^H} \xi - \frac{7}{\mu^H} \right] = \frac{\rho}{\rho + \mu^R} \left( \xi - 7 \right) > 0.$
**Condition for \( \tilde{V}(X, K) > 0 \)**

\[- (\phi^L - i^L) < \alpha (\phi^H, i^H) \equiv \frac{\phi^H - \alpha}{\rho + \mu^L - i^H} \mu^H \quad \text{(Eq. 35)}\]

Point C: \( \alpha (i^H, i^H) = 0 \) and Point A: \( \alpha (\rho + \mu^L, i^H) = \mu^H \)

**Conditions for \( x_B^* = x_I^* = 0 \)**

\[\frac{\phi^H}{\rho + \mu^L} \geq 1 \quad \text{(Proposition 11 Statement 1)} \]

\[- \phi^L > \beta (\phi^H, i^H) \equiv \frac{\phi^H - \alpha}{\rho + \mu^L - i^H} \mu^L \quad \text{(Proposition 11 Statement 1a)} \]

Point C: \( \beta (i^H, i^H) = 0 \) and Point B: \( \beta (\rho + \mu^L, i^H) = \mu^L \)

**Conditions for \( x_B^* > x_I^* = 0 \)**

\[\frac{\phi^H}{\rho + \mu^L} < 1 \quad \text{(Proposition 11 Statement 2)} \]

\[- \phi^L < \beta (\phi^H, 0) \equiv \frac{\phi^H - \alpha}{\rho + \mu^L - i^H} \mu^L \quad \text{(Proposition 11 Statement 2a)} \]

Point 0: \( \beta (0, 0) = 0 \) and Point B: \( \beta (\rho + \mu^L, 0) = \mu^L \)

**Conditions for \( x_I^* > x_B^* = 0 \)**

\[\frac{\phi^H}{\rho + \mu^L} < 1 \quad \text{(Proposition 11 Statement 2)} \]

\[- \phi^L > \beta (\phi^H, 0) \equiv \frac{\phi^H - \alpha}{\rho + \mu^L - i^H} \mu^L \quad \text{(Proposition 11 Statement 2b)} \]

Point 0: \( \beta (0, 0) = 0 \) and Point B: \( \beta (\rho + \mu^L, 0) = \mu^L \)

**Condition for Zero Investment Whenever Regime \( L \) Preval**

\[- \phi^L > \gamma (\phi^H) \equiv \frac{\mu^H}{\rho + \mu^L} (\phi^H - \xi) \quad \text{where} \quad \xi \equiv \left( 1 + \frac{\mu^L}{\mu^H} \right) \left( 1 + \frac{\mu^L}{\rho + \mu^L} \right) \rho \quad \text{Proposition 10} \]

Point D: \( \gamma (\xi) = 0 \) and, if \( \xi - \rho > \frac{\mu^L}{\rho + \mu^L} \)

Point E: \( \gamma (\rho + \mu^L) = \frac{\mu^H}{\rho + \mu^L} (\rho + \mu^L - \xi) < \beta (\rho + \mu^L, 7) \)

**Condition in Lemma 6**

\[\delta (\phi^H) \equiv \frac{\phi^H}{\rho + \mu^L} (\rho + \mu^H) - (\rho + \mu^H) + \frac{\mu^L}{\rho + \mu^L} \mu^H \]

\[\delta (\rho) = -\frac{\mu^L}{\rho + \mu^L} \rho < 0 \; ; \; \text{Point B:} \; \delta (\rho + \mu^L) = \frac{\mu^L}{\rho + \mu^L} \mu^H ; \; \delta (\xi) > \frac{\rho}{\rho + \mu^L} (\xi - \xi) > 0 \]

Restriction on \( \xi \): \( \xi < \rho \leq \xi \)

**Table 1: Linear Boundaries**

---

31
Figure 2: Four regions in parameter space

Arrows pointing down (DE, GH and JL) Regime L with i = 0 and d = 0
Arrows pointing straight up (AB and HI): Regime H with i = 0 and d = 0
Arrows pointing up with slope between $x_1^*$ and $x_D^*$ (BC, EF, UJ, and LM): Regime H with $i = I$, $d=0$
Arrows pointing up with slope $x_D^*$ (CD and FG): Regime H with $i = I$, $d>0$

Assumes $x_1^* < x_D^*$ and $i < \frac{\phi^H}{1+x_1^*}$

Figure 3: Evolution of $X_t$ and $K_t$. 
that the firm has zero gross investment in Regime $L$, and (3) $\bar{t} < \frac{\phi^H}{1 + x_D^*} < \frac{\phi^H}{1 + x_D}$, which not only implies that the constraint $i_t \leq \bar{t}$ binds when capital investment is positive, but also, as explained below, implies that the firm will eventually pay positive dividends if it remains in Regime $H$ long enough.

Figure 3 shows an illustrative sample path of $X_t$ and $K_t$. The slopes, $s_t$, of the directed line segments equal $\frac{\ddot{X}}{K_t}$, where $\dot{X}_t = \phi_t K_t - I_t - D_t$ and $\dot{K}_t = I_t$, so

$$s_t = \frac{\phi_t - d_t}{i_t} - 1$$

with $d_t \equiv \frac{D_t}{K_t}$ and $i_t \equiv \frac{i_t}{K_t}$. When Regime $L$ prevails, the firm neither pays dividends nor undertakes gross investment. Since the depreciation rate of capital is zero by assumption, the capital stock is constant. Since the cash flow, $\phi^L K_t$, is negative, the firm decumulates its stock of cash on hand, $X_t$, so the directed line segments in Figure 3 during Regime $L$ ($\overrightarrow{DE}, \overrightarrow{GH}$, and $\overrightarrow{JL}$) point vertically downward.

When Regime $H$ prevails, the slopes of the directed line segments depend on $\bar{t}$ and $x_t$. Figure 3 is drawn under the assumption that $\bar{t} < \frac{\phi^H}{1 + x_D}$ so that $i_t = \bar{t} < \frac{\phi^H}{1 + x_D}$ and the slopes of the directed line segments when $d_t = 0$ are $s_t = \frac{\phi^H}{1 + x_D} - 1 > x_D^*$. When $0 \leq x_t < x_D^*$, optimal gross investment and dividends are both zero so the directed line segments ($\overrightarrow{AB}$, $\overrightarrow{HI}$) point vertically upward as the firm uses all of its cash flow to accumulate $X_t$ while its capital stock, $K_t$, remains constant. When $x_D^* \leq x_t < x_D^*$, the firm uses some of its cash flow $\phi^H K_t$ to undertake gross investment at rate $i_t K_t$ and uses the remaining cash flow $(\phi^H - \bar{t}) K_t$ to accumulate $X_t$; dividends equal zero in this situation. Thus the directed line segments ($\overrightarrow{BC}$, $\overrightarrow{EF}$, $\overrightarrow{IJ}$, $\overrightarrow{LM}$) have slope $s_t = \frac{\phi^H}{1 + x_D} - 1 > x_D^*$. Therefore, as time proceeds along any of these directed line segments, $x_t$ approaches $x_D^*$. If the Regime $H$ prevails long enough, $x_t$ equals $x_D^*$. If $x_t = x_D^*$, the firm undertakes gross investment $i_t = \bar{t} < \frac{\phi^H}{1 + x_D^*}$ and accumulates cash on hand to maintain $x_t = x_D^*$, by setting $\dot{X}_t = x_D^* \ddot{K}_t = x_D^* \ddot{K}_t$, and dividends per unit of capital $d_t = \phi^H - (1 + x_D^*) i_t > 0$. The directed line segments ($\overrightarrow{CD}, \overrightarrow{FG}$) lie along the dashed line $x = x_D^*$.

Figure 3 shows the evolution of $X_t$ and $K_t$ in the $(K, X)$ plane, with movements over time indicated by directed line segments. Figure 4 illustrates these same movements in a time series graph of operating profits, investment, dividends, and accumulation of cash through retention of earnings. The parameter configuration underlying this figure is that same as

---

22 If $x_t > x_D^*$, the firm immediately pays a lump-sum dividend $(x_t - x_D^*) K_t$ so that $x$ jumps to $x_D^*$ immediately. An ongoing firm will never find itself in a situation with $x_t > x_D^*$. 

33
for Figure 3 so that $x^*_I > x^*_D > 0$. In this case, there are three scenarios for the uses of funds by an ongoing firm in Regime $H$: (1) for $x < x^*_I$, the firm uses all of its operating profits to accumulate cash on hand, so both investment and dividends are zero; (2) for $x^*_I \leq x < x^*_D$, the firm uses some of its funds from operating profits to undertake capital investment and uses the remainder of the operating profits to accumulate cash, so that dividends are zero; and (3) for $x = x^*_D$, the firm simultaneously undertakes capital investment, pays positive dividends, and accumulates cash on hand, so that capital stock and cash on hand both grow at rate $\gamma$.

7 Average $q$ and Marginal $q$

In the traditional $q$ theory of investment based on convex costs of adjustment, the optimal rate of investment equates the marginal value of a unit of installed capital with the marginal cost of a unit of capital composed of the purchase price of a unit of uninstalled capital and the marginal adjustment cost of capital. In that framework, the optimal rate of investment is finite because the convexity of the adjustment cost function increases the marginal cost of investment as the rate of investment increases. The current model does not include convex adjustment costs so the rate of investment is made finite by imposing a finite upper bound,
Optimal investment is positive when the marginal value of a unit of installed capital is greater than or equal to the marginal cost of a unit of installed capital. The marginal value of a unit of installed capital is $V_K \equiv v - xv'$. Although a unit of capital can be purchased at a price equal to one unit of output, the relevant marginal cost of a unit of capital is $V_X \equiv v' \geq 1$, because a one-unit reduction in cash on hand reduces the value of the firm by $V_X \equiv v' \geq 1$. This cost is the \textit{static cost of funds} introduced earlier. Using this terminology, positive investment is triggered when the level of cash on hand is such that the marginal value of a unit of capital, $v - xv_x$, equals the static cost of funds, $v_x$. Thus, the investment trigger, $x^{*}_I$, satisfies

$$v (x^{*}_I) = (1 + x^{*}_I) v_x (x^{*}_I). \quad (57)$$

Define marginal $q$ as

$$q^m (x) \equiv V_K \equiv v (x) - xv_x (x). \quad (58)$$

The characterization of the investment trigger, $x^{*}_I$, in equation (57) can be expressed as the equality of marginal $q$ and the static cost of funds as

$$q^m (x^{*}_I) = v_x (x^{*}_I). \quad (59)$$

For the parameter configuration in Statement 2(b)ii of Proposition 11, which underlies Figures 3 and 4, equality of marginal $q$ and the static cost of funds at the investment trigger in equation (59), implies that marginal $q$ is greater than one at the investment trigger. There is an interval of values of $x$ ranging from $\tilde{x}$ to $x^{*}_I$, for which marginal $q$ exceeds one and yet optimal investment is zero.

Marginal $q$ is not directly observable empirically. A common practice is to use average $q$, the ratio of the value of the firm to the replacement cost of its capital stock, in place of marginal $q$. In conventional models without any financial market constraints or imperfections, average $q$ is identically equal to marginal $q$ for a competitive firm with a technology that is linearly homogeneous in a single type of capital, the rate of investment, and any variable factors of production. Under these conditions, the firm’s maximized net flow of profit is simply proportional to the capital stock. In the current model, the firm has a profit function that is proportional to the capital stock, but the financial friction, modeled as the inability of shareholders to inject additional funds into the firm (or to borrow), breaks the identity of average $q$ and marginal $q$. Effectively, cash on hand is a second type of capital asset. Although the value of the firm is linearly homogeneous in $K$ and $X$, it is not linearly

35
homogeneous in $K$ alone. That is, the value of the firm is not proportional to the stock of physical capital, $K$. In light of this added complexity, it is useful to consider the following three definitions of average $q$.

$$q^a_1(x) \equiv \frac{V}{K} \equiv \bar{v}(x) \quad [\text{ignores cash}] \quad (60)$$

$$q^a_2(x) \equiv \frac{V - X}{K} \equiv v(x) - x \quad [\text{"corrects" $V$ for presence of cash}] \quad (61)$$

$$q^a_3(x) \equiv \frac{V}{K + X} \equiv \frac{v(x)}{1 + x} \quad [\text{treats cash same as physical capital}] \quad (62)$$

Figure 5 illustrates marginal $q$ and the three measures of average $q$ as functions of $x$, and compares their behavior. Several properties of the measures of $q$ are needed to draw Figure 5, and these properties are listed in Proposition 12 in Appendix E.

To compare the three measures of average $q$ to marginal $q$, it is helpful to observe that the linear homogeneity of $V(X, K)$ in $X$ and $K$ implies

$$q^m_n(x) = V_K(X, K) = \frac{V - XV}{K}. \quad (63)$$

The first measure of average $q$, $q^a_1(x)$, ignores cash in the value of the firm and simply divides the value of the firm by the replacement cost of the capital stock, $K$. This measure of average $q$ can be described as \textit{naive average $q$} because it attributes all of the firm’s value
to its physical capital stock, $K$, ignoring the fact that cash on hand also contributes to the firm’s value. As shown in Figure 5, the naive measure of average $q$ exceeds marginal $q$, $q^m$, for all $x > 0$. The second and third measures of average $q$ take account of the role of cash on hand in the firm’s value. Empirical estimates of average $q$ often reduce the numerator of average $q$ by the value of cash on hand and inventories, which contribute to firm’s value but are not part of the physical capital stock. In our framework, in which cash is the only asset other physical capital owned by firms, this measure of average $q$ is $q_2^o (x) \equiv \frac{V-X}{K}$. As shown in Figure 5, $q_2^o (x)$ is greater than marginal $q$ for all $0 < x < x^{\ast}_D$. The reason $q_2^o (x)$ overstates $q^m$ is that when $x < x^{\ast}_D$ a dollar of cash on hand is worth more than a dollar to the firm because the financial constraint that prevents negative dividends is binding, so $V_X > 1$. To construct a measure like $q_2^o (x)$ that correctly measures $q^m$, the valuation of cash $X V_X \geq X$ should be subtracted from $V$ in the numerator $q$, so that this alternative calculation of average $q$ would be $\frac{V-X V_X}{K}$, which equals $V_K$ in equation (63). The failure to multiply $X$ by $V_X$ in the numerator of $q_2^o (x)$ implies that $q_2^o (x) \equiv \frac{V-X}{K}$ overstates $V = K = q^m$ when the financial constraint is binding with $V_X > 1$.

Rather than subtract cash on hand, $X$, from the numerator of $q$, the value of cash on hand can be added to the denominator, which is the approach implemented in $q_3^o (x) \equiv \frac{V}{K+X}$. This measure of average $q$ is the ratio of the total value of the firm to the replacement cost of total assets. As shown in Figure 5, $q_2^o (\bar{x}) = q_3^o (\bar{x})$ at point A, where both $q_2^o (\bar{x})$ and $q_3^o (\bar{x})$ are equal to one. At point $A$, the numerators of $q_2^o (x)$ and $q_3^o (x)$ are equal to their respective denominators. The equality of the numerator and denominator of $q_2^o (x)$ implies $V = K$ and the equality of the numerator and denominator of $q_3^o (x)$ implies $V = K + X$, which is the same condition.

Unlike $q_2^o (x)$ and $q_3^o (x)$, $q_3^o (x)$ is not always greater than marginal $q$, $q^m (x)$. As shown in Figure 5, $q_3^o (x) \geq q^m (x)$ as $x \leq x^{\ast}_I$. Therefore, when $x$ equals the investment trigger $x^{\ast}_I$, both $q_3^o$ and $q^m (x)$ equal the static cost of funds, $v^f (x)$. Finally, note that $q_3^o (x)$ attains its maximum at the investment trigger, $x^{\ast}_I$.

8 Interaction of Financial Constraints and Capital Investment

The financial imperfection confronting the firm is that the firm cannot raise any additional external funds, either by issuing new debt or new equity. Formally, the financial imper-
Infection is modeled as a non-negativity constraint on dividends, since negative dividends are equivalent to equity issuance. The non-negativity constraint binds, and dividends are zero, when the marginal value of cash on hand, $V_X$, exceeds one. When the constraint does not bind and the firm pays positive dividends, marginal value of cash on hand, $V_X$, equals one.

The specification of the model is rich enough to incorporate parameter configurations for the financial constraint impacts optimal capital investment in Regime $H$ as well as parameter configurations for which optimal capital investment in Regime $H$ is completely unaffected by the financial constraint. It turns out that the myopic value of a unit of capital in Regime $H$, $\frac{\phi^H}{\rho + \mu}$, is the key determinant of whether the financial constraint impacts investment. Specifically, if the myopic value of capital exceeds one, which is the case in Regions 1a and 1b in Figure 2, then optimal investment in Regime $H$ is always $7$, regardless of the value of $V_X$ and regardless of the amount of cash on hand. When the myopic value of capital exceeds one in Regime $H$, the expected present value of profits accruing to a unit of capital over the remainder of the current Regime $H$ is greater than the cost of acquiring a unit of capital. In effect, the firm expects to earn back its investment in capital before the next arrival of Regime $L$, when the firm could be forced to close if that Regime $L$ persists long enough. In Region 1a the dividend trigger, $x^*_D$, is zero, so the firm never accumulates cash. The marginal value of cash on hand equals one in this case, and the financial constraint never binds in Regime $H$. When the parameter configuration is in region 1a, the firm terminates immediately upon the arrival of the next Regime $L$ because the absence of cash on hand makes the firm unable to pay the required outflow $-\phi^L K > 0$.

In region 1b, the required outflow per unit of capital in Regime $L$, $-\phi^L > 0$, is smaller than in region 1a (for a given $\phi^H$) and so cash on hand is depleted more slowly when $\phi_t = \phi^L$ in region 1b than it would be depleted in region 1a. The slower depletion of cash on hand increases the incentive to accumulate cash and the dividend trigger, $x^*_D$, is positive in region 1b. For $x < x^*_D$, the financial constraint is binding with $V_X > 1$, so optimal dividends are zero in this situation. Nevertheless, optimal investment is $7$, despite the binding financial constraint. In contrast to region 1a, the firm does not terminate immediately upon the arrival of the next Regime $L$; the firm’s cash on hand allows the firm to survive in an uninterrupted Regime $L$ for a length of time $\frac{x}{\phi^L}$, if it enters Regime $L$ with $x = x_t$. Of course, there is a chance that the firm will survive even longer than $\frac{x}{\phi^L}$ because there is a chance that a new Regime $H$ will arrive before that time.

The financial constraint affects optimal investment if the myopic value of capital in Regime $H$, $\frac{\phi^H}{\rho + \mu}$, is less than one. This situation arises whenever the parameter config-
uration is in region 2a or 2b. In region 2a \( x^*_I > 0 = x^*_D \) so the firm never accumulates cash and the optimal rate of investment is also zero. In region 2a, cash would be depleted so quickly so during Regime \( L \) (for a given \( \phi^H \)) that the firm has no incentive to accumulate cash. Instead the firm pays out its entire cash flow as dividends whenever it is in Regime \( H \). With no cash on hand, the firm terminates immediately upon the arrival of the next Regime \( L \). Since the firm will not survive longer than the current Regime \( H \), the expected present value of a unit of capital over the firm’s remaining lifetime is simply the myopic value of capital, \( \frac{\phi^H}{\rho + \mu} \), which is less than one. Therefore, optimal investment is zero.

In region 2b, both the dividend trigger, \( x^*_D \), and the investment trigger, \( x^*_I \), are positive. If the investment trigger, \( x^*_I \), is lower than the dividend trigger, \( x^*_D \), then the optimal amount of investment depends on the amount of cash on hand. If \( x < x^*_I \), optimal investment is zero, but if \( x \geq x^*_I \), the optimal amount of investment is \( \overline{i} \). Thus, if \( x^*_I \leq x < x^*_D \), the optimal amount of investment is \( \overline{i} \), even though the financial constraint is binding and \( V_X > 1 \). Elsewhere in region 2b, the investment trigger, \( x^*_I \), is greater than the dividend trigger, \( x^*_D \), and an ongoing firm never invests in physical capital.
**A Proofs**

**Proof of Proposition 2:** See paragraph immediately preceding Proposition 2 in the main text.

**Proof of Proposition 3:** Since \( V_{XX}^H (X^*) = 0 \), it follows that if \( V_{XX}^H (0) < 0 \), then \( X^* \neq 0 \), so \( X^* > 0 \). Also, if \( X^* > 0 \), then \( V_{XX}^H (X) < 0 \) for \( 0 \leq X < X^* \), so \( V_{XX}^H (0) < 0 \). Now evaluate the ODEs in equations (8) and (9) at \( X = 0 \) and use \( V^L (0) = 0 \) from equation (10) to obtain \( V_X^H (0) = \frac{\rho + \mu^L}{\phi^H} V^H (0) \) and \( V^H (0) = -\frac{\phi^H}{\rho^L} V_X^L (0) \), respectively, so that \( V_X^H (0) = \frac{\rho + \mu^L}{\phi^H} V^H (0) \). Evaluate equation (14) at \( X = 0 \) to obtain \( V_X^L (0) = \frac{1}{\rho^L} [ (\rho + r + \mu^L) V_X^H (0) - \phi^H V_{XX}^H (0) ] \), which can be substituted into the previous equation to obtain \( V_X^H (0) = \frac{\rho + \mu^L}{\phi^H} V^H (0) \). Therefore, since \( V_X^H (0) < 0 \) and \( \phi^H \phi^L \phi^H \phi^L < 0 \), it follows that \( V_{XX}^H (0) < 0 \) if and only if \( \frac{\phi^H}{\rho + \mu^L} > \frac{\phi^L}{\rho + \mu^L} > 0 \). ■

**Proof of Proposition 1:** Proposition 3 implies that if \( \Lambda = \frac{\phi^H}{\rho + \mu^L} \phi^L \phi^L = \frac{\phi^L}{\rho + \mu^L} = 1 \), then \( X^* = 0 \). Since \( X^* = 0 \), Proposition 1 implies that \( V_X^H (0) = 1 = \Lambda \). ■

**Proof of Proposition 4:** Starting from \( \Lambda (\rho, r, \phi^H, \phi^L, \mu^H, \mu^L) \equiv \frac{\phi^H}{\rho + \mu^L} \phi^L \phi^L = 1 \), any change in a parameter that increases \( \Lambda \) will increase \( X^* \) to a positive value. Inspection of the definition of \( \Lambda (\rho, r, \phi^H, \phi^L, \mu^H, \mu^L) \) immediately reveals that the following changes individually increase \( \Lambda \) and hence increase \( X^* \) to a positive value: a decrease in \( \rho \) (Statement 1); an increase in \( r \) (Statement 2); an increase in \( \mu^H \) (Statement 3); an increase in \( \phi^H \) (Statement 6); an increase in \( \phi^L \) since \( \phi^L \) is negative (Statement 7). A rightward translation of the unconditional distribution of \( \phi \) increases both \( \phi^H \) and \( \phi^L \) so Statements 6 and 7 imply that \( X^* \) increases (Statement 8).

To determine the impact of a small change in \( \mu^L \), differentiate \( \Lambda (\rho, r, \phi^H, \phi^L, \mu^H, \mu^L) \equiv \frac{\phi^H}{\rho + \mu^L} \phi^L \phi^L \) with respect to \( \mu^L \) to obtain \( \frac{\partial \Lambda}{\partial \mu^L} = -\frac{\Lambda}{\rho + \mu^L} + \frac{\Lambda}{\mu^L} + \frac{\Lambda}{\rho + \mu^L} = \frac{\Lambda}{\mu^L} \left[ -\mu^L (\rho - r + \mu^L) + (\rho + \mu^L) (\rho + r - \mu^L) - \mu^L (\rho + \mu^L) \right] \). Since \( \frac{\partial \Lambda}{\partial \mu^L} > 0 \), sign \( \left( \frac{\partial \Lambda}{\partial \mu^L} \right) \) is \( sign \left( -\mu^L (\rho - r + \mu^L) + (\rho + \mu^L) (\rho + r - \mu^L) - 2 \mu^L (\rho + \mu^L) \right) = sign \left( -\mu^L (\rho - r + \mu^L) \right) \). Therefore, if \( \mu^L < \sqrt{\left( \rho - r \right) \rho} \), then \( \frac{\partial \Lambda}{\partial \mu^L} > 0 \) so an increase in \( \mu^L \) increases \( X^* \) to a positive value (Statement 4). Alternatively, if \( \mu^L > \sqrt{\left( \rho - r \right) \rho} \), then \( \frac{\partial \Lambda}{\partial \mu^L} < 0 \) and an decrease in \( \mu^L \) increases \( X^* \) to a positive value (Statement 5). ■

**Proof of Proposition 5:** Rewrite \( \Lambda \equiv \frac{\phi^H}{\rho + \mu^L} \phi^L \phi^L \) in Definition 3 as \( \Lambda = \frac{\mu^L}{\rho + \mu^L} \phi^L \phi^L - \frac{\mu^L}{\rho + \mu^L} - \frac{\mu^H}{\rho + \mu^L} \phi^H \phi^L \phi^L \), where \( 0 < M \equiv \mu^L \phi^L + \mu^H \phi^L < \mu^H \phi^L \) is unchanged by a mean-
preserving change in $\phi^H$ and $\phi^L$. Starting from an initial parameter configuration in which $\Lambda = 1$, a mean-preserving decrease in $\phi^H$ and increase in $\phi^L$ decreases $\mu^H \phi^H$ while maintaining $M$ unchanged which increases the ratio $\frac{\mu^H \phi^H}{\mu^L \phi^L - M}$ and hence increases $\Lambda$ to a value greater than one. Therefore, $X^*$ increases to a positive number. ■

**Proof of Lemma 1:** Statement 1: Use $\eta_1 \equiv \frac{\rho + \mu^H}{\phi^H}$, $\eta_2 \equiv \frac{\rho + \mu^L}{\phi^L}$, and $\Gamma \equiv \frac{\mu^L}{\rho + \mu^L} \frac{\mu^H}{\rho + \mu^H}$ to rewrite the characteristic equation associated with the matrix $A$ in equation (70), $(\eta_1 - \omega)(\eta_2 - \omega) - \frac{\mu^L}{\phi^L} \frac{\mu^H}{\phi^H} = 0$, as $(\frac{1}{\nu^2} - \omega)(\frac{1}{\nu^2} - \omega) - \frac{1}{\nu^2} \frac{1}{\nu^2} = 0$, so that $\omega^2 - (\frac{1}{\nu^2} + \frac{1}{\nu^2}) \omega + \frac{1}{\nu^2} \frac{1}{\nu^2} - \frac{1}{\nu^2} \frac{1}{\nu^2} = 0$. The sum of the roots of this quadratic equation is the negative of the coefficient on the linear term in $\omega$. Therefore, $\omega_1 + \omega_2 = \frac{1}{\nu^2} + \frac{1}{\nu^2} > 0$, where the inequality follows from equation (2).

Statement 2: Divide both sides of the characteristic equation, $\omega^2 - (\frac{1}{\nu^2} + \frac{1}{\nu^2}) \omega + \frac{1}{\nu^2} \frac{1}{\nu^2} - \frac{1}{\nu^2} \frac{1}{\nu^2} = 0$, by $(\frac{1}{\nu^2} + \frac{1}{\nu^2} - \frac{1}{\nu^2} \frac{1}{\nu^2}) \omega^2$ to obtain a quadratic equation in $\omega^{-1}$, which is $\omega^{-2} - \frac{1}{\nu^2} + \frac{1}{\nu^2} \frac{1}{\nu^2} \frac{1}{\nu^2} \frac{1}{\nu^2} = \frac{1}{\nu^2} \frac{1}{\nu^2} > 0$, where the inequality follows from equation (2). ■

**Proof of Lemma 2:** Since $\omega_2 - \omega_1, x_m > 0$ if and only if $D \equiv \omega_1^2 (\eta_2 - \omega_2) - \omega_2^2 (\eta_2 - \omega_1) > 0$. Rewrite $D$ to obtain $D = (\omega_1^2 - \omega_2^2) \eta_2 - \omega_2 \omega_2 (\omega_1 - \omega_2) = (\omega_1 - \omega_2) [(\omega_1 + \omega_2) \eta_2 - \omega_2 \omega_2]$. Use $\omega_1 + \omega_2 = \eta_1 + \eta_2$ and $\omega_1 \omega_2 = \eta_1 \eta_2 - \frac{\mu^L}{\phi^L} \frac{\mu^H}{\phi^H}$ to obtain $D = (\omega_1 - \omega_2) [(\eta_1 + \eta_2) \eta_2 - \eta_1 \eta_2 + \frac{\mu^L}{\phi^L} \frac{\mu^H}{\phi^H}]$

$= (\omega_1 - \omega_2) \left[ \eta_2^2 + \frac{\mu^L}{\phi^L} \frac{\mu^H}{\phi^H} \right]$ and then use $\eta_2 \equiv \frac{\rho + \mu^L}{\phi^L}$ to obtain $D = (\omega_1 - \omega_2) \eta_2^2 \left[ 1 + \frac{\phi^H}{\rho + \mu^L} \frac{\phi^L}{\rho + \mu^L} \right]$

$= (\omega_2 - \omega_1) \eta_2^2 \left[ \frac{\phi^H}{\rho + \mu^L} \frac{\phi^L}{\rho + \mu^L} - 1 \right]$. When $r = 0$, the definition $\Lambda$ in Definition 3 implies that $\Lambda = \frac{\phi^H}{\rho + \mu^L} \frac{\phi^L}{\rho + \mu^L}$ so $D = (\omega_2 - \omega_1) \eta_2^2 \left[ \Lambda - 1 \right]$. Since $(\omega_2 - \omega_1) \eta_2^2 > 0$, $\text{sign} (D) = \text{sign} (\Lambda - 1)$ so $\text{sign} (x_m) = \text{sign} (\Lambda - 1)$. To prove the final equality in this lemma, use the definition of $\Lambda$ to obtain $\Lambda - 1 = \left( \frac{\phi^H}{\rho + \mu^L} \frac{\phi^L}{\rho + \mu^L} \right) \frac{\rho + \mu^L}{\rho + \mu^L} = \left( \frac{\phi^H}{\rho + \mu^L} \frac{\phi^L}{\rho + \mu^L} \right) \frac{\rho + \mu^L}{\rho + \mu^L}$. Then use the definition of $\Gamma \equiv \frac{\mu^H}{\rho + \mu^L} \frac{\mu^L}{\rho + \mu^L}$ to obtain $\Lambda - 1 = \left( \frac{\phi^H}{\rho + \mu^L} \frac{\phi^L}{\rho + \mu^L} \right) \frac{\rho + \mu^L}{\rho + \mu^L} \text{sign} (\nu^H \Gamma + \nu^L)$ since $\frac{\rho + \mu^L}{\rho + \mu^L} > 0$. ■

**Proof of Proposition 6:** Statement 1: If the firm is in Regime H and has cash on hand $X$ that exceeds $X^*$, it will immediately pay a dividend of $X - X^*$ and will then have an ex-dividend value of $V^H (X^*)$. Therefore, the expected present value of dividends accruing to shareholders when $X > X^*$ is $(X - X^*) + V^H (X^*)$. Statement 2: When the firm is in Regime H with cash on hand $X$, it can pay an immediate dividend equal to $X$ and, in addition, have an expected present value of cash flows equal to the myopic value of a unit of capital, $\frac{\phi^H}{\rho + \mu^L}$. Therefore, $V^H (X) \geq \frac{\phi^H}{\rho + \mu^L} + X$. If firm has cash on hand
in Regime $L$, it can simply pay a liquidating dividend $X$ and then immediately terminate. Therefore, $V^L(X) \geq X$. Statement 3: Assume that $x_m \geq 0$. Evaluate the first row of equation (22) at $X = X^*$ to obtain $V^H(X^*) = c_1 + c_2$. Use the expressions for $c_1$ and $c_2$ from equations (24) and (25), respectively, to obtain $V^H(X^*) = \frac{\omega_2}{\omega_2 - \omega_1} \left( \frac{\omega_2}{\omega_1} - \frac{\omega_1}{\omega_2} \right) = \frac{\omega_2 - \omega_1^2}{\omega_2 - \omega_1} = \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} = \frac{1}{\omega_1} + \frac{1}{\omega_2}$. Use Statement 2 of Lemma 1, $\frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{1}{1 - \Gamma} (\nu^H + V^L_\Gamma) > 0$, to obtain $V^H(X^*) = \frac{1}{1 - \Gamma} (\nu^H + V^L_\Gamma)$. Rewrite the ODE in equation (8), setting $r = 0$, as $(\rho + \mu L) V^H(X) = \phi^H V^H(X) + \mu L V^L(X)$. Evaluate this ODE at $X = X^*$, use the boundary condition $V^H_X(X^*) = 1$, and divide both sides of the equation by $\rho + \mu L$ using the definition of the myopic value $\nu^H \equiv \frac{\phi^H}{\mu + \rho}$ to obtain $V^H(X^*) = \nu^H + \frac{\mu L}{\mu + \rho} V^L(X^*)$. Therefore, $V^L(X^*) = \frac{\rho + \mu L}{\mu + \rho} (V^H(X^*) - \nu^H) = \frac{\rho + \mu L}{\mu + \rho} \left( \frac{1}{1 - \Gamma} (\nu^H + \nu^L) - \nu^H \right) = \frac{\rho + \mu L}{\mu + \rho} \frac{1}{1 - \Gamma} (\Gamma \nu^H + \nu^L) = \frac{1}{1 - \Gamma} \left( \frac{\mu L}{\mu + \rho} \nu^H + \frac{\rho + \mu L}{\mu + \rho} \nu^L \right) < \frac{1}{1 - \Gamma} (\nu^H + \nu^L) = V^H(X^*)$. Statement 4: If the firm is in Regime $H$, and knows that Regime $H$ will persist forever, it could pay an immediate dividend of $X_t$ at time $t$ and then pay a constant flow of dividends $\phi^H$ forever. In this situation, which is more favorable than the firm’s actual situation, the expected present value of dividends, including the dividend $X_t$ at time $t$, would be $\frac{\phi^H}{\rho} + X_t$. Therefore, $V^H(X_t) < \frac{\phi^H}{\rho} + X_t$. ■ Proof of Proposition 7: Statements 1 and 2 follow from setting $\mu^H = \mu^L = \mu$ in Proposition 3 to obtain $X^* > 0$ if and only if $(1 + \frac{\omega}{\rho})^2 < \frac{\phi^H}{\sigma^2}$, therefore $X^* > 0$ if and only if $\frac{\rho^2}{\mu} < \sqrt{\frac{\phi^H}{\sigma^2}} - 1$. To prove Statement 3, set $\mu^L = \mu^H = \mu$, the characteristic equation of $A$, $q(\omega) \equiv (\eta_2 - \omega)(\eta_1 - \omega) - \frac{\mu^L}{\phi^H} \frac{\mu^H}{\phi^H} = 0$, to obtain $\omega^2 - (\eta_1 + \eta_2) \omega + \eta_1 \eta_2 - \frac{\mu^L}{\phi^H} \frac{\mu^H}{\phi^H} = 0$. Use $\eta_1 = \frac{\mu + \rho}{\phi^H}$ and $\eta_2 = \frac{\mu + \rho}{\phi^H}$ to obtain $\eta_1 \eta_2 = \frac{\mu^L}{\phi^H} \frac{\mu^H}{\phi^H} = \eta_1 \eta_2 - \left( \frac{\mu}{\mu + \rho} \right)^2 \eta_1 \eta_2 = \left[ 1 - \left( \frac{\mu}{\mu + \rho} \right)^2 \right] \eta_1 \eta_2 = 0$. Therefore, the characteristic equation is $\omega^2 - (\eta_1 + \eta_2) \omega + \left[ 1 - \left( \frac{\mu}{\mu + \rho} \right)^2 \right] \eta_1 \eta_2 = 0$. Therefore, the eigenvalues of $A$ are $\omega_i = \frac{1}{2} \left( \eta_1 + \eta_2 \pm \sqrt{(\eta_1 + \eta_2)^2 - 4 \left[ 1 - \left( \frac{\mu}{\mu + \rho} \right)^2 \right] \eta_1 \eta_2} \right)$. Therefore, $\frac{\omega_1}{\mu} = \frac{1}{2} \left( \frac{n_1 + n_2}{\mu} \pm \sqrt{\left( \frac{n_1 + n_2}{\mu} \right)^2 - 4 \left[ 1 - \left( \frac{\mu}{\mu + \rho} \right)^2 \right] n_1 n_2 \mu \mu} \right)$. Since $\lim_{\mu \to \infty} \frac{n_1}{\mu} = 1/\sigma^2$ and $\lim_{\mu \to \infty} \frac{n_2}{\mu} = 1/\sigma^2$, it follows that $\lim_{\mu \to \infty} \frac{\omega_1}{\mu} = \frac{n_1}{\mu} + \frac{n_2}{\mu} = \frac{1}{\sigma^2} + \frac{1}{\sigma^2} < 0$ and $\lim_{\mu \to \infty} \frac{\omega_2}{\mu} = 0$. Also, since $\lim_{\mu \to \infty} \frac{\omega_1}{\mu} = \frac{1}{\sigma^2} + \frac{1}{\sigma^2} < 0$, $\lim_{\mu \to \infty} \omega_1 = -\infty$. Therefore, $\lim_{\mu \to \infty} (\omega_2 - \omega_1) = \lim_{\mu \to \infty} \mu \left( \frac{\omega_2}{\mu} - \frac{\omega_1}{\mu} \right) = -\lim_{\mu \to \infty} \mu \left( \frac{1}{\sigma^2} + \frac{1}{\sigma^2} \right) = +\infty$.

We next prove that $\lim_{\mu \to \infty} \omega_2 = \frac{2 \rho}{\phi^H} > 0$. Rewrite the characteristic equation, after dividing both sides by $\eta_1 + \eta_2$, as $\frac{\omega_1 + \eta_1 \eta_2}{\eta_1 + \eta_2} \omega + \frac{\eta_1 \eta_2 - \eta_1 \eta_2}{\eta_1 + \eta_2} \frac{\mu}{\phi^H} = 0$. Since $\lim_{\mu \to \infty} \frac{\omega_2}{\mu} = 0$, it follows that $\lim_{\mu \to \infty} \frac{\omega_2}{\eta_1 + \eta_2} = \lim_{\mu \to \infty} \frac{\omega_2}{\mu} = \lim_{\mu \to \infty} \frac{\omega_2}{\eta_1 + \eta_2} \times \lim_{\mu \to \infty} \frac{1}{\mu + \rho} = \lim_{\mu \to \infty} \frac{\omega_2}{\mu}$.
× \lim_{\mu \to \infty} \frac{1}{(1+\frac{1}{\mu}) \theta + (1+\frac{1}{\mu}) \phi} = 0 \times \frac{1}{\theta + \phi} = 0. \) Now take the limit of the rewritten characteristic function evaluated at \( \omega = \omega_2 \), to obtain \( \left( \lim_{\mu \to \infty} \frac{\omega_2}{\eta_1 + \eta_2} - 1 \right) \lim_{\mu \to \infty} \omega_2 + \lim_{\mu \to \infty} \frac{\eta_1 \eta_2 - \frac{\mu}{\eta_1 + \eta_2} \mu}{\eta_1 + \eta_2} = 0 \), which implies \( \lim_{\mu \to \infty} \omega_2 = \lim_{\mu \to \infty} \frac{\eta_1 \eta_2 - \frac{\mu}{\eta_1 + \eta_2} \mu}{\eta_1 + \eta_2} = \lim_{\mu \to \infty} \frac{\eta_1 \eta_2 - \frac{\mu}{\eta_1 + \eta_2} \mu}{\eta_1 + \eta_2} \).

Define \( Z \equiv \frac{\omega_1^2}{\omega_2 \omega_1 - \omega_1}. \) Since \( 0 < \frac{\omega_2}{\omega_2 - \omega_1} < 1 \) (because \( \eta_1 < \omega_1 < 0 < \omega_2 < \eta_2 \)), it follows that \( 0 < Z < \frac{\omega_2}{\omega_1} \). Therefore, \( \lim_{\mu \to \infty} Z \leq \lim_{\mu \to \infty} \frac{\ln Z}{\omega_2 - \omega_1} \leq \lim_{\mu \to \infty} \frac{2 \ln(-\omega_1) - 2 \ln(\omega_2)}{\omega_2 - \omega_1} = 2 \lim_{\mu \to \infty} \frac{-2 \ln(-\omega_1)}{\omega_2 - \omega_1}, \) where the final inequality follows from \( \lim_{\mu \to \infty} \omega_2 = 2 \frac{\rho}{\phi + \phi}, \) which is finite, and \( \lim_{\mu \to \infty} (\omega_2 - \omega_1) = +\infty. \) Observe that \( \lim_{\mu \to \infty} \frac{\ln Z}{\omega_2 - \omega_1} = \lim_{\mu \to \infty} \frac{1}{\mu} \ln(-\omega_1) = \mu \lim_{\mu \to \infty} \frac{1}{\mu} \ln(\mu) \left( \ln \left( -\frac{\omega_1}{\mu} \right) \right) = \lim_{\mu \to \infty} \frac{1}{\mu} \ln \mu, \) where the final equality follows from the fact that \( \lim_{\mu \to \infty} \ln \left( -\frac{\omega_1}{\mu} \right) \) is finite. L'Hopital's Rule implies that \( \lim_{\mu \to \infty} \frac{1}{\mu} \ln \mu = \frac{\lim_{\mu \to \infty} \frac{1}{\mu}}{1} = 0. \) Therefore, \( 0 \leq \lim_{\mu \to \infty} \frac{\ln Z}{\omega_2 - \omega_1} \leq 0, \) so \( \lim_{\mu \to \infty} \ln Z = 0. \) Finally, \( \lim_{\mu \to \infty} X^* = \lim_{\mu \to \infty} \max \left[ \ln Z, 0 \right] = 0. \)

**Proof of Lemma 3:** Average profits over a long interval of time from 0 to \( \hat{t} \) are \( \bar{\phi}_{\hat{t}} \approx \mu^H \phi^H + \mu^L \phi^L. \) The firm pays positive dividends only when \( \phi_t = \phi^H, \) but not always when \( \phi_t = \phi^L \). When the firms pays positive dividends, \( D_t = \phi^H. \) Let \( \alpha \) be the fraction of the time that the firm pays positive dividends so \( \bar{D}_{\hat{t}} \approx \alpha \phi^H. \) Setting average profits, \( \bar{\phi}_{\hat{t}}, \) approximately equal to average dividends, \( \bar{D}_{\hat{t}} \), yields

\[
\alpha \approx \frac{\mu^H + \mu^L \phi^L}{\mu^L + \mu^H} < \frac{\mu^H}{\mu^L + \mu^H},
\]

where the inequality follows from \( \mu^L \phi^L > \phi^L > \mu^H \phi^H. \)

**Proof of Proposition 8:** Use the facts that \( \phi_t - \bar{\phi}_{\hat{t}} \approx \mu^L \phi^L - \phi^L \phi_t \) when \( \phi_t = \phi^H \) and \( \phi_t - \bar{\phi}_{\hat{t}} \approx \mu^H \phi^H + \phi^L \phi_t \) when \( \phi_t = \phi^L \) to obtain the variance of profits as \( \text{Var}(\phi) \approx \mu^H \mu^L \left( \frac{\phi^L}{\mu^L + \mu^H} \right)^2. \) Use \( \text{Var}(D) = \alpha \left( \phi^H \right)^2 - \left( \alpha \phi^H \right)^2 \) along with the expression for \( \alpha \) in Lemma 3 to obtain the variance of dividends \( \text{Var}(D) \approx \left( \phi^H - \phi^L \right) \left( \mu^H \phi^H + \mu^L \phi^L \right) \frac{\mu^L}{\mu^L + \mu^H}. \) Dividing the variance of dividends by the variance of profits yields \( \frac{\text{Var}(D)}{\text{Var}(\phi)} \approx \frac{1 + \frac{\mu^L \phi^L}{\mu^L + \mu^H}}{1 - \frac{\phi^L}{\mu^H}} < 1, \) where the inequality follows from the fact that \( \frac{\phi^L}{\mu^H} < 0 \) so that the numerator of the expression for \( \frac{\text{Var}(D)}{\text{Var}(\phi)} \) is smaller than 1 and the denominator in that expression is greater than 1.
Proof of Lemma 4: Define $W^L (i^L) \equiv \frac{1}{\rho_0} E_0 \left\{ \int_0^{t_H} (\phi^L - i^L) K^t e^{-\rho^L dt} \right\}$ where $t_H \equiv \min \left\{ t \geq 0 : \phi_t = \phi^H \right\}$. Since Regime L prevails until $t_H$, $K_t = K_0 e^{i^L t}$ for all $t \in [0, t_H]$, so $W^L (i^L) = \frac{\phi^L - i^L}{\rho - i^L} E_0 \left\{ 1 - e^{-(\rho - i^L) t_H} \right\} = \frac{\phi^L - i^L}{\rho - i^L} \left( 1 - \frac{\mu}{\rho + \mu - i^L} \right)$. Therefore, $W^L (i^L) = \frac{\phi^L - i^L}{\rho + \mu - i^L} \left( 1 - \frac{\mu}{\rho + \mu - i^L} \right)$. Similarly, define $W^H (i^H) \equiv \frac{1}{\rho_0} E_0 \left\{ \int_0^{t_H} (\phi^H - i^H) K^t e^{-\rho^H dt} \right\}$ where $t_H \equiv \min \left\{ t \geq 0 : \phi_t = \phi^L \right\}$. Since $K_t = K_0 e^{i^H t}$ when Regime H prevails for all $t \in [0, t_H]$, $W^H (i^H) = \frac{\phi^H - i^H}{\rho - i^H} E_0 \left\{ 1 - e^{-(\rho - i^H) t_H} \right\} = \frac{\phi^H - i^H}{\rho + \mu - i^H} \left( 1 - \frac{\mu}{\rho + \mu - i^H} \right)$. Therefore, $W^H (i^H) = \frac{\phi^H - i^H}{\rho + \mu - i^H} \left( 1 - \frac{\mu}{\rho + \mu - i^H} \right)$.

Observe that $\tilde{V}^L (0, K_0) = E_0 \left\{ K_0 W^L (i^L) + e^{-\rho^L t} K t_H W^L (i^L) + e^{-\rho^L (t_H + t)} \right\}$ since $\tilde{V}^L (0, K)$ is proportional to $K$. Therefore, $\tilde{V}^L (0, K_0) = K_0 E_0 \left\{ W^L (i^L) + e^{-\rho^L (t_H + t)} \right\}$ + $E_0 \left\{ e^{-(\rho - i^L) t_H} e^{-\rho^L (t_H + t)} \right\} | \mathcal{F}_t$, which implies $\tilde{V}^L (0, K_0) = K_0 \left\{ W^L (i^L) + e^{-\rho^L (t_H + t)} \right\}$ + $E_0 \left\{ e^{-(\rho - i^L) t_H} e^{-\rho^L (t_H + t)} \right\} | \mathcal{F}_t$. Similarly, define $\tilde{V}^H (0, K_0) = K_0 \left\{ W^H (i^H) + e^{-\rho^H (t_H + t)} \right\}$ + $E_0 \left\{ e^{-(\rho - i^L) t_H} e^{-\rho^H (t_H + t)} \right\} | \mathcal{F}_t$. Therefore, $\tilde{V}^H (0, K_0) = K_0 \left\{ W^H (i^H) + e^{-\rho^H (t_H + t)} \right\}$ + $E_0 \left\{ e^{-(\rho - i^L) t_H} e^{-\rho^H (t_H + t)} \right\} | \mathcal{F}_t$.

Now use $W^L (i^L) = \frac{\phi^L - i^L}{\rho + \mu - i^L}$ and $W^H (i^H) = \frac{\phi^H - i^H}{\rho + \mu - i^H}$ defined above to obtain $\tilde{V}^L (0, K_0) = K_0 \left\{ \frac{\phi^L - i^L}{\rho + \mu - i^L} + \mu^L \right\}$. In the absence of any financial constraint that prevent negative dividends, $\tilde{V}^L (X, K_0) = \tilde{V}^L (0, K_0) + X$, so $\tilde{V}^L (X, K) = X + K \left\{ \frac{\phi^L - i^L}{\rho + \mu - i^L} + \mu^L \right\}$.

Similarly, define $\tilde{V}^H (0, K_0) = K_0 \left\{ \frac{\phi^H - i^H}{\rho + \mu - i^H} + \mu^H \right\}$. Therefore, $\tilde{V}^H (0, K_0) = K_0 \left\{ \frac{\phi^H - i^H}{\rho + \mu - i^H} + \mu^H \right\}$ + $E_0 \left\{ e^{-(\rho - i^L) t_H} e^{-\rho^H (t_H + t)} \right\} | \mathcal{F}_t$. Now use $W^L (i^L) = \frac{\phi^L - i^L}{\rho + \mu - i^L}$ and $W^H (i^H) = \frac{\phi^H - i^H}{\rho + \mu - i^H}$ defined above to obtain $\tilde{V}^H (0, K_0) = K_0 \left\{ \frac{\phi^H - i^H}{\rho + \mu - i^H} + \mu^H \right\}$. In the absence of any financial constraint that prevent negative dividends, $\tilde{V}^H (X, K_0) = \tilde{V}^H (0, K_0) + X$, so $\tilde{V}^H (X, K) = X + K \left\{ \frac{\phi^H - i^H}{\rho + \mu - i^H} + \mu^H \right\}$.

\[ \] Proof of Corollary 3: Since $(\rho + \mu^L - i^L) (\rho + \mu^L - i^H) - \mu^L \mu^H > 0$, Lemma 4 implies that $\tilde{V}^L (X, K) > X \geq 0$ for $X \geq 0$ and $K > 0$ if $(\rho + \mu^L - i^H) (\phi^L - i^L) + \mu^L (\phi^H - i^H) > 0$. Next observe from Lemma 4 that $\tilde{V}^H (X, K) - \tilde{V}^L (X, K) = \left( \frac{\phi^H - i^H}{\rho + \mu^L - i^H} - \frac{\phi^L - i^H}{\rho + \mu^L - i^H} \right) K > 0$.

\[ \] Proof of Lemma 5: Equation (51) immediately implies $\frac{\mu^H (i^L)}{\mu^H (i^L)} = \frac{\phi^H - i^H}{\rho + \mu^L - i^L}$, where the numerator and the denominator on the right hand side of this equation are both positive because $i^* < \rho < \min \left\{ \phi^L, \rho + \mu^L \right\}$. Therefore, $\frac{\phi^H - i^H}{\rho + \mu^L - i^H} \geq 1$ as $\eta_2^{-1} \equiv \frac{\phi^H - i^H}{\rho + \mu^L - i^H} \geq 1$.

\[ \] Proof of Proposition 9: If $\eta_2 < 1$, then Lemma 5 implies that $\mu^H (i^L) > \mu^H (i^H)$, so that
the optimal investment is positive when \( x = 0 \) in Regime \( H \); therefore, \( x_I^* = 0 \). If \( \eta_2 = 1 \), then Lemma 5 implies that \( v^H(0) = v_x^H(0) \), so that the optimal investment is positive when \( x > 0 \) in Regime \( H \), and the rate of investment is indeterminate for \( x = 0 \); therefore, \( x_I^* = 0 \). Finally, if \( \eta_2 > 1 \), then Lemma 5 implies that \( v^H(0) < v_x^H(0) \), so that the optimal investment is zero when \( x = 0 \) in Regime \( H \); therefore, \( x_I^* > 0 \). ■

**Proof of Corollary 4:** The definition \( \eta_2 \equiv \frac{\phi^H}{\rho + \mu^e} \) implies that if \( \eta_2 = 1 \), then \( \frac{\phi^H}{\rho + \mu^e} = 1 \) and \( x_I^* = 0 \). Anything that decreases \( \frac{\phi^H}{\rho + \mu^e} \) from its initial value of one will cause optimal investment to be zero at \( x = 0 \), which is equivalent to making \( x_I^* > 0 \). The three statements in the corollary follow immediately from this impact on \( \frac{\phi^H}{\rho + \mu^e} \). ■

**Proof of Proposition 10:** It suffices to prove that the right hand side of equation (55) is negative when \( -\phi^L > \gamma(\phi^H) \). Lemma 4 implies that \( \tilde{v}^H(x) \equiv \frac{\tilde{v}^H(X,K)}{R} = x + \frac{(\rho + \mu^L - i^L)}{\mu^L - i^L} \left( \phi^H - i^L \right) + \phi^L \phi^H \left( \rho + \mu^L - i^L \right) + \mu^L \phi^H \left( \rho + \mu^L - i^L \right) \). Since \( v^H(x) \leq \tilde{v}^H(x) \), the right hand side of equation (55) will be negative if \( (\rho + \mu^L - i^L)(\phi^H - i^L) + \phi^H \left( \rho + \mu^L - i^L \right) \left( \phi^H - i^L \right) < 0 \). This condition is equivalent to \( (\rho + \mu^L - i^L) \left( \phi^H - i^L \right) + \mu^L \left( \phi^H - i^L \right) < 0 \), which is equivalent to \( -\phi^L > \frac{\mu^H}{\rho^L - \mu^L} \phi^H \), which can be rewritten as \( -\phi^L > \frac{\mu^H}{\rho^L - \mu^L} \phi^H \), which is equivalent to \( -\phi^L > \frac{\mu^H}{\rho^L - \mu^L} \phi^H \), which implies that \( \phi^L > \gamma(\phi^H) \).

The following lemma is useful in the proof of Proposition 11. ■

**Lemma 7** Assume that \( \eta_2 > 1 \) and \( x_D^* > 0 \). Define \( \alpha \equiv \eta_2 - \omega_1 > \eta_2 - \omega_2 \equiv \beta > 0 \) and assume that \( \beta \omega_1^2 > \alpha \omega_2^2 \). Define \( f(x) \equiv \alpha(1 - (1 + x) \omega_2) e^{\omega_2 x} - \beta (1 - (1 + x) \omega_1) e^{\omega_1 x} \) and \( x_m \equiv \frac{1}{\omega_2 - \omega_1} \ln \frac{\beta \omega_2}{\alpha \omega_1} \) and assume that \( x_m > 0 \).

1. If \( \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 > x_m \), then \( f(x) = 0 \) has a unique root in \((0, x_m)\).

2. If \( \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 = x_m \), then \( x_m \) is the unique root of \( f(x) = 0 \) in \((0, x_m)\).

3. If \( \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 < x_m \), then \( f(x) < 0 \) for all non-negative \( x \) so \( f(x) = 0 \) has no real roots.
Proof of Lemma 7: \( f(0) = (\eta_2 - \omega_1)(1 - \omega_2) - (\eta_2 - \omega_2)(1 - \omega_1) = (\omega_2 - \omega_1)(1 - \eta_2) < 0 \). Differentiate \( f(x) \) with respect to \( x \) to obtain \( f'(x) = [-\alpha \omega_2^2 e^{\omega_2 x} + \beta \omega_1^2 e^{\omega_1 x}] (1 + x) \). If \( x_m \) is a root of \( f'(x) = 0 \), then \( e^{(\omega_2 - \omega_1)x_m} = \frac{\beta \omega_2}{\alpha \omega_2} \) so that \( x_m = \frac{1}{\omega_2 - \omega_1} \ln \frac{\beta \omega_2}{\alpha \omega_2} \) is a root of \( f'(x) = 0 \). Differentiate \( f'(x) \) with respect to \( x \) and evaluate \( f''(x) \) at \( x = x_m \), and use \( f'(x_m) = 0 \), to obtain \( f''(x_m) = [-\alpha \omega_2^2 e^{\omega_2 x_m} + \beta \omega_1^2 e^{\omega_1 x_m}] (1 + x_m) < 0 \), which implies that \( f'(x) > 0 \) for positive \( x < x_m \) and that \( x_m \) is the unique positive root of \( f'(x) = 0 \). Now evaluate \( f(x) \) at \( x = x_m \) to obtain \( f(x_m) = \alpha (1 - (1 + x_m) \omega_2) e^{\omega_2 x_m} - \beta (1 - (1 + x_m) \omega_1) e^{\omega_1 x_m} \), and use \( \alpha \omega_2^2 e^{\omega_2 x_m} = \beta \omega_1^2 e^{\omega_1 x_m} \) from \( f'(x) = 0 \) to obtain \( f(x_m) = \left[ \alpha (1 - (1 + x_m) \omega_2) e^{\omega_2 x_m} - (1 - (1 + x_m) \omega_1) \omega_2 e^{\omega_2 x_m} \right] [1 - \frac{\omega_2}{\omega_1}] \omega_2 (1 + x_m) \) \( \alpha e^{\omega_2 x_m} \), which has the same sign as \( \frac{1}{\omega_2} - \frac{1}{\omega_1} - 1 - x_m \) (since \( \omega_1 + \omega_2 = \eta_1 + \eta_2 < 0 \) which implies \( \omega_2 < -\omega_1 \) so \( \omega_2 - \omega_1 > -1 \) and hence \( 1 + \frac{\omega_2}{\omega_1} > 0 \)). Therefore, (Statement 1) if \( \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 > x_m \), then \( f(0) < 0 \) and \( f(x_m) > 0 \) so \( f(x) = 0 \) has a unique root in \((0, x_m) \). (Statement 2) If \( \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 = x_m \), then \( f(x_m) = 0 \) so \( x_m \) is the unique root of \( f(x) = 0 \) in \((0, x_m) \). Finally, (Statement 3), if \( \frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 < x_m \), then \( f(x_m) < 0 \), and since \( x_m \) maximizes \( f(x) \) for \( x \geq 0 \), \( f(x) = 0 \) does not have a nonnegative real root. 

Proof of Proposition 11

Proof of Statement 1: Lemma 5 implies that since \( \eta_2 \leq 1 \), \( v^H(0) \geq v^H_x(0) \), which implies that \( v^H(x) - (1 + x) v^H_x(x) \geq 0 \) at \( x = 0 \). Therefore, \( x^*_x = 0 \).

Proof of Statement 1a: If \( x^*_x = 0 \), then \( i = \bar{i} \) in Regime \( H \) and equation \((85)\) implies that the ODE in Regime \( H \) is \( 0 = \phi^H v^H_x(x) - (\rho + \mu^L) v^H(x) + \mu^L v^L(x) + (v^H(x) - (1 + x) v^H_x(x)) \) and in Regime \( L \), \( 0 = \phi^L v^L_x(x) - (\rho + \mu^H) v^L(x) + \mu^H v^H(x) \), so \( v^L(0) = 0 \) implies \( v^L_x(0) = \frac{\mu^H}{\rho + \mu^L} \).

Evaluate the ODE for \( v^H(x) \) at \( x = 0 \) to obtain \((\phi^H - \bar{i}) v^H_x(0) = (\rho + \mu^L - \bar{i}) v^H(0) \), equivalently, \( v^H(0) = \frac{\phi^H - \bar{i}}{\rho + \mu^L} v^H_x(0) \).

Now assume that \( x^*_D = 0 \) so that \( v^H_x(0) \leq 1 \) to obtain \( v^H(0) \leq \frac{\phi^H - \bar{i}}{\rho + \mu^L} v^H_x(0) \), which implies \( v^L_x(0) \leq \frac{\mu^H}{\rho + \mu^L} v^H(0) \).

Now differentiate the ODE in Regime \( H \) with respect to \( x \) and evaluate the resulting equation at \( x = 0 \) to obtain \( 0 = \phi^H v^H_x(0) - (\rho + \mu^L) v^H_x(0) + \mu^L v^L(0) - v^H(0) \). Using the boundary condition (since \( x^*_x = x^*_D = 0 \)) \( v^H_x(0) = 0 \) to obtain \( v^H(0) = \frac{\mu^L}{\rho + \mu^L} v^L(0) \leq \frac{\mu^L}{\rho + \mu^L} \frac{\phi^H - \bar{i}}{\rho + \mu^L} \leq 1 \) if and only if \( \frac{\phi^H - \bar{i}}{\rho + \mu^L} \leq \left(1 + \frac{\rho}{\mu^L}\right) \frac{\phi^H - \bar{i}}{\mu^L} \). Therefore, \( x^*_D \).

Proof of Statement 1b: If \( \frac{\phi^H - \bar{i}}{\rho + \mu^L} \geq \left(1 + \frac{\rho}{\mu^L}\right) \frac{\phi^H - \bar{i}}{\mu^L} \), then the proof of Statement 1a implies that \( v^H(0) > 1 \) so that \( x^*_D > 0 \). Since \( x^*_x = 0 \) and \( x^*_D > 0 \), it follows that \( v^H_x(0) < 0 \) and 23Since \( \eta_2 \leq \frac{\rho + \mu^L}{\mu^L} \leq 1 \), it follows that (recall \( \bar{i} < \rho + \mu^L \)) \( \frac{\phi^H - \bar{i}}{\rho + \mu^L} \frac{\rho + \mu^L}{\mu^L} \geq 1 \) and \( v^H(0) \geq v^H_x(0) \) as implied by Lemma 5.
\(v^H_{xx}(x^*_D) = 0\) where \(v^H(x)\) is given by equation (106), so that as derived in equation (119) in Appendix D,

\[
v^H_{xx}(x) = -k_1\eta_1^3 a (a+1) U(a+2, b+3, z(x)) + k_2\eta_1^3 \frac{a-b}{b+1} a + 1 \frac{a}{b+1} b + 2 M(a+2, b+3, z(x)).
\]

Therefore, \(v^H_{xx}(0) < 0\) implies

\[
\frac{U(a+2, b+3, z(0))}{M(a+2, b+3, z(0))} < -\frac{k_2}{k_1} \frac{b-a}{b+1} (b+2)
\]

and \(v^H_{xx}(x^*_D) = 0\) implies

\[
\frac{U(a+2, b+3, z(x^*_D))}{M(a+2, b+3, z(x^*_D))} = -\frac{k_2}{k_1} \frac{b-a}{b+1} (b+2).
\]

Equation (66) and Lemma 10, which states that \(\frac{U(a+2, b+3, z(x))}{M(a+2, b+3, z(x))}\) is strictly increasing in \(x\) and \(\lim_{x\to\phi^H-1} \frac{U(a+2, b+3, z(x))}{M(a+2, b+3, z(x))} = \infty\) implies that \(\frac{U(a+2, b+3, z(x))}{M(a+2, b+3, z(x))} = -\frac{k_2}{k_1} \frac{b-a}{b+1} (b+2)\) has a unique root in \(\left(0, \frac{\phi^H}{\phi^H-1}\right)\). That root is \(x^*_D\). Finally, since \(x^*_I = 0\), then \(Y(x) = k_1 e^{-\eta_1 x} U(a, b, z(x)) + k_2 e^{-\eta_1 x} M(a, b, z(x))\) (from equation (115) in Appendix D and the condition \(Y(0) = 0\) implies

\[
-\frac{k_2}{k_1} = \frac{U(a, b, z(0))}{M(a, b, z(0))}.
\]

Substituting equation (68) into equation (67) completes the proof that \(x^*_D\) is the unique root in \(\left(0, \frac{\phi^H}{\phi^H-1}\right)\) of

\[
\frac{U(a+2, b+3, z(x^*_D))}{M(a+2, b+3, z(x^*_D))} = \frac{b-a}{b+1} (b+2) \frac{U(a, b, z(0))}{M(a, b, z(0))}.
\]

Proof of Statement 2: Since \(\frac{\phi^H}{\phi^H-1} < 1\), Lemma 5 implies that \(v^H(0) < v^H(0)\) so that \(v^H(x) - (1+x) v^H_x(x) < 0\) at \(x = 0\). Therefore, optimal investment is zero at \(x = 0\) so \(x^*_I > 0\).

Proof of Statement 2a: Assume that \(x_m < 0\). Since \(\omega_2 - \omega_1 > 0\), \(x_m < 0\) implies that \(\frac{\eta_2 - \omega_2}{\eta_1 - \omega_1} \frac{\omega_1^3}{\omega_2^3} < 1\). Use \(\eta_1 + \eta_2 = \omega_1 + \omega_2\), to rewrite \(\frac{\eta_2 - \omega_2}{\eta_1 - \omega_1} \frac{\omega_1^3}{\omega_2^3}\) as \(\frac{\omega_1 - \eta_1}{\eta_2 - \omega_1} \frac{\omega_1^2 - \eta_1^2}{\omega_2^2 - \eta_1^2}\) > 0 since \(\eta_1 < \omega_1 < 0 < \omega_2 < \eta_2\). Therefore, \(\frac{\eta_2 - \omega_2}{\eta_1 - \omega_1} \frac{\omega_1^3}{\omega_2^3} < 1\) implies \(g(\omega) > 0\) where

\[
g(\omega) \equiv (\omega - \eta_1) \omega_2 - (\eta_2 - \omega)(\eta_1 + \eta_2 - \omega)^2.
\]

The function \(g(\omega)\) is a third-degree polynomial, which can be written as \(g(\omega) = 2\omega^3 - 3(\eta_1 + \eta_2) \omega^2 + (\eta_1 + 3\eta_2)(\eta_1 - \eta_2)\omega - \eta_2(\eta_1 + \eta_2)^2\) and then factored to \(g(\omega) = (2\omega - (\eta_1 + \eta_2))[(\omega^2 - (\eta_1 + \eta_2)(\omega + 2(\eta_1 + \eta_2))].\)

Use the definition of \(g(\omega)\) in the characteristic polynomial in equation (70) to obtain \(g(\omega)\)
Proof of Statement 2b: Since $x_I^* > 0$, the ODE in equation (17) holds in a positive neighborhood of $x = 0$. Therefore, if $x_m > 0$, the firm will not pay dividends when $x = 0$ in Regime $H$. Therefore, $x_D^* > 0$.

Proof of Statement 2(b)i: Assume that $x_m > 0$ and $\frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 < x_m$. Suppose that, contrary to what is to be proved, $x_I^* < x_D^*$ so that $f(x_I^*) = 0$. Therefore, $f(x) = 0$ has a root in $(0, x_D^*)$ but Statement 3 of Lemma 7 is that $f(x) = 0$ has no real roots. Therefore, $x_D^* < x_I^*$, and the firm’s dividend trigger is the same as in the absence of investment, so $x_D^* = x_m$. An ongoing firm will never have $x > x_D^* = x_m$ and the firm will never invest in capital.

Proof of Statement 2(b)ii: The function $f(x)$ defined in Lemma 7 equals $v^H(x) - (1 + x)v_x^H(x)$ for $0 \leq x \leq \min \{x_I^*, x_D^*\}$. Assume that $\frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 < x_m > 0$. Suppose that, contrary to what is to be proved, $x_I^* \geq x_D^*$. Therefore, $x_D^* = x_m$ but Statement 1 of Lemma 7 implies that $f(x) = 0$ has a root in $(0, x_m)$ where $v^H(x) - (1 + x)v_x^H(x) = 0$ so that root is $x_I^* < x_m = x_D^*$, which contradicts the supposition that $x_I^* \geq x_D^*$. Therefore, $x_I^* < x_D^*$. The definition of $f(x) \equiv (\eta_2 - \omega_1)(1 - (1 + x)\omega_2)\varepsilon_{m+1}x - (\eta_2 - \omega_2)(1 - (1 + x)\omega_1)\varepsilon_m x$ implies that the root of $f(x) = 0$, $x_I^*$, is the unique root of $e^{(\omega_2 - \omega_1)x} = \frac{\eta_2 - \omega_2}{\eta_2 - \omega_1} \frac{1 - (1 + x)\omega_1}{1 - (1 + x)\omega_2}$. Now observe that $\frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 > 0$ implies $\frac{1}{\omega_2} - 1 > -\frac{1}{\omega_1} > 0$. Since $\omega_1 < 0 < \omega_2 < \eta_2$ and $x_I^* > 0$, the expression $e^{(\omega_2 - \omega_1)x_I^*} = \frac{\eta_2 - \omega_2}{\eta_2 - \omega_1} \frac{1 - (1 + x_I^*)\omega_1}{1 - (1 + x_I^*)\omega_2}$ implies that $(1 + x_I^*)\omega_2 < 1$, or equivalently, $0 < x_I^* < \frac{1}{\omega_2} - 1$.

To calculate $x_D^*$, use the fact that equations (106), (118), and (118) hold for $x_I^* \leq x < x_D^*$. Since $v^H(x_I^*) - (1 + x_I^*)v_x^H(x_I^*) = 0$, equations (106), (118) imply that

$$
-k_2
-k_1
-b
-aM(a, b + 1, z(x_I^*))
-U(a, b + 1, z(x_I^*))
+(1 + x_I^*)\eta_1aU(a + 1, b + 2, z(x_I^*))
-(1 + x_I^*)\eta_1\frac{a}{b + 1}M(a + 1, b + 2, z(x_I^*))
$$

Equation (119) implies that $0 = v_{xx}^H(x_D^*) = -k_1\eta_1^3a(a + 1)U(a + 2, b + 3, z(x_D^*)) + k_2\eta_1^3\frac{a - b}{b + 1, b + 2}M(a + 2, b + 3, z(x_D^*)) = \frac{k_2}{k_1}\frac{b - a}{b + 1, b + 2}$. Using the ex-
expression for \(-\frac{k_2}{k_1}\) immediately above yields

\[
\frac{U(a + 2, b + 3, z(x_D^*))}{M(a + 2, b + 3, z(x_D^*))} = \frac{1}{b + 1} \frac{U(a, b + 1, z(x_D^*)) + (1 + x_D^*) \eta_1 a U(a + 1, b + 2, z(x_D^*))}{b + 1 + b + 2 M(a, b + 1, z(x_D^*)) - (1 + x_D^*) \eta_1 \frac{a}{b+1} M(a + 1, b + 2, z(x_D^*))}.
\]

Equation (119) also implies that \(0 > \frac{\mu H}{x_d(x_D^*)} = -k_1 \eta_1^3 \eta (a + 1) U(a + 2, b + 3, z(x_D^*)) + k_2 \eta_1^3 \frac{a - b}{b + 1} \eta_2 \frac{a + 1}{b + 2} M(a + 2, b + 3, z(x_D^*))\) so

\[
\frac{U(a + 2, b + 3, z(x_D^*))}{M(a + 2, b + 3, z(x_D^*))} < \frac{-k_2 b - a}{k_1 b} \frac{1}{b + 1} \frac{1}{b + 2} = \frac{U(a + 2, b + 3, z(x_D^*))}{M(a + 2, b + 3, z(x_D^*))}
\]

Therefore, Lemma 10, which states that \(U(a + 2, b + 3, z(x_D^*))\) is strictly increasing in \(x\) and \(\lim_{x \to \frac{\phi_H}{T} - 1} \frac{U(a + 2, b + 3, z(x_D^*))}{M(a + 2, b + 3, z(x_D^*))} = \infty\) implies that

\[
\frac{U(a + 2, b + 3, z(x_D^*))}{M(a + 2, b + 3, z(x_D^*))} = \frac{1}{b + 1} \frac{U(a, b + 1, z(x_D^*)) + (1 + x_D^*) \eta_1 a U(a + 1, b + 2, z(x_D^*))}{b + 1 + b + 2 M(a, b + 1, z(x_D^*)) - (1 + x_D^*) \eta_1 \frac{a}{b+1} M(a + 1, b + 2, z(x_D^*))}
\]

has a unique root in \(\left(x_D^*, \frac{\phi_H}{T} - 1\right)\).

**Proof. of Lemma 6:** The sum of the eigenvalues of the matrix \(A\) in equation (18) is

\[
\omega_1 + \omega_2 = trA = \eta_1 + \eta_2\]

and the product of the eigenvalues is \(\omega_1 \omega_2 = \eta_1 \eta_2 - \frac{\mu H}{\phi L} \frac{\mu L}{\mu H} = \eta_1 \eta_2 - \frac{\mu H}{\phi L} \frac{\mu L}{\mu H} \frac{\mu H}{\phi L} \frac{\mu L}{\mu H} = \left(1 - \frac{\mu H}{\phi L} \frac{\mu L}{\phi L}\right) \eta_1 \eta_2 - \frac{\mu H}{\phi L} \frac{\mu L}{\phi L} \frac{\mu H}{\phi L} \frac{\mu L}{\phi L}.\] Therefore, \(\frac{1}{\omega_1} + \frac{1}{\omega_2} - 1 = \frac{1}{\omega_1} \omega_2 + \frac{1}{\omega_2} \omega_1 - \frac{\mu H}{\phi L} \frac{\mu L}{\phi L} \frac{\mu H}{\phi L} \frac{\mu L}{\phi L} = \frac{\eta_1 \eta_2}{\omega_1 \omega_2} \left(\frac{\phi L}{\mu H} + \frac{\phi H}{\mu L} - (1 - \frac{\mu H}{\phi L} \frac{\mu L}{\phi L}) \eta_1 \eta_2\right) = \frac{\eta_1 \eta_2}{\omega_1 \omega_2} \left(\frac{\phi L}{\mu H} + \frac{\phi H}{\mu L} - (1 - \frac{\mu H}{\phi L} \frac{\mu L}{\phi L})\right) = \frac{\eta_1 \eta_2}{\omega_1 \omega_2} \frac{1}{\omega_1 \omega_2} \left(-\phi L - \delta(\phi_H)\right),\]

where \(\delta(\phi_H)\)

\[
\equiv \frac{\phi H}{\mu H} \left(\frac{\mu H}{\phi H}\right).
\]

49
The solution to the system of ODEs in equation (17) can be expressed in terms of the eigenvalues and eigenvectors of the $2 \times 2$ matrix $A$ defined in equation (18). The eigenvalues, $\omega_1$ and $\omega_2$ are the roots of the following characteristic equation

$$q(\omega) \equiv (\eta_2 - \omega)(\eta_1 - \omega) - \frac{\mu^L}{\phi^H} \frac{\mu^H}{\phi^L} = 0.$$  \hspace{1cm} (70)$$

and satisfy$^{24}$

$$\omega_1 + \omega_2 = \eta_1 + \eta_2 < 0.$$ \hspace{1cm} (71)$$

and

$$\eta_1 < \omega_1 < 0 < \omega_2 < \eta_2.$$ \hspace{1cm} (72)$$

It is straightforward to verify that$^{25}$ \[1 \frac{\phi^H}{\mu^I} (\eta_2 - \omega_i) \] is an eigenvector of $A$ with eigenvalue $\omega_i$, $i = 1, 2$.

The general solution to a two-equation system of constant-coefficient homogeneous first-order linear ODEs is the sum of the product of $e^{\omega_i X}$ and an eigenvector corresponding to the eigenvalue $\omega_i$, $i = 1, 2$, so

$$\begin{bmatrix} V^H(X) \\ V^L(X) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \frac{\phi^H}{\mu^I} (\eta_2 - \omega_1) \end{bmatrix} e^{\omega_1 (X - X^*)} + c_2 \begin{bmatrix} 1 \\ \frac{\phi^H}{\mu^I} (\eta_2 - \omega_2) \end{bmatrix} e^{\omega_2 (X - X^*)}$$ \hspace{1cm} (73)$$

Differentiate the expression for $V^H(X)$ in equation (73) twice with respect to $X$, evaluate $V^H_X(X)$ and $V^H_{XX}(X)$ at $X = X^*$, and use the boundary conditions in equations (11a) and (12) to obtain

$$V^H_X(X^*) = c_1 \omega_1 + c_2 \omega_2 = 1$$ \hspace{1cm} (74)$$

$^{24}$The facts that $q(0) = \eta_2 \eta_1 - \frac{\mu^L}{\phi^H} \frac{\mu^H}{\phi^L} < 0$ and $q''(\omega) = 2$ imply that the characteristic equation has two real roots, $\omega_1 < 0 < \omega_2$. Also, $q(\eta_2) = q(\eta_1) = -\frac{\mu^L}{\phi^H} \frac{\mu^H}{\phi^L} > 0$, so $\eta_2 > \omega_2$ and $\eta_1 < \omega_1$.

$^{25}$The first element of $A \begin{bmatrix} 1 \frac{\phi^H}{\mu^I} (\eta_2 - \omega_i) \end{bmatrix}'$ is $\eta_2 - (\eta_2 - \omega_i) = \omega_i$. The second element of $A \begin{bmatrix} 1 \frac{\phi^H}{\mu^I} (\eta_2 - \omega_i) \end{bmatrix}'$ is $-\frac{\mu^L}{\phi^H} + \eta_1 \frac{\phi^H}{\mu^I} (\eta_2 - \omega_i) = \frac{\phi^H}{\mu^I} \left[ -\frac{\mu^L}{\phi^H} + \eta_1 (\eta_2 - \omega_i) \right]$. The characteristic equation (70) implies that $\frac{\mu^L}{\phi^H} \frac{\mu^H}{\phi^L} = (\eta_2 - \omega_i)(\eta_1 - \omega_i)$ so the second element of $A \begin{bmatrix} 1 \frac{\phi^H}{\mu^I} (\eta_2 - \omega_i) \end{bmatrix}'$ is $\frac{\phi^H}{\mu^I} \left[ - (\eta_2 - \omega_i)(\eta_1 - \omega_i) + \eta_1 (\eta_2 - \omega_i) \right] = \omega_i \frac{\phi^H}{\mu^I} (\eta_2 - \omega_i)$. 

50
and

\[ V_{XX}^{H} (X^*) = c_1 \omega_1^2 + c_2 \omega_2^2 = 0. \]  

(75)

Equations (74) and (75) are two linear equations in the constants \( c_1 \) and \( c_2 \). Equation (75) implies

\[ c_1 \omega_1^2 = -c_2 \omega_2^2, \]  

(76)

which, along with equation (74), implies

\[ c_1 = \frac{1}{\omega_2 - \omega_1} \frac{\omega_2}{\omega_1} < 0 \]  

(77)

and

\[ c_2 = \frac{1}{\omega_1 - \omega_2} \frac{\omega_1}{\omega_2} > 0. \]  

(78)

Evaluate \( V^L (X) \) in equation (73) at \( X = 0 \), and then use the boundary condition \( V^L (0) = 0 \) from equation (10) to obtain

\[ V^L (0) = c_1 \frac{\phi^H}{\mu L} (\eta_2 - \omega_1) e^{-\omega_1 X^*} + c_2 \frac{\phi^H}{\mu L} (\eta_2 - \omega_2) e^{-\omega_2 X^*} = 0. \]  

(79)

Rearrange equation (79) using \( c_1 \omega_1^2 = -c_2 \omega_2^2 \) from equation (76) to obtain

\[ e^{(\omega_2 - \omega_1) X^*} = \frac{\omega_1^2}{\omega_2^2} \frac{\eta_2 - \omega_1}{\eta_2 - \omega_2} \]  

(80)

where

\[ Z \equiv \frac{\omega_1^2}{\omega_2^2} \frac{\eta_2 - \omega_1}{\eta_2 - \omega_2} > 0. \]  

(81)

Equations (80) and (81) imply that the target value of cash on hand is

\[ X^* = \max \left[ \frac{1}{\omega_2 - \omega_1} \ln Z, 0 \right] \]  

(82)

The fact that \( Z > 0 \) in equation (81) follows from \( \omega_1 < \omega_2 < \eta_2 \) in equation (23). Since \( \omega_2 - \omega_1 > 0 \), \( X^* \) will be positive if and only if \( Z > 1 \). Since both the numerator and the denominator of \( Z \) in equation (81) are positive, \( Z \) will be greater than one if and only if its numerator is larger than its denominator. Replacing \( \omega_2 \) in the definition of \( Z \) by \( \eta_1 + \eta_2 - \omega_1 \), the numerator of \( Z \) minus the denominator of \( Z \) can be written as \( g (\omega_1) \) where

\[ g (\omega) \equiv \omega^2 (\omega - \eta_1) - (\eta_2 - \omega) (\eta_1 + \eta_2 - \omega)^2. \]  

(83)

**Lemma 8** \( g (\omega_1) = [\eta_1 + \eta_2 - 2 \omega_1] \zeta \) where \( \zeta \equiv -\frac{\eta_2^2 - \mu L \mu H}{\phi \phi^*}. \)
Proof. of Lemma 8: \( g(\omega) = \omega^2 (\omega - \eta_1) + (\omega - \eta_2) \left( \omega^2 - 2(\eta_1 + \eta_2)\omega + (\eta_1 + \eta_2)^2 \right) \), which can be written as the following third-order polynomial in \( \omega \): 
\[
g(\omega) = 2\omega^3 + [-\eta_1 - 2(\eta_1 + \eta_2) - \eta_2] \omega^2 + [(\eta_1 + \eta_2)^2 + 2\eta_2(\eta_1 + \eta_2)] \omega - \eta_2(\eta_1 + \eta_2)^2,
\]
which can be simplified to 
\[
g(\omega) = 2\omega^3 - 3(\eta_1 + \eta_2)\omega^2 + (\eta_1 + 3\eta_2)(\eta_1 + \eta_2)\omega - \eta_2(\eta_1 + \eta_2)^2
\]
and then factored to 
\[
g(\omega) = (2\omega - (\eta_1 + \eta_2))\times \left[ \omega^2 - (\eta_1 + \eta_2)\omega + \eta_2(\eta_1 + \eta_2) \right].
\]
Use the definition of \( \theta(\omega) \) in the characteristic polynomial in equation (70) to obtain 
\[
\theta(\omega) = \omega^2 - (\eta_1 + \eta_2)\omega + \eta_1 \eta_2 - \frac{\mu^L \mu^H}{\sigma^2 \sigma^T},
\]
and then rewrite \( g(\omega) \) as 
\[
g(\omega) = [2\omega - (\eta_1 + \eta_2)] \times \left[ \omega^2 - (\eta_1 + \eta_2)\omega + \eta_1 \eta_2 - \frac{\mu^L \mu^H}{\sigma^2 \sigma^T} + \frac{\mu^L \mu^H}{\sigma^2 \sigma^T} + \eta_2^2 \right] = [2\omega - (\eta_1 + \eta_2)] 
\]
\[
\times \left[ q(\omega) + \frac{\mu^L \mu^H}{\sigma^2 \sigma^T} + \eta_2^2 \right].
\]
Since the characteristic polynomial equals zero when evaluated at either of the roots \( \omega_1 \) or \( \omega_2 \), 
\[
g(\omega) = (\eta_1 + \eta_2 - 2\omega) \times \left( -\frac{\mu^L \mu^H}{\sigma^2 \sigma^T} - \eta_2^2 \right) = (\eta_1 + \eta_2 - 2\omega_1) \zeta.
\]
Since \( \eta_1 + \eta_2 - 2\omega_1 > 0 \), \( g(\omega) \) has the same sign as \( \zeta \). Therefore, the numerator of \( Z \) is larger than the denominator of \( Z \) if and only if \( \zeta > 0 \).
C Optimal Dividends and Investment in Regime $H$

The value function in Regime $H$, $v^H(x)$, is an important component of the decisions to invest in capital and to pay dividends. This value function is characterized by a second-order ODE. This section first derives the general solution to this ODE. This ODE and the associated value function have different forms depending on whether the firm is in a situation in which it is optimal to have zero capital investment or positive capital investment. The first subsection below presents the value function for situations in which optimal capital investment turns out to be zero; the second subsection presents the value function for situations in optimal capital investment turns out to be positive.

Suppose that whenever an ongoing firm is in Regime $L$ optimal dividends and investment are both zero (Proposition 10 provides a sufficient condition for optimal investment to be zero.). Therefore, the ODE in equation (50b) is $v_x^L \phi^L + \mu^H v^H - (\rho + \mu^H) v^L = 0$. Imposing the boundary condition $v^L(0) = 0$ and recalling that $\eta_1 \equiv \frac{\alpha + \mu^H}{\phi^L}$, the solution to this ODE is

$$v^L(x) = -\frac{\mu^H}{\phi^L} \int_0^x e^{-\eta_1(u-x)} v^H(u) \, du.$$ (84)

In Regime $H$, the ODE in equation (50a) is

$$v_x^H \phi^H + \mu^L v^L - (\rho + \mu^L) v^H + \tau [v^H - (1 + x) v_x^H]^+ = 0,$$ (85)

where $[z]^+ \equiv \max \{0, z\}$.

To solve equations (84) and (85) for $v^H(x)$, it is convenient to rewrite this pair of equations as a second-order ODE in $Y(x)$, where

$$Y(x) \equiv \int_0^x e^{-\eta_1 u} v^H(u) \, du$$ (86)

has the following properties

$$Y(0) = 0$$ (87)

$$Y''(x) = e^{-\eta_1 x} v^H(x)$$ (88)

and

$$Y'''(x) = [v_x^H(x) - \eta_1 v^H(x)] e^{-\eta_1 x}.$$ (89)

The expression for $v^L(x)$ in equation (84) and the definition of $Y(x)$ in equation (86) imply

$$v^L(x) = -\frac{\mu^H}{\phi^L} e^{\eta_1 x} Y(x).$$ (90)
Equation (88) implies 

\[ v^H(x) = e^{\eta_1 x}Y'(x) \]  

(91)

and equations (89) and (91) imply 

\[ v^H_x(x) = [Y''(x) + \eta_1 Y'(x)]e^{\eta_1 x}. \]  

(92)

Substitute the expressions for \( v^L(x), v^H(x), \) and \( v^L_x(x) \) from equations (90), (91), and (92), respectively, into equation (85), divide both sides of the resulting equation by \( \phi^H e^{\eta_1 x} \) and use the definition \( \eta_2 = \frac{\phi^H}{\phi^L} \) to obtain 

\[ Y''(x) - (\eta_2 - \eta_1) Y'(x) - \frac{\mu^L \mu^H}{\phi^H \phi^L} Y(x) + \frac{1}{\phi^H} (Y'(x) - (1 + x) (Y''(x) + \eta_1 Y'(x)))^+ = 0 \]  

(93)

The solution to the ODE in equation (93) depends on whether \( x \leq x_I^* \) so that optimal investment is zero or \( x > x_I^* \) so that optimal investment is positive. These two cases are analyzed separately in the two subsections that follow.

### C.1 Zero Investment in Regime \( H \): \( x \leq x_I^* \)

Consider the case in which \( \eta_2 > 1 \) so that (Proposition 9) \( x_I^* > 0 \). Suppose that the firm is in Regime \( H \) and that \( x < x_I^* \), which implies that optimal investment is zero and that the term \( []^+ \) in equation (93) is zero. Therefore, the solution of this ODE is 

\[ Y(x) = e^{-\eta_1 x} (B_1 e^{\omega_1 x} + B_2 e^{\omega_2 x}) \]  

(94)

where \( \eta_1 < \omega_1 < 0 < \omega_2 < \eta_2 \) are the roots of the matrix \( A \) in equation (18).

The requirement \( Y(0) = 0 \) implies that \( -B_1 = B_2 \), so 

\[ Y(x) = e^{-\eta_1 x} B_1 (e^{\omega_1 x} - e^{\omega_2 x}) \]  

(95)

and hence 

\[ Y'(x) = e^{-\eta_1 x} B_1 [(\omega_1 - \eta_1) e^{\omega_1 x} - (\omega_2 - \eta_1) e^{\omega_2 x}]. \]  

(96)

Multiplying both sides of equation (96) by \( e^{\eta_1 x} \) and using equation (91) immediately yields 

\[ v^{(H)}(x) = B_1 [(\omega_1 - \eta_1) e^{\omega_1 x} - (\omega_2 - \eta_1) e^{\omega_2 x}]. \]  

(97)

Now suppose that there exists some positive \( 0 < x_I^* < x_D^* \) such that\(^{26} \)

\[ v^{(H)}(x_I^*) - (1 + x_I^*) v^{(H)'(x_I^*)} = 0. \]  

(98)

\(^{26}\)Proposition 11 Statement 2(b)ii presents conditions for \( 0 < x_I^* < x_D^* \).
Substituting equation (97) into equation (98) yields

\[
[1 - (1 + x_1^*) \omega_1] (\omega_1 - \eta_1) e^{\omega_1 x_1^*} = [1 - (1 + x_1^*) \omega_2] (\omega_2 - \eta_1) e^{\omega_2 x_1^*}
\]  (99)

or

\[
\frac{[1 - (1 + x_1^*) \omega_1] (\omega_1 - \eta_1)}{[1 - (1 + x_1^*) \omega_2] (\omega_2 - \eta_1)} = e^{(\omega_2 - \omega_1) x_1^*},
\]  (100)

which implies that \( x_1^* \frac{1}{\omega_2 - 1} < 1 \).

**C.2 Positive Investment in Regime \( H \): \( x > x_I^* \)**

Suppose that the firm is in Regime \( H \) and that \( x > x_I^* \), which implies that optimal investment is positive and that the term \([ + \] in equation (93) is positive. Then the ODE in equation (93) becomes

\[
\left( \frac{\phi_H^I}{i} - (1 + x) \right) Y''(x) - \left( \eta_2 - \eta_1 \right) \frac{\phi_H^I}{i} - (1 - (1 + x) \eta_1) Y'(x) - \frac{1}{i} \frac{\mu^I \mu^H}{\phi^L} Y(x) = 0.
\]  (101)

To express the solution to the ODE in equation (101) define

\[
a \equiv \frac{\rho + \mu^L \left( 1 - \frac{\mu^H}{\rho + \mu^H} \right) - \bar{\tau}}{\bar{\iota}}
\]  (102)

and assume that \( a > 0 \). Also define

\[
b \equiv \frac{\rho + \mu^L - \bar{\tau}}{\bar{\iota}} = a + \frac{\mu^L \mu^H}{\rho + \mu^H} \frac{1}{\bar{\iota}} > a > 0
\]  (103)

and

\[
z(x) \equiv \eta_1 \left( 1 + x - \frac{\phi_H^I}{i} \right) > 0,
\]  (104)

so that

\[
z'(x) = \eta_1 < 0.
\]  (105)

Appendix D derives the solution to the ODE in equation (101) and uses the relationship between \( Y(x) \) and \( v^H(x) \) in equation (86) to derive the following expression for \( v^H(x) \)

\[
v^H(x) = -k_1 \eta_1 U(a, b + 1, z(x)) + k_2 \frac{a - b}{b} M(a, b + 1, z(x)), \quad x_I^* \leq x < x_D^*
\]  (106)

where \( U(a, b + 1, z(x)) \) and \( M(a, b + 1, z(x)) \) are Kummer equations and \( k_1 \) and \( k_2 \) are constants.
Lemma 9. $k_1 > 0 > k_2$ in the expression for $v^H(x)$ in equation (106).

Proof. of Lemma 9: We require that for $x_1^* < x < x_D^*$, $v^H(x) > 0$, $v^H_x(x) > 0$, and $v^H_{xx}(x) < 0$. In equation (117) $k_1$ and $k_2$ both multiply positive numbers, so $v^H(x) > 0$ implies that at least one of $k_1$ and $k_2$ is positive. In equation (119) $k_1$ and $k_2$ also both multiply positive numbers, so $v^H_{xx}(x) < 0$ implies that at least one of $k_1$ and $k_2$ is negative. In equation (118), $k_1$ multiplies a positive number and $k_2$ multiplies a negative number, so $v^H_x(x) > 0$ implies that $k_1 > 0$. □

Lemma 10. Define $z(x) \equiv \eta_1 \left( 1 + x - \frac{\phi^H}{T} \right)$ for $0 \leq x < \frac{\phi^H}{T} - 1$. Then for $a > 0$ and $b > 0$, $\frac{U(a+2,b+3,z(x))}{M(a+2,b+3,z(x))}$ is increasing in $x$ for $0 < x < \frac{\phi^H}{T} - 1$ and $\lim_{x \to \frac{\phi^H}{T} - 1} \frac{U(a+2,b+3,z(x))}{M(a+2,b+3,z(x))} = \infty$.

Proof. of Lemma 10: Since $U(a+2,b+3,z(x))$ is strictly decreasing in $z(x)$, $M(a+2,b+3,z(x))$ is strictly increasing in $z(x)$, and $z'(x) = \eta_1 < 0$, it follows that $\frac{U(a+2,b+3,z(x))}{M(a+2,b+3,z(x))}$ is strictly increasing in $x$. Since $z(x) \equiv \eta_1 \left( 1 + x - \frac{\phi^H}{T} \right) > 0$, $\lim_{x \to \frac{\phi^H}{T} - 1} z(x) = 0$. Therefore, $\lim_{x \to \frac{\phi^H}{T} - 1} \frac{U(a+2,b+3,z(x))}{M(a+2,b+3,z(x))} = \infty$ (Abramowitz and Stegun 13.5.6) and $\lim_{x \to \frac{\phi^H}{T} - 1} M(a+2,b+3,z(x)) = 1$ (Abramowitz and Stegun 13.5.5). Therefore, $\lim_{x \to \frac{\phi^H}{T} - 1} \frac{U(a+2,b+3,z(x))}{M(a+2,b+3,z(x))} = \infty$. □
D Solution to ODE in equation (101) for $x \geq x^*_I$

This appendix derives the solution to the ODE in equation (101) and then uses the relationship between $\varphi(x)$ and $\psi(x)$ in equation (86) to derive $v^H(x)$ and its first two derivatives.

Since $\eta_2 \equiv \frac{\rho + \mu^L - \frac{7}{i}}{\phi^H}$, the definition of $b$ in equation (103) implies

$$\eta_2 \phi^H = 1 = \frac{\rho + \mu^L - \frac{7}{i}}{\phi^H} \equiv b \quad (107)$$

and since $\eta_1 \equiv \frac{\rho + \mu^H}{\phi^H}$, equation (103) implies

$$\frac{1}{\phi^L} \phi^H \equiv \frac{1}{\phi^L} \frac{\rho + \mu^H}{\phi^H} = \eta_1 \frac{\rho + \mu^L}{\phi^L} = \eta_1 (b - a). \quad (108)$$

Use equations (104), (107), and (108) to rewrite the ODE in equation (101) as

$$-\frac{z(x)}{\eta_1} Y''(x) - (b + z(x)) Y'(x) - \eta_1 (b - a) Y(x) = 0 \quad (109)$$

It will be convenient to define the function $f(z(x))$ such that

$$Y(x) = e^{-\eta_1 x} f(z(x)) \quad (110)$$

which implies

$$Y'(x) = \eta_1 e^{-\eta_1 x} [-f(z(x)) + f'(z(x))] \quad (111)$$

and

$$Y''(x) = \eta_1^2 e^{-\eta_1 x} [f(z(x)) - 2f'(z(x)) + f''(z(x))] \quad (112)$$

Now substitute equations (110), (111), and (112), for $Y(x)$, $Y'(x)$, and $Y''(x)$ in equation (109), and then divide both sides by $-\eta_1 e^{-\eta_1 x}$ to obtain

$$z(x) [f(z(x)) - 2f'(z(x)) + f''(z(x))] + (b + z(x)) [-f(z(x)) + f'(z(x))] + (b - a) f(z(x)) = 0. \quad (113)$$

Next collect terms in $f''(z(x))$, $f'(z(x))$, and $f(z(x))$ to obtain

$$z(x) f''(z(x)) + (b - z(x)) f'(z(x)) - af(z(x)) = 0. \quad (114)$$

Equation (114) is Kummer’s equation (Abramowitz and Stegun, equation 13.1.1) and the Kummer functions $M(a, b, z(x))$ and $U(a, b, z(x))$ are solutions for $f(x)$, so

$$Y(x) = k_1 e^{-\eta_1 x} U(a, b, z(x)) + k_2 e^{-\eta_1 x} M(a, b, z(x)). \quad (115)$$
Differentiate equation (115) with respect to \( \xi \), use \( \zeta (\xi) = \xi \), and \( v^H(x) = e^{\eta_1 x} Y'(x) \) from equation (91) to obtain

\[
v^H(x) = k_1 \eta_1 [U_z(a, b, z(x)) - U(a, b, z(x))] + k_2 \eta_1 [M_z(a, b, z(x)) - M(a, b, z(x))].
\]

Using Abramowitz and Stegun equations 13.4.12 \( (\frac{b-a}{b} M(a, b+1, z) = M(a, b, z) - M_z(a, b, z)) \) and 13.4.25 \( (U(a, b+1, z) = U(a, b, z) - U_z(a, b, z)) \) yields

\[
v^H(x) = -k_1 \eta_1 U(a, b+1, z(x)) + k_2 \eta_1 \frac{a-b}{b} M(a, b+1, z(x)).
\]

The definitions of \( a \) and \( b \) imply that \( \frac{a-b}{b} < 0 \) and \( b + 1 = \frac{\mu^+ \mu^-}{\eta_1} > 1 \).

Using Abramowitz and Stegun equations 13.4.9 \( (\frac{\nu}{\partial z^n} \{M(a, b, z)\} = \frac{(a)_n}{(b)_n} M(a + n, b + n, z)) \) and 13.4.22 \( (\frac{\nu}{\partial z^n} \{U(a, b, z)\} = (-1)^n(a)_n U(a + n, b + n, z)) \)

\[
v^H_x(x) = k_1 \eta_1^2 a U(a + 1, b + 2, z(x)) + k_2 \eta_1^2 \frac{a-b}{b+1} M(a + 1, b + 2, z(x))
\]

\[
v^H_{xx}(x) = -k_1 \eta_1^3 a (a + 1) U(a + 2, b + 3, z(x)) + k_2 \eta_1^3 \frac{a-b}{b+1} \frac{a+1}{b+2} M(a + 2, b + 3, z(x)).
\]
E Appendix: Properties of Marginal and Average q

Proposition 12 Define $x_D^*$ and $x_I^*$ so that $v'(x_D^*) = 1$ and $v(x_I^*) = (1 + x_I^*) v'(x_I^*)$. Assume that $0 < x_I^* < x_D^*$. If $v(0) \leq 1$, define $\hat{x}$ by $\hat{x} \in [0, x_D^*)$ and $q^a(\hat{x}) = 1$; if $v(0) > 1$, then $\hat{x} = 0$. Then

1. $q_I^a(0) = q_2^a(0) = q_3^a(0) = q^m(0) = v(0) > 0$.
2. $q_1^a(x) > q_2^a(x)$, for $x > 0$.
3. $q_2^a(x) > q^m(x)$, for $0 < x < x_D^*$, and $q_2^a(x) = q^m(x)$, for $x \geq x_D^*$
4. $q_3^a(x) < q_1^a(x)$ for $x > 0$
5. $q_3^a(x_I^*) = v'(x_I^*)$ and $x_I^* = \arg \max_{x \geq 0} q_3^a(x)$
6. $\hat{x}$ is unique. For $0 \leq x < x_D^*$, $q_3^a(x) \leq 1$ as $0 < x \leq \hat{x}$.
7. $q_3^a(x) \geq q_2^a(x)$ as $0 < x \leq \hat{x}$.
8. $q_3^a(x) \geq q^m(x)$ as $0 < x \leq x_I^*$.
9. If $x \geq x_D^* \geq x_I^* > 0$, then $q_1^a(x) = v(x_D^*) + (x - x_D^*)$, $q_2^a(x) = v(x_D^*) - x_D^* = q^m(x_D^*)$, $q_3^a(x) = \frac{v(x_D^*) + (x - x_D^*)}{x_D^*} > 1$, $q_3^a(x) < 0$, and $\lim_{x \to \infty} q_3^a(x) = 1$.

Proof of Proposition 12: Statement 1: The definitions of $q_1^a(x)$, $q_2^a(x)$, and $q_3^a(x)$ directly imply that $q_1^a(0) = q_2^a(0) = q_3^a(0) = v(0)$. Evaluate equation (50a) at $x = 0$ and use $v^L(0) = 0$ to obtain $(\rho + \mu_L - i^{H^+}) v^H(0) = (\phi^H - i^{H^+}) v^H(0)$. Therefore, since $\rho + \mu_L - i^{H^+}$, $v^H(0)$, and $\phi^H - i^{H^+}$ are all finite, $v^H(0)$ is finite, so that $x v_x(x) = 0$ when $x = 0$. Therefore, since $q^m(x) \equiv v(x) - x v_x(x)$, $q^m(0) = v(0)$. Finally, $v(0) \geq \frac{\phi^H}{\rho + \mu_L} > 0$.

Statement 2: $q_1^a(x) = q_2^a(x) = v(x) - (v(x) - x) = x > 0$.

Statement 3: $q_2^a(x) - q^m(x) = (v(x) - x) - (v(x) - x v_x(x)) = (v_x(x) - 1) x$, which is positive for $0 < x < x_D^*$ since $v_x(x) > 1$ for $0 < x < x_D^*$, and equals 0 for $x \geq x_D^*$ since $v_x(x) = 1$ for $x \geq x_D^*$.

Statement 4: $q_1^a(x)$ and $q_3^a(x)$ are both positive and $\frac{q_3^a(x)}{q_1^a(x)} = \frac{1}{1 + x}$, $< 1$ for $x > 0$.

Statement 5: Substitute $x_I^*$ into the definition of $q_3^a(x)$ to obtain $q_3^a(x_I^*) = \frac{v(x_I^*)}{1 + x_I^*} = \frac{v(x_I^*)}{(1 + x_I^*) v_x(x_I^*)} = v_x(x_I^*)$, where the second equality follows from the definition of the investment trigger $x_I^*$ in equation (57). Differentiate the definition $q_3^a(x) \equiv \frac{v(x)}{1 + x}$ with respect to $x$ and
evaluate the derivative at $x = x_1^*$ to obtain $q_3''(x_1^*) = \frac{(1+x_1^*)''(x_1^*) - v(x_1^*)}{(1+x_1^*)^2} = 0$, where the second equality follows from the definition of the investment trigger $x_1^*$ in equation (57).

Alternatively, the first derivative of $q_3''(x)$ with respect to $x$ can be expressed as $q_3''(x) = \frac{1}{1+x}(v'(x) - q_3''(x))$ and the second derivative with respect to $x$ is $q_3'''(x) = -\frac{1}{1+x}q_3''(x) + \frac{1}{1+x}(v''(x) - q_3''(x)) = \frac{1}{1+x}(v''(x) - 2q_3''(x))$. Therefore, if $q_3''(x_0) = 0$ for $0 \leq x_0 \leq x_D$, then $q_3'''(x_0) = \frac{1}{1+x}v''(x_0) < 0$, so $x_0$ is the unique local maximum for $0 \leq x \leq x_D$. Therefore, since $q_3''(x_1^*) = 0$, it follows that $x_1^* = \arg \max_{0\leq x \leq x_D} q_3''(x)$.

Statement 6: Since $q_3''(x) = 0$ at a unique $x$ in $0 \leq x \leq x_D$, and $q_3''(x_1^*) = 0$, it follows that $q_3''(x)$ is strictly increasing for $0 \leq x < x_1^*$ and strictly decreasing for $x_1^* < x < x_D$. Case I: $v(0) > 1$. By definition $\hat{x} = 0$, so $\hat{x}$ is unique. Case II: $v(0) \leq 1$. In this case, since $q_3''(x_1^*) = v'(x_1^*) > 1$, there is a unique $\hat{x}$ in $[0, x_1^*]$ for which $q_3''(\hat{x}) = 1$. Moreover, since $q_3''(x)$ is strictly decreasing for $x_1^* < x < x_D$, it follows that $q_3''(x) = \frac{1}{1+x}(v'(x) - q_3''(x)) < 0$, so $q_3''(x) > v'(x) > 1$ for $x_1^* < x < x_D$. Therefore, there is one and only one value of $x \in [0, x_1^*)$ for which $q_3''(x) = 1$ and for $0 \leq x < x_D$, $q_3''(x) \leq 1$ as $0 < x \leq \hat{x}$.

Statement 7: $q_3''(x) - q_3''(x) = q_3''(x) - (1 + x) q_3''(x) + x = x (1 - q_3''(x))$. Statement 6 implies that $1 - q_3''(x) \geq 0$ as $0 < x \leq \hat{x}$, so $q_3''(x) \leq q_3''(x)$ as $0 < x \leq \hat{x}$.

Statement 8: $q_3''(x) - v''(x) = q_3''(x) - (1 + x) q_3''(x) - x v'(x) = -x q_3''(x) + x v'(x) = x (v'(x) - q_3''(x))$. For $0 \leq x < x_D$, $v'(x) > v(x) = q_3''(x) > q_3''(x)$, so $q_3''(x) > v''(x)$. The proof of Statement 6 proves that for $x_1^* < x < x_D$, $q_3''(x) > v'(x)$, so $q_3''(x) - v''(x) = x (v'(x) - q_3''(x)) < 0$.

Statement 9: If $x \geq x_D > x_1^* > 0$, then $v(x) = v(x_D) + x - x_D$, so that $q_3''(x) = v(x_D) - v(x_D) = v(x_D) - v(x_D) = v(x_D) - v(x_D)$, since $v(x_D) = 1$ for $x > x_D$. Also, $q_3''(x) = q_3''(x) = v(x_D) - q_3''(x)$. To prove that $q_3''(x) > 1$, it suffices to show that $v(x_D) - v(x_D) > 1$. Define $f(x) = v(x_D) - (1 + x) v'(x)$ and differentiate $f(x)$ to obtain $f(x) = - (1 + x) v''(x) > 0$ for $0 \leq x < x_D$. Therefore, $f(x_D) > f(x_D) = 0$ since $v(x_D) = (1 + x_D) v'(x_D)$. Hence, $q_3''(x) > 1$ for $x \geq x_D$. From the proof of Statement 5, $q_3''(x) = q_3''(x) = v(x_D) - q_3''(x)$. For $x \geq x_D$, $v'(x) = 1$ and $q_3''(x) > 1$, so $q_3''(x) < 0$. Finally, $\lim_{x \to \infty} q_3''(x) = \lim_{x \to \infty} v'(x)$, so that L'Hopital's rule implies $\lim_{x \to \infty} q_3''(x) = \lim_{x \to \infty} v'(x) = 1$. ■
F Proof of Concavity of Value Function without Investment

\[ V^H_{X,X} (X) = c_1 \omega_1^2 e^{\omega_1 (X - X^*)} + c_2 \omega_2^2 e^{\omega_2 (X - X^*)}. \]

For \( X < X^* \), \( c_1 \omega_1^2 e^{\omega_1 (X - X^*)} < c_1 \omega_1^2 < 0 \) and \( 0 < c_2 \omega_2^2 e^{\omega_2 (X - X^*)} < c_2 \omega_2^2 \), so \( V^H_{X,X} (X) < c_1 \omega_1^2 + c_2 \omega_2^2 = V^H_{X,X} (X^*) = 0 \) for \( X < X^* \).

For \( X \leq X^* \), \( V^L_{X,X} (X) = \frac{\partial H}{\partial \mu} \left[ (\eta_2 - \omega_1) c_1 \omega_1^2 e^{\omega_1 (X - X^*)} + (\eta_2 - \omega_2) c_2 \omega_2^2 e^{\omega_2 (X - X^*)} \right] \leq \frac{\partial H}{\partial \mu} \left[ (\eta_2 - \omega_1) c_1 \omega_1^2 + (\eta_2 - \omega_2) c_2 \omega_2^2 \right] = \frac{\partial H}{\partial \mu} (\eta_2 - \omega_2) \times \left[ c_1 \omega_1^2 + c_2 \omega_2^2 + \left( \frac{\eta_2 - \omega_1}{\eta_2 - \omega_2} - 1 \right) c_1 \omega_1^2 \right] = \frac{\partial H}{\partial \mu} (\eta_2 - \omega_1) \left( \frac{\eta_2 - \omega_1}{\eta_2 - \omega_2} - 1 \right) c_1 \omega_1^2 \] where the final equality follows from \( c_1 \omega_1^2 + c_2 \omega_2^2 = 0 \). Note that \( \left( \frac{\eta_2 - \omega_1}{\eta_2 - \omega_1} - 1 \right) = \frac{\omega_2 - \omega_1}{\eta_2 - \omega_1} > 0 \), so \( \frac{\partial H}{\partial \mu} (\eta_2 - \omega_1) \left( \frac{\eta_2 - \omega_1}{\eta_2 - \omega_2} - 1 \right) c_1 \omega_1^2 < 0 \). Therefore, \( V^L_{X,X} (X) < 0 \) for \( X \leq X^* \).