Efficient and Incentive Compatible Liver Exchange

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Abstract

Liver is the second most transplanted organ from living donors. A living donor can usually donate either his smaller left lobe or larger right lobe. Left-lobe donation is substantially less risky for the donor. Because size compatibility is required besides blood-type compatibility for liver transplantation, doctors often utilize right-lobe donation due to organ shortage. To remedy the shortage, living donor liver exchange has already been utilized in some countries. We model liver exchange as a matching market design problem. We first introduce an algorithm to find a two-way efficient matching when only left-lobe donation is feasible and there are only two sizes of donors and patients. We then introduce a Pareto-efficient, individually rational, and incentive-compatible mechanism to elicit the right-lobe donation willingness of donors of the patients in the liver exchange pool and extend this approach to any number of individual patient and donor sizes. This approach is quite general and introduces a new class of mechanisms for bilateral exchange problems with weak preferences induced by multi-dimensional vector partial order. By simulations, we show that the decrease of number of transplants because of the incentive compatibility requirement is very small, while the number of transplants can increase substantially as liver exchange is utilized.

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1 Introduction

Following the kidney, liver is the second most common organ for transplantation worldwide. For the case of the US, more than 7000 of nearly 31000 organ transplants in 2015 were liver transplants. Transplantation is the only potential treatment for end-stage liver disease, unlike the end-stage kidney disease where there is an alternative (although inferior) treatment of dialysis. As in the case of kidneys, transplants from deceased donors and living donors are both possible (and widespread) for liver transplantation.[1] Unlike kidney transplantation, however, a living donor can donate only a part of his liver -henceforth referred as a *lobe*- going through a liver resection operation called *hepatectomy*. Based on the anatomy of the liver, the main options are donating either the smaller left lobe (normally 30-40% of the liver) with a *left hepatectomy* or the larger right lobe (normally 60-70% of the liver) with a *right hepatectomy*. Following the transplantation, the remnant liver of a living donor typically regenerates within a month. Assuming the donor and the patient are blood-type compatible,[2] which of these two options is preferred (or even feasible) depends on the relative liver volumes of the patient and the donor. In order to provide adequate liver function for the patient, at least 40% of the standard liver volume of the patient is required. The metabolic demands of a larger patient will not be met by the smaller left lobe from a relatively small donor. This phenomenon is known as *small-for-size syndrome*. The primary solution to avoid this syndrome has been harvesting the larger right lobe of the liver for transplantation. This procedure, however, involves considerably higher risks for the donor than harvesting the smaller left lobe. While donor mortality is approximately 0.1% for left hepatectomy, it is in the range 0.4-0.5% for right hepatectomy (Lee, 2010). Other risks, referred to as donor *morbidity*, are also much higher with the right hepatectomy. Hence one of the main challenges for living-donor liver transplantation is that, the much safer left-lobe transplantation is not a viable option for a majority of patients with willing donors. As an implication many patients with potential donors cannot receive a transplant since either their donors hesitate to go through the higher-risk right hepatectomy, or their doctors recommend against this procedure. The high risks associated with the right-lobe liver transplantation also affects the public perception for living-donor liver transplantation. The number of annual living-donor liver transplants in the US peaked in 2001 with 524 transplants, increasing eight-folds in the period 1996-2001. The

[1] The attitude towards living-donor liver transplantation differs considerably between western countries and Asian countries. In contrast to western countries, donations for liver transplantation in much of Asia come from living donors. For example in 2015, while only 359 of 7127 liver transplants were from living donors in the US, 942 of 1398 liver transplants were from living donors in South Korea.

[2] Each human is of one of the following four blood-types: O, A, B, or AB. While a blood-type O donor is blood-type compatible with any blood-type patient, a blood-type A donor is blood-type compatible with patients of blood types A and AB, a blood-type B donor is blood-type compatible with patients of blood types B and AB, and a blood-type AB donor is blood-type compatible with only patients of blood type AB.

highly publicized death of a right-lobe liver donor in the US in 2002 not only brought an end this remarkable increase, but also resulted in a 40-50% reduction from its peak in the US since then.[3]

As the shortage of transplant organs keep increasing annually worldwide, living-donor exchanges emerged as an important source for these potentially life-saving resources, especially for the case of kidneys. In its most basic form, a living-donor organ exchange involves two patients with willing donors who exchange donors either because direct donation is not an option due to an immunological barrier or because one or both patients receive a more favorable outcome through the exchange. The concept was originally proposed for kidneys by Rapaport (1986), and they became widespread over the last 15 years with the introduction of optimization and market design techniques to kidney exchange (Roth, Sönmez, and Ünver, 2004, 2005, 2007). A vast majority of these exchanges are conducted between incompatible kidney patient-donor pairs where a donor cannot directly donate to his patient due to immunological barriers.[4] Liver exchanges between incompatible patient-donor pairs are also conducted in modest numbers in several Asian countries, most notably in South Korea. Our focus in this paper is the design of a liver exchange mechanism that not only includes the incompatible pairs, but also a subset of compatible pairs whose only direct donation option to their patients is through the higher risk right hepatectomy. Under an efficient and incentive-compatible mechanism we introduce, these compatible pairs participate in exchange only if they strictly benefit from doing so by reducing the risks to their donors through a left hepatectomy. As such, our proposed mechanism not only increases the number of living-donor liver transplants, but also increases the reliance on the lower risk left-lobe liver transplantation in the spirit of the central tenet of the hippocratic oath "first do no harm."

While the practice of kidney exchange has flourished worldwide over the last fifteen years, inclusion of compatible pairs to exchange pools has proved to be a challenge since benefits to these pairs from joining kidney exchange pools are either non-present or weak. In contrast the benefits from joining to liver exchange pools can be considerable for a significant fraction of compatible pairs, if it means their donors can have a left hepatectomy rather than a right hepatectomy. And the welfare gains from their inclusion can be potentially very high. Consider a large blood-type A liver patient, who in the absence of exchange has to receive a right liver lobe from his small blood-type O donor. While this is a feasible medical procedure, an alternative arrangement of an exchange of donors with a small blood-type O patient with a large blood-type A donor will not only significantly reduce the risks to his donor (by replacing the donor's right hepatectomy with a left hepatectomy), but also enable a second

[3]See Grady (2002).

[4]For the case of kidney transplantation, these immunological barriers are blood-type incompatibility and tissue-type incompatibility.

patient to receive a potentially life-saving liver transplant. The possibility of offering a less risky procedure to such pairs provides an opportunity to increase the size of the liver exchange pool in a way that includes the much needed blood-type O donors.

In the above example, the large blood-type A patient with the small blood-type O donor would only be willing to participate in exchange if the pair benefits from exchange through an assurance that their donor goes through the much less risky procedure of left hepatectomy. However, not all cases are this straightforward. Consider a blood-type A patient with a blood-type B donor. Since they are blood-type incompatible to start with, not only they can benefit from exchange through a left-lobe donation but also through the less-desired right-lobe donation if the pair is willing to expose the donor to the higher mortality and morbidity risks of a right hepatectomy. This possibility is the primary reason why one cannot directly adopt to liver exchange the mechanisms and techniques developed for kidney exchange, unless the higher risk right hepatectomy is completely avoided. Not only a liver exchange mechanism has to determine which pairs are to be matched with each other to exchange donors, but it shall also determine which donors have to donate their right lobes rather than their left lobes. Of course, some pairs may not be willing to expose their donors to the more risky procedure of right hepatectomy, but a poorly designed exchange mechanism may also give them the incentives to hide their willingness to do so even if they are. As such, our focus is not only the design of an efficient mechanism, but at the same time the design of an *incentive compatible* liver exchange mechanism where a pair never receives a less favorable outcome by revealing their willingness to go through the less desired right hepatectomy.

For living-donor liver transplantation, size-compatibility is a requirement in addition to blood-type compatibility. A patient is size compatible with a donor if the volume of liver tissue transplanted from the donor is at least 40% of the patient's standard liver volume. Based on this requirement we define the *size of a patient* to be 40% of his standard liver volume, and the *size of a donor* to be the volume of the left lobe of his liver. Hence, assuming they are blood-type compatible, a patient can receive a left-lobe liver transplant from a donor who is at least of his own size and a right-lobe liver transplant from a smaller donor. A patient-donor type is thus characterized by the blood-types of the patient and donor, along with their respective sizes. The key types in the design of an efficient and incentive compatible mechanism are those who can participate in exchange both through left-lobe donation of their donor as well as through the less-preferred right-lobe donation. The challenge is determining when the donors of a particular type shall be considered for a right-lobe donation rather than a left-lobe donation. We refer this process as a *transformation* of a type. To assure incentive compatibility, a type should be transformed only after their left-lobe exchange possibilities are exhausted so that their announcement of whether they are willing for their donors to go through a right hepatectomy does not affect whether or not their donors go through the safer left hepatectomy. One simple approach might be first considering all such pairs for left-lobe donation, and next transforming them simultaneously once their left-lobe donation possibilities are exhausted. There are two problems with this simple approach. First, it is possible that an exchange between two such pairs might be possible with the transformation of only one of these pairs, say pair 1. If so, transforming both pairs and matching them for an exchange will result in a Pareto inferior outcome. Second, this possibility might encourage pair 1 to hide their willingness for a right-lobe donation of their donor. Hence, key in design is determining the order in which pairs of these critical types shall be transformed. We show that there is a well-defined ordering which assures that the resulting mechanism is not only Pareto efficient, but also incentive compatible. While we show this result for an arbitrary number of potential sizes of patients and donors making our mechanism practically relevant, the intuition of the mechanism is more clear for the case of two sizes. Hence first we develop a Pareto efficient and incentive compatible mechanism for this simpler case providing a geometric representation of our mechanism, and then extend our analysis to an arbitrary number of sizes.

2 A Model with Two Donation Modalities

Let $\mathcal{B} = \{O, A, B, AB\}$ denote the set of blood types. Let $S \geq 1$ be the number of possible sizes, and let $\mathcal{S} = \{0, 1, \ldots, S-1\}$ denote the set of possible sizes, where larger numbers correspond to larger sizes.[5] We will assume that the blood type and the size of each individual are observable physical attributes. We will refer to $\mathcal{B} \times \mathcal{S}$ as the set of individual types.

There are two donation modalities: A donor can donate the *left lobe* or the *right lobe* of his liver to a patient. Approximately, a left lobe is 40% of the total liver volume, and the right lobe is 60%. Each patient requires a graft that is at least 40% of his own liver volume. Therefore, for left-lobe donation, a donor is size-compatible with a patient if he is at least as large as the patient. Right-lobe donation enables donors to donate to larger patients, which we will discuss shortly.

A donor can donate his left-lobe to a patient if and only if the patient is blood-type compatible with the donor, and the donor is at least as large as the patient. Let $\trianglerighteq$ denote the left-lobe donation partial order on $\mathcal{B} \times \mathcal{S}$. For the case of two sizes ($S = 2$), Figure 1 illustrates the donation partial order $\trianglerighteq$ on $\mathcal{B} \times \mathcal{S}$, and the standard partial order $\geq$ over the corners of the three-dimensional cube $\{0, 1\}^3$. Note that more generally for arbitrary S, $(\mathcal{B} \times \mathcal{S}, \trianglerighteq)$ and

[5]These sizes refer to different things for donors and patients. More precisely, for a patient of size s, this size is the minimum size of the volume of a liver lobe she is allowed to receive. For a donor of size s, this is the volume of the left lobe of his liver. Moreover, Set $\mathcal{S}$ can be specific to each liver exchange pool, allowing for a continuum of sizes generically as long as each pool we analyze is finite.

$(\{0,1\}^2 \times \{0,1,\ldots,S-1\}, \geq)$ are order isomorphic, where the order isomorphism associates each individual type $T \in \mathcal{B} \times \mathcal{S}$ with the following vector $X \in \{0,1\}^2 \times \{0,1,\ldots,S-1\}$:

$$\begin{aligned} X_1 = 0 &\Leftrightarrow T \text{ has the } A \text{ antigen} \\ X_2 = 0 &\Leftrightarrow T \text{ has the } B \text{ antigen} \\ X_3 = s &\Leftrightarrow T \text{ is of size } s \end{aligned}$$

For convenience, we will work with the equivalent representation $\mathcal{T} = \{0,1\}^2 \times \{0,1,\ldots,S-1\}$ and $\geq$ of individual types: For any $X, Y \in \mathcal{T}$, a donor of type Y can donate his left lobe to a patient of type X if and only if $X \leq Y$. All of our conclusions can be rephrased in terms of blood types and size using the order isomorphism above.

As we briefly discussed above, right-lobe donation enables donors to donate to larger patients. We will model the technology of right-lobe donation though a function $\rho(\cdot)$ that determines the maximum size patient that each size donor can donate his right lobe to.

A **right-lobe size function** is a function $\rho : \mathcal{S} \to \mathcal{S}$ such that:

1. ρ is non-decreasing, and
2. $\rho(s) > s$ for all $s \in \mathcal{S} \setminus \{S-1\}$.

We will abuse notation and let $\rho(Y) := Y_1 Y_2 \rho(Y_3)$ for any type $Y \in \mathcal{T}$.

Otherwise, when right-lobe donation is feasible, we will assume that $\rho(s) > s$ for all $s \in \{0,1,...,S-1\}$. In this case, a size $s \in \mathcal{S}$ donor can donate his right lobe to patients who are blood-type compatible with the donor and are of size $\rho(s)$ or smaller: For any $X, Y \in \mathcal{T}$, a donor of type Y can donate his right lobe to a patient of type X if and only if $X \leq \rho(Y)$. Since the right-lobe size is larger than the left-lobe size, the right-lobe donation technology increases the set of potential exchanges and direct donations.

Each patient participates in the liver exchange with one donor. A patient and her donor are referred to as a **pair**. The observable characteristics of a pair is summarized by an ordered pair of individual types $X - Y \in \mathcal{T} \times \mathcal{T}$, where X denotes the type of the patient and Y denotes the type of the donor; $X - Y$ is called the **pair type**.

A **liver-exchange pool** is a tuple $\mathcal{E} = (\mathcal{I}, \tau)$ where

1. $\mathcal{I} = \{1,2,\ldots,I\}$ is a nonempty finite set of patient-donor pairs, and
2. $\tau : \mathcal{I} \to \mathcal{T} \times \mathcal{T}$ is a function such that for every pair $i \in \mathcal{I}$:
    (a) $\tau(i) = \tau_P(i) - \tau_D(i)$.[6]

[6]We refer to a pair type as $X - Y$ instead of (X, Y) as a convention.

(b) $\tau_P(i) \in \mathcal{T}$ is the type of the patient of the pair i.

(c) $\tau_D(i) \in \mathcal{T}$ is the type of the donor of the pair i.

For the rest of this section, we will take as fixed an arbitrary liver-exchange pool $(\mathcal{I}, \tau)$ and a right-lobe size function ρ.

In the rest of this paper, we will assume that only direct donations and two-way exchanges are allowed. We next define the *compatibility* graph, which lists all pairs i, j (possibly $i = j$) such that j's donor can donate his left or right lobe to i's patient, and i's donor can donate his left or right lobe to j's patient.

Suppose a subset of pairs $\mathcal{J}_{l\&r} \subseteq \mathcal{I}$ who can donate left or right lobe is given, while the rest of the pairs can donate only left lobe. Given $\mathcal{J}_{l\&r}$, the **compatibility graph** $G_c[\mathcal{J}_{l\&r}] = (\mathcal{I}, E_c[\mathcal{J}_{l\&r}])$ where $E_c[\mathcal{J}_{l\&r}]$, the set of feasible matches, is defined as:[7]

$$\{i,j\} \in E_c \quad \Longleftrightarrow \quad \left\{ \begin{array}{cc} \tau_P(i) \leq \rho(\tau_D(j)) & \text{if } j \in \mathcal{J}_{l\&r} \\ \tau_P(i) \leq \tau_D(j) & \text{if } j \notin \mathcal{J}_{l\&r} \end{array} \right\} \& \left\{ \begin{array}{cc} \tau_P(j) \leq \rho(\tau_D(i)) & \text{if } i \in \mathcal{J}_{l\&r} \\ \tau_P(j) \leq \tau_D(i) & \text{if } i \notin \mathcal{J}_{l\&r} \end{array} \right\}$$

for all $i, j \in \mathcal{I}$.[8]

A **matching** $M \subseteq E_c[\mathcal{J}_{l\&r}]$ of $G_c[\mathcal{J}_{l\&r}]$ is such that for any $e, e' \in M$, $e \cap e' \neq \emptyset \implies e = e'$, i.e., no pair participates in two distinct exchanges or direct transplants. Let $\mathcal{M}_c[\mathcal{J}_{l\&r}]$ be the set of matchings given the compatibility graph $G_c[\mathcal{J}_{l\&r}]$. When we fix the set $\mathcal{J}_{l\&r}$, we will suppress the arguments of this set.

We denote the match of pair $i \in \mathcal{I}$ in matching $M \in \mathcal{M}_c[\mathcal{J}_{l\&r}]$ as $M(i)$. If $M(i) = i$ (i.e., $\{i\} \in M$), then the pair participates in a **direct transplant**. If $M(i) = j$ for some $j \neq i$ (i.e., $\{i, j\} \in M$), then pairs i & j participate in a **(two-way) exchange**. If $M(i) = \emptyset$ (i.e., there is no $e \in M$ such that $i \in e$), then pair i is **unmatched**.

In the next section, we will focus on a simplification of this model in which only two sizes $\mathcal{S} = \{0, 1\}$ exist and only left-lobe donation is feasible. The model will be an introduction to the more general model and is also of independent theoretical interest, as it is a symmetric version of our more general model where all 3 dimensions of an individual type can have at most 2 different, i.e., binary, values: $\mathcal{T} = \{0, 1\}^3$. This symmetry turns out to simplify many of the graph theoretical complications.

[7] Graph theoretical preliminaries are stated formally in Appendix A. Some of our current definitions are restated for general graphs in this appendix, as well.

[8] Observe that this definition allows for a *loop* $\{i, i\} = \{i\}$ to be in E_c. This depicts that the donor of pair i can donate to the patient of the pair.

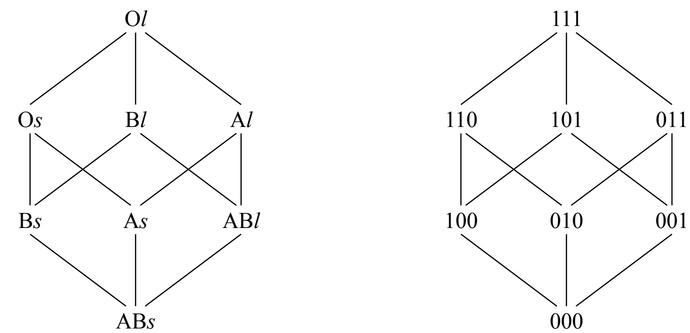


Figure 1: The Partially Ordered Sets $(\mathcal{B} \times \mathcal{S}, \trianglerighteq)$ and $(\{0,1\}^3, \geq)$.

3 Results in a Simple Symmetric Model: Two Sizes and Only Left-Lobe Donation

Suppose there are only two sizes, i.e., $\mathcal{S} = \{0, 1\}$. Then the compatibility relationship is given as a lattice on the 3-dimensional binary cube $\mathcal{T} = \{0, 1\}^3$. The order isomorphism between the blood types and sizes $\mathcal{B} \times \mathcal{S}$ and $\mathcal{T}$ are depicted in Figure 1.

Through this section, we assume that no pairs can donate right lobe, and thus, the set of pairs who can participate in left and right lobe donation is $\mathcal{J}_{l\&r} = \emptyset$. Hence, only left-lobe donation is possible.

Consistent with practical applications, pairs have endowment bias, in the sense that, they would prefer participating in a direct transplant, if it is feasible, rather than any two-way exchange: it usually requires time to pass for arranging an exchange, while a direct transplant is usually immediately feasible. Moreover, there could be other reasons for such a preference, such as emotional involvement between the donor and the patient. We say that a left-lobe-only matching $M \in \mathcal{M}_c[\emptyset]$ is **individually rational** if $\{i\} \in E_c$ then $\{i\} \in M$. Thus, no patient-donor pair $X - Y$ such that $Y \geq X$ participates in any two way-exchange in an individually rational matching. Given this, no patient-donor pair $X' - Y'$ such that $X' > Y'$ can take part in a two-way exchange either: If a pair $X - Y$ can be part of a two-way exchange with $X' - Y'$, then $Y \geq X' > Y' \geq X$, implying that $X - Y$ is out of the market. Therefore, the only patient-donor pairs that can be matched through two-way exchanges have incomparable types with respect to the partial order $\geq$. We summarize this observation in the following Lemma.

Lemma 1 *The only pair types that could participate in a left-lobe-only exchange in an individually rational matching are* $X - Y \in \mathcal{T} \times \mathcal{T}$ *such that* $X \not\geq Y$ *and* $Y \not\geq X$.

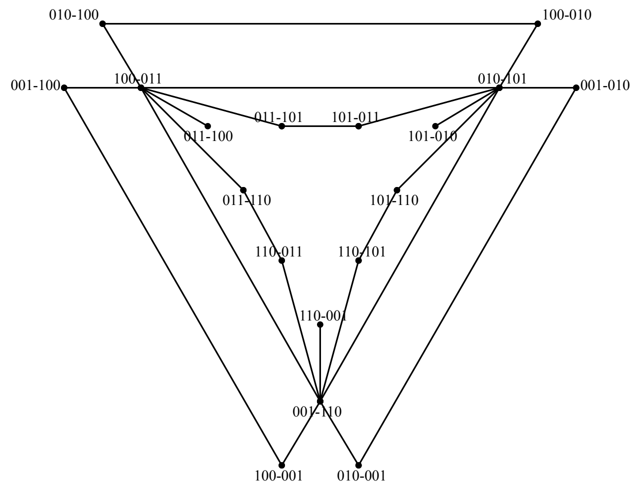


Figure 2: Possible Two-Way Exchanges among Pair Types

We summarize the possible two-way exchanges among the types in Lemma 1, as the edges of the graph in Figure 2.

We define the **value** of a pair type $X-Y \in \mathcal{T}\times\mathcal{T}$ as $v(X-Y) = \sum_{p=1}^{3} Y_p - \sum_{p=1}^{3} X_p$. The **waste** of a two-way exchange of pair types $U-V$ & $X-Y$ is defined as $v(U-V)+v(X-Y)$. As seen in Figure 2, all possible exchanges are of waste 0,1, or 2.

We will show that the following matching algorithm maximizes the number of transplants in any individually rational matching. The algorithm sequentially maximizes three subsets of two-way exchanges. The main innovation of the algorithm is using the wastes of exchanges as the sufficient information to prioritize among them.

The two-size left-lobe-only sequential matching algorithm:

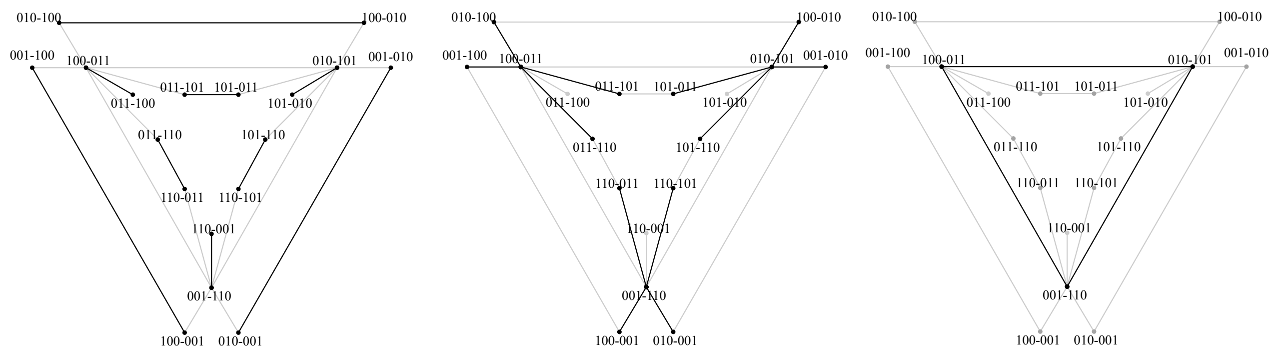
Step 0: Clear direct transplants: All compatible pairs (i.e., pairs with types $X-Y$ such that $X \leq Y$) participate in direct transplants.

Step 1: Clear waste 0 exchanges: Match the maximum number of $X-Y$ and $Y-X$ types for all $X, Y \in \{0,1\}^3$.

Step 2: Clear waste 1 exchanges: Match the maximum number of $100-011$, $010-101$, and $001-110$ types, without matching them to each other.

Step 3: Clear waste 2 exchanges: Match the maximum number of $100-011$, $010-101$, and $001-110$ types among each other.[9]

[9]The maximum number of pairs can be matched as follows in this step: Let $\{X_1 - Y_1, X_2 - Y_2, X_3 -$



(a) Step 1 **(b)** Step 2 **(c)** Step 3

Figure 3: Two-Way Exchanges Carried out in Steps 1,2,3

Figure 3 graphically illustrates the two-way exchanges that are maximized at each step of the sequential matching algorithm. The next theorem shows the optimality of this algorithm. It is straightforward to compute the maximum number of two-way exchanges using the algorithm.

Theorem 1 *When $S = 2$, the two-size left-lobe-only sequential matching algorithm maximizes the number of two-way exchanges in any individually rational matching.*[10]

We conclude this section with a simple example showing that why another prioritization using the wastes of exchanges may not be Pareto efficient in a sequential algorithm.

Example 1 Consider an exchange pool with pairs $\mathcal{I} = \{i_1, i_2, i_3, i_4\}$ such that

$$\tau(i_1) = 100 - 011, \qquad \tau(i_2) = 010 - 100,$$
$$\tau(i_3) = 100 - 010, \qquad \tau(i_4) = 001 - 110.$$

$Y_3\} = \{100 - 011, 010 - 101, 001 - 110\}$ be such that $n^*(X_1 - Y_1) \geq n^*(X_2 - Y_2) \geq n^*(X_3 - Y_3)$ where $n^*(X_k, Y_k)$ is the number of remaining type $X_k - Y_k$ pairs for each $X_k - Y_k$. Two cases are possible: Case (1) $n^*(X_1 - Y_1) > n^*(X_2 - Y_2) + n^*(X_3 - Y_3)$: Then the maximum number of transplants can be found by matching all $X_2 - Y_2$ and $X_3 - Y_3$ pairs exclusively with $X_3 - Y_3$ pairs. Case (2) $n^*(X_1 - Y_1) \leq n^*(X_2 - Y_2) + n^*(X_3 - Y_3)$: Then (a) all remaining pairs from these three types are matched if their total number is even; and (b) all remaining pairs but an arbitrary one from these three types are matched if their total number is odd. A way to match them in Case 2 is as follows: If Case 2(a) holds, *pick* all remaining pairs from these three types, and if Case 2(b) holds, *pick* all remaining pairs but an arbitrary pair from these three types. Then arbitrarily relabel these three types as $X_k - Y_k$, $X_\ell - Y_\ell$, and $X_m - Y_m$; match the picked $X_k - Y_k$ pairs with the picked pairs of the other two types, $X_\ell - Y_\ell$ and $X_m - Y_m$, such that an equal number of picked $X_\ell - Y_\ell$ and $X_m - Y_m$ pairs remain unmatched; and finally, match the remaining picked $X_\ell - Y_\ell$ pairs with picked $X_m - Y_m$ pairs.

[10] Since all left-lobe direct transplants are conducted in an individually rational left-lobe-only matching, it is well known that all Pareto-efficient left-lobe matchings match the same, maximum number of pairs (for example, see Roth, Sönmez, and Ünver, 2005.

There are three possible exchanges:

$$\{i_2, i_3\} \text{ (waste 0)}, \qquad \{i_1, i_2\} \text{ (waste 1)}, \qquad \{i_1, i_4\} \text{ (waste 2)}.$$

If we design a sequential algorithm that prioritizes waste 1 exchanges over waste 0 exchanges, then such an algorithm will find a matching only with the waste 1 exchange above. While our algorithm finds a matching with the waste 0 and waste 2 exchanges above, and this matching Pareto dominates the previous matching and matches all pairs.

Consider another exchange pool with pairs $\mathcal{I}' = \{j_1, j_2, j_3, j_4\}$ such that

$$\tau(j_1) = 100 - 011, \qquad \tau(j_2) = 001 - 100,$$
$$\tau(j_3) = 010 - 001, \qquad \tau(j_4) = 001 - 110.$$

There are three possible exchanges:

$$\{j_1, j_2\} \text{ (waste 1)}, \qquad \{j_3, j_4\} \text{ (waste 1)}, \qquad \{j_1, j_4\} \text{ (waste 2)}.$$

If we design a sequential algorithm that prioritizes waste 2 exchanges over waste 1 exchanges, then such an algorithm would find a matching with the waste 2 exchange above. While our algorithm finds a matching with the two waste 1 exchanges above, and this matching Pareto dominates the previous matching and matches all pairs. ■

4 Efficient and Incentive Compatible Exchange under Left&Right-Lobe Donation

We return our attention back to both right-lobe and left-lobe donation.

Since right-lobe donation is riskier for the donor, it is always less preferred to left-lobe donation. Therefore, for donors who can donate their left lobes to a patient (as long as she is blood-type compatible), we assume that right-lobe donation is not a viable option. Thus, we say that a patient and a donor are **left-lobe compatible** if they are blood-type compatible and the patient is not larger than the donor. A patient and a donor are **right-lobe-only compatible** if the donor can donate his right lobe to the patient, but not his left lobe.

We next define a function $e(\cdot)$, which, given any pairs i and j, determines whether the donor of the first pair i and the patient of the second pair j are left-lobe compatible (l), right-lobe-only compatible (r), or incompatible ($\emptyset$).

We define the **match type** function $e : \mathcal{I} \times \mathcal{I} \to \{l, r, \emptyset\}$ as follows:

$$e(i,j) = \begin{cases} l & \text{if } \tau_P(j) \leq \tau_D(i) \\ r & \text{if } \tau_P(j) \not\leq \tau_D(i) \ \& \ \tau_P(j) \leq \rho(\tau_D(i)) \\ \emptyset & \text{otherwise} \end{cases}$$

for any $(i,j) \in \mathcal{I} \times \mathcal{I}$. Let $e(i) = e(i,i)$ for all $i \in \mathcal{I}$.

For every matching $M \in \mathcal{M}_c[\mathcal{I}]$, we will use the function $e(\cdot)$ to interpret M as collection of direct transplants and two-way exchanges based on the donation lobe kind, left or right, for each pair. For all $\{i\} \in M$, if $e(i) = l$, then the pair i engages in *direct left-lobe transplant* in the matching M. Even though in this case, the donor of i can also donate his right lobe to the patient of i, we rule this out since direct left-lobe donation is feasible and less risky for the pair. For all $\{i\} \in M$, if $e(i) = r$, then the pair i engages in *direct right-lobe transplant*. Similarly, for any exchange $\{i,j\} \in M$, the pairs i and j engage in a two-way exchange, where the donor of i donates his left lobe to the patient of i if $e(i,j) = l$, and his right lobe if $e(i,j) = r$.

We next define the preferences of a pair i. We will assume that each pair i ranks different transplantation options in the following order:

1. i's donor directly donating his left lobe to i's patient (if possible)
2. indifferent among participating in any two-way exchange where i's donor donates his left lobe.
3. i's donor directly donating his right lobe to i's patient (if possible)
4. indifferent among participating in any two-way exchange where i's donor donates his right lobe.

Now imagine a situation where the patient of the pair i can receive a transplantation only if the donor of the pair i donates his right-lobe (directly or via a two-way exchange). Since right-lobe donation is significantly riskier than left-lobe donation for the donor, it is conceivable that the pair i weighs the tradeoff between the benefit of transplantation for the patient versus the risk of right-lobe donation for the donor in either direction. We will say that the pair i is *willing (w)* is they prefer the transplantation option in such a situation. We will say that the pair i is *unwilling (u)* is they prefer the no-transplantation option. Willing pairs will rank "no transplantation" option below option 4 above. The unwilling types rank the "no transplantation" option in between options 2 and 3 above. Whether a pair is willing or not is their private information.

For every pair i, the two possible rankings of i's transplantation options above, naturally induce a willing preference ranking R_i^w, and an unwilling preference ranking R_i^u over the set of potential matches of i.

For any pair $i \in \mathcal{I}$, we define two transitive and complete binary relations over $\{j : \{i, j\} \in E_c[\mathcal{I}]\} \cup \{\emptyset\}$: The **willing preference** R_i^w, and the **unwilling preference** R_i^u, which rank the potential matches of i in the following order:

R_i^w	R_i^u
i if $e(i) = l$	i if $e(i) = l$
indifferent among $\{j \neq i : e(i, j) = l\}$	indifferent among $\{j \neq i : e(i, j) = l\}$
i if $e(i) = r$	$\emptyset$
indifferent among $\{j \neq i : e(i, j) = r\}$	i if $e(i) = r$
$\emptyset$	indifferent among $\{j \neq i : e(i, j) = r\}$

Let $\mathcal{R}_i = \{R_i^w, R_i^u\}$ denote the set of possible preferences of the pair i. For each $R_i \in \mathcal{R}_i$, let P_i, denote the asymmetric part of R_i. We will abuse notation and also let R_i denote the induced preference over the matchings $\mathcal{M}_c$ defined through:

$$MR_iM' \iff M(i)R_iM'(i).$$

Finally, let $\mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_I$ denote the set of preference profiles.

Although the types of the participating pairs are observable, their preferences are not. A (direct) mechanism determines a matching as a function of the reported preference profile. Given $R \in \mathcal{R}$ we will some times denote the type of a pair i with type $X - Y$ as $X - Yw$ if $R_i = R_i^w$ and $X - Yu$ if $R_i = R_i^u$.

Since we fix an exchange pool $(\mathcal{I}, \tau)$ throughout, we define **mechanism** as a function $f : \mathcal{R} \to \mathcal{M}_c[\mathcal{I}]$.

A matching M is individually rational if no pair i has an incentive to leave the exchange in order to stay unmatched, nor (if possible) to arrange for direct donation outside of the exchange.

A matching $M \in \mathcal{M}_c[\mathcal{I}]$ is **individually rational (IR) at** a preference profile $R \in \mathcal{R}$ if for every $i \in \mathcal{I}$: $M(i)R_i\emptyset$, and if $e(i) \neq \emptyset$ then $M(i)R_ii$. A mechanism f is **individually rational (IR)** if $f(R)$ is individually rational at R for any $R \in \mathcal{R}$.

We next give the definitions of Pareto efficiency and (dominant-strategy) incentive compatibility, which are both standard.

A matching M is **Pareto efficient (PE) at** a preference profile $R \in \mathcal{R}$ if there does not exist a matching $M' \in \mathcal{M}_c$ such that $M' R_i M$ for all $i \in \mathcal{I}$ and $M' P_i M$ for some $i \in \mathcal{I}$. A mechanism f is **Pareto efficient (PE)** if $f(R)$ is Pareto efficient at R for any $R \in \mathcal{R}$.

A mechanism f is **incentive compatible (IC)** if for all $i \in \mathcal{I}$, $R_{-i} \in \prod_{j \neq i} \mathcal{R}_j$ and $R_i, \hat{R}_i \in \mathcal{R}_i$:

$$f(R_i, R_{-i}) R_i f(\hat{R}_i, R_{-i}).$$

4.1 Preliminary Results

We start with the following generalization of Lemma 1:

Lemma 2 *In any individually rational matching, a pair of type $X - Y \in \mathcal{T} \times \mathcal{T}$ can participate in an exchange*

- *by donating a left lobe only if $X \ngeq Y$ and $Y \ngeq X$, and*
- *by donating a right lobe only if $X \ngeq \rho(Y)$ and $\rho(Y) \ngeq X$.*

Proof: Suppose a pair $i \in \mathcal{I}$ of type $X - Y$ participates in an exchange with $j \in \mathcal{I} \setminus \{i\}$ in an individually rational matching $]M \in \mathcal{M}_c$. Suppose first, $e(i,j) = l$. Suppose $\tau(j) = U - V$. Then $X \nleq Y$. Otherwise, i should have been in a direct transplant. Suppose to the contrary of the claim $X > Y$. Two cases are possible, $e(j,i) = l$ or $e(j,i) = r$. Suppose first, $e(j,i) = l$. Then $U - V$ has to satisfy $V \geq X > Y \geq U$. But then, j has to participate in a direct transplant in M since $e(j) = l$. A contradiction. Now suppose $e(j,i) = r$. Then $U - V$ has to satisfy $\rho(V) \geq X > Y \geq U$. But then j can participate in direct transplant in M as $e(j) = r$ or $e(j) = l$, a contradiction again. Thus, $X \ngeq Y$ should also hold. The second claim is proven using $\rho(Y)$ instead of Y in the proof of first claim.

■

We continue by presenting a Lemma that partitions the set of pair types into six categories. The categorization is made in terms of whether in an individually rational matching, a pair can be matched, and if so how: (a) via direct left-lobe donation, (b) via direct right-lobe donation, (c) by taking part in a two-way exchange by donating left lobe, or (d) by taking part in a two way exchange by donating right lobe. For pairs in some of these categories (I, II, and III), the individual rationality condition and the preference profile alone pin down whether or not the pair is matched and if so how. For pairs in the other categories (IV, V, and VI), the individual rationality condition and the preference profile merely provide a simplification of alternative ways in which the pair can be matched. Each category is discussed in detail following the Lemma.

Lemma 3 *Fix a preference profile $R \in \mathcal{R}$ and a matching $M \in \mathcal{M}_c$. The matching M is individually rational at R if and only if for any pair of any type $X - Y \in \mathcal{T} \times \mathcal{T}$:*

I *If $X \leq Y$, then the pair directly donates left lobe.*

II *If $X > \rho(Y)$, then the pair is unmatched.*

III *If $Y < X \leq \rho(Y)$, then the pair is unwilling and unmatched, or is willing and directly donates right lobe.*

IV *If $X > Y$, $X \ngeq \rho(Y)$ & $X \nleq \rho(Y)$, then the pair is unmatched, or is willing and participates in a two-way exchange by donating right lobe.*

V *If $X \nleq \rho(Y)$ & $X \ngeq Y$, then the pair is unmatched, or participates in a two-way exchange by donating left lobe, or is willing and participates in a two-way exchange by donating right lobe.*

VI *If $X < \rho(Y)$, $X \ngeq Y$, & $X \nleq Y$, then the pair is unwilling and unmatched, or participates in a two-way exchange by donating left lobe, or is willing and directly donates right lobe.*

From now on, we will refer to pair types in Lemma 3 as Category I–VI types. Similarly, any pair whose pair type lies in a given category, will also be referred to as a member of the same category.

Note that the only pairs who could be part of both a two-way exchange by donating left lobe or right lobe are of Category V. This will make the way we handle how Category V pairs are matched in our mechanism of special importance.

The next Lemma provides a characterization of incentive compatibility for individually rational mechanisms. Specifically, an individually rational mechanism is incentive compatible if and only if for every pair i, whether i participates in a two-way exchange where by donating left-lobe is independent of i's willingness announcement.

Lemma 4 *Let f be an individually rational mechanism. Then, f is incentive compatible if and only if for all $i \in \mathcal{I}$, and $R_{-i} \in \prod_{j \neq i} \mathcal{R}_j$, the following equivalence holds:*

Under $f(R_i^w, R_{-i})$, i participates in a two-way exchange by donating left lobe.

$$\Updownarrow$$

Under $f(R_i^u, R_{-i})$, i participates in a two-way exchange by donating left lobe.

Note that by Lemma 3, in an individually rational matching, the only pairs that could be part of a two-way exchange by donating left lobe are of Category V and VI. As a result, a mechanism is incentive compatible if it is individually rational and such that for any Category V or VI pair i, whether i participates in a two-way exchange by donating left-lobe is independent of i's willingness announcement.

In the next section, we continue the analysis of the symmetric model with two sizes under feasible right-lobe donation.

5 Two Sizes and Right-Lobe Donation

We assume once again in this section $\mathcal{S} = \{0, 1\}$ and continue our analysis we started in Section 3 for left-lobe donation in the symmetric model. This will also illustrate what needs to be taken care of in the general model with multiple sizes. Now we assume right lobe donation is also feasible, i.e., $\rho(0) = 1$. Thus a small donor can donate to a large patient his right lobe. On the other hand a large donor does not need to donate his right lobe to be matched with a large or small patient (i.e., $\rho(1) = 1$ by definition of ρ).

Observe that when an $X - Yw$[11] (and hence, $Y_3 = 0$,) type pair donates right lobe, it mimics type $X - Y_1Y_21$. Thus, we will treat such $X_1X_2X_3 - Y_1Y_20w$ types like $X_1X_2X_3 - Y_1Y_21$ types whenever necessary, and we will refer to this operation as a *transformation*. After a transformation of this sort, the type of such a pair is treated as $X_1X_2X_3 - Y_1Y_21$ for description purposes. When such a transformation is done, we refer to such types as *auxiliary types* meaning that type $X - Y_1Y_21$ refers to both non-transformed $X - Y_1Y_21$ type and transformed $X - Yw$ type. On the other hand, we refer to the non-transformed $X - Y_1Y_21$ type as *native*.[12] Given a (native or auxiliary) type we refer to the (native or auxiliary) types that its pairs can participate in a two-exchange as *neighbor* types.

The above observation is crucial in determining who benefits from exchange by left-&right-lobe donation and who does not. We state it as a lemma formally as follows:

All possible two-way exchanges that involve both left-lobe and right-lobe transplants are given in Figure 4. Each solid line in the figure refers to feasible left-lobe-only exchange

[11] Recall that the suffix w means that such a pair is willing for right lobe donation. The suffix u, on the other hand, means that such a pair is unwilling.

[12] When we do not mention whether a type is auxiliary or native, we mean its native version.

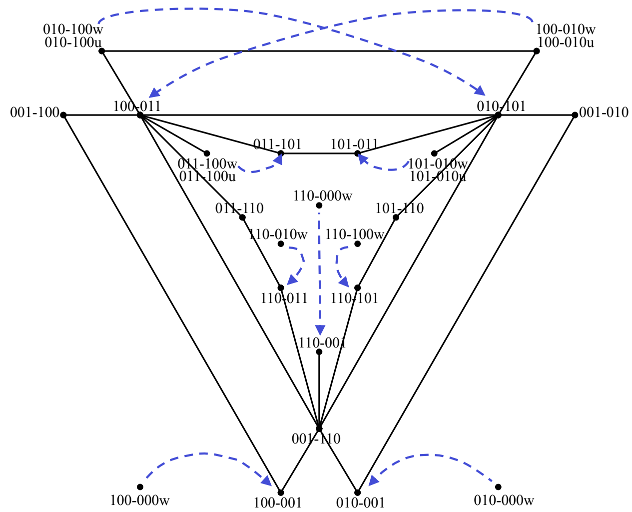


Figure 4: Possible Two-Way Exchanges with Left- and Right-Lobe Transplants

(as in Figure 2). Additionally, any type $X_1X_2X_3 - Y_1Y_20w$ pair can participate in every two-way exchange that a type $X_1X_2X_3 - Y_1Y_21$ can by donating a right lobe. The possible transformations that result with a possible two-way exchange are described by a dashed-line ending with an arrow in the figure. These are for Category IV and Category V willing types (Category IV willing types have no left-lobe exchange option while Category V willing types do). There are additional transformations that do not lead to two-way exchanges, and hence, they are not marked on Figure 4. These make a type right-lobe-only compatible. Some of these types can participate in exchanges by donating left lobe (Category VI willing types) and some cannot (Category III willing types).

As our main contribution in this section, we will introduce an incentive-compatible and Pareto-efficient exchange mechanism building on the sequential algorithm that was introduced in Section 3 to find optimal matchings for two individual sizes.

The two-size left&right-lobe sequential matching mechanism: Fix a priority order over pairs. The outcome of the mechanism is found by the following iterative algorithm. Suppose R is the reported willingness profile.

> **Step 0:** Match each Category I pair by a direct left-lobe transplant.
> Match each Category III willing type pair by a direct right-lobe transplant.
>
> **Step 1:** Transform Category IV willing types, and obtain a new auxiliary pool.
> Category IV willing types will be matched to donate right lobe from now on.

Clear waste 0 exchanges in the new pool: Match the maximum number of reciprocal auxiliary type pairs, i.e., $X-Y$ and $Y-X$ auxiliary types for all $X-Y \in \mathcal{T}\times\mathcal{T}$, according to the *priority order*.[13]

After this either $010-100w$ or $100-010w$ (or both) type pairs are depleted.

Assume without loss of generality $100-010w$ types are depleted.[14]

Step 2: Clear waste 1 exchanges of $010-100$.

Step 3: If there are $010-100w$ pairs remaining, *transform* them to auxiliary type $010-101$ (i.e., from now on, they can only participate in exchanges where they give right lobe) and obtain a new auxiliary exchange pool.

Clear waste 0 exchanges in the new pool: Match the maximum number of reciprocal auxiliary-type pairs according to the priority order.[15]

Step 4: If there are $011-100w$ and $101-010w$ type pairs remaining, transform them to types $011-101$ and $101-011$, respectively, and obtain a new auxiliary exchange pool.

Clear waste 0 exchanges in the new pool: Match the maximum number of reciprocal auxiliary-type pairs according to the priority order.[16]

Step 5: Clear waste 1 exchanges in the new pool: Match the maximum number of $100-011$, $010-101$, and $001-110$ auxiliary types, without matching them to each other, according the priority order.

[13]By maximizing the number of transplants with respect to the priority order in a step, we mean throughout the algorithm that a (two-way) *priority matching* is found among the pairs considered in that step as in Roth, Sönmez, and Ünver (2005). This matching has a simple algorithmic implementation for Steps 1-5, while its algorithmic implementation for Step 6 is slightly more complicated. In each of Steps 1-5, we have several *trivial two-sided matching* problems with pairs from one auxiliary type (call their set A) need to be matched with pairs from some other (possibly multiple) auxiliary types (call their set B). Then we can simply match the highest priority pair in A with the highest priority pair in B and continue iteratively in this manner with the remaining pairs until one set is depleted. We explain how to implement Step 6's priority matching in Footnote 18.

[14]Otherwise, Steps 2 and 3 are symmetrically defined below swapping 1st and 2nd binary digits for both patient and donor in all relevant auxiliary types. Moreover, if $010-100w$ types are depleted as well, types $011-101$ and $101-011$ can be symmetrically treated in the rest of the algorithm as efficiency or incentive compatibility constraints do not require an asymmetry between these two types. Our algorithm has this asymmetry built in it for brevity. To prevent this asymmetric treatment, in this case we can skip Steps 2 and 3 and continue with Step 4, and our algorithm will still be Pareto efficient and incentive compatible.

[15]Note that the only type of exchange cleared is between $101-010$ & $010-101$ types such that the latter entirely consists of the transformed $010-100w$ type pairs participating in right-lobe donations.

[16]Note that the only type of exchange cleared is between $101-011$ & $011-101$ types, who can both involve transformed types donating right lobes, i.e., $101-010w$ and $011-100w$, respectively, such that one of these types is definitely a transformed type.

Step 6: Clear waste 2 exchanges in the new pool: Match the maximum number of $100-011$, $010-101$, and $001-110$ auxiliary types with each other,[17]according the priority order.[18]

Step 7: Remove each remaining w willingness type right-lobe-only compatible pair by giving its donor's right lobe to its patient (these are exclusively Category VI willing pairs).

We depict Steps 1-6 in Figure 5 when in Step 1 $100-010w$ pairs are depleted.

Main innovation in this new algorithm over the left-lobe-only algorithm of the previous section is aligning incentives of pairs to report their true willingness type while making sure that the outcome is Pareto efficient.

For incentive compatibility, our first modification is introducing a fixed priority order to clear pairs in exchanges involving the same auxiliary types. Therefore, a pair with right-lobe donation opportunity cannot receive precedence over another pair of the same type regardless of how it reports its willingness type.[19]

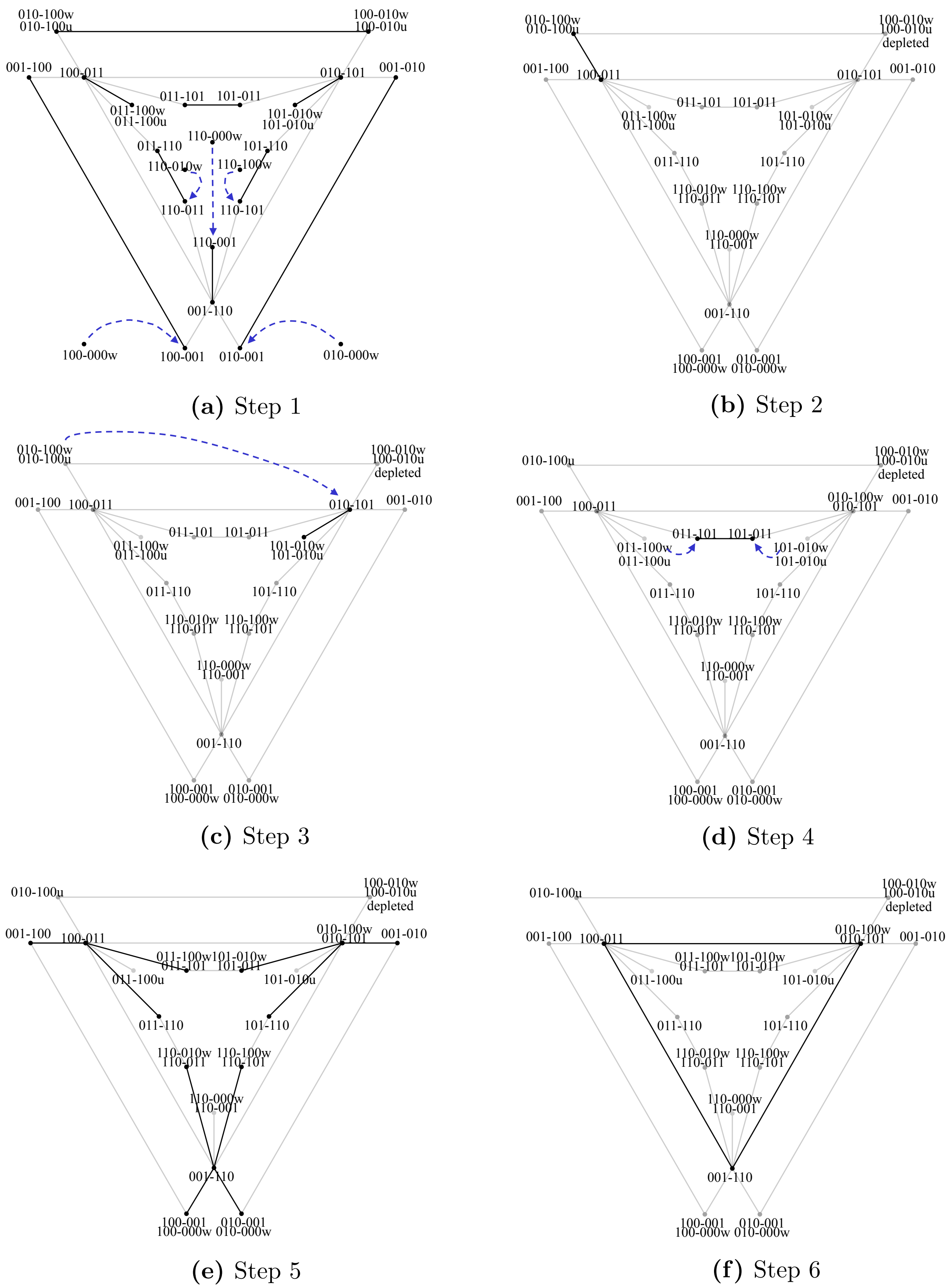
Once a priority order is fixed, we need to take care of the exchange order of Category V type willing pairs (4 types: $010-100w$, $100-010w$, $011-100w$, $101-010w$), because by Lemma 3 these are the only willing types that can participate in exchanges by donating left or right lobes.

We need to transform them only after *all exchange possibilities for their native types* are exhausted to prevent them reporting their types as unwilling. This is the incentive-compatibility constraint for exchange, which introduces a "natural order" in which willing types should be transformed as follows: As shown in Figure 4, because $100-010$ types can be matched with auxiliary $010-101$ types (i.e., $010-101$ and $010-100w$) to donate their left

[17]Note that the only right-lobe donations possible in Step 5 and Step 6 exchanges are the ones by $010-100w$ type pairs, which were transformed to $010-101$ in Step 3.

[18]The priority matching works as follows in this case (also see Footnote 9): Let auxiliary type set $\{X_1-Y_1, X_2-Y_2, X_3-Y_3\} = \{100-011, 010-101, 001-110\}$ be such that $n^*(X_1-Y_1) \geq n^*(X_2-Y_2) \geq n^*(X_3-Y_3)$ where $n^*(X_k, Y_k)$ is the number of pairs of auxiliary type X_k-Y_k pairs remaining at the beginning of Step 6 for each X_k-Y_k. Two cases are possible: Case (1) $n^*(X_1-Y_1) > n^*(X_2-Y_2)+n^*(X_3-Y_3)$: Then the algorithm matches all pairs of auxiliary types X_2-Y_2 and X_3-Y_3 exclusively with those of auxiliary type X_3-Y_3 according to the latter group's priority order such that $n^*(X_1-Y_1) - \big(n^*(X_2-Y_2)+n^*(X_3-Y_3)\big)$ lowest priority pairs of auxiliary type X_3-Y_3 remain unmatched. Case (2) $n^*(X_1-Y_1) \leq n^*(X_2-Y_2)+n^*(X_3-Y_3)$: Then all remaining pairs of these three auxiliary types are matched if their total number is even; and the lowest priority one among them remains unmatched if their total number is odd. A way to implement the matching in the latter case is after determining which pairs will be matched, (a) match auxiliary type X_k-Y_k pairs for any k with the pairs of the other two auxiliary types such that an equal number of the other two auxiliary type pairs remain unmatched, and then (b) match the remaining two groups with each other.

[19]The proof of Proposition 1 below shows how objectives incompatible with priority clearance may lead to obvious manipulation possibilities.



(a) Step 1

(b) Step 2

(c) Step 3

(d) Step 4

(e) Step 5

(f) Step 6

Figure 5: Transformations Followed by Conducted Two-Way Exchanges in Steps 1-6

lobes, $100-010w$ types have to be transformed after $010-100w$ types. The same argument is true for $011-100w$ and $100-010w$, respectively.

We state this result as follows:

Lemma 5 *The two-size left&right-lobe sequential matching mechanism is incentive compatible.*

On the other hand, with both right- and left-lobe donation options, Pareto efficiency is trickier to guarantee than incentive compatibility. The main intuition regarding Pareto efficiency in the left-lobe-only algorithm is that we should clear waste 0 exchanges before waste 1, and waste 1 before waste 2 if we are matching them in a greedy manner. We respect this intuition to the degree that it does not conflict with the IC constraint. That is, as $010-100w$ type needs to be transformed after its all left-lobe donation opportunities are exhausted and before transforming $101-010w$ type, we clear the exchanges involving left-lobe neighbors of $010-100$ as soon as possible. That is the reason Step 2 seems out of its natural order. In Steps 1 and 2, by clearing only waste 0 and certain waste 1 exchanges, we guarantee that either $010-100$ pairs are exhausted or its neighbor types, $100-010$ and $100-011$, are.

Moreover, in Step 2, while trying to exhaust $010-100$ type's left-lobe donation opportunities, we only clear some of the waste 1 exchanges that its neighbor $100-011$ type pairs could be part of (see Figure 5(b)), namely those with $010-100$. In particular, we do not clear possible $100-011$ & $011-101$ exchanges, immediately. It turns out that if we do not make such a deliberate delay, $011-101$ type pairs can be prematurely exhausted, and some waste 0 exchanges that they can help to facilitate (with transformed $101-010w$ pairs in Step 4) may not be realized (see Figure 5(d)). On the other hand $100-011$ type pairs to be used to match $011-101$ type pairs prematurely could have been matched in other ways causing a delicate inefficiency. The following example demonstrates this latter point:

Example 2 Consider an exchange pool with pairs $\mathcal{I}=\{i_1,i_2,i_3,i_4\}$ such that

$$\begin{aligned}\tau(i_1)&=100-011, &\quad \tau(i_2)&=101-010,\\ \tau(i_3)&=011-101, &\quad \tau(i_4)&=001-110.\end{aligned}$$

Suppose i_2 is type w, i.e., $R_{i_2}=R^w_{i_2}$.
Suppose that we modify Step 2 of the algorithm such that we clear all waste 1 matches involving $100-011$ type pairs before we proceed with other steps. Then the algorithm finds

a matching with only one exchange in Step 2 as

$$M = \Big\{\{i_1, i_3\}\Big\},$$

while the following matching with two exchanges Pareto-dominates it (and it is found by our unmodified left&right-lobe sequential matching algorithm such that the first exchange is fixed in Step 4 and the second exchange is fixed in Step 6):

$$M' = \Big\{\{i_1, i_2\}, \{i_3, i_4\}\Big\}.$$

$\blacksquare$

As our second result in this subsection, we show that the left&right-lobe sequential matching mechanism is Pareto efficient:

Lemma 6 *The two-size left&right-lobe sequential matching mechanism is Pareto efficient.*

Thus, we obtain the following result, which Lemmas 5 and 2 imply together with the fact that the construction of its algorithm ensures its individual rationality.

Theorem 2 *For any priority order of pairs, the two-size left&right-lobe sequential matching mechanism is individually rational, Pareto efficient, and incentive compatible.*

Although the left&right-lobe sequential matching mechanism is Pareto efficient, it may not maximize the number of transplants. If we would like to maximize the number of transplants or the number of left-lobe transplants, we need to sacrifice incentive compatibility as the following proposition shows.

Proposition 1 *There is no incentive compatible mechanism that maximizes the number of transplants or the number of left-lobe transplants.*

Proof: Consider an exchange pool with $\mathcal{I} = \{i_1, i_2, i_3, i_4\}$ and

$$\begin{aligned} \tau(i_1) &= 101 - 011, \qquad & \tau(i_2) &= 100 - 011, \\ \tau(i_3) &= 011 - 100, \qquad & \tau(i_4) &= 011 - 100. \end{aligned}$$

Suppose i_3 and i_4 are type w.
Any left-lobe donation or total transplant maximizing matching (two of which exists such

that i_3 and i_4 can be swapped) generates two exchanges. Consider such matchings:

$$M = \Big\{\{i_1, i_3\}, \{i_2, i_4\}\Big\} \qquad M' = \Big\{\{i_1, i_4\}, \{i_2, i_3\}\Big\}.$$

Observe that $e(i_3, i_1) = e(i_4, i_1) = l$ while $e(i_3, i_2) = e(i_4, i_2) = r$. Any (probabilistic) mechanism that chooses a matching with the maximum number of transplants or the maximum number of left-lobe transplants chooses at least one of these two matchings in its support. W.l.o.g., suppose M' is that matching. Then i_3 has an incentive to announce its type as u by revealing $R'_{i_3} = R^u_{i_3}$, as the mechanism will choose M' in this case with probability 1. Hence, there is no incentive compatible mechanism that maximizes the total number of transplants or left-lobe transplants. ■

6 Efficient and Incentive Compatible Mechanism in the General Model with Multiple Sizes

For the purposes of this section, we assume there are possibly multiple sizes, i.e., $\mathcal{S} = \{0, 1, ..., S-1\}$.

We need to handle to difficulties to modify the two-size mechanism to the general model. First, it may be difficult to find efficient matchings by a straightforward sequential algorithm even under only left-lobe donation in the general model; thus, we introduce a non-sequential approach. Second, it is not immediately clear we need what kind of a priority order for *transformations* of the willing pair types to make sure the mechanism is incentive compatible and efficient; thus, we construct such a priority order.

The first challenge is well known in the graph theory literature. In a general graph (not necessarily the compatibility graph of a liver exchange pool), we need to expand the set of simultaneously *matchable* pairs in a recursive way to find an efficient matching (e.g., as in the cardinality matching algorithm of Edmonds, 1965). We define matchability formally. Suppose pairs in $\mathcal{J}_{l\&r} \subseteq \mathcal{I}$ are willing to participate in both left- and right-lobe donation. We say that a subset of pairs $\mathcal{J} \subseteq \mathcal{I}$ **matchable** in compatibility graph $G_c[\mathcal{J}_{l\&r}]$ if there exist a matching $M \in \mathcal{M}_c[\mathcal{J}_{l\&r}]$ such that $M(j) \neq \emptyset$ for all $j \in \mathcal{J}$.[20] We can check whether a subset of pairs is matchable or not in polynomial time.[21]

[20]Our definition of matchability differs from the standard definition in graph theory, which requires the subgraph induced by $\mathcal{J}$ to have a perfect matching. See for instance Schrijver (2003, Vol I, p59).

[21]For example, we can use an algorithm by Edmonds (1965), called weighted matching algorithm, for checking matchability. This algorithm works on a graph, in which each exchange e is assigned a non-negative

We illustrate the second generalization as follows. The left&right-lobe sequential matching mechanism that we introduced for the case of two sizes was individually rational. Also in our algorithm, the decision to whether match a Category V pair via a two-way exchange where it donates left lobe, was made without using that pair's willingness announcement which guaranteed incentive compatibility. We will preserve these features in our general mechanism for arbitrary sizes, which will imply individual rationality and incentive compatibility (by Lemma 4).

In the left&right-lobe sequential matching mechanism that we introduced for the case of two sizes, the transformation of Category V pair types played a key role in guaranteeing Pareto efficiency and incentive compatibility of the mechanism. Intuitively, we wanted to transform each Category V pair only after their left-lobe exchange possibilities were exhausted, subject to the IC restriction that the algorithm decides whether or not the pair takes part in an exchange by donating left lobe, without using the pair's willingness announcement.

A related challenge was the specification of the sequence in which we transform Category V types. As an example, consider two Category V pair types $X-Y$ and $U-V$ such that the two pairs cannot form a left-lobe only exchange, but can form an exchange where Y donates his right lobe to U and V donates his left-lobe to X. In such a case, we would like to transform the $X-Y$ pair types before the $U-V$ pair types, because after transforming the $X-Y$ pair types, the left lobe exchange possibilities of the $U-V$ pair types may increase. This motivates the following definition.

Define a directed graph on the set of Category V pair types, that we will call the **precedence digraph**, where for any Category V pair types $X-Y$ and $U-V$:

$$X-Y \longrightarrow U-V \quad \iff \quad X \leq V,\ U \not\leq Y\ \&\ U \leq \rho(Y).$$

If $X-Y \longrightarrow U-V$, we will also say that $X-Y$ **precedes** $U-V$. For any Category V pairs i and j, we will also write $i \longrightarrow j$ and say that i **precedes** j if $\tau(i) \longrightarrow \tau(j)$.

In the above definition, $X-Y$ precedes $U-V$, if V can donate his left lobe to X, and Y cannot donate his left lobe but can donate his right lobe to U. In this case the two pair types $X-Y$ and $U-V$ cannot form a left-lobe only exchange, but can participate in an

weight $\Pi(e)$, while there are no direct transplants. The algorithm finds a matching that maximizes the sum of weights of the exchanges. Suppose each pair in $j \in \mathcal{J}$ is assigned a weight $\pi(j) > 0$ where no two agents are assigned the same weight, and each pair $i \in \mathcal{I}\backslash\mathcal{J}$ is assigned a weight $\pi(i) = 0$. Each exchange $\{i,j\} \in E_c[J_{l\&r}]$ is assigned the weight $\Pi(\{i,j\}) = \pi(i) + \pi(j)$. Using Edmonds' weighted matching algorithm to maximize the weights (designated by Π) on this new graph chooses a matching that matches as many pairs in $\mathcal{J}$ as possible (i.e., in the descending order of weights $\pi(.)$). If all pairs in $\mathcal{J}$ are matched in this matching then $\mathcal{J}$ is matchable in $G_c[J_{l\&r}]$ (Okumura, 2014). This method can also be used to determine a maximal subset of pairs that are matchable for any compatibility graph (see for example Roth, Sönmez, and Ünver, 2005).

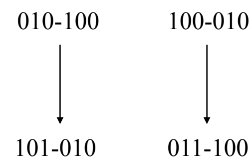


Figure 6: Precedence Digraph on Category V Types with Two Sizes ($S = 2$)

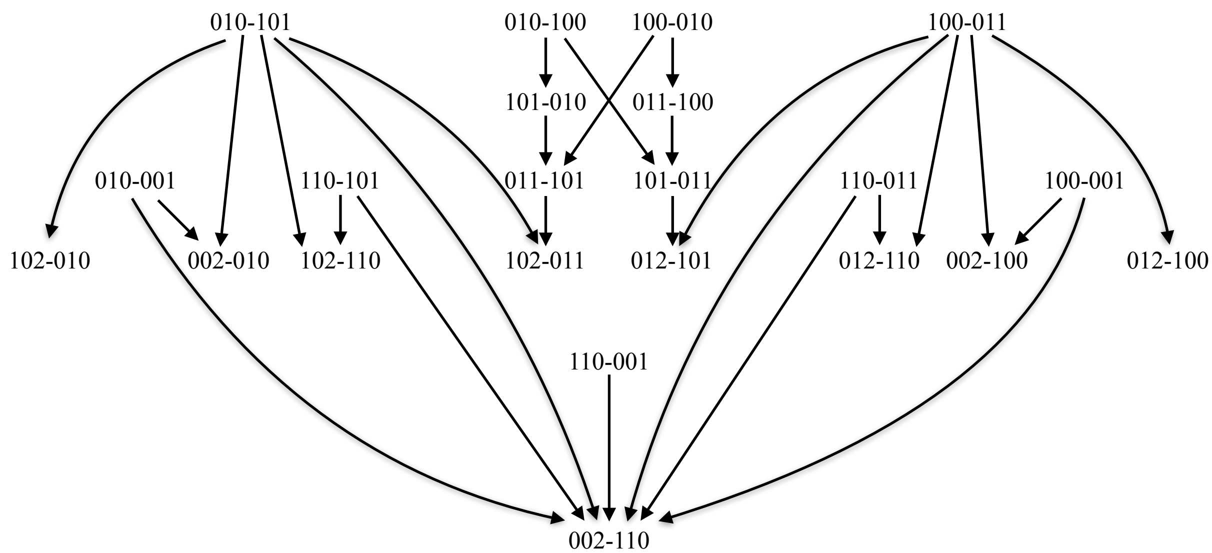


Figure 7: Precedence Digraph on Category V Types with Three Sizes ($S = 3$)

exchange after transforming the $X - Y$ pair type. Figure 6 depicts the precedence digraph over Category V pairs in the case of two sizes ($\mathcal{S} = \{0, 1\}$, $\rho(0) = \rho(1) = 1$). Figure 7 depicts the precedence digraph over Category V pairs in the case of three sizes ($\mathcal{S} = \{0, 1, 2\}$, $\rho(0) = 1$, $\rho(1) = \rho(2) = 2$).

In defining our mechanism more generally, we would like to transform Category V pair types in a sequence such that for any Category V pair types $X-Y$ and $U-V$: $X-Y \longrightarrow U-V$ implies that $X - Y$ pair types are transformed before the $U - V$ pair types. We next argue that it is indeed possible to find such a transformation order.

Note that the precedence digraphs in Figure 6 and Figure 7 are acyclic. It turns out that this observation is not specific to these two examples. As shown in the next Lemma, the precedence digraph for any liver exchange pool is acyclic. By Lemma 8 in Section A, this implies that the precedence digraph has a topological order, which will be the desired transformation sequence of Category V pair types.

Lemma 7 *The precedence digraph on Category V pair types is acyclic, and thus, it induces a linear order $\mathbb{L}$ consistent with the precedence digraph over the Category V pairs, i.e., for any two Category V pair types $X-Y$ and $U-V$, $X-Y \to U-V$ implies $X-Y \ \mathbb{L} \ U-V$. This linear order $\mathbb{L}$ is referred to as a topological order.*

We are ready to state our mechanism. An expanded equivalent definition is in Appendix E.1 and will be used in the proof of our main result of this section.

The General Left&Right-Lobe Priority Mechanism:

The outcome of the mechanism is found by the following iterative algorithm:

Fix an initial priority order over all pairs and a topological order of Category V pairs induced by the precedence digraph.
Let R be the reported willingness profile.

Step 0: Match each Category I pair by a direct left-lobe transplant.
Match each willing Category III pair by a direct right-lobe transplant.
Leave unwilling Category III and IV pairs and all Category II pairs unmatched.

Step 1: Let $\mathcal{I}_0$ be the set of pairs not handled in Step 1. Transform Category IV willing pairs, and obtain a new auxiliary pool. Let $G_0 = (\mathcal{I}_0, E_0)$ be the compatibility graph of this pool. Let $\mathcal{J}_0 := \emptyset$. Step 1's substeps proceed inductively.

Step 1.(k)*:* Let i be the k'th highest priority Category V pair in the initial topological order. If $\mathcal{J}_{k-1} \cup \{i\}$ is matchable in G_{k-1}, then let $\mathcal{J}_k := \mathcal{J}_{k-1} \cup \{i\}$ and $G_k := G_{k-1}$. Otherwise let $\mathcal{J}_k := \mathcal{J}_{k-1}$, and

- if i is not willing: let $G_k := G_{k-1}$.
- if i is willing: let $G_k = (\mathcal{I}_0, E_k)$ be the compatibility graph obtained from G_{k-1} by transforming i, i.e., by including to E_{k-1} the two-way exchanges in which i can donate right lobe.

Proceed with Step 1.(k+1).

When Step 1 ends in some step K, set $\mathcal{J}_K$ is matchable in G_K such that all pairs in $\mathcal{J}_K$ donate left lobe.

Step 2: Inductively, we continue with the remaining pairs $\mathcal{I}_0 \setminus \mathcal{J}_K$ starting with $J_0^* := \emptyset$.

Step 2.(n)*:* Let i be the n'th highest priority pair in $\mathcal{I}_0 \setminus \mathcal{J}^*_{n-1}$ according to the initial priority order over $\mathcal{I}$. If $\mathcal{J}_K \cup \mathcal{J}^*_{n-1} \cup \{i\}$ is matchable in G_K, then let $\mathcal{J}^*_n := \mathcal{J}^*_{n-1} \cup \{i\}$. Otherwise, $\mathcal{J}^*_n := \mathcal{J}^*_{n-1}$. Proceed with Step 3.(n+1).

When Step 2 ends in some step N, set $\mathcal{J}_K \cup \mathcal{J}^*_N$ is a maximal matchable set in G_K.

Step 3: Transform and match all willing Category VI pairs in $\mathcal{I}_0 \setminus (\mathcal{J}_K \cup \mathcal{J}^*_N)$ in direct right-lobe transplants.

A matching of graph G_K that match all pairs in $\mathcal{J}_K \cup \mathcal{J}^*_N$ in two-way exchanges together with the direct transplants determined in Steps 1 and 4 is the outcome of the mechanism.

To illustrate what we accomplish with this algorithm, first observe that Step 0 of this algorithm is identical to the beginning (Step 0 and the beginning of Step 1) of the two-size left&right lobe sequential matching algorithm.

In Step 1, once we transform all willing Category IV pairs, the two algorithms slightly diverge. Here the general algorithm uses an inductive approach to determine a maximal matchable set of Category V pairs who can be matched donating left lobe. These pairs are prioritized according to the chosen topological order of the precedence order. First, we try to match a Category V pair by left-lobe donation in addition to the previously committed pairs. If it is possible, this pair is added to the committed set of pairs. If it is not possible and the pair is willing, then we transform it, i.e., make it possible to be matched through right-lobe donation. We continue with the next Category V pair in the new expanded compatibility graph. Step 2 ends with a maximal set of Category V pairs that are committed to be matched through donating their left lobes only and an expanded graph such that remaining willing Category V pairs can potentially donate right lobe.

In Step 2, we consider all remaining pairs according to the chosen priority order over all pairs. Inductively, we try to match the next pair in addition to the previously committed pairs of this step and Step 2. If it is possible, this pair is added to the committed set of pairs, otherwise we skip it. Step 3 ends with a maximal set of pairs that can are matchable in the expanded graph of Step 2. We pick any matching that matches these pairs.

Finally, in Step 3 we match the remaining unmatched, willing Category VI pairs by direct right-lobe transplants, similar to the ending of the two-size algorithm explored in the previous section.

There is another important similarity between the two-size mechanism and the general mechanism. Suppose $S = 2$. As seen in Figure 6, the precedence digraph consists of only two directed edges in this case. Thus, type $010 - 100$ is ordered before type $101 - 010$ in the topological order (and in the final priority order of the general mechanism). Hence, the transformation of type $010 - 100$ pairs are always done before the transformation of type $101 - 010$ pairs in Step 1 of the general algorithm. This is consistent with the two-size algorithm of the previous section: First in Step 3, we transform $010 - 100$ type pairs, and then in Step 4, we transform $101 - 010$ type pairs.

The general mechanism depicted here has the desired good properties for the general case. We state our main theorem for this section as follows:

Theorem 3 *For any priority order over pairs, any topological order of the precedence digraph of Category V types, and any given willingness profile, the matchings that can be chosen by the general left&right-lobe priority mechanism are Pareto indifferent.*[22]
Moreover, this mechanism is individually rational, Pareto efficient, and incentive compatible.

7 Simulations

In this section, we report the results of computer simulations using South Korean aggregate statistics to determine the potential gains of our approach and the cost of incentive compatibility. Recall that our mechanism potentially does not maximize the number of transplants. An individually rational and incentive compatible mechanism that always maximizes the number of transplants does not exist as shown in Proposition 1.

Table 1 summarizes the calibration parameters we use for our study. We assume that each patient is paired with a donor. We determine the blood type, gender, and height characteristics for patients and their donors independently and randomly.[23].

A donor and patient are deemed left-lobe compatible if they are blood-type compatible and the donor's left-lobe volume is at least 40% of the total liver volume of the patient.

A donor and patient are deemed right-lobe-only compatible if they are not left-lobe-only

[22] I.e., they match the same pairs through the same types of transplants.

[23] Then, we use the following weight determination formula as a function of height: $w = a\ h^b$, where w is weight in kg, h is height in meters, and constants a and b are set as $a = 26.58, b = 1.92$ for males and $a = 32.79, b = 1.45$ for females (Diverse Populations Collaborative Group, 2005). The body surface area (BSA in m^2) of an individual is determined through the Mostellar formula given in Um et al. (2015) as $BSA = \frac{\sqrt{h\ w}}{6}$, and the liver volume ($l_v$ in ml) of Korean adults is determined through the estimated formula in Um et al. (2015) as $l_v = 893.485\ BSA - 439.169$. Each patient and donor have a height drawn independently from the truncated normal distribution using the mean and variance reported in this table. We assume that the left-lobe of each donor is 35% of his all liver, as this is reported as the mean of the left-lobe volume in Korea (Um et al., 2015)

Live-Liver Donation Recipients in 2010-2014		
Female	1492 (34.55%)	
Male	2826 (64.45%)	
Total	4318 (100.0%)	
Live-Liver Donors in 2010-2014		
Female	1149 (26.61%)	
Male	3169 (73.39%)	
Total	4318 (100.0%)	
Adult Height (cm.)		
Female	**Mean:** 157.4	**Std Dev:** 5.99
Male	**Mean:** 170.7	**Std Dev:** 6.4
Blood-Type Distribution		
O	37%	
A	33%	
B	21%	
AB	9%	

Table 1: Calibration statistics from South Korea for liver-exchange simulations . This table reflects the parameters used in calibrating the simulations. We obtained the blood-type distribution for South Korea from `http://bloodtypes.jigsy.com/East_Asia-bloodtypes` on 04/10/2016. The South Korean adult height distribution's mean and standard deviation were obtained from the Korean Agency for Technology and Standards (KATS) website `http://sizekorea.kats.go.kr` on 04/10/2016. The transplant data were obtained from the Korean Network for Organ Sharing (KONOS) 2014 Annual Report, retrieved from `https://www.konos.go.kr/konosis/index.jsp` on 04/10/2016.

compatible, they are blood-type compatible, and the donor's right-lobe volume is at least 40% of the total liver volume of the patient.

We generate $I = 50, 100, 250$ patient-donor pairs in three sets of simulations. Since we do not have empirical statistics on the willingness of donors for right-lobe donation, we consider 6 scenarios for each population size in which on average $0, 20, 40, 60, 80$, and 100% of all pairs are willing (we refer to this as $w-$rate). According to these w probabilities, we randomly determine each pair's willingness.

We consider three treatments:

- *No exchange*: Only direct transplants take place.
- *PE & IC mechanism*: An outcome of the general left&right-lobe priority mechanism is determined for arbitrary topological and priority orders.
- *Maximum matching*: We find a maximum individually rational matching as follows: We assume that all willing Category V pairs can participate in right-lobe transplants right away and transform them at the beginning of Step 1 of the general mechanism using the same topological and priority orders above.

Pop. Size	w Rate	All Treatments: L. Lobe Direct (Cat. I)	All Treatments: R. Lobe Direct Cat. III	No exchange: R. Lobe Direct Cat. VI	No exchange: *Total*	Our general IR, PE, & IC mechanism: *Exchange* L. Lobe Trans.	*Exchange* R. Lobe Trans.	*Exchange* *Total Trans.*	Remain. R. Lobe Direct (Cat. VI)	*Total Direct*	*Total*	A maximum IR matching: *Exchange* L. Lobe Trans.	*Exchange* R. Lobe Trans.	*Exchange* *Total Trans.*	Remain. R. Lobe Direct (Cat. VI)	*Total Direct*	*Total*
	0	6.204	0	0	6.204	1.918	0	1.918	0	6.204	8.122	1.918	0	1.918	0	6.204	8.122
		(2.280)	(0)	(0)	(2.280)	(1.908)	(0)	(1.908)	(0)	(2.280)	(2.831)	(1.908)	(0)	(1.908)	(0)	(2.280)	(2.831)
	0.2	6.204	2.349	2.180	10.733	2.670	1.822	4.492	1.869	10.422	14.914	2.625	1.967	4.592	1.972	10.525	15.117
		(2.280)	(1.496)	(1.445)	(2.912)	(2.023)	(1.636)	(2.758)	(1.396)	(2.920)	(3.598)	(2.004)	(1.683)	(2.846)	(1.415)	(2.917)	(3.694)
	0.4	6.204	4.662	4.419	15.285	3.286	3.532	6.818	3.539	14.405	21.223	3.164	3.934	7.098	3.726	14.592	21.690
		(2.280)	(2.011)	(2.071)	(3.330)	(2.085)	(2.226)	(3.314)	(1.946)	(3.351)	(3.961)	(2.051)	(2.362)	(3.497)	(1.983)	(3.338)	(4.139)
50	0.6	6.204	7.050	6.615	19.869	3.816	5.028	8.844	4.898	18.152	26.996	3.605	5.697	9.302	5.081	18.335	27.637
		(2.280)	(2.415)	(2.441)	(3.491)	(2.153)	(2.509)	(3.573)	(2.330)	(3.632)	(3.930)	(2.075)	(2.676)	(3.799)	(2.365)	(3.625)	(4.096)
	0.8	6.204	9.446	8.855	24.505	4.336	6.404	10.740	6.190	21.840	32.580	3.968	7.340	11.308	6.198	21.848	33.156
		(2.280)	(2.761)	(2.785)	(3.581)	(2.211)	(2.704)	(3.756)	(2.683)	(3.837)	(3.840)	(2.107)	(2.933)	(3.987)	(2.723)	(3.855)	(3.924)
	1	6.204	11.739	11.131	29.074	4.767	7.521	12.288	7.387	25.330	37.618	4.196	8.822	13.018	6.985	24.928	37.946
		(2.280)	(2.954)	(3.045)	(3.507)	(2.237)	(2.795)	(3.776)	(2.988)	(3.973)	(3.581)	(2.106)	(3.115)	(4.052)	(2.988)	(4.029)	(3.630)
	0	12.497	0	0	12.497	5.498	0	5.498	0	12.497	17.995	5.498	0	5.498	0	12.497	17.995
		(3.367)	(0)	(0)	(3.367)	(3.229)	(0)	(3.229)	(0)	(3.367)	(4.526)	(3.229)	(0)	(3.229)	(0)	(3.367)	(4.526)
	0.2	12.497	4.741	4.356	21.594	7.031	4.425	11.456	3.631	20.869	32.325	6.875	5.015	11.890	3.916	21.154	33.044
		(3.367)	(2.162)	(2.099)	(4.255)	(3.300)	(2.609)	(4.385)	(1.952)	(4.254)	(5.540)	(3.299)	(2.691)	(4.594)	(2.025)	(4.263)	(5.752)
	0.4	12.497	9.401	8.795	30.693	8.303	8.303	16.606	6.725	28.623	45.229	7.979	9.639	17.618	7.313	29.211	46.829
		(3.367)	(2.969)	(2.983)	(4.605)	(3.337)	(3.469)	(5.006)	(2.698)	(4.647)	(5.681)	(3.299)	(3.696)	(5.454)	(2.854)	(4.658)	(6.082)
100	0.6	12.497	14.023	13.254	39.774	9.543	11.611	21.154	9.244	35.764	56.918	8.891	13.751	22.642	9.823	36.343	58.985
		(3.367)	(3.443)	(3.571)	(4.814)	(3.379)	(3.797)	(5.271)	(3.364)	(4.963)	(5.570)	(3.295)	(4.127)	(5.781)	(3.550)	(5.048)	(5.944)
	0.8	12.497	18.703	17.699	48.899	10.693	14.447	25.140	11.424	42.624	67.764	9.486	17.572	27.058	11.337	42.537	69.595
		(3.367)	(3.939)	(4.026)	(5.032)	(3.488)	(4.217)	(5.713)	(3.871)	(5.380)	(5.350)	(3.277)	(4.584)	(6.196)	(3.986)	(5.506)	(5.572)
	1	12.497	23.392	22.169	58.058	11.691	16.869	28.560	13.311	49.200	77.760	9.718	20.916	30.634	12.112	48.001	78.635
		(3.367)	(4.389)	(4.370)	(5.062)	(3.488)	(4.490)	(5.847)	(4.207)	(5.708)	(5.226)	(3.076)	(4.969)	(6.290)	(4.135)	(5.835)	(5.255)
	0	31.031	0	0	31.031	19.652	0	19.652	0	31.031	50.683	19.652	0	19.652	0	31.031	50.683
		(5.236)	(0)	(0)	(5.236)	(5.844)	(0)	(5.844)	(0)	(5.236)	(7.681)	(5.844)	(0)	(5.844)	(0)	(5.236)	(7.681)
	0.2	31.031	11.812	11.083	53.926	23.315	13.141	36.456	8.711	51.554	88.010	22.765	15.515	38.280	9.917	52.760	91.040
		(5.236)	(3.354)	(3.379)	(6.572)	(5.862)	(4.393)	(7.585)	(3.140)	(6.488)	(9.000)	(5.799)	(4.682)	(8.123)	(3.270)	(6.532)	(9.579)
	0.4	31.031	23.456	22.044	76.531	26.512	23.792	50.304	15.589	70.076	120.380	25.158	28.952	54.110	18.156	72.643	126.753
		(5.236)	(4.608)	(4.629)	(7.263)	(5.980)	(5.700)	(8.634)	(4.355)	(7.280)	(8.967)	(5.806)	(6.248)	(9.420)	(4.625)	(7.353)	(9.827)
250	0.6	31.031	35.196	33.191	99.418	29.478	32.736	62.214	21.195	87.422	149.636	26.945	40.897	67.842	23.301	89.528	157.370
		(5.236)	(5.622)	(5.400)	(7.639)	(6.076)	(6.290)	(8.940)	(5.227)	(7.992)	(8.543)	(5.567)	(7.077)	(9.877)	(5.793)	(8.230)	(9.107)
	0.8	31.031	46.986	44.308	122.325	32.249	40.423	72.672	25.693	103.710	176.382	27.534	51.802	79.336	25.212	103.229	182.565
		(5.236)	(6.401)	(6.021)	(7.777)	(6.138)	(6.958)	(9.305)	(6.069)	(8.602)	(8.322)	(5.454)	(7.718)	(10.068)	(6.522)	(8.841)	(8.449)
	1	31.031	58.667	55.417	145.115	34.656	46.546	81.202	29.709	119.407	200.609	26.999	61.005	88.004	25.448	115.146	203.150
		(5.236)	(7.176)	(6.482)	(7.744)	(6.147)	(7.186)	(9.131)	(6.570)	(9.074)	(7.805)	(5.309)	(8.281)	(9.942)	(6.409)	(9.176)	(7.726)

Table 2: Simulation results for $I = 50, 100, 250$ and willingness rates $0, 0.2, 0.4, 0.6, 0.8, 1$.x Standard deviations of the populations are reported below the averages in parentheses for 500 simulations.

The results of the simulations are given in Table 2. About 12.5% of all pairs are left-lobe compatible and never enter exchange. We see 0% to 45.5% of all pairs become right-lobe-only compatible as a linear function of $w-$rate (see the no exchange treatment in the table). The IC mechanism, on the other hand, matches 18% to 77.75% of all pairs, a seemingly "concave" increasing function of $w-$rate for $I = 100$ (see the IC treatment in the table). Thus, exchange's marginal contribution ranges from 45% to 25% of all transplants as $w-$rate increases.

We also inspect the cost of using a PE & IC mechanism in terms of number of potential transplants lost. As willingness rate increases the IC mechanism increases the number of right-lobe transplants (through exchange and direct transplant), slower than maximum matching does. However, it increases the number of left-lobe transplants faster than maximum matching does. These can be seen in Figure 8 as a summary graph for the IC mechanism and maximum matching when $I = 100$ (in this figure, the numbers of left-lobe, right-lobe, and total transplants are given on the vertical axis as a function of the willingness rates in

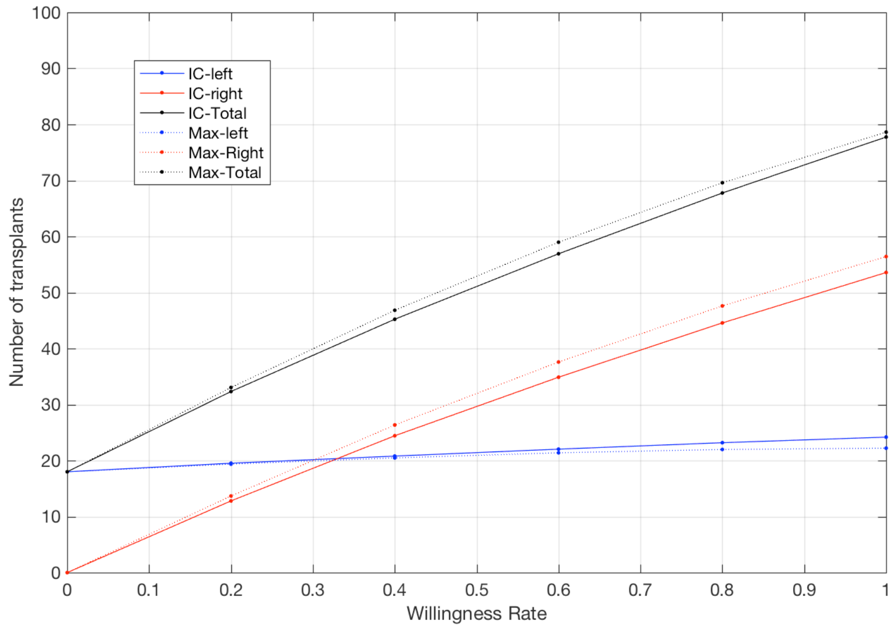


Figure 8: Summary of simulations for $I = 100$ pairs: Averages of the left-lobe, right-lobe, and total transplants in two treatments as a function of willingness rate: our general IR, PE, & IC mechanism (denoted as IC in the legend) and a maximum IR matching (denoted as Max in the legend).

the horizontal axis). The right-lobe transplants ranges from 0% to 53.5% as a function of willingness rate under the IC mechanism, while it ranges from 0% to 56.5% under the maximum matching (each function is seemingly "concave"). The absolute difference between the two is monotonically increasing and seemingly "convex." On the other hand for the left-lobe donations the ranking changes: for the IC mechanism they range from 18% to 22%, while for the maximum matching they range from 18% to 22%. Both curves are seemingly "concave." The difference between them is also monotonically increasing and concave. Thus, from 18% to 77.75% of the pairs get matched under the IC mechanism as a function of willingness. Those range from 18% to 78.5% for the maximum matching. As a result, the absolute difference between the two regimes ranges from 0% for $w-$rate $= 0$ to 2% for $w-$rate $= 0.6$ and back to 0.75% for $w-$rate $= 1$. Thus, the cost of incentive compatibility seems to be small. [24]

[24]Results are similar for $I = 50$ and $I = 250$.

Appendix A Mathematical Preliminaries

In this section, we will state some definitions and a result from graph theory that will be used in subsequent sections.

A tuple $G = (\mathcal{V}, E)$ is a **graph** if $\mathcal{V}$ is a nonempty set and $E \subset \Big\{\{x, y\} : x, y \in \mathcal{V}\Big\}$. The elements of $\mathcal{V}$ are called **vertices**. The elements of E are called **edges**.

Note that in the definition of a graph, we are allowing for loops, i.e., edges $\{x, y\}$ such that $x = y$.[25]

A **matching** in a graph $G = (\mathcal{V}, E)$ is a subset $M \subseteq E$ of pairwise disjoint edges, i.e., $e, e' \in M$ such that $e \cap e' \neq \emptyset \implies e = e'$. Given a matching M in G, we will abuse notation and also define the function $M : \mathcal{V} \to \mathcal{V} \cup \{\emptyset\}$ by:

$$M(x) = \begin{cases} y & \text{if there exists } y \in V \text{ such that } \{x, y\} \in M \\ \emptyset & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{V}$. We call $M(x)$ the **match of x in M**. We will say that a subset $\mathcal{W} \subseteq \mathcal{V}$ is **matchable in G**, if there is a matching M in G such that $M(x) \neq \emptyset$ for all $x \in \mathcal{W}$.

The **size** of a matching M is defined by $\sum_{e \in M} |e|$.

In a graph, the vertices corresponding to each edge $e = \{x, y\}$ are unordered. We will also need the notion of a *directed graph* where the order of the vertices does matter.

A tuple $G = (\mathcal{V}, E)$ is a **directed graph (digraph)** if $\mathcal{V}$ is a nonempty set and $E \subseteq \{(x, y) \in \mathcal{V} \times \mathcal{V} : x \neq y\}$. When the digraph is understood, we will also use $x \to y$ to denote $(x, y) \in E$.

Note that as opposed to our definition of a compatibility graph, in the definition of a digraph, we are ruling out loops. i.e., directed edges (x, y) such that $x = y$.[26]

Given a digraph $G = (\mathcal{V}, E)$, a **topological order on** G is a linear order $\mathbb{L}$ on $\mathcal{V}$ such that: $x \to y$ implies $x \mathbb{L} y$, for all $x, y \in \mathcal{V}$.

[25] In some texts, a *simple undirected graph with loops*, is what we call a graph here. See for example Korte and Vygen (2011, p13-14).

[26] In some texts, a *simple directed graph without loops*, is what we call a digraph here. See again Korte and Vygen (2011, p13-14).

A digraph $G = (\mathcal{V}, E)$ is **acyclic** if there do not exist an integer $n \geq 2$ and $v_1, \ldots, v_n \in \mathcal{V}$ such that: $v_1 \to v_2 \to \ldots \to v_n \to v_1$.

The following Lemma is a standard result in graph theory.[27]

Lemma 8 *Given a digraph $G = (\mathcal{V}, E)$, there exists a topological order on G if and only if G is acyclic.*

Appendix B Proofs for Section 3

Proof of Theorem 1: We say that a matching is *maximum* if it maximizes the number of two-way exchanges in an individually rational matching. A maximum matching M_0 exists by finiteness of the problem. Fix an exchange pool.

Step 0: In all individually rational matchings all compatible pairs participate in direct transplants.

Step 1: Suppose that M_0 does not maximize the two-way exchanges between $100-011$ and $011-100$ types. That is, the number of $100-011$ and $011-100$ types matched in M_0 is strictly less than $\min\{n(100-011), n(011-100)\}$. Let $\Delta > 0$ be the difference. Since $011-100$ types can only be matched to $100-011$ types, at least Δ many $011-100$ types are unmatched at M_0 and at least Δ many $100-011$ types are matched to other types. We can define a new matching M_0' by unmatching Δ many of those $100-011$ types and rematching them to the unmatched $011-100$'s. Then, M_0' continues to be maximum and maximizes the two-way exchanges between $100-011$ and $011-100$ types.

Suppose that M_0' does not maximize the two-way exchanges between $010-100$ and $100-010$ types. Then, the number of $010-100$ and $100-010$ types matched in M_0' is strictly less than $\min\{n(010-100), n(100-010)\}$. Let $\Delta > 0$ be the difference. From Figure 2 and optimality of M_0', at least Δ many $010-100$ types are matched to $100-011$ types, and at least Δ many $100-010$ types are matched to $010-101$ types. We can define a new matching M_0'' by undoing these matches, rematching those $010-100$ and $100-010$ types to each other, and rematching those $100-011$ and $010-101$ to each other. Then, M_0'' continues to be maximum and also maximizes the two-way exchanges between $010-100$ and $100-010$ types.

By applying the above arguments to the other two-way exchanges in Figure 3(a), we obtain a maximum matching M_1 that maximizes the two-way exchanges in Step 1 of the matching algorithm.

[27]For example, see Proposition 2.9 in Korte and Vygen (2011, p20).

Step 2: Fix the matches maximized in Step 1, and consider any submatching M^* among remaining types that maximizes the two-way exchanges between the remaining $100 - 011$ types and the remaining types in $T = \{010 - 100, 001 - 100, 011 - 101, 011 - 100\}$. Let k and m denote the number of two-way exchanges between $100 - 011$ types and types in T at the submatching M^* and at the matching M_1, respectively. Since the submatching M^* maximizes these two-way exchanges after Step 1, $k \geq m$. At the matching M_1, unmatch the m matches between $100-011$ types and types in T, and unmatch an additional $k-m$ many $100 - 011$ types matched to $010 - 100$ or $011 - 110$ types. Then, rematch those k $100 - 011$ types with types in T as in the submatching M^*. The new matching M_1' obtained in this way, is maximum and maximizes the two-way exchanges between the remaining $100 - 011$ types and the remaining types in T.

By applying the above argument to the other two-way exchanges in Figure 3(b), we obtain a maximum matching M_2 that sequentially maximizes the two-way exchanges in Steps 1 and 2 of the matching algorithm.

Step 3: Take any matching M that agrees with M_2 in the matches created in the first two steps. Since the only remaining two-way exchanges after Steps 1 and 2 of the algorithm are those in Figure 3(c), the matching M is maximum if and only if it maximizes the exchanges in Figure 3(c) given the matches in the first two steps. ■

Appendix C Proofs for Section 4

Proof of Lemma 3: Let M be an individually rational matching. All types $X-Y$ fall into one of the following six mutually exclusive categories:

I $X \leq Y$: Then the pair is left-lobe compatible and in M it receives direct left-lobe transplant. On the other hand, in none of the remaining cases, a direct left-lobe transplant is possible in M for and $X - Y$ pair.

II $X > \rho(Y)$: This also implies, $X > Y$. Since $Y > X$, $Y - X$ or $\rho(Y) - X$ types do not participate in exchange but only direct donation. Thus, $X - Y$ types cannot not participate in exchange by donating left or right lobe.

III $Y < X \leq \rho(Y)$: These types can receive direct-right-lobe transplant in M. Moreover, as they satisfy $X \geq Y$, they cannot participate in a two-way exchange by donating left lobe in M as $Y-X$. Thus, they can only remain unmatched in M if they are unwilling.

IV $X > Y$, $X \not\geq \rho(Y)$ & $X \not\leq \rho(Y)$: By Lemma 2, they cannot participate in exchange by donating a left lobe. On the other hand, again by Lemma 2, they can participate in exchange to donate only right lobe. So this is their only option in M if they are willing or remaining unmatched.

V $X \not\leq \rho(Y)$ & $X \not\geq Y$: By Lemma 2, they can participate in exchange by donating a left lobe. On the other hand, by Lemma 2, they can participate in exchange to donate a right lobe as well if they are willing. Thus, they have two exchange options in M. They can also remain unmatched in M.

VI $X < \rho(Y)$, $X \not\geq Y$, & $X \not\leq Y$: In M they can participate in a direct right-lobe transplant if they are willing. As an exchange needs to be individually rational, they will never participate in an exchange to donate right lobe. We also have $X \not\geq Y$. Therefore, by Lemma 2, they can participate in exchange by donating left lobe. In M, they will only be unmatched if they are unwilling.

Conversely, suppose M is a matching consisting entirely of the types of transplants depicted in I, III, IV, V, and IV above. Then, by definition M is individually rational, concluding the proof. ■

Proof of Lemma 4: Throughout the proof, let f be an individually rational mechanism.

We start by proving that if f is incentive compatible, then the equivalence holds. Let $i \in \mathcal{I}$ and $R_{-i} \in \prod_{j \neq i} \mathcal{R}_j$. To see the "$\Leftarrow$" direction, suppose for a contradiction that i participates in a two-way exchange by donating left lobe at $f(R_i^u, R_{-i})$, but not at $f(R_i^w, R_{-i})$. By individual rationality, the pair i is not left-lobe compatible (i.e., not Category I), otherwise it would directly donate left-lobe at $f(R_i^u, R_{-i})$. Therefore, i cannot directly donate left-lobe also at $f(R_i^w, R_{-i})$. The only possible match possibilities of i at $f(R_i^w, R_{-i})$ are: i directly donates right lobe; i takes part in a two-way exchange by donating right lobe; or i is unmatched. By the definition of preferences, at R_i^w, i strictly prefers being part of a two-way exchange by donating left lobe to all the three match possibilities at $f(R_i^w, R_{-i})$, i.e.: $f(R_i^u, R_{-i}) P_i^w f(R_i^w, R_{-i})$, contradicting the incentive compatibility of f. The proof of the "$\Rightarrow$" direction is symmetric by switching the roles of R_i^u and R_i^w.

Now assume that the equivalence holds. We will show that this implies incentive compatibility of f. Take any $i \in \mathcal{I}$ and $R_{-i} \in \prod_{j \neq i} \mathcal{R}_j$. If i is left-lobe compatible (i.e., Category I), then individual rationality implies that i directly donates left-lobe at $f(R_i^u, R_{-i})$ and $f(R_i^w, R_{-i})$. Therefore, whether i is willing or unwilling, i is indifferent between $f(R_i^u, R_{-i})$ and $f(R_i^w, R_{-i})$, implying the incentive compatibility condition for i. Suppose next that i is not left-lobe compatible. By the equivalence, there are two cases to consider:

Case 1: In both $f(R_i^u, R_{-i})$ and $f(R_i^w, R_{-i})$, the pair i is part of a two-way exchange by donating left lobe. Therefore, whether i is willing or unwilling, i is indifferent between $f(R_i^u, R_{-i})$ and $f(R_i^w, R_{-i})$, implying the incentive compatibility condition for i holds.

Case 2: In neither $f(R_i^u, R_{-i})$ nor $f(R_i^w, R_{-i})$, the pair i is part of a two-way exchange by donating left lobe. By individual rationality, this implies that i is unmatched at $f(R_i^u, R_{-i})$. At $f(R_i^w, R_{-i})$, the only match possibilities of i are: i directly donates right lobe; i takes part in a two-way exchange by donating right lobe; or i is unmatched. Therefore, by the specification of preferences, $f(R_i^u, R_{-i}) R_i^u f(R_i^w, R_{-i})$ and $f(R_i^w, R_{-i}) R_i^w f(R_i^u, R_{-i})$, as desired. ■

Appendix D Proofs for Section 5

Proof of Lemma 5: First observe that willingness type revelation has no effect in the order of pairs of a given auxiliary type are matched within a step (after relevant transformations are done). These matches are utilized exclusively using an exogenous priority order.

Any pair that is unwilling is best off by reporting its willingness type truthfully, as otherwise they could sometimes be matched to donate right lobes, which is worse than being unmatched.

Consider willing pairs. Category IV willing pairs cannot do anything better than revealing their willingness types truthfully as they have no other exchange possibility other than those involving the donation of right lobe. This transformation is done in Step 1 of the algorithm. Category III type pairs receive right-lobe donation from themselves in Step 0. Since they cannot participate in exchange, it is best for them to reveal their types truthfully. Category VI types are only transformed after all left-lobe donation possibilities are exhausted in Step 7, therefore, they are best off by being truthful as well. Next, we explore Category V types separately. Without loss of generality assume that $n(010-100) \geq n(100-010)$.

- $100-010w$ types are exhausted in Step 1 by donating left lobe, so they are indifferent between being truthful or not.
- $010-100w$ types can donate left lobe only to $100-011$ and $100-010$ types in an exchange. They are transformed for right-lobe donation in Step 3 only after all of these two neighbor type pairs are exhausted in Steps 1 and 2. Therefore, they cannot do any better by being untruthful about their types.
- $011-100w$ types can donate left lobe only to $100-011$ and $100-010w$ types in an exchange. The pairs of the latter of these two types are exhausted in Step 3. As they

are transformed if all of $100-011$ types are matched in Step 2, they cannot do any better by being untruthful about their types.

- $101-010w$ types can donate left lobe only to $010-101$ and $010-100w$ types in an exchange. As they are transformed only after all of $100-011$ types are matched in Step 1 and all of $010-100w$ type are matched in Steps 1 and 3, they cannot do any better by being untruthful about their types.

■

Proof of Lemma 2: Let M be the outcome of the mechanism and M' be a Pareto-efficient matching such that either all pairs are indifferent between M and M' or M' Pareto-dominates M. We will show that M' cannot Pareto-dominate M. Our proof strategy will be sequentially constructing a set of matchings $M_0 := M', M_1, ..., M_5$ such that M_s agrees with M for all pairs matched in Steps 1,...,s of the algorithm and all pairs are indifferent between $M' = M_0$ and M_s. Then finally we will show that pairs should be Pareto-indifferent between M_5 and M, proving that M is Pareto efficient.

Without loss of generality assume that $n(010-100) \geq n(100-010)$ in the rest of this proof.

Construction of M_1: We construct M_1 as follows by changing the partners of pairs matched in Step 1. Recall that 0 waste exchanges are realized in this step. These are either (a) value -1 & value +1 exchanges or (b) value 0 & value 0 exchanges. Suppose we clear them in this order in Step 1:

Step 1.a. Value -1 & value +1 exchanges are cleared: Consider an exchange between pair i of type $101-010$ and pair j of type $010-101$ chosen by $M = M_6$ but not $M' = M_0$. Since i is matched to donate left lobe in M, it is also matched to donate left lobe in M'. So is j. Let $M'(i) = k$ and $M'(j) = \ell$ for some pairs k and ℓ.

Pair k is either (a) type $010-101$ or (b) type $010-100w$ and matched to donate a right lobe to i in M' (see Figure 4, $010-101$ is the only auxiliary neighbor type for $101-010$).

Thus, pair j and pair k are either same type or pair k can be transformed to pair j's type. Since $M'(j) = \ell$, k and ℓ can also be matched. Define

$$M'' := \Big[M' \setminus \Big\{\{i,k\},\{j,\ell\}\Big\}\Big] \cup \Big\{\{i,j\},\{k,\ell\}\Big\}.$$

Now all pairs are indifferent between $M' = M_0$ and M''. We use a similar construction to obtain a new matching from M'' (instead of M') for any other value -1 & value +1 exchange cleared in Step 1 and not picked in M' (i.e., those exchanges of auxiliary types $110 - 001$ & $001 - 110$ and $011 - 100$ & $100 - 011$), and iteratively continue.[28] Suppose $M_{1.a}$ is the outcome of this step. Now, all pairs are indifferent between $M_{1.a}$ and $M' = M_0$ and all value -1 & value 1 exchanges coincide those in M.

Step 1.b. Value 0 & value 0 exchanges are cleared: Consider an exchange between pair i of type $010 - 100$ and pair j of type $100 - 010$ chosen by M but not $M_{1.a}$. As M' weakly Pareto-dominates M, both i and j are matched and donate left lobes in M' as well. Suppose $M'(i) = k$ and $M'(j) = \ell$ for some pairs k and ℓ. Then k can be of type $100-011$ or $100-010$; and ℓ can be of type $010-100$ or $010-101$ (see Figure 4). Observe that k and ℓ are left-lobe compatible with each other's donors. Therefore, we can form a matching

$$M''' := \Big[M_{1.a} \setminus \Big\{\{i,k\},\{j,\ell\}\Big\}\Big] \cup \Big\{\{i,j\},\{k,\ell\}\Big\}$$

such that all pairs are indifferent between M''' and $M_{1,a}$ (and hence, M' by Step 1.a).

We use a similar construction to obtain a new matching from M''' (instead of $M_{1,a}$) for any other value 0 & value 0 exchange cleared in Step 1 (i.e., exchanges involving $001 - 100$, $001 - 010$, $011 - 101$, $011 - 110$, and $101 - 110$ and their reciprocal auxiliary types) and not picked in M', and iteratively continue. Suppose M_1 is the outcome of this step. Now, all pairs are indifferent between M_1 and $M' = M_0$ and all 0-waste exchanges coincide those in M.

Construction of M_2: Remaining highest priority pairs of types $010 - 100$are matched with remaining highest priority $100 - 011$ type pairs in M.

Take the highest priority i of type $010 - 100$ such that $M(i) = j$ for some pair j of type $100-011$ and $M_1(i) \neq j$. Now $M_1(i) = k$ and $M_1(j) = \ell$ for some k and ℓ such that i donates a left lobe (as well as j) because M' weakly Pareto-dominates M. As $X - Y$ type pairs are matched with $Y - X$ type pairs in Step 1 of the algorithm and as i is available in Step 2, all $100 - 010$ type pairs should have been exhausted in Step 1. Because M_1 and M coincide for all pairs matched in Step 1 by construction of M_1, no $100 - 010$ type pairs could be matched with i in M_1. Thus, the only feasible type for k is $100 - 011$ (this is a native type, i.e., k is not of type $100 - 010w$, because such pairs are exhausted in Step 1; see Figure **??**), i.e. the same type of j. Thus the following is a feasible matching:

$$M'' := \Big[M_1 \setminus \Big\{\{i,k\},\{j,\ell\}\Big\}\Big] \cup \Big\{\{i,j\},\{k,\ell\}\Big\}$$

such that all pairs are indifferent between M'' and M_1 (and thus, $M_0 = M'$), and M'' agrees

[28]The only difference in the argument is that an auxiliary $110 - 001$ type can only be matched with a pair of its reciprocal type in M'.

with M for all pairs matched before i and j in the algorithm and also for pairs i and j.

We repeat the above procedure for all such remaining i starting with M'' instead of M_1. Let M_2 be the final outcome of this procedure.

Construction of M_3: Suppose that some pair i of type $010-100w$ is matched to donate its right lobe to some pair j of type $101-010$ in Step 3 of the algorithm so that $M(i)=j$ but $M_2(i)=k\neq j$ (see Figure 5(c)). Moreover, let i be the highest priority such pair. By the construction of M_1 and M_2, all of the pairs that i can be matched to donate its left lobe are matched with other pairs in M_2. Thus, i has to be right-lobe-matched in M_2, too. Hence k is one of the types that is neighbor of $010-101$.

Suppose $M_2(j)=\ell$. Since j is left-lobe matched in M, it should also be left-lobe matched in M_2. Thus, the only option for ℓ's type is $010-101$ or $010-100w$, since j's type $101-010$ can donate left lobe in an exchange with a pair of only one of these types (see Figure 4). Hence, k and ℓ can be matched with each other. We have

$$M'' := \Big[M_2 \setminus \Big\{\{i,k\},\{j,\ell\}\Big\}\Big] \cup \Big\{\{i,j\},\{k,\ell\}\Big\}$$

as a feasible matching such that all pairs are indifferent between M' and M'', and M'' agrees with M for all pairs matched before i and j in the algorithm and also for pairs i and j.

We repeat the above procedure for all such remaining i starting with M'' instead of M_2. Let M_3 be the final outcome of this procedure.

Construction of M_4: Observe that pairs of both types $101-011$ and $011-101$ could not have remained in the pool simultaneously until Step 4 of the algorithm (those would have been matched in Step 1). Thus, either $101-011$ or $011-101$ type pairs (or both) have been depleted in Step 1.

Take a pair i matched in Step 4 with the highest priority such that $j=M(i)\neq M_3(i)=k$ for some j,k. Let $M(k)=\ell$ for some ℓ.

Without loss of generality let i be of auxiliary type $011-101$ and j be of auxiliary type $101-011$ such that at least one is matched to donate a right lobe. Because i and j remained unmatched until Step 4 and because we showed that all pairs matched in Steps 1-3 have the same partners in both M and M_3, we have the following cases:

1. i can be of two types:
   (a) $011-101$: Then all pairs of its neighbor type $101-011$ are matched in Step 1 in M, and thus with other partners than i in M'. Hence, k has to be of type $101-010w$ or $100-011$.

(b) $011 - 100w$: Then all pairs if its neighbor types $100 - 011$ and $100 - 010w$ are matched in Step 1 in M, and hence with other partners than i in M'. Thus, k has to be of type $101 - 010w$ or $101 - 011$.

2. j can be of two types:

   (a) $101-011$: Then all pairs of its neighbor type $011-101$ are matched in Step 1 in M, and thus with other partners than j in M'. Hence, ℓ has to be of type $011 - 100w$, $010 - 100w$, or $010 - 101$.

   (b) $101 - 010w$: Then all pairs of its neighbor type $010 - 101$ are matched in Step 1 and those of $010 - 100w$ are matched in Steps 1 and 3 in M, and thus with other partners than j in M' . Thus, ℓ has to be of type $011 - 101$ or $011 - 100w$.

Now, regarding i and j together, 3 of the above possible 4 combinations can occur at the same time: 1.(a) and 2.(b), 1.(b) and 2.(a), or 1.(b) and 2.(b). Thus, in all cases $\{k, \ell\}$ is a feasible exchange such that either k and ℓ each improve or remain indifferent with respect to M_3. Since M_3 is Pareto efficient

$$M'' := \Big[M_3 \setminus \Big\{\{i,k\},\{j,\ell\}\Big\}\Big] \cup \Big\{\{i,j\},\{k,\ell\}\Big\}$$

leaves every pair indifferent with respect to M'. Moreover, M'' agrees with M for all pairs matched before i and j in the algorithm and also for pairs i and j. We repeat the above procedure for all such remaining i starting with M'' instead of M_3. Let M_4 be the final outcome of this procedure.

Construction of M_5: Take pair i of one of the auxiliary types $100 - 011$, $010 - 101$, or $001 - 110$ such that $M(i) = j \neq k = M_4(i)$ and it is the highest such priority pair. Let $M_4(j) = \ell$. Observe that by construction of $M_1, ..., M_4$, all pairs matched in Steps 1,...,4 in M have the same partners in M_4. Three cases are possible (See Figure **??**):

1. i is of type $100 - 011$: Then j is of types $011 - 101/011 - 100w$, $011 - 110$ or $001 - 100$. If j is of type $011 - 101/011 - 100w$, then pairs of types $101 - 010w$ and $101 - 011$, the other neighboring types of j besides i's type, are exhausted in Steps 1,3, and 4. If j is of type $011 - 110$ or $001 - 100$, then the other neighboring auxiliary types of j besides i's type, $110 - 011$ and $100 - 001$, are exhausted in Step 1. Thus ℓ has to be of the same type as i, that is type $100 - 011$.

2. i is of type $010 - 100w$ or $010 - 101$: Then j is of type $001 - 010$, $101 - 110$, or $101 - 010w/101 - 011$; observe in Figure **??** that all possible partners of j have been

exhausted in Step 1, Step 1, or Steps 1,3, and 4, respectively, except auxiliary type $010-101$. Thus, ℓ's and i's auxiliary types are the same.

3. i is of type $001-110$: Then j is of auxiliary type $110-011$, $110-101$, $100-001$, or $010-001$. Thus, all of its potential partners except those of type $001-110$ has been exhausted in Step 1. Thus i and ℓ have the same type.

In each case, $\{k,\ell\}$ is a feasible match, and

$$M'' := \Big[M_4 \setminus \Big\{\{i,k\},\{j,\ell\}\Big\}\Big] \cup \Big\{\{i,j\},\{k,\ell\}\Big\}$$

leaves every pair indifferent with respect to M'. Moreover, M'' agrees with M for all pairs matched before i and j in the algorithm and also for pairs i and j. We repeat the above procedure for all such remaining i starting with M'' instead of M_3. Let M_5 be the final outcome of this procedure.

We are ready to finish the proof of the theorem. By the algorithm, if a pair i of auxiliary type $X-Y \in \{100-011, 010-101, 001-110\}$ was not matched in Steps 1,...,5 then none of its neighbors except the ones belonging to types in $\{100-011, 010-101, 001-110\} \setminus \{X-Y\}$, i.e., the other two auxiliary types, have remained available until Step 6. Thus, to create a Pareto-efficient matching among the pairs remaining in this step, we need to maximize the number of exchanges among them. As the algorithm exactly does this and M_5 weakly Pareto-dominates M while all the matches prior to Step 6 are identical between M_5 and M, M_5 should be Pareto indifferent to M, proving that M is Pareto efficient. ■

Appendix E Formal Statement of General Mechanism and Proofs of Results in Section 6

Proof of Lemma 7: Suppose for a contradiction that the precedence digraph has a cycle:

$$X^0 - Y^0 \longrightarrow X^1 - Y^1 \longrightarrow \ldots \longrightarrow X^{n-1} - Y^{n-1} \longrightarrow X^0 - Y^0$$

where $n \geq 2$.

Note that for each $k \in \{0,1,\ldots,n-1\}$:

$$X^k - Y^k \longrightarrow X^{\bmod_n(k+1)} - Y^{\bmod_n(k+1)} \longrightarrow X^{\bmod_n(k+2)} - Y^{\bmod_n(k+2)}$$

implies that $X_3^k \leq Y_3^{\text{mod }_n(k+1)}$ and $Y_3^{\text{mod }_n(k+1)} < X_3^{\text{mod }_n(k+2)}$. Therefore $X_3^k < X_3^{\text{mod }_n(k+2)}$. That is, a patient along the cycle has a smaller size than the patient two steps ahead in the cycle. This can be used to obtain a contradiction in two separate cases:

Case 1 "n is even": $X_3^0 < X_3^2 < \ldots < X_3^{n-2} < X_3^0$.

Case 2 "n is odd": $X_3^0 < X_3^2 < \ldots < X_3^{n-1} < X_3^1 < X_3^3 < \ldots < X_3^{n-2} < X_3^0$. ■

E.1 Formal Mechanism

We will now formally describe the algorithm for the general sizes that will define an individually rational, Pareto efficient, and incentive compatible mechanism.

In the rest of the section, fix a liver-exchange pool $(\mathcal{I}, \tau)$, a right-lobe size function ρ, a preference profile $R \in \mathcal{R}$, and an arbitrary linear order over $\mathcal{I}$ that we will interpret as the priority ranking of the pairs. Let $\mathcal{I}_V$ denote the set of Category V pairs. By Lemma 7, the precedence digraph on $\mathcal{I}_V$ is acylic. By Lemma 8, we can fix a topological ordering of the precedence digraph on Category V pairs. Let $\mathcal{I}_V = \{i_1, \ldots, i_K\}$ be an enumeration of Category V pairs with respect to the topological ordering.

The algorithm is separated into four main steps in four subsections. The second one is the key step. The succinct version of the algorithm presented in Section 6 and the algorithm below are equivalent. The algorithm in the text have some redundancy as some willing pairs cannot participate in any exchange when transformed or they can participate in direct transplants before or after transformation as explained in Lemma 3.

E.1.1 Step 0

One of our objectives is to ensure that the outcome of the algorithm is individually rational. As an implication, we can immediately conclude using Lemma 3 that some categories of pairs have to be unmatched and some have to be involved in direct donation, independently of the rest of the liver-exchange pool. In Step 1 of the algorithm, we determine these matches:

1. Let all Category I pairs directly donate left lobe.
2. Leave all Category II pairs unmatched.
3. For any Category III pair i
   (a) If i is unwilling, leave i unmatched.

(b) If i is willing, let i directly donate right lobe.

4. Leave all unwilling Category IV pairs unmatched.

Let M_0 be the matching in $G_c[\mathcal{I}]$ that corresponds to the above direct donations. Formally:

$$M_0 = \Big\{\{i\} : i \in \mathcal{I} \text{ is Category I}\Big\} \cup \Big\{\{i\} : i \in \mathcal{I} \text{ is Category III and } R_i = R_i^w\Big\}.$$

E.1.2 Step 1

Let $\mathcal{I}_0$ denote the set of pairs whose match (possibly $\emptyset$) is not determined in Step 0. That is, $\mathcal{I}_0$ consists of all willing Category IV pairs, all Category V pairs, and all Category VI pairs.

In this part of the algorithm, we will define a sequence of graphs $G_0 = (\mathcal{I}_0, E_0)$, $G_1 = (\mathcal{I}_0, E_1)$,..., $G_K = (\mathcal{I}_0, E_K)$ on $\mathcal{I}_0$, where the edge sets are nondecreasing: $E_0 \subset E_1 \subset \ldots \subset E_K$. We will also define two nondecreasing sequences of subsets $\emptyset = \mathcal{J}_0 \subset \mathcal{J}_1 \subset \ldots \subset \mathcal{J}_K$ and $\emptyset = \tilde{\mathcal{J}}_0 \subset \tilde{\mathcal{J}}_1 \subset \ldots \subset \tilde{\mathcal{J}}_K$ of Category V pairs.

The initial graph G_0 has edges that represent two-way exchanges in which all participating Category V and VI pairs donate left-lobe, and all (willing) Category IV pairs donate right lobe. Formally,

$$E_0 = \left\{ \{i,j\} \in E_c[\mathcal{I}] : \begin{array}{c} i,j \in \mathcal{I}_0,\ i \neq j \\ e(i,j) = r \ \Rightarrow\ i \text{ is Category IV} \\ e(j,i) = r \ \Rightarrow\ j \text{ is Category IV} \end{array} \right\}$$

We will construct a maximal subset $\mathcal{J}_K$ of $\mathcal{I}_V$ matchable in G_K, the final graph of this step. We will commit to every pair in $\mathcal{J}_K$ that it will participate in a two-way exchange by donating left lobe, without specifying until later (the end of Step 3) which pair they enter the exchange with. The maximality of $\mathcal{J}_K$ means that, in addition to pairs in $\mathcal{J}_K$, we cannot match any pair $i \in \mathcal{I}_V \setminus \mathcal{J}_K$ via a two-way exchange where i donates left lobe. Therefore, given our commitment to pairs in $\mathcal{J}_K$, the only match possibility for the willing Category V pairs in $\mathcal{I}_V \setminus \mathcal{J}_K$ is when they donate right lobe. We let $\tilde{\mathcal{J}}_K \subseteq \mathcal{I}_V \setminus \mathcal{J}_K$ be such willing Category V pairs. We iteratively transform each of such pairs to include the exchanges they can donate right lobe to the compatibility graph. We proceed inductively for this construction:

We next formally present the algorithm for Step 2. Below, k runs though $1, \ldots, K$.

Step $1.k$**:** Is $\mathcal{J}_{k-1} \cup \{i_k\}$ matchable in G_{k-1}?

$$
\begin{array}{lllllllll}
\textbf{YES} & \text{Let} & \mathcal{J}_k & := & \mathcal{J}_{k-1} \cup \{i_k\} & \text{and} & \tilde{\mathcal{J}}_k & := & \tilde{\mathcal{J}}_{k-1} \\
\textbf{NO} & \text{Let} & \mathcal{J}_k & := & \mathcal{J}_{k-1} & \text{and} & \tilde{\mathcal{J}}_k & := & \begin{cases} \tilde{\mathcal{J}}_{k-1} \cup \{i_k\} & \text{if } R_{i_k} = R^w_{i_k} \\ \tilde{\mathcal{J}}_{k-1} & \text{otherwise.} \end{cases}
\end{array}
$$

Define the graph $G_k = (\mathcal{I}_0, E_k)$ by:

$$
E_k = \left\{ \{i,j\} \in E_c : \begin{array}{c} i, j \in \mathcal{I}_0, \ i \neq j \\ e(i,j) = r \ \Rightarrow \ i \text{ is Category IV or } i \in \tilde{\mathcal{J}}_k \\ e(j,i) = r \ \Rightarrow \ j \text{ is Category IV or } j \in \tilde{\mathcal{J}}_k \end{array} \right\}
$$

If $k < K$, go to Step 2.$(k+1)$. If $k = K$, Step 1 of the algorithm is over.

Note that the sets $\mathcal{J}_k$, $\tilde{\mathcal{J}}_k$, E_k defined inductively by the above algorithm are indeed nondecreasing in k as stated earlier in the section. At the end, the subalgorithm outputs a graph G_K and a partition of the Category V pairs into three sets: $\mathcal{J}_K$, $\tilde{\mathcal{J}}_K$, and $\mathcal{I}_V \setminus (\mathcal{J}_K \cup \tilde{\mathcal{J}}_K)$. The interpretation of these objects within the context of our mechanism is as follows:

We commit to all pairs in $\mathcal{J}_K$ that they will be part of a two-way exchange where they will donate left-lobe. Given this commitment, it is impossible for any other Category V pairs to be part of a two-way exchange where they will donate left-lobe (See Lemma 10 in Section E.2.1). Among members of $\mathcal{I}_V \setminus \mathcal{J}_K$, $\tilde{\mathcal{J}}_K$ is the set of pairs who are willing to donate right lobe; and $\mathcal{I}_V \setminus (\mathcal{J}_K \cup \tilde{\mathcal{J}}_K)$ is the set of pairs who are not willing to donate right lobe. As a result, we conclude with certainty at the end of Step 2 that pairs in $\mathcal{I}_V \setminus (\mathcal{J}_K \cup \tilde{\mathcal{J}}_K)$ will be unmatched. Pairs in $\tilde{\mathcal{J}}_K$ may be part of two-way exchanges in which they donate right lobe. Therefore, the relevant graph of pairwise exchanges becomes G_K obtained from G_0 by adding two-way exchanges in which right-lobe donation is allowed for members of $\tilde{\mathcal{J}}_K$ as well as willing Category IV pairs.

E.1.3 Step 2

In this step, we will determine a maximal subset of $\mathcal{I}_0$ that contains $\mathcal{J}_K$ and is matchable in G_K. Let N denote the number of pairs in $\mathcal{I}_0 \setminus \mathcal{J}_K$. Enumerate those pairs with respect to the priority order as $\{i^*_1, \ldots, i^*_N\}$. We will define an increasing sequence of subsets of this set $\mathcal{J}^*_0 \subset \mathcal{J}^*_1 \subset \ldots \mathcal{J}^*_N$. Let $\mathcal{J}^*_0 := \emptyset$. Below, n runs through $1, \ldots, N$.

Step $2.n$**:** Is $\mathcal{J}_K \cup \mathcal{J}^*_{n-1} \cup \{i^*_n\}$ matchable in G_K?

YES Let $\mathcal{J}^*_n := \mathcal{J}^*_{n-1} \cup \{i^*_n\}$

NO Let $\mathcal{J}_n^* := \mathcal{J}_{n-1}^*$

If $n < N$, go to Step 3.$(n+1)$. If $n = N$, Step 3 of the algorithm is over.

The algorithm above returns a set $\mathcal{J}_N^*$, with the property that $\mathcal{J}_K \cup \mathcal{J}_N^*$ is a maximal subset of $\mathcal{I}_0$ that contains $\mathcal{J}_K$ and is matchable in G_K.

Let M_2 be any matching in G_K such that $M_2(i) \neq \emptyset$ for all $i \in \mathcal{J}_K \cup \mathcal{J}_N^*$. The following Lemma summarizes how different pairs in $\mathcal{I}$ are matched under such an M_2:

Lemma 9 *Suppose that*

$$M_2 \text{ is a matching in } G_K \text{ s.t. } \forall i \in \mathcal{J}_K \cup \mathcal{J}_N^* : M_2(i) \neq \emptyset. \tag{1}$$

Then, under M_2, the pairs in $\mathcal{I}_0$ are matched as follows:

1. *$\forall i \in \mathcal{J}_K$: i takes part in a two-way exchange donating left lobe.*
2. *$\forall i \in \mathcal{J}_N^*$, one of the following holds:*
   (a) *i is a willing Category IV pair, and takes part in a two-way exchange donating right lobe.*
   (b) *$i \in \tilde{\mathcal{J}}_K$, and takes part in a two-way exchange donating right lobe.*
   (c) *i is a Category VI pair, and takes part in a two-way exchange donating left lobe.*
3. *$\forall i \in \mathcal{I}_0 \setminus (\mathcal{J}_K \cup \mathcal{J}_N^*)$: i is unmatched.*

There could be multiple matchings that satisfy Condition (1), but as implied by Lemma 9, all of them match pairs in $\mathcal{I}_0$ in the same way. An implication is that every pair in i is indifferent among all matchings that satisfy Condition (1). We leave it open how the algorithm selects the matching M_2 as long as it satisfies Condition (1).

Another implication of Lemma 9 is that in such an M_2, all Category V pairs in $\mathcal{J}_K$ take part in two way exchanges donating left lobe, some Category V pairs in $\tilde{\mathcal{J}}_K$ take part in two way exchanges donating right lobe, and all other Category V pairs are unmatched.

E.1.4 Step 3

In this step, we let any Category VI pair who is unmatched in previous steps directly donate right-lobe to themselves. Let M_3 be a matching in $G_c[\mathcal{I}]$ that corresponds to these direct donations. Formally:

$$M_3 = \Big\{\{i\} : i \in \mathcal{I}_0 \setminus \mathcal{J}_N^*, i \text{ is Category VI, and } R_i = R_i^w\Big\}.$$

The algorithm is over at the end of this step returning the matching:

$$M = M_0 \cup M_2 \cup M_3$$

Given a priority order over pairs, and a topological ordering of the precedence order over Category V types, define the **general left&right-lobe priority mechanism** $f^{l\&r} : \mathcal{R} \to \mathcal{M}_c[\mathcal{I}]$, by: For any preference profile $R \in \mathcal{R}$, $f^{l\&r}(R)$ is the matching computed by the algorithm in Section E.1.

E.2 Proofs

E.2.1 Proof of Lemma 9

We start by proving a Lemma that we will use in proving Lemma 9.

Lemma 10 *Let $j \in \mathcal{I}_V \setminus \mathcal{J}_K$. Then, there is no matching M in G_K such that $e(i, M(i)) = l$ for all $i \in \mathcal{J}_K \cup \{j\}$.*

Proof: Take any $j \in \mathcal{I}_V \setminus \mathcal{J}_K$. Since j is Category V, $j = i_k$ for some $k \in \{1, \ldots, K\}$. Since $j \notin \mathcal{J}_K$, by Step 1 of the algorithm, $\mathcal{J}_{k-1} \cup \{j\}$ is not matchable in G_{k-1}.

Suppose for a contradiction that there exists a matching M in G_K such that $e(i, M(i)) = l$ for all $i \in \mathcal{J}_K \cup \{j\}$. Define a (smaller) matching M' in G_K by:

$$M' := \Big\{\{i, M(i)\} : i \in \mathcal{J}_{k-1} \cup \{j\}\Big\}$$

Note that M' is not a matching in G_{k-1}, otherwise $\mathcal{J}_{k-1} \cup \{j\}$ would be matchable in G_{k-1}, contradicting the previous paragraph. Therefore, there exists $i \in \mathcal{J}_{k-1} \cup \{j\}$ such that:

$$\{i, M(i)\} \in E_K \setminus E_{k-1}.$$

Since $e(i, M(i)) = l$, this is only possible if $e(M(i), i) = r$ and $M(i) \in \tilde{\mathcal{J}}_K \setminus \tilde{\mathcal{J}}_{k-1}$.

Note that $i \in \mathcal{J}_{k-1} \cup \{j\}$and $M(i) \in \tilde{\mathcal{J}}_K \setminus \tilde{\mathcal{J}}_{k-1}$ imply that $i = i_l$ and $M(i) = i_m$ for some $l, m \in \{1, \ldots, K\}$ such that $l \leq k \leq m$.

Note that $e(i, M(i)) = l$ and $e(M(i), i) = r$ imply that $M(i) \to i$ in the precedence digraph over Category V pairs. Then, $M(i) = i_m$ must be ranked higher than $i = i_l$ with respect to the topological ordering, i.e., $m < l$, a contradiction. ■

In the following, fix a matching M_2 that satisfies Condition (1) in Lemma 9. We prove Lemma 9 in four parts.

Proof of Parts 1 and 2.(c) in Lemma 9

Proof: By definition of E_K, all edges in $G_K = (\mathcal{I}_0, E_K)$ correspond to two-way exchanges where the only pairs who might donate right lobe are those in $\tilde{\mathcal{J}}_K$ and those that are (willing) Category IV. Since M_2 is a matching in G_K and since $\mathcal{J}_K$ and $\tilde{\mathcal{J}}_K$ are disjoint, all pairs in $\mathcal{J}_K$ must be matched through two-way exchanges where they donate left lobe. Similarly, all matched Category VI pairs must be matched through two-way exchanges where they donate left lobe. ■

Proof of Part 2.(a) in Lemma 9

Proof: Suppose for a contradiction that i is a willing Category IV pair, and that in M_2, i takes part in a two-way exchange with j by donating left-lobe. Let $X - Y = \tau(i)$ and $U - V = \tau(j)$. Since i is Category IV, $X > Y$. Since $j \in \mathcal{I}_0$, it is enough to consider two cases:

Case 1 "j is a (willing) Category IV pair or a Category V pair": Since i and j take part in a two-way exchange where i donates left lobe, and j donates left or right lobe, we have $\rho(V) \geq X > Y \geq U$. This contradicts $\rho(V) \ngeq U$ since j is Category IV or Category V.

Case 2 "j is a Category VI pair": Since i and j take part in a two-way exchange where both i and j donate left lobe, we ave $V \geq X > Y \geq U$. This contradicts $V \ngeq U$ since j is Category VI. ■

Proof of Part 2.(b) and "Part 2 ⇒ 2.(a), 2(b), or 2.(c)" in Lemma 9

Proof: To see part 2.(b), let $j \in \mathcal{J}_N^* \cap \tilde{\mathcal{J}}_K$. Since $j \in \tilde{\mathcal{J}}_K$, we have that $j \in \mathcal{I}_V \setminus \mathcal{J}_K$. By Part 1 of Lemma 9, M_2 is a matching in G_K such that $e(i, M_2(i)) = l$ for all $i \in \mathcal{J}_K$. So by Lemma 10, $e(j, M_2(j)) \neq l$. Since $j \in \mathcal{J}_N^*$, j is matched in M_2, so j must be part of a two-way exchange by donating right lobe.

To see that "Part 2 ⇒ 2.(a), 2(b), or 2.(c)," it is enough to show that there is no $j \in \mathcal{J}_N^*$ such that j is Category V and $j \notin \tilde{\mathcal{J}}_K$. Suppose for a contradiction that there exists such a j. Then, $j \in \mathcal{I}_V \setminus \mathcal{J}_K$. By Part 1 of Lemma 9, M_2 is a matching in G_K such that $e(i, M_2(i)) = l$

for all $i \in \mathcal{J}_K$. So by Lemma 10, $e(j, M_2(j)) \neq l$. Since $j \in \mathcal{J}_N^*$, j is matched in M_2, so j must be part of a two-way exchange by donating right lobe, i.e.: $e(j, M(j)) = r$. Since $\{j, M(j)\} \in E_K$ and $j \in \mathcal{I}_V \setminus \tilde{\mathcal{J}}_K$, this contradicts the definition of E_K. ■

Proof of Part 3 in Lemma 9

Proof: Take any $i \in \mathcal{I}_0 \setminus (\mathcal{J}_K \cup \mathcal{J}_N^*)$. Since $i \in \mathcal{I}_0 \setminus \mathcal{J}_K$, there exists $n \in \{1, \ldots, N\}$ such that $i = i_n^*$. Since $i = i_n^* \notin \mathcal{J}_N^*$, at Step 2.$n$ of the algorithm, $\mathcal{J}_K \cup \mathcal{J}_{n-1}^* \cup \{i_n^*\}$ was not matchable in G_K. Since $\mathcal{J}_{n-1}^* \subset \mathcal{J}_N^*$, this also implies that $\mathcal{J}_K \cup \mathcal{J}_N^* \cup \{i_n^*\}$ is not matchable in G_K. Therefore, M_2 must leave $i = i_n^*$ unmatched. ■

E.2.2 Proof of Theorem 3

Given a priority order over all pairs, a topological order of the precedence digraph of Category V pairs, and a willing ness profile, the Pareto indifference of all pairs under all possible outcomes of $f^{l\&r}$ follows from Lemma 9. We will prove the other three properties of $f^{l\&r}$ in three separate Lemmas.

Lemma 11 *The general left&right-lobe priority mechanism is individually rational.*

Proof: Take any preference profile $R \in \mathcal{R}$ and let $M = f^{l\&r}(R)$. By Lemma 3, individual rationality condition is satisfied for all pairs whose match is determined in Step 0 of the algorithm, i.e., Category I–III pairs and unwilling Category IV pairs.

Next, take any willing Category IV or Category V pair i such that $M(i) \neq \emptyset$. Note that i must be matched in Step 2 of the algorithm, so $M R_i \emptyset$ by parts 1, 2(a), and 2(b) of Lemma 9. This is enough to conclude that individual rationality condition is satisfied for all Category IV and V pairs, since they cannot directly donate.

Finally, consider any Category VI pair i. Note that i is either matched in Step 2 by being part of a two-way exchange donating left-lobe (by part 2(c) of Lemma 9), or is willing and directly donates right lobe in Step 3, or is unwilling and left unmatched in Step 3. Therefore, the individual rationality condition is also satisfied for i. ■

Lemma 12 *The general left&right-lobe priority mechanism is incentive compatible.*

Proof: Note that all two-way exchanges are determined at Step 2 of the algorithm. By Lemma 9, only Category V and Category VI pairs may be part of a two-way exchange by donating left lobe. We will show that for any Category V or Category VI pair i and $R_{-i} \in \prod_{j\neq i} \mathcal{R}_j$:

$$\text{Under } f(R_i^w, R_{-i}),\ i \text{ participates in a two-way exchange by donating left lobe}$$

$$\Updownarrow$$

$$\text{Under } f(R_i^u, R_{-i}),\ i \text{ participates in a two-way exchange by donating left lobe,}$$

which will prove that $f^{l\&r}$ is incentive compatible by Lemmas 4 and 11. In the rest of the proof, denote the objects defined by the algorithm under the preference profile (R_i^w, R_{-i}) by using the superscript w, and those defined by the algorithm under the preference profile (R_i^u, R_{-i}) by using the superscript u.

First consider the case where i is Category V. Then $i = i_k$ for some $k \in \{1, \ldots, K\}$. Up to the end of Step 1.$(k-1)$, both versions of the algorithm run in exactly the same way, since they do not depend on the willingness announcement of $i = i_k$. This implies that $\mathcal{J}_{k-1}^w = \mathcal{J}_{k-1}^u$ and $G_{k-1}^w = G_{k-1}^u$. Then,

$$e(i, M_3^w(i)) = l \;\;\Leftrightarrow\;\; i \in \mathcal{J}_K^w \;\;\Leftrightarrow\;\; \mathcal{J}_{k-1}^w \cup \{i_k\} \text{ is matchable in } G_{k-1}^w$$

$$\Leftrightarrow\;\; \mathcal{J}_{k-1}^u \cup \{i_k\} \text{ is matchable in } G_{k-1}^u \;\;\Leftrightarrow\;\; i \in \mathcal{J}_K^u \;\;\Leftrightarrow\;\; e(i, M_3^u(i)) = l$$

where the first and last equivalences above follow from Lemma 9 and i being Category V; the second and fourth equivalences follow from the definition of the Step 1.k of the algorithm and $i = i_k$; and the middle equivalence follows from $\mathcal{J}_{k-1}^w = \mathcal{J}_{k-1}^u$ and $G_{k-1}^w = G_{k-1}^u$.

Next consider the case where i is Category VI. Since the algorithm is independent of the willingness announcenment of Category VI pairs until the end of Step 2, we have $\mathcal{J}_N^{*w} = \mathcal{J}_N^{*u}$. Then,

$$e(i, M_3^w(i)) = l \;\;\Leftrightarrow\;\; i \in \mathcal{J}_N^{*w} \;\;\Leftrightarrow\;\; i \in \mathcal{J}_N^{*u} \;\;\Leftrightarrow\;\; e(i, M_3^u(i)) = l$$

where the the first and last equivalences above follow from Lemma 9 and i being Category VI; and the middle equivalence follows from $\mathcal{J}_N^{*w} = \mathcal{J}_N^{*u}$. ■

Lemma 13 *The general left&right-lobe priority mechanism is Pareto efficient.*

Proof: Take any preference profile $R \in \mathcal{R}$ and let $M = f^{l\&r}(R)$. Let M' be a matching in G_c such that $M' R_i M$ for all $i \in \mathcal{I}$. To conclude that $f^{l\&r}$ is Pareto efficient, it is enough to show that all pairs are indifferent between M and M'. Below, we will prove this indifference separately for pairs in $\mathcal{I} \setminus \mathcal{I}_0$, $\mathcal{J}_K$, $\mathcal{J}_N^*$, and $\mathcal{I}_0 \setminus (\mathcal{J}_K \cup \mathcal{J}_N^*)$. Note first that since all pairs weakly prefer M' to M and M is individually rational at R (by Lemma 11), the matching M' is also individually rational at R.

Claim 1: Pairs in $\mathcal{I} \setminus \mathcal{I}_0$ are indifferent between M and M'

By individual rationality of M' and Lemma 3, M' matches Category I–III pairs and unwilling Category IV pairs in exactly the same way as M, as determined in Step 0 of the algorithm.

Claim 2: Pairs in $\mathcal{J}_K$ are indifferent between M and M'

Take any pair $i \in \mathcal{J}_K$. By part 1 of Lemma 9, in M, i takes part of a two-way exchange donating left lobe. Since $M' R_i M$, and direct donation is not possible for Category V pairs, i must also take part in a two-way exchange donating left lobe at M'.

Let M'' be the edges in M' corresponding to the *two-way exchanges* among pairs in $\mathcal{I}_0$, i.e.:

$$M'' = \Big\{\{i,j\} \in M' : i,j \in \mathcal{I}_0 \text{ and } i \neq j\Big\}.$$

Since Category VI pairs are the only pairs in $\mathcal{I}_0$ that can directly donate, we have:

$$M''(i) = \begin{cases} \emptyset & \text{if } i \text{ is Category VI and } M'(i) = i \\ M'(i) & \text{otherwise.} \end{cases}$$

for all $i \in \mathcal{I}_0$. Note that since $M'' \subset M'$, M'' is also a matching in G_c.

Claim 3: M'' is a matching in G_K

To see that $M'' \subset E_K$, take any $\{i,j\} \in M''$. Then, $i \neq j$ by definition of M''; and $\{i,j\} \in E_c$, since M'' is a matching in G_c. Suppose that $e(i,j) = r$, i.e., in M'', i is part of a two-way exchange by donating right lobe. Then, $i \notin \mathcal{J}_K$ because as argued in Claim 2, the pairs in $\mathcal{J}_K$ take part in two-way exchanges donating left lobe at M', so also at M''. Also $i \notin \mathcal{I}_V \setminus (\mathcal{J}_K \cup \tilde{\mathcal{J}}_K)$, because by Step 1 of the algorithm any pair in $\mathcal{I}_V \setminus (\mathcal{J}_K \cup \tilde{\mathcal{J}}_K)$ is unwilling, so individual rationality of M' implies that in M', hence in M'', they cannot be part of a two-way exchange donating right-lobe. Finally, note that i is not Category VI since by individual rationality of M' and Lemma 3, in M', hence in M'', Category VI pairs cannot be part of a two-way exchange donating right-lobe. Therefore, i is a (willing) Category IV pair or belongs to $\tilde{\mathcal{J}}_K$. We proved that

$$e(i,j) = r \;\Rightarrow\; i \text{ is Category IV or } i \in \tilde{\mathcal{J}}_K$$

The proof of the other implication:

$$e(j,i) = r \;\Rightarrow\; j \text{ is Category IV or } j \in \tilde{\mathcal{J}}_K$$

is exactly symmetric, hence omitted. So $\{i,j\} \in E_K$.

Claim 4: Pairs in $\mathcal{J}_N^*$ are indifferent between M and M'

We will prove the Claim separately for Category IV, V and VI pairs:

- *Claim 4.IV:* Take any Category IV pair $j \in \mathcal{J}_N^*$. By part 2 of Lemma 9, j is a willing Category IV pair taking part in a two-way exchange donating right lobe at M. By individual rationality of M' and Lemma 3, j is unmatched or takes part in a two-way exchange donating right lobe at M'. Therefore, $M' R_j M$ implies that j is part of a two-way exchange donating right lobe also at M'.

- *Claim 4.V:* Take any Category V pair $j \in \mathcal{J}_N^*$. By part 2 of Lemma 9, $j \in \tilde{\mathcal{J}}_K \cap \mathcal{J}_N^*$ and j is part of two-way exchange donating right lobe at M. As argued in Claim 2, the pairs in $\mathcal{J}_K$ take part in two-way exchanges donating left lobe at M', hence also at M''. By Claim 3, M'' is a matching in G_K, so by Lemma 10 and $j \in \mathcal{I}_V \setminus \mathcal{J}_K$, j is not part of a two-way exchange donating left lobe at M'', hence also in M'. Being Category V, j cannot directly donate, so $M' R_j M$ implies that j is part of a two-way exchange donating right lobe also at M'.

- *Claim 4.VI:* Take any Category VI pair $j \in \mathcal{J}_N^*$. By part 2 of Lemma 9, j is part of two-way exchange donating left lobe at M. Being Category VI, j cannot directly donate left lobe, so $M' R_j M$ implies that j is part of a two-way exchange donating right lobe also at M'.

The argument of Claim 4 also establishes that M'' is a matching in G_K such that $M''(i) \neq \emptyset$ for all $i \in \mathcal{J}_K \cup \mathcal{J}_N^*$. So by Lemma 9, $M''(i) = \emptyset$ for all $i \in \mathcal{I}_0 \setminus (\mathcal{J}_K \cup \mathcal{J}_N^*)$.

Claim 5: Pairs in $\mathcal{I}_0 \setminus (\mathcal{J}_K \cup \mathcal{J}_N^*)$ are indifferent between M and M'

Take any $i \in \mathcal{I}_0 \setminus (\mathcal{J}_K \cup \mathcal{J}_N^*)$. If i is not Category VI, then i cannot directly donate so $M''(i) = \emptyset$ implies $M'(i) = \emptyset$. Note that i is also unmatched under M, since i is not matched in Step 2 (by Lemma 9) nor Step 4 (since i is not Category VI) of the algorithm. Therefore, such pairs are indifferent between M and M'. Next suppose that i is Category VI. Then, if i is unwilling $M''(i) = \emptyset$ and individual rationality of M' imply that $M'(i) = \emptyset$. Note that i is also unmatched under M, since i is not matched in Step 2 (by Lemma 9) nor Step 3 (since i is not willing) of the algorithm. Therefore, such pairs are also indifferent between M and M'. Finally, suppose that i is a willing Category VI pair. Then, $M''(i) = \emptyset$ and individual rationality of M' imply that $M'(i) = i$, i.e., i directly donates right lobe. Note that i also directly donates right lobe under M, since i is not matched in Step 2 (by Lemma 9), but is matched in Step 3 (since i is willing) of the algorithm. Therefore, such pairs are also indifferent between M and M'. ■

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